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On a type of generalized closed sets

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ABSTRACT: The purpose of this paper is to introduce and study a new class of generalized closed sets in a topological space X, defined in terms of a grill $\mathcal{G}$ on X. Explicit characterization of such sets along with certain other properties of them are obtained. As applications, some characterizations of regular and normal spaces are achieved by use of the introduced class of sets.

Key Words: Grill, g-closed, topology $\tau_{\mathcal{G}}$, operator Φ, $\mathcal{G}$-g-closed sets.

Contents

1. Introduction and prerequisites

It is found from literature that during recent years many topologists are interested in the study of generalized types of closed sets. For instance, a certain form of generalized closed sets was initiated by Levine [4], whereas the notion of α-generalized closed sets was studied by Maki et al. [5]. Following the trend, we have introduced and investigated a kind of generalized closed sets, the definition being formulated in terms of grills. The concept of grill was first introduced by Choquet [3] in 1947. From subsequent investigations it is revealed that grills can be used as an extremely useful device for investigation of a number of topological problems, like extension of spaces, theory of proximity spaces and so on (see for instance, [1], [2], [12] for details). The definition of grill goes as follows.

Definition 1.1 *[3] A nonempty collection $\mathcal{G}$ of nonempty subsets of a topological space X is called a grill if*
(i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$, and (ii) $A, B \subseteq X$ and $A \bigcup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Let $\mathcal{G}$ be a grill on a topological space (X, τ). In [9] an operator $\Phi : P(X) \rightarrow P(X)$ (where $P(X)$ stands for the power set of X) was defined by $\Phi(A) = \{ x \in X : U \bigcap A \in \mathcal{G}$ for all open set U containing $x\}$. It was also shown in the same paper that the map $\psi : P(X) \rightarrow P(X)$, given by $\psi(A) = A \bigcup \Phi(A)$ (for $A \in P(X)$), is a Kuratowski closure operator determining a topology $\tau_{\mathcal{G}}$ (say) on X, strictly finer

2000 *Mathematics Subject Classification:* 54A05, 54A10, 54D10, 54D15 than τ. Thus a subset A of X is $\tau_{\mathcal{G}}$-closed (resp. $\tau_{\mathcal{G}}$-dense in itself) if $\psi(A) = A$ or equivalently if $\Phi(A) \subseteq A$ (resp. $A \subseteq \Phi(A)$).

In the next section we introduce and study a new class of generalized closed sets, termed $\mathcal{G}$-g-closed, in terms of a given grill $\mathcal{G}$ on the ambient space, the definition having a close bearing to the above operator Φ. This class of $\mathcal{G}$-g-closed sets will be seen to properly contain the class of g-closed sets as introduced in [4]. An explicit form of such a $\mathcal{G}$-g-closed set is also obtained . As applications, some formulations of certain separation axioms in terms of $\mathcal{G}$-g-closed sets and associated concepts will be established in Section 3.

Throughout the paper, by a space X we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, we shall adopt the usual notations $\text{int}(A)$ and $\text{cl}(A)$ respectively for the interior and closure of A in (X, τ). Again, $\tau_{\mathcal{G}}$-cl(A) and $\tau_{\mathcal{G}}$-int(A) will respectively denote the closure and interior of A in $(X, \tau_{\mathcal{G}})$. Similarly, whenever we say that a subset A of a space X is open (or closed), it will mean that A is open (resp. closed) in (X, τ). For open and closed sets with respect to any other topology on X, e.g. $\tau_{\mathcal{G}}$, we shall write '$\tau_{\mathcal{G}}$-open' and '$\tau_{\mathcal{G}}$-closed'. The collection of all open neighbourhoods of a point x in (X, τ) will be denoted by $\tau(x)$. A subset A of a space (X, τ) is said to be α-open [7] (preopen [6]) if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ (resp. $A \subseteq \text{int}(\text{cl}(A))$). The family of all α-open sets in (X, τ), denoted by τ^α, is known [7] to be a topology on X finer than τ, and the closure of A in (X, τ^α) is denoted by α-cl(A).

We now append a few definitions and results that will be frequently used in the sequel.

Definition 1.2 *A subset A of a space (X, τ) is said to be g-closed [4] (αg-closed [5]) if $cl(A) \subseteq U$ (resp. α-$cl(A) \subseteq U$) whenever $A \subseteq U$ and U is open.*

The complement of a g-closed (αg-closed) set is called a g-open (resp. an αg-open) set.

Theorem 1.3 *[9] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for any A, $B \subseteq X$ the following hold:*
(a) $A \subseteq B \Rightarrow \Phi(A) \subseteq \Phi(B)$.
(b) $\Phi(A \bigcup B) = \Phi(A) \bigcup \Phi(B)$.
(c) $\Phi(\Phi(A)) \subseteq \Phi(A) = cl(\Phi(A)) \subseteq cl(A)$, and hence $\Phi(A)$ is closed in (X, τ), for all $A \subseteq X$.

