# On a theorem of Paley and the Littlewood conjecture

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## 1. Introduction

Given a subset $E$ of the set $\mathbf{Z}$ of all integers, denote by $L_E^1$ the space of all integrable functions $f$ on the unit circle $\mathbf{T}$ whose Fourier coefficients $\hat{f}(n)$ vanish for all integers $n$ outside the set $E$; normalize the measure on $\mathbf{T}$ so that

$$\|f\|_1 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|\, d\theta.$$

We prove the following analogue of a classical theorem of Paley, and we use it to improve on the known estimates concerning the Littlewood conjecture.

**Theorem 1.** *If $E$ is infinite and bounded below, then there is a strictly increasing sequence $\{h_k\}_{k=0}^{\infty}$ of elements of $E$ such that, for each index $k$ there are fewer than $4^k$ elements of $E$ less than $h_k$, and such that*

(1) $$\sum_{k=0}^{\infty} |\hat{f}(h_k)|^2 \leqq 8(\|f\|_1)^2,$$

*for all functions $f$ in $L_E^1$.*

**Corollary 1.** *If $F$ is a finite subset of the integers having $N$ elements, then*

(2) $$\|f\|_1 \geqq [(\log_4 N)/8]^{1/2} \min_{n\in F} |\hat{f}(n)|,$$

*for all functions $f$ in $L_F^1$.*

A similar estimate was proved under added assumptions about the size of the coefficients $\hat{f}(n)$ by S. K. Pichorides [25]. We shall discuss other work on the Littlewood conjecture at the end of this section and at the end of the paper.

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Paley's theorem [21] asserts that for each real number $\lambda>1$ there is a constant $C(\lambda)$ so that, if $\{h_k\}_{k=0}^{\infty}$ is a strictly increasing sequence of nonnegative integers with the property that $h_{k+1}\geqq\lambda h_k$ for all $k$, and if $\hat{f}(n)=0$ for all $n<0$, then

$$(3) \qquad (\textstyle\sum_{k=0}^{\infty}|\hat{f}(h_k)|^2)^{1/2} \leqq C(\lambda)\|f\|_1.$$

In Section 2, we state Theorem 2, a common generalization of Theorem 1 and Paley's theorem, and we show how the latter two theorems follow from Theorem 2. In Section 3, we prove Theorem 2 by giving a direct proof of a dual assertion; we also show that when $\lambda\geqq 2$ the best value of the constant $C(\lambda)$ in Paley's theorem is $\sqrt{2}$.

The idea that theorems about thin sets in Fourier analysis have equivalent dual formulations goes back to Banach [1]. In 1956, W. Rudin [30] observed that Paley's theorem is equivalent to the assertion that, if $\lambda$ and $\{h_k\}_{k=0}^{\infty}$ are as in the statement of that theorem, then for every square-summable sequence $\{v_k\}_{k=0}^{\infty}$, there exists a bounded function $g$, with $\|g\|_\infty\leqq C(\lambda)\|v\|_2$, and such that

$$\hat{g}(h_k) = v_k \quad \text{for all } k, \text{ while } \quad \hat{g}(n) = 0 \quad \text{for all other} \quad n \geqq 0.$$

Rudin's methods did not yield an explicit procedure for obtaining such a function $g$ given the sequences $\{h_k\}_{k=0}^{\infty}$ and $\{v_k\}_{k=0}^{\infty}$. Such a procedure was discussed by Z. Nehari [19], in a paper on bounded bilinear forms that appeared in 1957, but the connection with Rudin's paper was apparently not noticed at the time.

In his paper, Nehari used the Schur algorithm, a procedure that was discovered by various mathematicians just before World War I. In Section 3, we prove Theorem 2 by applying the Schur algorithm to functions of several complex variables, and then using some elementary properties of finite Riesz products. We also use the Schur algorithm in one variable to verify that $\sqrt{2}$ is the best value of the constant $C(\lambda)$ when $\lambda\geqq 2$. We discuss other applications of this method in Section 4. First we consider an analogue of Paley's theorem that was discovered by Gundy and Varopoulos [12]; we give a new proof of this analogue and of some related results. Then we answer a question raised in [9] by proving a theorem about Fourier coefficients before gaps. Finally, we give a new, constructive proof of Grothendieck's inequality.

Other proof of Paley's theorem have appeared in [13], [33], [20], and [17, p. 274]. The last of these proofs is based on properties of Hankel operators, and it was one of the clues that led the present author to Nehari's paper.

Finally, we comment briefly on the history of the Littlewood problem. In [14], Hardy and Littlewood conjectured that there is a constant $A$ so that, if $F$ is a finite subset of $Z$ having $N$ elements, and if $f$ is the function in $L_F^1$ with the property that $\hat{f}(n)=1$ for all $n$ in $F$, then $\|f\|_1\geqq A\log N$. In [4], Paul Cohen showed that

$$\|f\|_1 \geqq A(\log N/\log\log N)^{1/8}$$

for all sufficiently large values of $N$; this lower bound was soon improved to $A(\log N/\log\log N)^{1/4}$ by $H$. Davenport [5]. Recently, S. K. Pichorides [23] modified the method of Cohen and Davenport to obtain the lower bound $A(\log N/\log\log N)^{1/2}$. The methods used in these papers also work for all functions $f$ in $L^1_F$ with the property that $|\hat{f}(n)|\geqq 1$ for all $n$ in $F$.

In 1977, Pichorides [26] used a completely new method to show that $\|f\|_1\geqq A(\log N)^{1/2}$ if $f\in L^1_F$, and $|\hat{f}(n)|=1$ for all $n$ in $F$; this method also yields that if $f\in L^1_F$, and $1\leqq|\hat{f}(n)|\leqq B$ for all $n$ in $F$, then $\|f\|_1\geqq A(B)(\log N)^{1/2}$, where $A(B)\to 0$ as $B\to\infty$. By Corollary 1, this estimate actually holds with a constant $A$ that is independent of $B$.

In many cases, better lower bounds than $A(\log N)^{1/2}$ are known to exist. Enumerate the set $F$ as $\{n_k\}_{k=1}^N$. A special case of a theorem of Bockarev [2] on uniformly bounded, orthonormal systems is that

$$\frac{1}{N}\sum_{K=1}^{N}\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1}^{K}\exp(in_k\theta)\right|d\theta\geqq A\log N;$$

that is, Littlewood's conjecture holds on the average. In another direction, P. G. Dixon has shown [6] that if we merely add the assumption that the sequence $\{n_{k+1}-n_k\}_{k=1}^{N-1}$ is nondecreasing, then $\|f\|_1\geqq A(\log N/\log\log N)$ for all sufficiently large values of $N$, and all functions $f$ in $L^1_F$ with the property that $|\hat{f}(n)|\geqq 1$ for all $n$ in $F$. Finally, there are interesting connections between the Littlewood conjecture and the *cosine problem.* Assume that $n>0$ for all $n$ in $F$, let $f$ be a function in $L^1_F$ with the property that $|\hat{f}(n)|\geqq 1$ for all $n$ in $F$, and let $M(f)=|\min_\theta \operatorname{Re} f(\theta)|$; one can ask how $M(f)$ behaves as $N\to\infty$. As in [23] it follows easily from Corollary 1 that $M(f)\geqq A(\log N)^{1/2}$. Pichorides has shown [24] that

$$M(f)\log M(f)+\|f\|_1\geqq A\log N.$$

I am grateful to Dr. Pichorides for sending me a copy of his interesting survey [25] on norms of exponential sums. I have also benefited from conversations about this subject with Grahame Bennett.

