15. *On a Theorem of Landau*

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§1. **Introduction.** Let $\rho=\beta+i\gamma$ run over the non-trivial zeros of the Riemann zeta function $\zeta(s)$. Landau [7] has shown that for fixed $x>1$,

$$\sum_{0<\gamma\leqq T} x^{\rho}=-\frac{T}{2\pi}\Lambda(x)+O(\log T),$$

where $T>T_0$, $\Lambda(x)=\log p$ if $x=p^k$ with a prime number p and an integer $k\geqq 1$ and $\Lambda(x)=0$ otherwise. Recently, Gonek [5], [6] has clarified the dependence on x in Landau's theorem as follows:

$$\sum_{0<\gamma\leqq T} x^{\rho}=-\frac{T}{2\pi}\Lambda(x)+O(x\log(2x)\cdot\log\log(3x))+O(x\log(2T))$$
$$+O(\log x\cdot \operatorname{Min}(T, x/\langle x\rangle))+O(\operatorname{Min}(\log T/\log x, T\log T)),$$

where $T, x>1$ and $\langle x\rangle$ is the distance from x to the nearest prime power other than x itself. On the other hand, in Corollary 3 of [1], the author has refined Landau's theorem under the Riemann Hypothesis as follows; for fixed $x>1$,

$$\sum_{0<\gamma\leqq T} x^{(1/2)+i\gamma}=-\frac{T}{2\pi}\Lambda(x)+\frac{x^{(1/2)+iT}\log(T/2\pi)}{2\pi i\cdot\log x}+O\left(\frac{\log T}{\log\log T}\right).$$

The author has also given in Theorem 1′ of [2] a result on the dependence on x which has been suitable for our applications. The purpose of the present article is to refine all of these results under the Riemann Hypothesis, which we shall assume below. We shall prove the following theorem by improving the author's proof in [1].

Theorem. *For $x>1$ and $T>T_0$, we have*

$$\sum_{0<\gamma\leqq T} x^{(1/2)+i\gamma}=-\frac{T}{2\pi}\Lambda(x)+\sqrt{x}\cdot M(x,T)-\frac{x}{2\pi i}F(x,T)$$
$$+O(x\log(2x))+O\left(\log x\operatorname{Min}\left(T,\frac{x}{\langle x\rangle}\right)\right)+O\left(x\sqrt{\frac{\log T}{\log\log T}}\right)$$
$$+O\left(x^{(1/2)+(1/\log\log T)}\cdot\log(2x)\cdot\frac{\log T}{\log\log T}\right),$$

where

$$M(x,T)\equiv\frac{1}{2\pi}\int_1^T x^{it}\log\frac{t}{2\pi}\,dt$$
$$=\begin{cases}\dfrac{x^{iT}\log(T/2\pi)}{2\pi i\log x}+O\left(\dfrac{1}{\log x}+\dfrac{1}{\log^2 x}\right) & \textit{if } \dfrac{1}{\log T}\ll\log x\\[2ex] O\left(\dfrac{\log T}{\log x}\right) & \textit{if } \dfrac{1}{T}\ll\log x\ll\dfrac{1}{\log T}\\[2ex] O(T\log T) & \textit{if } \log x\ll\dfrac{1}{T}\end{cases}$$

and

$$F(x,T)\equiv\int_1^{x^a}\Big(\sum_{x-y\le k\le x}\Lambda(k)\Big(\Big(\frac{x}{k}\Big)^{iT}-1\Big)-\sum_{x\le k\le x+y}\Lambda(k)\Big(\Big(\frac{x}{k}\Big)^{iT}-1\Big)\Big)\frac{dy}{y^2}$$
$$=O(\log(2x)\log\log(3x)),$$

a being any positive number <1.

§2. **Proof of Theorem.** By the Riemann- von Mangoldt formula, we get as in p. 103 of [1],

$$\sum_{0<\gamma\le T}x^{i\gamma}=M(x,T)-i\log x\int_C^T\cos(t\log x)S(t)dt$$
$$+\log x\int_C^T\sin(t\log x)S(t)dt+x^{iT}S(T)+O(1)$$
$$=M(x,T)-i\log x\,S_1+\log x\,S_2+x^{iT}S(T)+O(1),$$

say, where we put $S(t)=(1/\pi)\arg\zeta((1/2)+it)$ as usual, C is some positive constant and we suppose that $T>T_0$. As in p. 104 of [1],

$$S_1=\mathrm{Im}\Big(\frac{1}{\pi i}\int_{(1/2)+iC}^{(1/2)+iT}\cos\Big(-i\Big(z-\frac{1}{2}\Big)\log x\Big)\log\zeta(z)dz\Big)$$
$$=\mathrm{Im}\Big(\frac{1}{\pi i}\Big(\int_{1+\delta+iC}^{1+\delta+iT}-\int_{(1/2)+iT}^{1+\delta+iT}+\int_{(1/2)+iC}^{1+\delta+iC}\Big)\cos\Big(-i\Big(z-\frac{1}{2}\Big)\log x\Big)\cdot\log\zeta(z)dz\Big)$$
$$=\mathrm{Im}\Big(\frac{1}{\pi i}(S_3+S_4+S_5)\Big),\ \text{say, where we put } \delta=1/\log(3x).$$