Theorem 1.4 *[9] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then $\beta(\mathcal{G}, \tau) = \{V \backslash A \colon V \in \tau \text{ and } A \notin \mathcal{G}\}$ is an open base for $\tau_{\mathcal{G}}$.*

2. Generalized closed sets with respect to a grill

We begin by introducing a new class of generalized closed sets in terms of grills as follows.

Definition 2.1 *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then a subset A of X is said to be g-closed with respect to the grill $\mathcal{G}$ ($\mathcal{G}$-g-closed, for short) if $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*

A subset A of X is said to be $\mathcal{G}$-g-open if $X \setminus A$ is $\mathcal{G}$-g-closed.

Remark 2.2 *For a topological space (X,τ) and a grill $\mathcal{G}$ on X, we obtain as follows.*
(a) Every closed set in X is $\mathcal{G}$-g-closed.
(b) As $\Phi(\Phi(A)) \subseteq \Phi(A)$, $\Phi(A)$ is $\mathcal{G}$-g-closed for any subset A of X, and hence every $\tau_{\mathcal{G}}$-closed set is $\mathcal{G}$-g-closed.
(c) It is known [9] that for any $A \notin \mathcal{G}$, $\Phi(A) = \phi$. Thus any non-member of $\mathcal{G}$ is $\mathcal{G}$-g-closed.
(d) Every g-closed set is $\mathcal{G}$-g-closed; that the converse is false is shown by the following example.

Example 2.3 *Let $X=\{a,b,c\}$, $\tau=\{\phi, \{b\}, \{b,c\}, X\}$ and $\mathcal{G} = \{\{a\},\{c\},\{a,c\},\{a,b\}, \{b,c\},X\}$. Then (X,τ) is a topological space and $\mathcal{G}$ is a grill on X. Suppose $A = \{b\}$. Then it is easy to verify that A is not g-closed but is $\mathcal{G}$-g-closed. In fact, $A \subseteq \{b,c\}$ but $cl(A)=X \not\subseteq \{b,c\}$, and $\Phi(A) = \phi$.*

Corresponding to any nonempty subset A of X, a typical grill $[A]$ on X was defined in [10] in the following manner.

Definition 2.4 *Let X be a space and $(\phi \neq)A \subseteq X$. Then*

$$[A] = \{B \subseteq X : A \bigcap B \neq \phi\}$$

is a grill on X, called the principal grill generated by A.

Remark 2.5 *In the case of principal grill $[X]$ generated by X, it is known [10] that $\tau = \tau_{[X]}$, so that any $[X]$-g-closed set becomes simply a g-closed set and vice-versa.*

Next we observe that in a space X, $\mathcal{G}_\delta = \{A \subseteq X : \mathrm{int}(\mathrm{cl}(A)) \neq \phi\}$ is a grill on X, and for this grill we have:

Theorem 2.6 *Let (X,τ) be a topological space and $A \subseteq X$. Then $\tau_{\mathcal{G}_\delta} = \tau^\alpha$ and hence a subset A of X is $\mathcal{G}_\delta$-g-closed iff A is αg-closed.*

Proof: It is well known that for any subset A of a space (X,τ), α-cl$A = A \bigcup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$. Now, with the grill $\mathcal{G}_\delta$ we have for any $A \subseteq X$,
$\Phi_\delta(A)= \{x \in X : U \bigcap A \in \mathcal{G}_\delta, \forall\, U \in \tau(x)\} = \{x \in X : int(cl(U \bigcap A)) \neq \phi, \forall\, U \in \tau(x)\} = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$. Thus $\tau_{\mathcal{G}_\delta}$-cl$(A) = A \bigcup \Phi(A) = A \bigcup cl(int(cl(A))) = \alpha$-cl$(A)$. Hence $\tau_{\mathcal{G}_\delta} = \tau^\alpha$. □

In what follows in this section, we derive certain characterizations and properties of $\mathcal{G}$-g-closed sets.

Theorem 2.7 *Let (X,τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for a subset A of X, the following are equivalent:*
(a) A is $\mathcal{G}$-g-closed.
(b) $\tau_{\mathcal{G}}$-cl$(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
(c) For all $x \in \tau_{\mathcal{G}}$-cl(A), cl$(\{x\}) \bigcap A \neq \phi$.
(d) $\tau_{\mathcal{G}}$-cl$(A) \setminus A$ contains no nonempty closed set of (X,τ).
(e) $\Phi(A) \setminus A$ contains no nonempty closed set of (X,τ).