## 2. A counting argument

Before stating Theorem 2, we need some more notation and terminology. By a *multiindex* we mean a finite sequence $\alpha=\{\alpha_k\}_{k=0}^K$ of integers. We say that a nontrivial multiindex $\alpha$ begins at $k_1$ and ends at $k_2$ if $k_1$ and $k_2$ are the smallest and largest values of $k$ for which $\alpha_k\neq 0$. We call $\alpha$ *supplementary* if it has the following four properties:

(i) $\sum_{k=0}^{K} \alpha_k = 1$,
(ii) If $\alpha$ begins at $k_1$, and $k' \geqq k_1$, then $\sum_{k=0}^{k'} \alpha_k \geqq 1$,
(iii) $\sum_{k=0}^{K} |\alpha_k| \geqq 3$,
(iv) $\alpha_k \in \{-1, 0, 1\}$ for all $k$.

Given a multiindex $\alpha = \{\alpha_k\}_{k=0}^{K}$ and a finite sequence $h = \{h_k\}_{k=0}^{K}$ of integers, we let

$$\alpha \cdot h = \sum_{k=0}^{K} \alpha_k h_k.$$

We call an integer *n supplementary to the sequence h* if $n = \alpha \cdot h$ for some supplementary multiindex $\alpha$, and we let $S(h)$ denote the set of all integers that are supplementary to $h$. Similarly, if $h = \{h_k\}_{k=0}^{\infty}$ is an infinite sequence of integers, then we let $S(h)$ consist of all integers that are supplementary to some finite partial subsequence $\{h_k\}_{k=0}^{K}$.

**Theorem 2.** *Let* $h = \{h_k\}_{k=0}^{K}$ *be a finite or infinite sequence of integers. Suppose that an integrable function f has the property that* $\hat{f}(n) = 0$ *for all integers n in the set* $S(h)$. *Then*

$$\sum_{k=0}^{K} |\hat{f}(h_k)|^2 \leqq 8(\|f\|_1)^2.$$

Before deriving Theorem 1 from this assertion, we show how Paley's theorem follows from Theorem 2. First, choose an integer $m$ so that $\lambda^m \geqq 2$, and split the given sequence $\{h_k\}_{k=0}^{\infty}$ into $m$ subsequences, $\{h_k^{(j)}\}_{k=0}^{\infty}$ say, each with the property that $h_{k+1}^{(j)} \geqq 2h_k^{(j)}$ for all $k$. Now if $n$ is supplementary to one of these subsequences, then $n < 0$; one way to see this is to use the fact that, in addition to property (iv), each supplementary mulitindex $\alpha$ that ends at $k$ has the property that $\alpha_k = -1$. If $\hat{f}(n) = 0$ for all $n < 0$, then, by Theorem 2,

$$\sum_{k=0}^{\infty} |\hat{f}(h_k^{(j)})|^2 \leqq 8(\|f\|_1)^2,$$

for all $j$. Thus Paley's theorem holds with $C(\lambda) = (8m)^{1/2}$. We shall see at the end of Section 3 that we can in fact let $C(\lambda) = (2m)^{1/2}$.

*Proof of Theorem 1.* Fix a set $E$ of integers that is infinite and bounded below. Given an integer $n$, let $P(n)$ be the number of elements $m$ of $E$ for which $m < n$. To derive Theorem 1 from Theorem 2, we merely have to construct an increasing sequence $h = \{h_k\}_{k=0}^{\infty}$ so that $S(h)$ is disjoint from $E$, and so that $P(h_k) < 4^k$ for all $k$.

To do this, we proceed inductively. Let $h_0$ be the smallest element of $E$, and let $h_1$ be the next smallest element. Suppose that $h_0, h_1, \ldots, h_{k-1}$ have been chosen so that, for all $j < k$,

(a) $h_j \in E$
(b) $P(h_j) < 4^j$
(c) $\sum_{i=0}^{j} \alpha_i h_i \notin E$, for all supplementary multiindices $\alpha$ that end at $j$.

We shall let $h_k$ be the smallest element of $E$ that is greater than $h_{k-1}$, and has the three properties above with $j=k$; first, however, we have to verify that $E$ has such an element.

Given an integer $m$ in $E$, and a supplementary multiindex $\alpha$ that ends at $k$, we say that an integer $n$ is *disqualified by the pair* $(m, \alpha)$ if $n > h_{k-1}$ and

$$\alpha_k n + \sum_{i<k} \alpha_i h_i = m. \tag{4}$$

Of course, an integer $n$ may be disqualified by several different pairs $(m, \alpha)$; we say that $n$ is *disqualified by an integer* $j$ if $n$ is disqualified by some pair $(m, \alpha)$ for which $\alpha$ starts at $j$.

We consider the number $D(j)$ of integers that are disqualified by a given integer $j$. Let $N(j)$ be the number of supplementary multiindices $\alpha$ that begin at $j$ and end at $k$. By properties (iii) and (iv), every such $\alpha$ must have at least 3 nonzero terms; hence, $N(k-1)=0$, and $D(k-1)=0$. Now let $j \leqq k-2$; then exactly $3^{k-j-1}$ multiindices begin at $j$ with the value $+1$ and end at $k$ with the value $-1$. Every supplementary multiindex $\alpha$ must begin with the value $+1$, end with the value $-1$, and have property (i) as well. Hence $N(j) \leqq \frac{1}{2} 3^{k-j-1}$. On the other hand, if $m$ is given by formula (4) above, for some supplementary multiindex $\alpha$ that begins at $j$ and ends at $k$, and some integer $n > h_{k-1}$, then $m < h_j$. This follows from properties (i), (ii), and (iii) above, and the fact that $h_j < h_{j+1} < \ldots < h_{k-1} < n$. If $m$ and $\alpha$ are given, then $n$ is determined by formula (4). Hence

$$D(j) \leqq P(h_j) N(j) < \frac{1}{2} 4^j 3^{k-j-1}.$$

The total number of integers disqualified at the $k$-th stage is therefore smaller than

$$\begin{aligned}\frac{1}{2} \sum_{j=0}^{k-2} 4^j 3^{k-j-1} &= \frac{1}{6} 4^k \sum_{j=0}^{k-2} \left(\frac{3}{4}\right)^{k-j} \\ &< \frac{1}{6} 4^k \left(\frac{3}{4}\right)^2 \Big/ \left(1-\frac{3}{4}\right) \\ &= \frac{3}{2} 4^{k-1}.\end{aligned}$$

Let $h_k$ be the smallest element of $E$ that is greater than $h_{k-1}$ and not disqualified. Then

$$P(h_k) < P(h_{k-1}) + 1 + \frac{3}{2} 4^{k-1} < 4^k.$$

Hence assertions (a), (b) and (c) above hold with $j=k$, and the induction is complete.

