On a theorem of Mazur and Orlicz

by

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MAZUR and ORLICZ proved the following theorem[1]):

(M-O) *Let X be a linear space, let $\omega(x)$ be a functional defined on X such that*

(1) $$\omega(x+y)\leqslant\omega(x)+\omega(y),$$

(2) $$\omega(tx)=t\omega(x) \quad \text{for} \quad t\geqslant 0,$$

let $x(\tau)$ be a mapping of an abstract set T into X, and let $c(\tau)$ be a finite real function on T. In order that there exist an additive and homogeneous functional $\xi(x)$ defined on X, such that

(3) $$\xi(x)\leqslant\omega(x) \quad \text{for} \quad x\in X,$$

(4) $$c(\tau)\leqslant\xi\big(x(\tau)\big) \quad \text{for} \quad \tau\in T,$$

it is necessary and sufficient that

(5) $$\sum_{k=1}^{n} t_k c(\tau_k)\leqslant\omega\Big(\sum_{k=1}^{n} t_k x(\tau_k)\Big)$$

for every finite sequence $\tau_1,\dots,\tau_n\in T$ and for arbitrary non-negative real numbers $t_1,\dots,t_n$.

In this paper I shall show that Theorem (M-O) follows in a very simple way from the following well known theorem of Banach:

(B) *Let $\omega(x)$ be a functional defined on a linear space X and satisfying conditions (1) and (2). Then there is an additive and homogeneous functional $\xi(x)$ defined on X, such that $\xi(x)\leqslant\omega(x)$ for $x\in X$.*

We shall first formulate Theorem (B) in another more convenient form (see Theorem (B′) below).

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[1]) S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires (II)*, this volume, p. 137-179. See p. 174, Theorem 2. 41.

Let X be a linear space and let Y denote the set of all real numbers. The linear space $Z=X\times Y$ is the set of all pairs $z=(x,y)$, where $x\in X$ and $y\in Y$.

A set $S\subset Z$ is said to be a *positive cone* if

(α) S is a cone, i. e. $t_1z_1+t_2z_2\in S$ whenever $z_1,z_2\in S$, $t_1,t_2\geqslant 0$;

(β) $(0,y)\in S$ for $y\geqslant 0$;

(γ) for every $x\in X$ there is a y such that $(x,y)\in S$.

A positive cone $S\subset Z$ is said to be *proper* if $S\neq Z$.

(i) *The following conditions are equivalent for each positive cone $S\subset Z$:*

(a) *S is proper;*

(b) *if $(0,y)\in S$, then $y\geqslant 0$;*

(c) *the number $\omega(x)=\inf[y\,|\,(x,y)\in S]$ is finite for every $x\in X$.*

This equivalence holds in the case where X is at most two-dimensional. This equivalence holds also in the general case since each of the above conditions (a),(b),(c) is satisfied if and only if, for every at most two-dimensional linear subspace $X'\subset X$, it is satisfied by the positive cone $S'=S\cdot(X'\times Y)$ in the space $Z'=X'\times Y$.

(ii) *If the positive cone $S\subset Z$ is proper, then the functional $\omega(x)$ defined by* (c) *satisfies conditions* (1) *and* (2). *Conversely, if $\omega(x)$ is a functional satisfying* (1) *and* (2), *then the set*

$$S=[(x,y)\,|\,\omega(x)\leqslant y]\subset Z$$

is a proper positive cone.

The first part of (ii) is true if X is at most two-dimensional. It is also true in the general case since ω satisfies (1) and (2) in X if and only if ω satisfies these conditions in every at most two-dimensional linear subspace $X'\subset X$.

The second part of (ii) is obvious.

(B′) *If $S\subset Z$ is a proper positive cone, then there is an additive and homogeneous functional $\xi(x)$ in X such that $\xi(x)\leqslant y$ whenever $(x,y)\in S$.*

It is sufficient to apply Theorem (B) to the functional $\omega(x)$ defined by (c).

Clearly the condition that S is proper is also necessary for the existence of ξ.

Proof of (M-O). Only the sufficiency should be proved.

Suppose that $\omega(x)$, $x(\tau)$, $c(\tau)$ satisfy conditions (1), (2) and (5). Let S be the set of all pairs

$$\left(x-\sum_{k=1}^{n} t_k\, x(\tau_k), y\right) \epsilon\, X\times Y=Z,$$

where

$$y \geqslant \omega(x)-\sum_{k=1}^{n} t_k\, c(\tau_k),$$

$\tau_1,\ldots,\tau_n \,\epsilon\, T$, and $t_1,\ldots,t_n$ are non-negative real numbers. It follows from (1) and (2) that S is a positive cone viz. the least cone containing all elements $\left(x,\omega(x)\right)$ and $\left(-x(\tau),-c(\tau)\right)$. It follows from (5) that $(0,y)\,\epsilon\, S$ implies $y\geqslant 0$. Thus S is proper.

By (B′) there is an additive and homogeneous functional $\xi(x)$ defined on X, such that $\xi(x)\leqslant y$ for $z=(x,y)\,\epsilon\, S$. Putting $z=\left(x,\omega(x)\right)$ or $z=\left(-x(\tau), -c(\tau)\right)$ we obtain the inequalities (3) and (4) respectively, q. e. d.

The above proof explains the geometrical sense of (5). The condition (5) is an analytic formulation of the assertion that the positive cone S defined above is proper.

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Sur une formule de Efros

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Introduction. A. M. Efros[1]) a établi la formule

$$L^{-1}\{v(s)F[q(s)]\}=\int_0^\infty f(\tau)\,L^{-1}\{v(s)\,e^{-\tau q(s)}\}\,d\tau, \tag{1}$$

où F est la transformée de Laplace de f,

$$F(s)=\int_0^\infty e^{-st} f(t)\,dt, \tag{2}$$

et L^{-1} désigne généralement la transformation inverse à celle de Laplace. L'auteur n'a pas précisé les conditions pour f, q et v, dans lesquelles la formule (1) est valable. Or, ces fonctions ne peuvent pas être arbitraires.

Si, par exemple, $q(s)=s$ et $v(s)=1$, l'expression $L^{-1}\{v(s)\,e^{-\tau q(s)}\}$, c'est-à-dire $L^{-1}\{e^{-\tau s}\}$, est dépourvue de sens. En effet, supposons que cette expression soit égale à une fonction $\varphi(t)$. Alors

$$\int_0^\infty e^{-st}\varphi(t)\,dt=e^{-\tau s}, \tag{3}$$

d'où, en dérivant par rapport à s,

$$-\int_0^\infty e^{-st}\,t\varphi(t)\,dt=-\tau e^{-\tau s}. \tag{4}$$

Il vient de (3) et (4)

$$\int_0^\infty e^{-st}(t-\tau)\,\varphi(t)\,dt=0,$$

d'où $\varphi(t)=0$ pour presque tout $t\geqslant 0$, ce qui est en contradiction avec (3).

W. Ditkin et P. Kuznecov[2]) ont introduit certains restrictions pour les fonctions en question qui rendent la formule

[1]) А. М. Эфрос, *О некоторых применениях операторного исчисления к анализу*, Математический Сборник 42 (1935), p. 699 - 705.

[2]) В. А. Диткин и П. И. Кузнецов, *Справочник по операционному исчислению*, Москва 1951, p. 62 - 67.