Proof: (a)⇒(b): Suppose A is $\mathcal{G}$-g-closed and $A \subseteq U$ where U is open in (X, τ). Then $\Phi(A) \subseteq U$ so that $\tau_{\mathcal{G}}$-cl$(A)= A \bigcup \Phi(A) \subseteq U$.
(b)⇒(c): Suppose $x \in \tau_{\mathcal{G}}$-cl(A). If cl$(\{x\}) \bigcap A = \phi$, then $A \subseteq X\setminus$ cl$(\{x\})$ and using (b), $\tau_{\mathcal{G}}$-cl$(A) \subseteq X\setminus$ cl$(\{x\})$, a contradiction since $x \in \tau_{\mathcal{G}}$-cl(A).
(c)⇒(d): Suppose F is a closed set of (X, τ) contained in $\tau_{\mathcal{G}}$-cl$(A) \setminus A$ and $x \in F$. Since $F \bigcap A = \phi$, we have cl$(\{x\}) \bigcap A = \phi$. Again since $x \in \tau_{\mathcal{G}}$-cl(A), by (c) we have cl$(\{x\}) \bigcap A \neq \phi$, a contradiction. This proves (d).
(d)⇒(e): It follows from the fact that $\tau_{\mathcal{G}}$-cl$(A) \setminus A = \Phi(A) \setminus A$.
(e)⇒(a): Suppose that $A \subseteq U$ and U is open in (X, τ). Since $\Phi(A)$ is closed (by Theorem 1.3.) and $\Phi(A) \bigcap (X \setminus U) \subseteq \Phi(A) \setminus A$ holds, $\Phi(A) \bigcap (X \setminus U)$ is a closed set in (X, τ) contained in $\Phi(A) \setminus A$. Then by (e), $\Phi(A) \bigcap (X \setminus U) = \phi$ which gives $\Phi(A) \subseteq U$. Hence A is $\mathcal{G}$-g-closed. □

Corollary 2.8 *Let (X, τ) be a T_1-space and $\mathcal{G}$ be a grill on X. Then every $\mathcal{G}$-g-closed set is $\tau_{\mathcal{G}}$-closed.*

Proof: Follows from Theorem 2.7 ((a)⇒(c)). □

Corollary 2.9 *Let (X, τ) be a T_1-space and $\mathcal{G}$ be a grill on X. Then A $(\subseteq X)$ is $\mathcal{G}$-g-closed iff A is $\tau_{\mathcal{G}}$-closed.*

Corollary 2.10 *Let $\mathcal{G}$ be grill on a space (X, τ) and A be a $\mathcal{G}$-g-closed set. Then the following are equivalent:*
(a) A is $\tau_{\mathcal{G}}$-closed.
(b) $\tau_{\mathcal{G}}$-cl$(A) \setminus A$ is closed in (X, τ).
(c) $\Phi(A) \setminus A$ is closed in (X, τ).

Proof: (a)⇒(b): If A is $\tau_{\mathcal{G}}$-closed, then $\tau_{\mathcal{G}}$-cl$(A) \setminus A = \phi$ and so $\tau_{\mathcal{G}}$-cl$(A) \setminus A$ is a closed set.
(b)⇔(c): It is clear, since $\tau_{\mathcal{G}}$-cl$(A) \setminus A = \Phi(A) \setminus A$.
(c)⇒(a): If $\Phi(A) \setminus A$ is closed in (X, τ) and A is $\mathcal{G}$-g-closed, then by Theorem 2.7, $\Phi(A) \setminus A = \phi$ and so A is $\tau_{\mathcal{G}}$-closed. □

Theorem 2.11 *Let $\mathcal{G}$ be a grill on a space (X, τ). Then A $(\subseteq X)$ is $\mathcal{G}$-g-open iff $F \subseteq \tau_{\mathcal{G}}$-int$(A)$ whenever $F \subseteq A$ and F is closed.*

Proof: Let A be $\mathcal{G}$-g-open and $F \subseteq A$, where F is closed in (X, τ). Then $X \setminus A \subseteq X \setminus F \Rightarrow \Phi(X \setminus A) \subseteq (X \setminus F) \Rightarrow \tau_{\mathcal{G}}$-cl$(X \setminus A) \subseteq (X \setminus F) \Rightarrow F \subseteq \tau_{\mathcal{G}}$-int$(A)$. □

Conversely, $X \setminus A \subseteq U$, where U is open in $(X, \tau) \Rightarrow X \setminus U \subseteq \tau_{\mathcal{G}}$-int$A \Rightarrow \tau_{\mathcal{G}}$-cl$(X \setminus A) \subseteq U$. Thus $(X \setminus A)$ is $\mathcal{G}$-g-closed and hence A is $\mathcal{G}$-g-open.