To derive Corollary 1 from Theorem 1, simply let $E$ consist of the given finite set $F$ together with all integers that lie strictly to the right of $F$. The estimate (2) then holds not only for all functions $f$ in $L^1_F$, but also for all $f$ in $L^1_E$. The new method of Pichorides [26] does not seem to apply to general functions $f$ in $L^1_E$; it was observed, however, by I. Kessler [16] that the method of Cohen and Davenport does apply to such functions, and the same is true for the version of this method used by Pichorides in [23]. The counting argument used above will also remind some readers of Cohen's method.

Finally we comment on the possible rates of growth of the sequence $\{P(h_k)_{k=0}^{\infty}\}$. If the set $E$ is very thin, consisting only of the powers of 2 for instance, then the conclusion of Theorem 1 holds with $h_k$ equal to the $k$-th element of $E$ for each $k$; in this case, $P(h_k)=k-1$ for all $k$. On the other hand, when $E$ consists of all non-negative integers, then condition (c) above forces $P(h_k)$ to grow rapidly; moreover [30], if $\sum_{k=0}^{\infty}|\hat{f}(h_k)|^2<\infty$ for all integrable functions $f$ with the property that $\hat{f}(n)=0$ for all $n<0$, then there is a uniform bound on the number of terms in the sequence $\{h_k\}_{k=0}^{\infty}$ that lie between successive powers of 2. Finally rapid growth of $P(h_k)$ alone can fail to imply that $\sum_{k=0}^{\infty}|\hat{f}(h_k)|^2<\infty$ for all $f$ in $L^1_E$. For instance, there are sets $E$ so that if each $h_k$ is chosen to be the $4^k$-th element of $E$, then the sequence $\{h_k\}_{k=0}^{\infty}$ contains arbitrarily long arithmetic progressions; in this case [31, Theorem 4.1], there are functions $f$ in $L^1_E$ for which $\sum_{k=0}^{\infty}|\hat{f}(h_k)|^2=\infty$.

## 3. The Schur algorithm

In this section, we construct certain bounded functions so that their Fourier coefficients have various prescribed properties, and so that the essential suprema of these functions are not much larger than their $L^2$-norms. Our main goal is to prove Theorem 3, a dual assertion to Theorem 2. We state Theorem 3 below, and derive Theorem 2 from it by a duality argument. Next we consider, in a very special case, the problem of constructing a reasonably small, analytic function of several complex variables, given the "initial segment" of its power series; we use the Schur algorithm to solve the relevant special case of this problem. We then state and prove Theorem 4, a version of Theorem 3 for functions of several complex variables, and we derive Theorem 3 from Theorem 4. To determine the best constant in Paley's theorem, we study the Schur algorithm in one complex variable. Finally, we compare the construction used here with the ones used for similar purposes in [32] and [8].

Denote the $l^2$-norm of a square-summable sequence $v=\{v_k\}_{k=0}^{K}$ by $\|v\|_2$.

**Theorem 3.** *Let* $h=\{h_k\}_{k=0}^K$ *be a sequence of integers with the property that the set* $S(h)$ *is disjoint from the range of* $h$. *Then for every square-summable sequence* $v=\{v_k\}_{k=0}^K$ *there is a bounded function* $g$ *so that*

$\|g\|_\infty \leqq 8^{1/2}\ \|v\|_2$

$\hat{g}(n)=0$ *unless* $n\in S(h)$ *or* $n=h_k$ *for some* $k$,

*and* $\hat{g}(h_k)=v_k$ *for all* $k$.

*If only a finite number of the terms* $v_k$ *are nonzero, then the function* $g$ *can be chosen to be a trigonometric polynomial.*

*Proof of Theorem 2.* It is enough to deal with the case where the sequence $h$ is finite. Suppose that $f$ is integrable, and that $\hat{f}(n)=0$ for all $n$ in $S(h)$. If some term $h_k\in S(h)$, then $\hat{f}(h_k)=0$, so that we can delete this term from the sequence $h$ without affecting the conclusion of Theorem 2; we can therefore assume that $S(h)$ is disjoint from the range of $h$.

Let $v=\{v_k\}_{k=0}^K$ be a sequence of complex numbers. Since $K$ is finite, there is a trigonometric polynomial $g$ with the properties listed in the statement of Theorem 3. In particular, $\overline{\hat{g}(n)}\hat{f}(n)=0$ unless $n=h_k$ for some $k$. Thus

$$\begin{aligned}\left|\sum_{k=0}^K \hat{f}(h_k)\bar{v}_k\right| &= \left|\sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}\right| \\ &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\overline{g(\theta)}\,d\theta\right| \\ &\leqq 8^{1/2}\|f\|_1\|v\|_2 .\end{aligned}$$

Since this inequality holds for all sequences $v$, we conclude that

$$\sum_{k=0}^K |\hat{f}(h_k)|^2 \leqq 8(\|f\|_1)^2,$$

as asserted in Theorem 2. This completes the proof.

We introduce some notation, mostly taken from [29], for our discussion of the Schur algorithm. Fix a positive integer $N$. The symbol $w$ will always denote an element $\{w_n\}_{n=1}^N$ of the product $\mathbf{T}^N$, and the symbol $\beta$ an element $\{\beta_n\}_{n=1}^N$ of $\mathbf{Z}^N$. By $w^\beta$ we mean the product $w_1^{\beta_1}w_2^{\beta_2}\dots w_N^{\beta_N}$. Every integrable function $f$ on $\mathbf{T}^N$ has a Fourier series

$$\sum_\beta \hat{f}(\beta)\,w^\beta. \tag{5}$$

We define a partial order on $\mathbf{Z}^N$ by declaring that $\beta\geqq 0$ if and only if $\beta_n\geqq 0$ for all $n$, and that $\beta'\geqq\beta$ if and only if $\beta'-\beta\geqq 0$. We call a series (5) a *power series* if $\hat{f}(\beta)=0$ unless $\beta\geqq 0$, and we call a function $f$ *analytic* if its Fourier series is a power series.

