The Annals of Mathematical Statistics
1968, Vol. 39, No. 6, 2094–2097

NOTES

ON A THEOREM OF SKOROHOD[1]

BY LESTER E. DUBINS

University of California, Berkeley

1. Skorohod in [2], p. 180, found for each L_2-martingale $X_1, X_2, \cdots$ of mean 0 with independent increments, a sequence of stopping times $\tau_1 \leqq \tau_2 \leqq \cdots$ for standard Brownian motion $B(t)$ such that (i): $B(\tau_1), B(\tau_2), \cdots$ has the same joint distribution as the martingale and (ii): each τ_j has a finite expectation. (And David Freedman and Strassen [3], p. 318, noted that the assumption of independent increments may be dropped.) The stopping times Skorohod found depend upon a random variable independent of the Brownian motion $B(t)$. The point of this note is to exhibit equally effective stopping times τ_j whose moment of stopping, $\tau_j(\omega)$, depends only on the path ω, and not on a random variable independent of the Brownian motion. The construction incidentally realizes the martingale $X_1, X_2, \cdots$ inside Brownian motion in a natural way even if the X_j do not have finite second moments, and indeed sometimes even if they have no first moment. What is essential is that $E[X_{n+1} - X_n \mid X_1, \cdots, X_n]$ be 0, in which event, the process is *fair*; but it is not essential that the increment $X_{n+1} - X_n$ itself have a mean. Moreover, as will also be evident, the *same* stopping times τ_j embed the discrete-time martingale $X_1, X_2, \cdots$ inside any continuous-time martingale $M(t)$ that resembles Brownian motion in having continuous, unbounded paths with $M(0) \equiv 0$.

2. Let $\mathcal{C}$ be the set of all continuous, real-valued functions ω defined for $0 \leqq t < \infty$ which are unbounded from above and from below. Of course, if $\mathcal{C}$ is endowed with its natural σ-field, on which a countably-additive probability is given, then $\omega(t)$, or, more precisely, the set of evaluation maps $\omega \rightarrow \omega(t)$, becomes a stochastic process. All continuous processes in this note are to be understood to be of this form.

A *lottery* is a probability measure on the real line.

The program is to define for every lottery μ with a finite expectation $E(\mu)$, and every $\omega \varepsilon \mathcal{C}$, a nonnegative real number $\tau(\mu, \omega) = \tau(\mu)(\omega)$, so that $\tau(\mu)$ is a stopping time such that, for every martingale $\omega(t)$ with unbounded, continuous paths, and with $\omega(0) \equiv E(\mu)$, the map $\omega \rightarrow \omega(\tau(\mu, \omega))$ has μ for its distribution.

To define $\tau(\mu)$, it is convenient to introduce μ^+ and μ^- for the conditional

Received 13 February 1968.

[1] Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant 1312-67.

distribution of μ given $[E(\mu), \infty)$ and $(-\infty, E(\mu))$ respectively. (In the trivial case that μ has only one point in its support, let $\mu^- = \mu^+ = \mu$.)

As another preliminary, for any set K of n-tuples, let $(m; K)$ be the set of $(n+1)$-tuples of the form (m, x) where x is an element of K. Now introduce, for each μ and positive integer n, a finite set of n-tuples of real numbers $H_n(\mu)$, thus.

Let $H_1(\mu)$ contain only the 1-tuple $E(\mu)$; and, letting $K_n(\mu)$ be the union of $H_n(\mu^+)$ with $H_n(\mu^-)$, $H_{n+1}(\mu)$ is defined to be $(E(\mu); K_n(\mu))$.

Here is the main definition.

Let $\tau(\mu, \omega)$ be the least t such that for all positive integers n, there is an n-tuple $t_1 \leqq t_2 \leqq \cdots \leqq t_n \leqq t$ for which $(\omega(t_1), \cdots, \omega(t_n))$ is an element of $H_n(\mu)$. As is easily verified: for all μ with finite expectations, and all $\omega \varepsilon \mathcal{C}$, $\tau(\mu, \omega)$ is a finite, nonnegative, real number; and $\tau(\mu)$ is a stopping time, the *natural stopping time* for μ.

3. As is easily verified, for each n and $x \varepsilon H_n(\mu)$ there is a real number y, such that the $(n+1)$-tuple x followed by y, is an element of $H_{n+1}(\mu)$, and every element of $H_{n+1}(\mu)$ is of this form.