Lemma 2.12 *Let (X, τ) be a space and $\mathcal{G}$ be a grill on X. If A $(\subseteq X)$ is $\tau_{\mathcal{G}}$-dense in itself, then $\Phi(A) = cl(\Phi(A)) = \tau_{\mathcal{G}}$-$cl(A) = cl(A)$.*

Proof: A is $\tau_{\mathcal{G}}$-dense in itself $\Rightarrow A \subseteq \Phi(A) \Rightarrow \text{cl}(A) \subseteq \text{cl}(\Phi(A)) = \Phi(A) \subseteq \text{cl}(A)$ (using Theorem 1.3) $\Rightarrow \text{cl}(A) = \Phi(A) = \text{cl}(\Phi(A))$. Again, $\tau_{\mathcal{G}}\text{-cl}(A) = A \bigcup \Phi(A) = A \bigcup \text{cl}(A) = \text{cl}(A)$. Consequently, $\Phi(A) = \text{cl}(\Phi(A)) = \tau_{\mathcal{G}}\text{-cl}(A) = \text{cl}(A)$. □

Theorem 2.13 *Let $\mathcal{G}$ be a grill on a space (X, τ). If A $(\subseteq X)$ is $\tau_{\mathcal{G}}$-dense in itself and $\mathcal{G}$-g-closed, then A is g-closed.*

Proof: Follows at once from Lemma 2.12. □

Corollary 2.14 *For a grill $\mathcal{G}$ on a space (X, τ), let A $(\subseteq X)$ be $\tau_{\mathcal{G}}$-dense in itself. Then A is $\mathcal{G}$-g-closed iff it is g-closed.*

Theorem 2.15 *For any grill $\mathcal{G}$ on a space (X, τ) the following are equivalent:*
(a) Every subset of X is $\mathcal{G}$-g-closed.
(b) Every open subset of (X, τ) is $\tau_{\mathcal{G}}$-closed.

Proof: **(a)⇒(b):** Suppose A is open in (X, τ). Then by (a), A is $\mathcal{G}$-g-closed so that $\Phi(A) \subseteq A$. Then A is $\tau_{\mathcal{G}}$ -closed.
(b)⇒(a): Let $A \subseteq X$ and U be open in (X, τ) such that $A \subseteq U$. Then by (b), $\Phi(U) \subseteq U$. Again, $A \subseteq U \Rightarrow \Phi\ (A) \subseteq \Phi(U)$ (by Theorem 1.3) $\subseteq U$, which implies that A is $\mathcal{G}$-g-closed. □

Theorem 2.16 *For any subset A of a space (X, τ) and a grill $\mathcal{G}$ on X, the following are equivalent:*
(a) A is $\mathcal{G}$-g-closed.
(b) $A \bigcup (X \backslash \Phi(A))$ is $\mathcal{G}$-g-closed.
(c) $\Phi(A) \backslash A$ is $\mathcal{G}$-g-open.

Proof: **(a)⇒(b):** Suppose $A \bigcup (X \backslash \Phi(A)) \subseteq U$, where U is open in X. Then $X \backslash U \subseteq X \backslash (A \bigcup (X \backslash \Phi(A))) = \Phi(A) \backslash A$. Since A is $\mathcal{G}$-g-closed, by Theorem 2.7 we have $X \backslash U = \phi$, i.e., $X = U$. Since X is the only open set containing $A \bigcup (X \backslash \Phi(A))$, $A \bigcup (X \backslash \Phi(A))$ is $\mathcal{G}$-g-closed.
(b)⇒(a): Suppose $F \subseteq \Phi(A) \backslash A$ where F is closed in (X, τ). Then $A \bigcup (X \backslash \Phi(A)) \subseteq X \backslash F$ and so by (b), $\Phi(A \bigcup (X \backslash \Phi(A))) \subseteq X \backslash F \Rightarrow \Phi(A) \bigcup \Phi(X \backslash \Phi(A)) \subseteq X \backslash F \Rightarrow F \subseteq X \backslash \Phi(A)$. Again, since $F \subseteq \Phi(A)$ we have $F = \phi$. Hence by Theorem 2.7, A is $\mathcal{G}$-g-closed.
(b)⇔(c) : Follows from the fact that $X \backslash (\Phi(A) \backslash A) = A \bigcup (X \backslash \Phi(A))$. □

Theorem 2.17 *Let (X, τ) be a space, $\mathcal{G}$ be a grill on X and A, B be subsets of X such that $A \subseteq B \subseteq \tau_{\mathcal{G}}$-cl$(A)$. If A is $\mathcal{G}$-g-closed, then B is $\mathcal{G}$-g-closed.*

Proof: Suppose $B \subseteq U$, where U is open in X. Since A is $\mathcal{G}$-g-closed, $\Phi(A) \subseteq U \Rightarrow \tau_{\mathcal{G}}\text{-cl}(A) \subseteq U$. Now, $A \subseteq B \subseteq \tau_{\mathcal{G}}\text{-cl}(A) \Rightarrow \tau_{\mathcal{G}}\text{-cl}(A) \subseteq \tau_{\mathcal{G}}\text{-cl}(B) \subseteq \tau_{\mathcal{G}}\text{-cl}(A)$. Thus $\tau_{\mathcal{G}}\text{-cl}(B) \subseteq U$ and hence B is $\mathcal{G}$-g-closed. □