By the *initial segment* of a power series (5), we mean the sum of the terms $\hat{f}(\beta)w^{\beta}$ for which $0\leqq\beta\leqq(1, 1, \dots, 1)$. Given a sequence $c=\{c_n\}_{n=0}^N$ of complex numbers, we form the polynomial

$$p_N(w) = c_N + c_{N-1}w_N + c_{N-2}w_N w_{N-1} + \dots + c_0 w_N w_{N-1} \dots w_1.$$

Observe that, if $\hat{p}_N(\beta)\neq 0$, then the nonzero terms in the sequence $\{\beta_n\}_{n=1}^N$ *form a tail,* that is, if $\beta_n$ is nonzero, then so are $\beta_{n+1}, \beta_{n+2}, \dots$. We call a function $f_N$ *a good extension of* $p_N$ if $f_N$ is an analytic function on $\mathbf{T}^N$, its power series has initial segment $p_N$, the coefficients $\hat{f}_N(\beta)$ vanish unless the nonzero terms in the sequence $\beta$ form a tail, and $\|f_N\|_\infty<1$.

The Schur algorithm is an explicit procedure for constructing a good extension when $p_N$ has such an extension. Here is how it works.

**Lemma 1.** *If* $\|c\|_2\leqq 1/\sqrt{2}$, *then* $p_N$ *has a good extension.*

*Proof.* Suppose that $\|p_N\|_2=\|c\|_2\leqq 1/\sqrt{2}$. We can then define sequences of complex numbers $\{b_n\}_{n=0}^N$, and polynomials $\{p_n\}_{n=1}^N$ as follows. Let $b_N=c_N$; certainly $|b_N|<1$. We observe that

$$(p_N(w_1, w_2, \dots, w_N)-b_N)/(1-|b_N|^2)\,w_N$$

is actually independent of $w_N$, and we denote this polynomial by

$$p_{N-1}(w_1, w_2, \dots, w_{N-1}).$$

Now

$$\|p_{N-1}\|_2^2 = \frac{\|p_N\|_2^2-|b_N|^2}{(1-|b_N|^2)^2} \leqq \frac{(1/2)-|b_N|^2}{1-2|b_N|^2} = \frac{1}{2}.$$

Letting $b_{N-1}=\hat{p}_{N-1}(0)=c_{N-1}/(1-|b_N|^2)$, we have that $|b_{N-1}|<1$. We then let

$$p_{N-2}(w_1, w_2, \dots, w_{N-2}) = (p_{N-1}(w_1, w_2, \dots, w_{N-1})-b_{N-1})/(1-|b_{N-1}|^2)\,w_{N-1}$$

and continue. Finally, we let $b_0=c_0/\prod_{n=1}^N(1-|b_n|^2)$.

Next we use the sequence $\{b_n\}_{n=0}^N$ to generate a sequence $\{f_n\}_{n=1}^N$ of rational functions. Let

$$f_1(w_1) = \frac{b_0 w_1 + b_1}{1+\bar{b}_1 b_0 w_1};$$

then $\|f_1\|_\infty<1$, because $|b_0|<1$ and $|b_1|<1$. Given $f_{n-1}(w_1, w_2, \dots, w_{n-1})$, let

$$f_n(w_1, w_2, \dots, w_n) = \frac{w_n f_{n-1}(w_1, w_2, \dots, w_{n-1})+b_n}{1+\bar{b}_n w_n f_{n-1}(w_1, w_2, \dots, w_{n-1})};$$

again $\|f_n\|_\infty<1$. Observe that

$$p_1(w_1) = \frac{c_1+c_0 w_1}{\prod_{n>1}(1-|b_n|^2)} = b_1+(1-|b_1|^2)\,b_0 w_1,$$

while

$$f_1(w_1) = \frac{b_1(1+\bar{b}_1 b_0 w_1) - |b_1|^2 b_0 w_1 + b_0 w_1}{1+\bar{b}_1 b_0 w_1}$$
$$= b_1 + (1-|b_1|^2) b_0 w_1 [1 - \bar{b}_1 b_0 w_1 + (\bar{b}_1 b_0 w_1)^2 - \ldots].$$

Thus $p_1$ is the initial segment of the power series of $f_1$. Consider the function of two variables given by $(w_1, w_2) \mapsto w_2 f_1(w_1)$; the initial segment of its power series is the polynomial $(w_1, w_2) \mapsto w_2 p_1(w_1)$. It then follows, as above, that $p_2$ is the initial segment of the power series of $f_2$, and, by induction, that $p_N$ is the initial segment of the power series of $f_N$. Finally, the same analysis shows that if $\hat{f}_N(\beta) \neq 0$, then the nonzero terms in the sequence $\beta$ form a tail. Hence $f_N$ is a good extension of $p_N$, and the proof of the lemma is complete.

Much as in Section 2, we use the symbol $\alpha$ to denote an element $\{\alpha_n\}_{n=0}^N$ of $\mathbf{Z}^{N+1}$, and the symbol $z$ to denote an element of $\mathbf{T}^{N+1}$. Denote the multiindex whose $n$-th term is equal to 1 and whose other terms are equal to 0 by $\delta^n$. Let $S$ be the set of all supplementary multiindices.

**Theorem 4.** *For each sequence* $c = \{c_n\}_{n=0}^N$ *of complex numbers there is a trigonometric polynomial* $p$ *on* $\mathbf{T}^{N+1}$ *of the form*

$$p(z) = \sum_{n=0}^N c_n z_n + \sum_{\alpha \in S} \hat{p}(\alpha) z^\alpha \tag{6}$$

*with the property that* $\|p\|_\infty \leq \sqrt{8}\, \|c\|_2$.

*Proof.* Suppose initially that $\|c\|_2 = 1/\sqrt{2}$. When $1 \leq n \leq N$, let $w_n = z_{n-1}/z_n$. Form the polynomial $p_N$ as above, and let $f_N$ be the specific good extension of $p_N$ that was constructed in the proof of Lemma 1. Let

$$P(z) = z_N \cdot f_N(w_1, w_2, \ldots, w_N).$$

The function $P$ is unlikely to be a trigonometric polynomial, but it has most of the other properties that we require of the polynomial $p$. First of all,

$$z_N p_N(w_1, w_2, \ldots, w_N) = c_N z_N + c_{N-1} z_{N-1} + \ldots + c_0 z_0.$$

More generally, every nonzero term in the Fourier series of $P$ is of the form $z_N \hat{f}_N(\beta) w^\beta$, where $\hat{f}_N(\beta) \neq 0$. Rewriting this term as $\hat{P}(\alpha) z^\alpha$, we have that $\alpha_N = 1 - \beta_N$, that $\alpha_n = \beta_{n+1} - \beta_n$ when $1 \leq n < N$, and that $\alpha_0 = \beta_1$. Then $\sum_{n=0}^N \alpha_n = 1$; that is, $\alpha$ has property (i). Suppose that $\alpha$ begins at $n_1$. If $n_1 = N$, then $\alpha_N = 1$ by property (i); in this case, $\alpha = \delta^N$. If $n_1 < N$, then $\beta$ begins at $n_1 + 1$, and since the nonzero terms in $\beta$ are positive and form a tail, $\beta_n > 0$ for all $n > n_1$; it follows that if $n_1 \leq m < N$, then $\sum_{n=0}^m \alpha_n = \beta_{m+1} > 0$. Thus $\alpha$ has property (ii). Moreover, any multiindex that has properties (i) and (ii) and is not equal to $\delta^n$ for some $n$, must also have property (iii).