LEMMA[2] 1. *If $H_n(\mu) = H_n(\mu')$ for all n, then $\mu = \mu'$.*

PROOF. Let Q be the set of all x such that $\mu[x, \infty) = \mu'[x, \infty)$. Since $H_2(\mu) = H_2(\mu')$, $E(\mu) = E(\mu')$ and $E(\mu) \varepsilon Q$, as is easily verified. Let $S_n(\mu)$ be the set of x such that for some $(x_1, \cdots, x_n) \varepsilon H_n(\mu)$, $x_n = x$. Verify that because $H_{n+1}(\mu) = H_{n+1}(\mu')$, $S_n(\mu) = S_n(\mu')$ and $S_n(\mu) \subset Q$. So if $S_\infty(\mu)$ is the union over n of the $S_n(\mu)$, $S_\infty(\mu) \subset Q$. Moreover, $S_\infty(\mu)$ is dense in the support of μ, as is easily seen. The problem is to see that every x in the support of μ is in Q. This is now apparent if there is no, or at most one, point x of positive probability under μ; it becomes apparent in general by first verifying that if $x' < x''$ are two points of positive probability under μ, then $\exists\, x \varepsilon S_\infty(\mu)$ such that $x' \leqq x \leqq x''$. Since every x in the support of μ is in Q, $\mu = \mu'$, and the lemma is proven.

PROPOSITION 1. *Let μ be a lottery with a finite mean m. Then for every continuous martingale $\omega(t)$ with unbounded paths and with $\omega(0) \equiv m$, the stopped random variable $\omega \to \omega(\tau(\mu, \omega))$ has μ for its distribution. If the martingale $\omega(t)$ is standard Brownian motion, the expectation of $\tau(\mu)$ is the variance of μ.*

PROOF. To establish the first assertion, it suffices, according to Lemma 1, to show that $H_n(\mu) = H_n(\mu')$ for all n, where μ' is the distribution of $\omega(\tau(\mu, \omega))$. Details are given here only for μ with compact support for $n = 1$, and the modifications necessary for the general case are left to the reader.

As established by Doob [1], p. 382, almost all paths ω of the martingale are unbounded from above and from below, so $\tau(\mu, \omega)$ is well defined. Plainly, the infimum of t and $\tau(\mu, \omega)$, namely $t \wedge \tau(\mu, \omega)$ is $\tau(\mu, \omega)$ for all sufficiently large t. So $\omega^*(t) = \omega(t \wedge \tau(\mu, \omega)$ is $\omega(\tau(\mu, \omega))$ for all such t. By familiar theory, $\omega^*(t)$ is

[2] I thank Isaac Meilijson and Friedrich W. Scholz for helping me see the role of Lemma 1.

a martingale with expectation $E(\omega^*(t)) = E(\omega^*(0)) = m$. Under the assumption that μ has compact support, ω^* is uniformly bounded. Since it converges almost surely to $\omega(\tau(\mu, \omega))$, the limiting random variable also has mean m. So $H_1(\mu) = H_1(\mu')$.

For the proof of the second part of Proposition 1, a definition and a lemma are useful. Let $\tau(n) = \tau(n, \mu)$ be the least t such that there is an n-tuple $t_1 \leqq t_2 \leqq \cdots \leqq t_n = t$ for which $\omega(t_1), \cdots, \omega(t_n)$ is an element of $H_n(\mu)$. Since the $\tau(n)$ are stopping times with $\tau(n) \leqq \tau(n+1)$, and since for each n, $\omega(t \wedge \tau(n))$ is uniformly bounded in t, standard theory applies to show that $\omega(\tau(1)), \omega(\tau(2)), \cdots$ is a martingale.

As is easily verified, for any convex function φ, $E(\varphi(\omega(\tau(n))))$ is majorized by $\int \varphi \, d\mu$, which, according to the first part of this proposition, is the same as $E(\varphi(\omega(\tau)))$. So $E|\omega(\tau(n))| \leqq E|\omega(\tau)|$; and, of course $\omega(\tau(n)) \rightarrow \omega(\tau)$. Therefore one has

LEMMA 2. *$\omega(\tau(1)), \omega(\tau(2)), \cdots, \omega(\tau)$ is a martingale.*

Incidentally, the distribution of the discrete martingale $\omega(\tau(1)), \omega(\tau(2)), \cdots$ plainly depends only on μ, and in particular, not on the distribution of the continuous martingale $\omega(t)$. In contrast, the distribution of $\tau(n)$ and of τ, plainly do depend on the distribution of $\omega(t)$.