Corollary 2.18 *$\tau_{\mathcal{G}}$-closure of every $\mathcal{G}$-g-closed set is $\mathcal{G}$-g-closed.*

Theorem 2.19 *Let $\mathcal{G}$ be a grill on a space (X,τ) and A, B be subsets of X such that $A \subseteq B \subseteq \Phi(A)$. If A is $\mathcal{G}$-g-closed, then A and B are g-closed.*

Proof: $A \subseteq B \subseteq \Phi(A) \Rightarrow A \subseteq B \subseteq \tau_{\mathcal{G}}$-cl$(A)$, and hence by Theorem 2.17, B is $\mathcal{G}$-g-closed. Again, $A \subseteq B \subseteq \Phi(A) \Rightarrow \Phi(A) \subseteq \Phi(B) \subseteq \Phi(\Phi(A)) \subseteq \Phi(A)$ (by Theorem 1.3) $\Rightarrow \Phi(A) = \Phi(B)$. Thus A and B are $\tau_{\mathcal{G}}$-dense in itself and hence by Theorem 2.13, A and B are g-closed. □

The following result gives a precise form of a $\mathcal{G}$-g-closed set.

Theorem 2.20 *Let $\mathcal{G}$ be a grill on a space (X,τ). Then A ($\subseteq X$) is $\mathcal{G}$-g-closed iff $A = F\backslash N$ where F is $\tau_{\mathcal{G}}$-closed and N is a set not containing any nonempty closed set.*

Proof: Let A be $\mathcal{G}$-g-closed. Then by Theorem 2.7, $\Phi(A)\backslash A = N$ (say) contains no nonempty closed set. If $F = \tau_{\mathcal{G}}$-cl(A), then F is a $\tau_{\mathcal{G}}$-closed set such that $F\backslash N = A \bigcup \Phi(A)\backslash(\Phi(A)\backslash A) = A$.
Conversely, suppose that $A = F\backslash N$,where F is $\tau_{\mathcal{G}}$-closed and N cotains no nonempty closed set. Suppose that $A \subseteq U$ where U is open in X. Then $F\backslash N \subseteq U$ which implies that $F \bigcap (X\backslash U) \subseteq N$. Now, $A \subseteq F$ and $\Phi(F) \subseteq F$ imply that $\Phi(A) \bigcap (X\backslash U) \subseteq \Phi(F) \bigcap (X\backslash U) \subseteq F \bigcap (X\backslash U) \subseteq N$. Since $\Phi(A) \bigcap (X\backslash U)$ is closed, we have by hypothesis that $\Phi(A) \bigcap (X\backslash U) = \phi$ and hence $\Phi(A) \subseteq U$, proving A to be $\mathcal{G}$-g-closed. □

3. Some characterizations of regular and normal spaces

As already proposed, this section is meant for deriving certain applications of the study in the last section; some characterizations of regular and normal spaces are achieved here in terms of the introduced concept of $\mathcal{G}$-g-closed sets. To that end, we first mention as follows a result of Njastad [7], restated in our present setting.

Lemma 3.1 *For any space (X,τ), $\tau^\alpha = \{V\backslash A \colon V \in \tau$ and $A \notin \mathcal{G}_\delta\}$ where $\mathcal{G}_\delta = \{A \subseteq X :$ intcl $(A) \neq \phi\}$.*

Theorem 3.2 *[11] Let $\mathcal{G}$ be grill on a space (X,τ). Then $PO(X)\backslash\{\phi\} \subseteq \mathcal{G}$ iff $\mathcal{G}_\delta \subseteq \mathcal{G}$, where $\mathcal{G}_\delta = \{A \subseteq X :$ intcl$(A) \neq \phi\}$ and $PO(X)$ denotes the collection of all preopen sets in (X,τ).*

Theorem 3.3 *Let $\mathcal{G}$ be a grill on a space (X,τ) such that $PO(X)\backslash\{\phi\} \subseteq \mathcal{G}$. Then $\tau_{\mathcal{G}} \subseteq \tau^\alpha$.*

Proof: Let $V \in \tau_{\mathcal{G}}$. Then by Theorem 1.4, $V = \bigcup\{V_\alpha\backslash A_\alpha : V_\alpha \in \tau$, and $A_\alpha \notin \mathcal{G}\}$. By Theorem 3.2, $\mathcal{G}_\delta \subseteq \mathcal{G}$. So for each α, $A_\alpha \notin \mathcal{G}_\delta$ which gives, in view of Lemma 3.1 $V_\alpha\backslash A_\alpha \in \tau^\alpha$. Since τ^α is a topology, $V \in \tau^\alpha$. Hence $\tau_{\mathcal{G}} \subseteq \tau^\alpha$. □

The above result helps us to obtain the following characterizations of a normal space in terms of $\mathcal{G}$-g-open sets.