We now use a Riesz product to pass to a polynomial $p$ for which $\hat{p}(\alpha)=0$ unless $\alpha$ has property (iv). Let

$$R(z) = \Pi_{n=0}^{N}\left[1+\frac{1}{2}(z_n+\bar{z}_n)\right].$$

Then $R$ is a nonnegative trigonometric polynomial, $\|R\|_1=\hat{R}(0)=1$, and $\hat{R}(\delta^n)=1/2$ for all $n$; moreover $\hat{R}(\alpha)\neq 0$ if and only if $\alpha$ has property (iv). Suppose now that $\|c\|_2=1/2\sqrt{2}$, and let $c'_n=2c_n$ for all $n$. Use the sequence $c'$ in place of $c$ in the above construction to get a function $P'$. Let $p$ be the convolution $R*p'$. Then

$$\|p\|_\infty \leqq \|P'\|_\infty < 1 = \sqrt{8}\,\|c\|_2,$$

and $p$ has the form (6).

Finally, we indicate what to do if $\|c\|_2\neq 1/2\sqrt{2}$. If $\|c\|_2=0$, let $p\equiv 0$. Otherwise, let $c''_n=c_n/(2\sqrt{2}\,\|c\|_2)$ for all $n$, and let $p''$ be the trigonometric polynomial that we obtain by proceding as above with the sequence $c''$ in place of $c$. Let $p=2\sqrt{2}\|c\|_2 p''$; then $p$ has the desired properties.

*Proof of Theorem 3.* It suffices, by a weak-star compactness argument, to deal with the case when $K$ is finite. In this case, let $N=K$, let $\{c_n\}_{n=0}^{N}=\{v_k\}_{k=0}^{K}$, and let $p$ be the trigonometric polynomial constructed above. Let

$$g(\theta) = p\left(\exp(ih_0\theta), \exp(ih_1\theta), \ldots, \exp(ih_N\theta)\right);$$

then $g$ is a trigonometric polynomial on $\mathbf{T}$ with the desired properties. This completes the proof of the theorem.

By using the function $P$ in place of $p$, we could prove in the same way that Paley's theorem holds with $C(\lambda)=\sqrt{2}$ when $\lambda\geqq 2$; in order to see that the constant $\sqrt{2}$ is optimal, however, we work entirely with functions of one complex variable. Fix a finite, increasing sequence $h=\{h_k\}_{k=0}^{K}$ of nonnegative integers such that $h_{k+1}\geqq 2h_k$ for all $k<K$, and such that $h_0=0$; define another sequence $\{n_k\}_{k=1}^{K}$ by letting $n_k=h_k-h_{k-1}$ for all $k$. Later we shall need the facts that $n_k>0$ for all $k$, and that

$$n_{k+1} > n_1+n_2+\ldots+n_k \quad \text{for all} \quad k < K; \tag{7}$$

these assertions follow immediately from our assumptions about the sequence $h$. Denote by $H^1$ the space of all functions $f$ in $L^1(\mathbf{T})$ such that $\hat{f}(n)=0$ for all $n<0$, and denote the intersection $L^\infty(T)\cap H^1$ by $H^\infty$. Let $C$ be a positive constant. As in [30], the inequality

$$\left(\sum_{k=0}^{K}|\hat{f}(h_k)|^2\right)^{1/2} \leqq C\|f\|_1 \tag{8}$$

holds for all $f$ in $H^1$ if and only if for every complex-valued sequence $\{v_k\}_{k=0}^{K}$ there exists a function $g$ in $L^\infty(\mathbf{T})$ such that

$$\hat{g}(h_k) = v_k \quad \text{for all } k, \text{ and} \quad \hat{g}(n) = 0 \quad \text{for all other} \quad n \geqq 0,$$

and such that

$$\|g\|_\infty \leqq C\|v\|_2.$$

Following Nehari [19], we associate with every function $g$ having these properties the function $f$ given by the rule that

$$f(\theta) = \exp(ih_K\theta)g(-\theta) \quad \text{for all } \theta.$$

Then $f \in H^\infty$, and $\|f\|_\infty \leqq C\|v\|_2$. Moreover, the Fourier series of $f$ has the form

$$v_K + v_{K-1}\exp(in_K\theta) + \dots + v_0\exp[i(n_K+\dots+n_1)\theta] + \textstyle\sum_{n>h_k}\hat{f}(n)\exp(in\theta).$$

Let $z=e^{i\theta}$, and let

(9) $$p_K(z) = v_K + v_{K-1}z^{n_K} + \dots + v_0 z^{n_K+\dots+n_1};$$

then $p_K$ is a partial sum of the Fourier series of $f$. We call any function $F$ in $H^\infty$ such that $p_K$ is a partial sum of the Fourier series of $F$, and such that $\|F\|_\infty < 1$, a *good extension of* $p_K$. Let

$$C_K = \inf\{C > 0;\ \text{if } \|p_K\| < 1/C, \text{ then } p_K \text{ has a good extension}\}$$

then, by the discussion above, $C_K$ is also the smallest number for which inequality (8) holds, with $C = C_K$, for all $f$ in $H^1$.