$$S_4=-\frac{1}{2}\int_{1/2}^{1+\delta}(x^{(\sigma-(1/2)+iT)}+x^{-(\sigma-(1/2)+iT)})\log\zeta(\sigma+iT)d\sigma$$
$$=-\frac{1}{2}x^{-(1/2)+iT}\int_{1/2}^{1+\delta}x^{\sigma}\log\zeta(\sigma+iT)d\sigma+O\Big(\int_{1/2}^{1+\delta}|\log\zeta(\sigma+iT)|d\sigma\Big)$$
$$=-\frac{1}{2}x^{-(1/2)+iT}S_6+O(S_7),\quad\text{say.}$$

To estimate S_6 and S_7, we use the following lemma whose proof can be seen in p. 529 of [3].

Lemma.

$$\int_{1/2}^{3}|\log\zeta(\sigma+iT)|d\sigma\ll\frac{\log T}{\log\log T}.$$

We put $\sigma_1=(1/2)+(1/\log Y)$ with $Y=\log T\cdot\log\log T$.

$$S_6=\Big(\int_{1/2}^{\sigma_1}+\int_{\sigma_1}^{1+\delta}\Big)x^{\sigma}\log\zeta(\sigma+iT)d\sigma=S_8+S_9,\quad\text{say.}$$

$$S_8\ll x^{\sigma_1}\int_{1/2}^{\sigma_1}|\log\zeta(\sigma+iT)|d\sigma\ll x^{\sigma_1}\frac{\log T}{\log\log T}.$$

$$S_9=\frac{x^{1+\delta}}{\log x}\log\zeta(1+\delta+iT)-\frac{x^{\sigma_1}}{\log x}\log\zeta(\sigma_1+iT)$$
$$-\int_{\sigma_1}^{1+\delta}\frac{x^{\sigma}}{\log x}\frac{\zeta'}{\zeta}(\sigma+iT)d\sigma=S_{10}+S_{11}+S_{12},\quad\text{say.}$$

$$S_{10}\ll\frac{x}{\log x}\log\log(3x).$$

By 14.14.4 of [9], we get

$$S_{11}\ll\frac{x^{\sigma_1}}{\log x}\frac{\log T}{\log\log T}.$$

To treat S_{12}, we use Selberg's expression of $(\zeta'/\zeta)(s)$ (cf. [8] and 14.21.4 of [9]) as follows. For $\sigma\geqq\sigma_1$,

$$\frac{\zeta'}{\zeta}(\sigma+iT)=-\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma+iT}}+O\left(Y^{(1/2)-\sigma}\left|\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1+iT}}\right|\right)+O(Y^{(1/2)-\sigma}\log T),$$

where we put

$$\Lambda_Y(n)=\begin{cases}\Lambda(n) & \text{for } 1\leqq n\leqq Y\\ \Lambda(n)\dfrac{\log(Y^2/n)}{\log Y} & \text{for } Y\leqq n\leqq Y^2.\end{cases}$$

$$\log x\cdot S_{12}\ll\sum_{n<Y^2}\Lambda_Y(n)\int_{\sigma_1}^{1+\delta}(x/n)^\sigma d\sigma$$
$$+Y^{1/2}\left|\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1+iT}}\right|\left|\int_{\sigma_1}^{1+\delta}(x/Y)^\sigma d\sigma+Y^{1/2}\log T\cdot\int_{\sigma_1}^{1+\delta}(x/Y)^\sigma d\sigma\right.$$
$$=S_{13}+S_{14}+S_{15},\quad\text{say.}$$

$$S_{13}\ll x\sum_{n<Y^2,x\geqq n}\frac{\Lambda_Y(n)}{n^{1+\delta}}+x^{\sigma_1}\sum_{n<Y^2,n>x}\frac{\Lambda_Y(n)}{n^{\sigma_1}}=S_{16}+S_{17},\quad\text{say.}$$

$$S_{16}\ll x\log(3x).$$

$$x^{-\sigma_1}\cdot S_{17}\ll\sum_{n\leqq Y}\frac{\Lambda(n)}{n^{\sigma_1}}+\sum_{Y<n\leqq Y^2}\frac{\Lambda(n)\log(Y^2/n)}{n^{\sigma_1}\log Y}\ll\frac{Y}{\log Y}\ll\log x\frac{\log T}{\log\log T},$$

provided that $\log x\geqq(1/2)\log\log T$.