Theorem 3.4 *Let $\mathcal{G}$ be such a grill on a space (X,τ) that $PO(X)\backslash\{\phi\}\subseteq\mathcal{G}$. Then the following are equivalent:*
(a) X is normal.
(b) For each pair of disjoint closed sets F and K, there exist disjoint $\mathcal{G}$-g-open sets U and V such that $F\subseteq U$ and $K\subseteq V$.
(c) For each closed set F and any open set V containing F, there is a $\mathcal{G}$-g-open set U such that $F\subseteq U\subseteq\tau_{\mathcal{G}}$-cl$(U)\subseteq V$.

Proof: (a)⇒(b): It is clear, since every open set is $\mathcal{G}$-g-open.
(b)⇒(c): Let F be a closed set and V an open set in (X,τ) such that $F\subseteq V$. Then F and $X\backslash V$ are disjoint closed sets and so by (b), there exist disjoint $\mathcal{G}$-g-open sets U and W such that $F\subseteq U$ and $X\backslash V\subseteq W$. Since W is $\mathcal{G}$-g-open and $X\backslash V\subseteq W$ where $(X\backslash V)$ is closed, we have by Theorem 2.11, $X\backslash V\subseteq\tau_{\mathcal{G}}$-int$(W)$, and so $X\backslash\tau_{\mathcal{G}}$-int$(W)\subseteq V$. Again, $U\bigcap W=\phi\Rightarrow U\bigcap\tau_{\mathcal{G}}$-int$(W)=\phi$, and so $\tau_{\mathcal{G}}$-cl$(U)\subseteq X\backslash\tau_{\mathcal{G}}$- int$(W)\subseteq V$. Thus $F\subseteq U\subseteq\tau_{\mathcal{G}}$-cl$(U)\subseteq V$, where U is a $\mathcal{G}$-g-open set.
(c)⇒(a): Let F and K be any two disjoint closed sets in X. Then by (c), there is a $\mathcal{G}$-g-open set U such that $F\subseteq U\subseteq\tau_{\mathcal{G}}$-cl$(U)\subseteq X\backslash K$. Since U is $\mathcal{G}$-g-open and $F\subseteq U$ where F is closed in (X,τ), we have by Theorem 2.11 that $F\subseteq\tau_{\mathcal{G}}$-int$(U)$. Again, $PO(X)\backslash\{\phi\}\subseteq\mathcal{G}\Rightarrow\tau_{\mathcal{G}}\subseteq\tau^{\alpha}$ (by Theorem 3.3) $\Rightarrow\tau_{\mathcal{G}}$-int$(U)$, $X\backslash\tau_{\mathcal{G}}$-cl$(U)$ $\in\tau^{\alpha}$. Thus $F\subseteq\tau_{\mathcal{G}}$-int$(U)\subseteq$ int(cl(int($\tau_{\mathcal{G}}$-int(U)))) $=G$ (say), and $K\subseteq X\backslash\tau_{\mathcal{G}}$-cl$(U)\subseteq$ int(cl(int($X\backslash\tau_{\mathcal{G}}$-cl$(U)$))) $=H$ (say). To show that (X,τ) is normal, it now suffices to show that $G\bigcap H=\phi$. In fact, $x\in G\bigcap H\Rightarrow x\in$ int(cl(int($\tau_{\mathcal{G}}$-int(U)))) and $x\in H\Rightarrow x\in$ cl(int($\tau_{\mathcal{G}}$-int(U))) and $x\in H\in\tau\Rightarrow$ there exist $y\in$ int($\tau_{\mathcal{G}}$-int(U)) and $y\in H\subseteq$ cl(int($X\backslash\tau_{\mathcal{G}}$-cl$(U)$)) $\Rightarrow$ int($\tau_{\mathcal{G}}$-int(U)) $\bigcap$ int($X\backslash\tau_{\mathcal{G}}$-cl$(U)$) $\neq\phi\Rightarrow\tau_{\mathcal{G}}$-int$(U)$ $\bigcap(X\backslash\tau_{\mathcal{G}}$-cl$(U))\neq\phi$, a contradiction. □

If in the above theorem we take $\mathcal{G}=\mathcal{G}_{\delta}$, then by using Theorem 2.6 we arrive at the following known result (viz. Theorem 4.1 of [8]).

Corollary 3.5 *For a space (X,τ) the following are equivalent:*
(a) X is normal.
(b) For each pair of disjoint closed sets F and K, there exist disjoint αg-open sets U and V such that $F\subseteq U$ and $K\subseteq V$.
(c) For each closed set F and any open set V containing F, there exists an αg-open set U such that $F\subseteq U\subseteq\alpha$-cl$(U)\subseteq V$.