We now show that $C_K \leqq \sqrt{2}$ for all $K$, and that $\lim_{K\to\infty} C_K = \sqrt{2}$. To this end, we use the Schur algorithm much as in the proof of Lemma 1. Given a sequence $\{v_k\}_{k=0}^K$, we let $b_K = v_K$; if $|b_K| \geqq 1$, we stop, but, otherwise, we let $b_{K-1} = v_{K-1}/(1-|b_K|^2)$. We continue this process unless, at some stage, $|b_K| \geqq 1$. We call a sequence $v = \{v_k\}_{k=0}^K$ *good*, if, for all $k>0$, the numbers $b_k$ are well defined and satisfy the inequality $|b_k|<1$; we call $v$ *bad* if it is not good. We claim that a sequence $v$ is good if and only if the corresponding polynomial $p_K$ has a good extension. Indeed, suppose that $p_K$ has a good extension, $F_K$ say. Certainly $|b_K|<1$. Regard $F_K$ as a function of $z$, and let

$$F_{K-1}(z) = \frac{F_K(z) - b_K}{\left(z^{n_K}(1-\bar{b}_K F_K(z)\right)}.$$

Then $F_{K-1} \in H^\infty$, and $\|F_{K-1}\|_\infty < 1$; moreover, using property (7) of the sequence $\{n_k\}_{k=1}^K$, we can verify that the polynomial

$$p_{K-1}(z) = \left(p_K(z) - b_K\right)/(1-|b_K|^2)z^{n_K}$$

is a partial sum of the Fourier series of $F_{K-1}$. Now $b_{K-1} = \hat{F}_{K-1}(0)$, whence $|b_{K-1}|<1$; we can therefore define function $F_{K-2}$ and $p_{K-2}$ etc. Thus $v$ is good if $p_K$ has a good extension. Conversely, suppose that $v$ is good. Define a sequence of functions $\{f_k\}_{k=1}^K$ by letting

$$f_1(z) = \frac{b_0 z^{n_1} + b_1}{1 + \bar{b}_1 b_0 z^{n_1}},$$

and, given $f_{k-1}(z)$, letting

$$f_k(z) = \frac{b_k + z^{n_k} f_{k-1}(z)}{1+\bar{b}_k z^{n_k} f_{k-1}(z)}.$$

Much as in the proof of Lemma 1, we have that $f_K$ is a good extension of $p_K$; again, in verifying this, we need property (7) of the sequence $\{n_k\}_{k=1}^K$. Thus $p_K$ has a good extension if $v$ is good.

Hence

$$C_K = \inf\{C > 0;\ \text{if } \|v\|_2 < 1/C, \text{ then } v \text{ is good}\};$$

equivalently,

$$C_K = \sup\{1/\|v\|_2\colon v \text{ is bad}\}.$$

We saw in the proof of Lemma 1 that $v$ is good if $\|v\|_2 \leqq 1/\sqrt{2}$; therefore $C_K \leqq \sqrt{2}$. To get a lower bound for $C_K$, we construct a specific bad sequence $v$, and compute $\|v\|_2$. The sequence $v$ will be defined in terms of another sequence $b$ by the rule that

$$v_k = b_k \prod_{j=k+1}^K (1-|b_j|^2 \quad \text{for all } k. \tag{10}$$

We let $b_0=1$, and let $b_1$ be the positive number for which the $L^2$-norm of the polynomial

$$p_1(z) = (1-|b_1|^2)\, b_0 z^{n_1} + b_1$$

is minimal; we then let $b_2$ be the positive number for which the $L^2$-norm of

$$p_2(z) = (1-|b_2|^2) z^{n_2} p_1(z) + b_2$$

is minimal, and continue. Easy calculations show that $b_n=(n+1)^{-1/2}$, and $\|p_n\|_2 = [(n+2)/2(n+1)]^{1/2}$ for all $n$. Because $b_0=1$, the sequence $v$ defined by formula (10) is bad; hence

$$C_K \geqq 1/\|v\|_2 = [2(K+1)/(K+2)]^{1/2}. \tag{11}$$

In fact we have equality here, but all we need is inequality (11) and the inequality $C_K \leqq \sqrt{2}$, which imply that $C_K \to \sqrt{2}$ as $K \to \infty$.

It follows that Paley's theorem holds with $C(\lambda)=\sqrt{2}$ when $\lambda \geqq 2$, and that the constant $\sqrt{2}$ is best possible; in fact, if $\{h_k\}_{K=0}^\infty$ is any infinite sequence of non-negative integers for which there is a constant $C$ such that

$$\left(\sum_{k=0}^\infty |\hat{f}(h_k)|^2\right)^{1/2} \leqq C\|f\|_1 \quad \text{for all } f \text{ in } H^1,$$

then $C \geqq \sqrt{2}$. The same comments apply to a generalization, due to Rudin [28, p. 213], of Paley's theorem to the content of $H^1$-spaces on compact abelian groups with totally-ordered dual groups. Indeed, if one uses Rudin's theorem and Nehari's method in this setting, then one is led to Lemma 1. There is also a version of Paley's theorem for analytic functions of several complex variables [20];

in this setting, however, our methods seem to work only in certain very special cases.

Other constructions of functions with many of the properties of the function $g$ devised in Theorem 3 have been discovered by Salem and Zygmund [32], and by the present author [8]. To compare these constructions we fix a finite, complex-valued sequence $\{v_k\}_{k=0}^K$ and a sequence $\{h_k\}_{k=0}^K$ of nonnegative integers such that $h_{k+1} > 3h_k$ for all $k < K$. We call a multiindex $\alpha$ *small* if it has property (iv), and we observe that, if $\alpha$ and $\alpha'$ are distinct small multiindices, then the integers $\alpha \cdot h$ and $\alpha' \cdot h$ are distinct. If $\alpha$ is small, we let

$$v^\alpha = \prod_{k=0}^K |v_k|^{|\alpha_k|} (\operatorname{sgn} v_k)^{\alpha_k};$$

here we use the standard conventions that $\operatorname{sgn} z = z/|z|$ if $z \neq 0$, and $\operatorname{sgn} 0 = 0$, and that $0^0 = 1$. We denote $\sum_{k=0}^K |\alpha_k|$ by $|\alpha|$.

Let $G$ be the trigonometric polynomial such that $\hat{G}(\alpha \cdot h) = (-1)^{(|\alpha|-1)/2} v^\alpha$ for all small multiindices $\alpha$, and $\hat{G}(n) = 0$ for all other integers $n$. Salem and Zygmund showed that $\|G\|_\infty < \exp(2\|v\|_2^2)$. Using this fact, one can easily show [7, p. 163], that the same estimate holds for the $L^\infty$-norm of the polynomial $G'$ for which $\hat{G}'(\alpha \cdot h) = \hat{G}(\alpha \cdot h)$ if $\alpha$ has property (i), and $\hat{G}(n) = 0$ for all other integers $n$. Call a small multiindex *complementary* if it has properties (i) and (iii), and, in addition, its nonzero terms alternate in sign. Let $G''$ be the polynomial such that $\hat{G}''(\alpha \cdot h) = \hat{G}(\alpha \cdot h)$ if $\alpha = \delta^k$ for some $k$, or $\alpha$ is complementary, and such that $\hat{G}''(n) = 0$ for all other integers $n$. It was shown in [8] that $\|G''\|_\infty < \exp(\|v\|_2^2/2)$.