$$S_{14}=\begin{cases}O\left(\dfrac{Y^{3/2}}{\log Y}\right)=O\left(x\sqrt{\dfrac{\log T}{\log\log T}}\right) & \text{if } (1/2)Y\leqq x\leqq 2Y\\ O\left(\dfrac{Y^{(1/2)-\delta}x}{\log Y}\right)=O\left(x\sqrt{\dfrac{\log T}{\log\log T}}\right) & \text{if } x>2Y\\ O\left(\dfrac{Yx^{\sigma_1}}{\log Y\cdot\log(Y/x)}\right)=O\left(x^{\sigma_1}\cdot\log(3x)\cdot\dfrac{\log T}{\log\log T}\right) & \text{if } x<(1/2)Y.\end{cases}$$

$$S_{15}=\begin{cases}O(Y^{1/2}\log T)=O\left(x\sqrt{\dfrac{\log T}{\log\log T}}\right) & \text{if } (1/2)Y\leqq x\leqq 2Y\\ O(Y^{-(1/2)-\delta}x\log T)=O\left(x\sqrt{\dfrac{\log T}{\log\log T}}\right) & \text{if } x>2Y\\ O\left(\dfrac{x^{\sigma_1}\log T}{\log(Y/x)}\right)=O\left(x^{\sigma_1}\cdot\log(3x)\cdot\dfrac{\log T}{\log\log T}\right) & \text{if } x<(1/2)Y.\end{cases}$$

Hence we get if $\log x\geqq(1/2)\log\log T$,

$$S_{12}\ll x+x^{\sigma_1}\cdot\frac{\log T}{\log\log T}+\frac{x}{\log x}\sqrt{\frac{\log T}{\log\log T}}.$$

If $\log x<(1/2)\log\log T$, then we take $Y=\log T$ and we get by the same argument as above

$$S_{12}\ll\frac{1}{\log x}\left(x\log(3x)+x^{\sigma_1}\cdot\frac{\log T}{\log\log T}\right).$$

In any case, we get

$$S_4\ll x^{(1/\log\log T)}\cdot\frac{\log T}{\log\log T}\left(1+\frac{1}{\log x}\right)+\sqrt{x}\,\frac{\log(3x)}{\log x}+\frac{\sqrt{x}}{\log x}\sqrt{\frac{\log T}{\log\log T}}.$$

Similarly, S_5 has the same upper bound.

Finally, we shall evaluate S_3.

$$S_3=\frac{1}{2}ix^{(1/2)+\delta}\Big(\sum_{k\leq(1/2)x}+\sum_{(1/2)x<k\leq N-1}+\sum_{k=N}+\sum_{N+1\leq k<2x}+\sum_{k\geq 2x}\Big)$$
$$\times\frac{\Lambda(k)}{k^{1+\delta}\log k}\int_C^T (x/k)^{it}dt+O(1)=S_{18}+S_{19}+S_{20}+S_{21}+S_{22}+O(1),\quad \text{say,}$$

where N is the nearest integer to x.

$$S_{18}\ll\sqrt{x}\sum_{l=0}^{\infty}\Big(\frac{1}{\log x}\Big)^{l+1}\sum_{k\leq(1/2)x}\frac{\Lambda(k)}{k^{1+\delta}}(\log k)^{l-1}\ll\sqrt{x}\,\frac{\log\log(3x)}{\log x}.$$

Similarly, we get

$$S_{22}\ll\sqrt{x}\,\frac{\log\log(3x)}{\log x}.$$

$$S_{19}=\frac{1}{2}x^{(1/2)+\delta}\cdot\sum_{(1/2)x<k\leq N-1}\frac{\Lambda(k)}{k^{\delta}\log k}\cdot\frac{1}{x-k}\Big(\Big(\frac{x}{k}\Big)^{iT}-\Big(\frac{x}{k}\Big)^{iC}\Big)+O(\sqrt{x})$$
$$=\frac{1}{2}\frac{\sqrt{x}}{\log x}\int_1^{(1/2)x}\Big(\sum_{x-y\leq k\leq x}\Lambda(k)\Big(\Big(\frac{x}{k}\Big)^{iT}-\Big(\frac{x}{k}\Big)^{iC}\Big)\frac{dy}{y^2}+O(\sqrt{x})$$
$$=\frac{1}{2}\frac{\sqrt{x}}{\log x}\int_1^{x^a}\Big(\sum_{x-y\leq k\leq x}\Lambda(k)\Big(\Big(\frac{x}{k}\Big)^{iT}-1\Big)\frac{dy}{y^2}+O(\sqrt{x}),$$

because

$$\int_{x^a}^{(1/2)x}\sum_{x-y\leq k\leq x}\Lambda(k)\frac{dy}{y^2}\ll\log(2x)\cdot\Big(\log\log\Big(\frac{1}{2}x\Big)-\log\log x^a\Big)+1$$
$$\ll\log(2x)\qquad\text{for any positive } a<1.$$

Treating S_{21} similarly, we get

$$S_{19}+S_{21}=\frac{1}{2}\frac{\sqrt{x}}{\log x}F(x,T)+O(\sqrt{x}).$$

We notice that

$$S_{20}=\begin{cases}\dfrac{1}{2}i\dfrac{\Lambda(x)}{\sqrt{x}\log x}(T-C) & \text{if } x=N\\[2ex] O\Big(\dfrac{\Lambda(N)}{\sqrt{x}\log x}\,\mathrm{Min}\Big(T,\dfrac{x}{|x-N|}\Big)\Big) & \text{if } x\neq N.\end{cases}$$

Evaluating S_2 in a similar manner and using the estimate $S(T)\ll\log T/\log\log T$, we get our theorem as described in the introduction.

References

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