The following theorem gives characterizations of a normal space in terms of g-open sets, which is a consequence of Theorem 3.4 and Remark 2.5 if one takes $\mathcal{G}=[X]$.

Theorem 3.6 *For a space (X,τ) the following are equivalent:*
(a) X is normal.
(b) For each pair of disjoint closed sets F and K, there exist disjoint g-open sets

U and V such that $F \subseteq U$ and $K \subseteq V$.
(c) For each closed set F and any open set V containing F, there exists a g-open set U such that $F \subseteq U \subseteq cl(U) \subseteq V$.

It is now the turn, as above, to formulate regular spaces in terms of $\tau_{\mathcal{G}}$-open sets and associated concepts.

Theorem 3.7 *Let $\mathcal{G}$ be a grill on a space (X, τ) for which $PO(X) \backslash \{\phi\} \subseteq \mathcal{G}$. Then the following are equivalent:*
(a) X is regular.
(b) For each closed set F and each $x \in X \backslash F$, there exist disjoint $\tau_{\mathcal{G}}$-open sets U and V such that $x \in U$ and $F \subseteq V$.
(c) For each open set V of (X, τ) and each point $x \in V$, there exists a $\tau_{\mathcal{G}}$-open set U such that $x \in U \subseteq \tau_{\mathcal{G}}$-cl$(U) \subseteq V$.

Proof: (a)⇒(b): It is clear as $\tau \subseteq \tau_{\mathcal{G}}$.
(b)⇒(c): Let V be any open in (X, τ) containing a point x of X. Then by hypothesis, there exist disjoint $\tau_{\mathcal{G}}$-open sets U and W such that $x \in U$ and $X \backslash V \subseteq W$. Now, $U \bigcap W = \phi \Rightarrow \tau_{\mathcal{G}}$-cl$(U) \subseteq X \backslash W \subseteq V$. Thus $x \in U \subseteq \tau_{\mathcal{G}}$-cl$(U) \subseteq V$.
(c)⇒(a): Let F be closed and $x \notin F$. Then by hypothesis, there exists a $\tau_{\mathcal{G}}$-open set U such that $x \in U \subseteq \tau_{\mathcal{G}}$-cl$(U) \subseteq X \backslash F$. Let $V = X \backslash \tau_{\mathcal{G}}$-cl(U). Then U and V are disjoint $\tau_{\mathcal{G}}$-open sets. Since $PO(X) \backslash \{\phi\} \subseteq \mathcal{G}$, we have $\tau_{\mathcal{G}} \subseteq \tau^{\alpha}$ (by Theorem 3.3). Thus $x \in U \subseteq$ int(cl(int(U))) $= G$ (say) and $F \subseteq V \subseteq$ int(cl(int(V))) $= H$ (say). As $U \bigcap V = \phi$, we can prove as in '(c)⇒(a)' of Theorem 3.4 that $G \bigcap H = \phi$. Hence X is regular. □

If we set $\mathcal{G} = \mathcal{G}_{\delta}$ in the above theorem, we have the following result which gives characterizations of regular spaces in terms of α-open sets, and which incidentally proves Theorem 5.2 of [8].

Corollary 3.8 *For a space (X, τ) the following are equivalent:*
(a) X is regular.
(b) For any closed set F in (X, τ) and each $x \in X \backslash F$, there exist disjoint α-open sets U, V such that $x \in U$ and $F \subseteq V$.
(c) For each open set V in (X, τ) and each point $x \in V$, there exists an α-open set U such that $x \in U \subseteq \alpha$-cl$(U) \subseteq V$.

Theorem 3.9 *Let $\mathcal{G}$ be a grill on a T_1-space (X, τ) such that $PO(X) \backslash \{\phi\} \subseteq \mathcal{G}$. Then the following are equivalent:*
(a) X is regular.
(b) For each closed set F and each $x \in X \backslash F$, there exist disjoint $\mathcal{G}$-g-open sets U and V such that $x \in U$ and $F \subseteq V$.
(c) For each open set V of (X, τ) and each point $x \in V$, there exists a $\mathcal{G}$-g-open set U such that $x \in U \subseteq \tau_{\mathcal{G}}$-cl$(U) \subseteq V$.

Proof: (a)⇒(b): It is trivial as each open set is $\mathcal{G}$-g-open.
(b)⇒(c): Let V be open in (X, τ) and $x \in V$. Then by (b), there exist disjoint $\mathcal{G}$-g-open sets U and W such that $x \in U$ and $X \backslash V \subseteq W$. As W is $\mathcal{G}$-g-open and $X \backslash V$

is closed with $X \backslash V \subseteq W$, we have by Theorem 2.11 that $X \backslash V \subseteq \tau_{\mathcal{G}}$-int$(W)$, i.e., $X \backslash \tau_{\mathcal{G}}$-int$(W) \subseteq V$. Now, $U \bigcap W = \phi \Rightarrow U \bigcap \tau_{\mathcal{G}}$-int$(W) = \phi \Rightarrow \tau_{\mathcal{G}}$-cl$(U) \subseteq X \backslash \tau_{\mathcal{G}}$-int$(W) \subseteq V$. Thus $x \in U \subseteq \tau_{\mathcal{G}}$-cl$(U) \subseteq V$.