Since the transform $\hat{G}''$ coincides on its support with $\hat{G}$, and since $\hat{G}'$ has the same property, we say that $G'$ and $G''$ belong to the family generated by $G$. The function $g$ devised in Theorem 3 does not belong to this family. Consider the basic case where $\|v\|_2 = 1/2\sqrt{2}$. It is true that the support of $\hat{g}$ is included in that of $\hat{G}$, and that $\hat{g}(\alpha \cdot h) = v_k = \hat{G}(\alpha \cdot h)$ if $\alpha = \delta^k$ for some $k$, but, if $\alpha$ is supplementary, and if $v^\alpha \neq 0$, then $\hat{g}(\alpha \cdot h) \neq \hat{G}(\alpha \cdot h)$. For instance, let $\alpha_3 = \alpha_4 = 1$, let $\alpha_5 = -1$, and let $\alpha_k = 0$ otherwise; then

$$\hat{g}(\alpha \cdot h) = \frac{(-1)^{(|\alpha|-1)/2} v^\alpha}{(1-|b_5|)^2 \prod_{k>5} (1-|b_k|^2)^2},$$

where $\{b_k\}_{k=0}^K$ is the sequence associated with $2v$.

The construction used here and the construction used in [8] are related, however, in the following curious way. In [8], we defined sequences $\{g_n\}_{n=0}^\infty$ and $\{h_n\}_{n=0}^\infty$ of trigonometric polynomials, and we computed $|g_n|^2 + |h_n|^2$ using the complex identity

$$|a+vb|^2 + |b-\bar{v}a|^2 = (1+|v|^2)(|a|^2+|b|^2).$$

Consider the function $P$ used in the proof of Theorem 4, and denote it now by $P_N$.

It is a rational function; in fact $P_N = t_N/q_N$, where $t_N$ and $q_N$ are the trigonometric polynomials defined by letting $t_0 = b_0$, $q_0 = 1$,

$$t_n = t_{n-1} + b_n z_n q_{n-1}, \quad \text{and} \quad q_n = q_{n-1} + \bar{b}_n \bar{z}_n t_{n-1}.$$

This procedure differs only by a minus sign from one used in [8]; moreover, one way to verify that $\|P_N\|_\infty < 1$ is to show that $|q_n|^2 - |t_n|^2 > 0$ for all $n$, using the identity

$$|q + \bar{b}t|^2 - |t + bq|^2 = (1 - |b|^2)(|q|^2 - |t|^2).$$

## 4. Other applications

Gundy and Varopoulos [12] have discovered a new class of subspaces of $L^1(\mathbf{T})$ that have many of the properties of the classical space $H^1$. In particular, these subspaces are closed and translation invariant, and a version of Paley's theorem holds for them. To simplify the discussion, we describe only two of these spaces, but what we do also works for the other spaces considered by Gundy and Varopoulos.

Let

$$A = \{3^n + k3^{n+1} \colon k \in \mathbf{Z},\ n = 0, 1, 2, \ldots\},$$

and

$$B = \{m \in \mathbf{Z} \colon \text{either } m = 4^n + k4^{n+1},$$
$$\text{or } m = 2(4^n + k4^{n+1}),\ k \in \mathbf{Z},\ n = 0, 1, 2, \ldots\}.$$

Observe that if $m \neq 0$, then exactly one of the numbers $m$ and $-m$ belongs to the set $A$; the set $B$ also has this property.

**Theorem 5.** *If $f \in L^1_A$, then*

$$\left(\sum_{n=0}^{\infty} |\hat{f}(3^n)|^2\right)^{1/2} \leqq \sqrt{8}\,\|f\|_1. \tag{12}$$

*If $f \in L^1_B$, then*

$$\left(\sum_{n=0}^{\infty} |\hat{f}(2^n)|^2\right)^{1/2} \leqq 4\,\|f\|_1. \tag{13}$$

*Proof.* Define a finite sequence $\{h_k\}_{k=0}^{K}$ by letting $h_k = 3^{K-k}$ for all $k$. Then the set $S(h)$ is disjoint from the set $A$; to see this, one merely needs to know that every supplementary multiindex has property (iv), and ends with the value $-1$. By Theorem 2,

$$\left(\sum_{k=0}^{K} |\hat{f}(3^n)|^2\right)^{1/2} \leqq \sqrt{8}\,\|f\|_1$$

for all $f$ in $L^1_A$. To obtain assertion (12), simply let $K \to \infty$. Similar applications of Theorem 2 yield that, for all $f$ in $L^1_B$,

$$\left(\sum_{n=0}^{\infty} |\hat{f}(4^n)|^2\right)^{1/2} \leqq \sqrt{8}\,\|f\|_1,$$

and

$$\left(\sum_{n=0}^{\infty} |\hat{f}(2\cdot 4^n)|^2\right)^{1/2} \leqq \sqrt{8}\, \|f\|_1.$$

This completes the proof of the theorem.

Except for the presence of the explicit constant $\sqrt{8}$, the first part of Theorem 5 follows immediately from Theorems 2 and 3 in [12]; the corresponding assertion for $L_B^1$, however, does not seem to follow easily from the results in [12]. The spaces $L_A^1$ and $L_B^1$ are associated with certain backwards martingales, and we can prove a stronger assertion than Theorem 4 by looking at martingale differences. We deal with $L_B^1$ here. Given $f$ in $L^1(\mathbf{T})$, and an integer $n \geqq 0$, let

$$f_n(\theta) = 2^{-n} \sum_{k=1}^{2^n} f(\theta + 2\pi k/2^n),$$

and let $d_n = f_n - f_{n+1}$. Denote the set of all multiples of $2^n$ by odd numbers by $\mathcal{O}_n$; then the Fourier series of $d_n$ is just

$$\sum_{m\in\mathcal{O}_n} \hat{f}(m)\, e^{im\theta}.$$

Let $\delta_n(\theta) = \operatorname{sgn} d_n(\theta)$ if $d_n(\theta) \neq 0$, and let $\delta_n(\theta) = \exp(i2^n\theta)$ otherwise; then

$$\|d_n\|_1 = \frac{1}{2\pi}\int_0^{2\pi} d_n(\theta)\overline{\delta_n(\theta)}\, d\theta,$$

$|\delta_n(\theta)| = 1$ for all $\theta$, and $\hat{\delta}_n(m) = 0$ unless $m \in \mathcal{O}_n$. Moreover, $\delta_n \in L_B^1$ if $d_n \in L_B^1$. Using Theorem 4, these properties of the functions $\delta_n$, and a duality argument, we can show that

$$\left(\sum_{n=0}^{\infty} \|d_n\|_1^2\right)^{1/2} \leqq 4\, \|f\|_1,$$

for all functions $f$ in $L_B^1$. A similar inequality holds for functions in $L_A^1$, but, in that case, more is known, namely that

$$\left(\sum_{n=0}^{\infty} |d_n|^2\right)^{1/2} \in L^1 \quad \text{for all } f \text{ in } L_A^1;$$

this follows from Theorems 1 and 2 in [12]. It may be possible to prove the corresponding statement for $L_B^1$ using the methods of [15].