Now suppose that $\omega(t)$ is standard, Brownian motion and, for simplicity, assume that μ has mean 0. Then, as observed by Paul Lévy, $\omega^2(t) - t$ is a martingale of mean 0. The immediate problem is to see that, for each n,

$$E(\omega^2(\tau)(n)) = E(\tau(n)). \tag{1}$$

To see (1), notice first that

$$E(\omega^2(t \wedge \tau(n)) - t \wedge \tau(n)) = 0 \quad \text{for each} \quad t. \tag{2}$$

Of course, $t \wedge \tau(n) \uparrow \tau(n)$, and $\omega^2(t \wedge \tau(n)) \rightarrow \omega^2(\tau(n))$ as $t \rightarrow \infty$. Since the first convergence is monotone in t, $E(t \wedge \tau(n)) \uparrow E(\tau(n))$. And since $\omega^2(t \wedge \tau(n))$ is uniformly bounded in t, $E(\omega^2(t \wedge \tau(n)))$ converges to $E(\omega^2(\tau(n)))$. As now follows from (2), (1) does indeed hold.

Since $\omega(\tau(1)), \omega(\tau(2)), \cdots \rightarrow \omega(\tau)$, Lemma (2) implies that $E(\omega^2(\tau(n))) \rightarrow E(\omega^2(\tau))$. Plainly, $E(\tau(n)) \rightarrow E(\tau)$. Hence, in view of (1), $E(\omega^2(\tau)) = E(\tau)$. And, by the first part of this proposition, $E(\omega^2(\tau)) = \int x^2 \, d\mu(x)$. Hence $E(\tau)$ is the variance of μ.

Let $X_1, X_2, \cdots$ be a stochastic process. If μ_0, the distribution of X_1, as well as $\mu_n(x_1, \cdots, x_n)$, a regular conditional distribution of $X_{n+1} - X_n$ given $X_j = x_j$, $1 \leqq j \leqq n$, have mean 0, the process is *fair*.

Let $\tau(\mu_0)$ and $\tau(\mu_n(x_1, \cdots, x_n))$ be the natural stopping times associated with μ_0 and $\mu_n(x_1, \cdots, x_n)$, and let their values at ω be written as $\tau(\mu_0; \omega)$ and $\tau(\mu_n(x_1, \cdots, x_n); \omega)$.

Now define an increasing sequence of stopping times for $\mathcal{C}$, $\tau_0 \leqq \tau_1 \leqq \cdots$, thus. Let $\tau_0 = \tau(\mu_0)$, and let

$$\tau_n(\omega) = \tau_{n-1}(\omega) + \tau(\mu_n(x_1', \cdots, x_n'); \omega'), \tag{3}$$

where $x'_{j+1} = \omega(\tau_j(\omega))$, and $\omega'(s)$, or more fully, $\omega_n'(s)$, is $\omega(s + \tau_{n-1}(\omega)) - \omega(\tau_{n-1}(\omega))$.

THEOREM. *Let $X_1, X_2, \cdots$ be a martingale of mean 0, or more generally, a fair stochastic process, and let $\omega(t)$ be a martingale with continuous, unbounded paths for $0 \leqq t < \infty$ and $\omega(0) \equiv 0$. Then $\omega(\tau_0), \omega(\tau_1), \cdots$ has the same joint distribution as does $X_1, X_2, \cdots$. Moreover, if each X_{j+1} has a variance $v_j < \infty$, and if the martingale ω is standard Brownian motion, then the expected value of τ_j is finite and equals v_j.*

PROOF. That $\omega(\tau_0)$ has the same distribution as X_1 is immediate from Proposition 1. Turn now to the conditional distribution of the increment $\omega(\tau_{n+1}) - \omega(\tau_n)$ given the history of the martingale ω up to time τ_n. As is not difficult to verify, for any stopping time s, the conditional distribution of the future of a martingale (with continuous and unbounded paths), given its past up to time s, is again almost surely that of a martingale (with continuous and unbounded paths). From this fact, together with Proposition 1 and standard-type argumentation involving conditional distributions, it follows that the conditional distribution of $\omega(\tau_{n+1}) - \omega(\tau_n)$ given the past of ω until time τ_n is $\mu_{n+1}(x_1', \cdots, x'_{n+1})$, where x'_{j+1} is $\omega(\tau_j(\omega))$. Since the conditional distribution of $X_{n+1} - X_n$ given $X_1, \cdots, X_n$ is $\mu_n(X_1, \cdots, X_n)$, the unconditional joint distribution of $\omega(\tau_0)$, $\omega(\tau_1), \cdots$ is the same as that of $X_1', X_2, \cdots$. This completes the proof of the first assertion of Theorem 1. The second assertion is provable by an argument similar to the one which proves the second assertion of Proposition 1.

REFERENCES

[1] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[2] SKOROHOD, A. B. (1965). *Studies in the Theory of Random Processes.* Addison-Wesley, Reading.

[3] STRASSEN, VOLKER (1967). Almost sure behavior of sums of independent random variables and martingales. *Proc. Fifth Berkeley Symp. Math. Statist.* **2** 315–343 Univ. of California Press.