(c)⇒(a): Let F be any closed set not containing a point $x \in X$. Then by (c), there exists a $\mathcal{G}$-g-open set U such that $x \in U \subseteq \tau_{\mathcal{G}}$-cl$(U) \subseteq X \backslash F$. Since (X, τ) is T_1, we have using Theorem 2.11, $\{x\} \subseteq \tau_{\mathcal{G}}$-int(U). Since $PO(X) \backslash \{\phi\} \subseteq \mathcal{G}$, $\tau_{\mathcal{G}} \subseteq \tau^{\alpha}$ (by Theorem 3.3) and so $\tau_{\mathcal{G}}$-int(U) and $X \backslash \tau_{\mathcal{G}}$-cl$(U)$ are α-open sets. Now $x \in \tau_{\mathcal{G}}$-int$(U) \subseteq$ int(cl(int($\tau_{\mathcal{G}}$-int(U)))) $= G$ (say) and $F \subseteq X \backslash \tau_{\mathcal{G}}$-cl$(U) \subseteq$ int(cl(int($X \backslash \tau_{\mathcal{G}}$-cl$(U)$))) $= H$ (say). Since G and H are disjoint open sets in X such that $x \in G$ and $F \subseteq H$, X is regular. □

If we take $\mathcal{G} = \mathcal{G}_{\delta}$ in the above theorem, then by using Theorem 2.6 we have the following result:

Corollary 3.10 *For a T_1-space (X, τ) the following are equivalent:*
(a) X is regular.
(b) For each closed set F in (X, τ) and each $x \in X \backslash F$, there exist disjoint αg-open sets U and V such that $x \in U$ and $F \subseteq V$.
(c) For each open set V and each point $x \in V$, there exists an αg-open set U in X such that $x \in U \subseteq \alpha$-cl$(U) \subseteq V$.

If the principal grill $[X]$ takes the role of $\mathcal{G}$ in Theorem 3.9, then we obtain the following characterizations of a regular space in terms of g-open sets (refer to Remark 2.5).

Corollary 3.11 *For a T_1-space (X, τ) the following are equivalent:*
(a) X is regular.
(b) For each closed set F in (X, τ) and each $x \in X \backslash F$, there exist disjoint g-open sets U and V such that $x \in U$ and $F \subseteq V$.
(c) For each open set V in (X, τ) containing a point x , there exists a g-open set U in X such that $x \in U \subseteq$ cl$(U) \subseteq V$.

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References

1. K. C. Chattopadhyay, O. Njastad and W. J. Thron; *Merotopic spaces and extensions of closure spaces*, Can. J. Math., (No 4) (1983), 613-629.

2. K. C. Chattopadhyay and W. J. Thron; *Extensions of closure spaces*, Can. J. Math., (No 6) (1977), 1277-1286. .

3. G. Choquet, *Sur les notions de filtre et grille*, Comptes Rendus Acad. Sci. Paris, 224 (1947), 171-173.

4. N. Levine; *Generalised closed sets in topology*, Rent. Cire. Mat. Palermo,(2) 19 (1970), 89-96.

5. H. Maki, R. Devi and K. Balachandan; *Associated topologies of generalized α-closed sets and α-generalized closed sets*, Mem. Fac. Sci. Kochi Univ. Ser. A, Math., 15 (1994), 51-63.

6. A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb ; *On precontinuous and weak precontinuous mappings*, Proc. Math. and Phys. Soc. Egypt, 53 (1982), 47-53.

7. O. Njastad; *On some classes of nearly open sets*, Pacific J. Math., 15 (1965) 961-970.

8. T. Noiri; *Almost αg-closed functions and separation axioms*, Acta Math. Hungar., 82(3)(1999), 193-205.

9. B. Roy and M. N. Mukherjee; *On a typical topology induced by a grill*, Soochow J. Math., 33(4)(2007), 771-786.

10. B. Roy and M. N. Mukherjee; *Concerning topologies induces by principal grills*, An. Stiint. Univ. AL. I. Cuza Iasi. Mat.(N. S.), 55 (2) (2009), 285-294.

11. B. Roy, M. N. Mukherjee and S. K. Ghosh ; *On a subclass of preopen sets via grills*; Stud. Si Cercet. Stiint. Ser. Mat. Univ. Bacau, 18 (2008), 255-266.

12. W. J. Thron; *Proximity structures and grills*, Math. Ann., 206 (1973), 35-62.

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