Next we consider the properties of Fourier coefficients before and after gaps. Let $\{n_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ be sequences of nonnegative integers such that $n_0 \leqq m_1 < n_1 \leqq m_2 < n_2 \leqq \dots$; for each $k \geqq 1$, let $l_k = n_k - m_k - 1$. We consider regular Borel measures $\mu$ such that $\hat{\mu}(n) = 0$ whenever $m_k < n < n_k$ for some $k$; that is, $\hat{\mu}$ vanishes in the gap of length $l_k$ before each $n_k$. Denote the set of such measures $\mu$ by $M_l$. It was shown in [10] that if there is a positive number $\varepsilon$ such that $l_{k+1} \geqq \varepsilon n_k$ for all $k$, then $\sum_{k=0}^{\infty} |\hat{\mu}(n_k)|^2 < \infty$ for all measures $\mu$ in $M_l$. We can now prove that the coefficients before the gaps must also be square-summable in this case.

**Theorem 6.** *Suppose that there is a positive number* $\varepsilon$ *such that* $l_{k+1} \geqq \varepsilon n_k$ *for all* $k \geqq 1$. *Then there is a constant* $C$ *so that*

(14) $$\sum_{k=1}^{\infty} |\hat{\mu}(m_k)|^2 \leqq C\|\mu\|^2$$

*for all measures* $\mu$ *in* $M_l$.

*Proof.* As in [10], it is easy to reduce matters to the case where $\varepsilon \geqq 1$, and where $d\mu(\theta) = f(\theta)d\theta/2\pi$ for some function $f$ in $L^1(\mathbf{T})$. Inequality (14) will follow if it can be shown that

(15) $$\sum_{k=1}^{K+1} |\hat{f}(m_k)|^2 \leqq 8\,\|f\|_1^2$$

for all $K < \infty$. To prove the latter inequality let $h_k = m_{K+1-k}$ when $0 \leqq k \leqq K$. If a supplementary multiindex $\alpha$ begins at $k$, then $\alpha_k$ and the next nonzero term of $\alpha$ are both equal to $+1$; using this fact, property (iv), and the lacunarity assumption, one can show that $\alpha \cdot h$ falls in the gap between $m_k$ and $n_k$. It then follows immediately from Theorem 2 that inequality (15) holds for all functions $f$ in $M_l$. This completes the proof of the theorem.

Fix an increasing sequence $\{h_k\}_{k=0}^{\infty}$ of nonnegative integers such that $h_{k+1} \geqq 2h_k$ for all $k$. In 1956, Rudin [30] used Paley's theorem to show that, for each square-summable sequence $\{v_k\}_{k=0}^{\infty}$, there exists a bounded function $g$ such that $\hat{g}(h_k) = v_k$ for all $k$, and $\hat{g}(n) = 0$ for all other $n \geqq 0$. During the 1960's, it was observed, in [9, Theorem 12] for instance, that, in this situation, there also exists a bounded function $G$ such that $\hat{G}(h_k) = v_k$ for all $k$, and $\hat{G}(n)$ for all $n < 0$. The interesting fact that both kinds of functions exist was discussed by Goes in [11].

A specific function $G$ with the properties mentioned above was constructed in [8]; it has the further property that $\|G\|_\infty \leqq \sqrt{e}\,\|v\|_2$. The methods used in the proof of Paley's theorem given near the end of Section 3 yield a specific function $g$, as above, for which $\|g\|_2 \leqq \sqrt{2}\,\|v\|_2$. By combining the two constructions, we get an apparently new proof of Grothendieck's inequality [18]. One of the many equivalent formulations of this inequality asserts that there is a constant $K_G^C$ such that if $A$ is a square matrix, of any size, with the property that

$$\left|\sum_{i,j} A_{ij} a_i \bar{b}_j\right| \leqq 1$$

whenever $\|a\|_\infty \leqq 1$ and $\|b\|_\infty \leqq 1$, then, for all sequences $\{x^i\}$ and $\{y^i\}$ in the unit ball of a complex Hilbert space,

$$\left|\sum_{i,j} A_{ij}(x^i, y^j)\right| \leqq K_G^C.$$

To prove this, we take the Hilbert space to be $l^2$, and we construct functions $g_i$ and $G_j$, as above, such that $\hat{g}_i(h_k) = x_k^i$ for all $k$, and $\|g\|_\infty \leqq \sqrt{2}\,\|x^i\|_2$, while

$\hat{G}_j(h_k)=y_k^j$ for all $k$, and $\|G\|_\infty \leqq \sqrt{e}\,\|y^j\|_2$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} g_i(\theta)\overline{G_j(\theta)}\,d\theta = (x^i, y^j),$$

because $\hat{g}_i(n)\hat{G}_j(n)=0$ unless $n=h_k$ for some $k$. Therefore,

$$\left|\sum_{i,j} A_{ij}(x^i, y^j)\right| = \left|\frac{1}{2\pi}\int_0^{2\pi} \sum A_{ij}\, g_i(\theta)\overline{G_j(\theta)}\,d\theta\right| \leqq \sqrt{2e}.$$

This proof is only apparently new, because it is just a dual version of a proof given by Pelczynski [22, p. 20]; our method is based on more explicit procedures, however, and it yields a better constant. A similar proof has been given by Blei [3]. Finally, we mention that we can modify the above procedure to obtain the inequality with $K_G^C=2$ rather than $\sqrt{2e}$; we omit the proof because it is known [27] that the inequality holds with $K_G^C \leqq e^{1-\gamma}$, where $\gamma$ is Euler's constant.

*Remark.* The following matters have come to my attention since this paper was written. First, Pichorides (*Bull. Greek Math. Soc.* **19** (1978), 247—277.) has modified the new method of [26] to obtain another proof of the estimate $\|f\|_1 \geqq A(\log N)^{1/2}$ when $f \in L_F^1$, where $F$ has $N$ elements, and $|\hat{f}(n)| \geqq 1$ for all $n$ in $F$. Next, A. Baernstein II and Eric Sawyer have independently improved on Dixon's estimate [6], when $n_{k+1}-n_k \geqq n_k - n_{k-1}$ for all $k$, by showing that, in this case, $\|f\|_1 \geqq A \log N$. The construction in [8], which is described briefly on pp.211—212 above, and used on p.214, turns out to have been discovered earlier by J.M. Clunie [*Proc. London Math. Soc.* **14A** (1965), 58—68.] Finally, the essential cases of Theorem 6 above and Theorem 1 of [10] were proved earlier, by other methods, by Y. Meyer (*Ann. scient. École Norm. Sup.* **(4)**1 (1968), 499—588. (See p.533)).

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