On a Uniform Framework for the Definition of Stochastic Process Languages*

Rocco De Nicola[1], Diego Latella[2], Michele Loreti[1], and Mieke Massink[2]

[1] Dipartimento di Sistemi e Informatica - Università di Firenze
[2] Istituto di Scienza e Tecnologie dell'Informazione "A. Faedo"- CNR

Abstract. In this paper we show how *Rate Transition Systems* (RTSs) can be used as a unifying framework for the definition of the semantics of stochastic process algebras. RTSs facilitate the *compositional* definition of such semantics exploiting operators on the next state functions which are the functional counterpart of classical process algebra operators. We apply this framework to representative fragments of major stochastic process calculi namely $TIPP$, $PEPA$ and IML and show how they solve the issue of transition multiplicity in a simple and elegant way. We, moreover, show how RTSs help describing different languages, their differences and their similarities. For each calculus, we also show the formal correspondence between the RTSs semantics and the standard SOS one.

1 Introduction

Several stochastic, and in particular Markovian, process algebras have been proposed in the recent past. An overview can be found in [16]. Examples include $TIPP$ [13,17], $PEPA$ [19], $EMPA$ [3], stochastic π-calculus [24] and, more recently, calculi for Mobile and Service Oriented Computing [10,6,7,23,4,8]. The main aim has been the integration of qualitative behavioural descriptions with non-functional ones, e.g. performance, in a single mathematical framework, namely that of process algebras. This has lead to the combination of two very successful approaches to concurrent systems modelling and analysis, namely Labeled Transition Systems (LTSs), widely used in the framework of process algebra, and Continuous Time Markov Chains (CTMCs), one of the most successful approaches to modelling and analysing system performance. The common feature of the most prominent stochastic process algebra proposals, including all the above mentioned ones, is that the actions used to label transitions are enriched with rates of exponentially distributed random variables (r.v.) characterising their duration[1]. Although all these languages relay on the same class of r.v., the underlying models and notions are significantly different, in particular with regards to the issue of the correct representation of the *race condition* principle for

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[1] Sometimes actions are assumed to have zero duration; then the associated r.v. is interpreted as a *delay*, before the action takes place.

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the choice operator, inherited from the theory of CTMCs. This principle implies that an expression like $(\alpha, \lambda).P + (\alpha, \lambda).P$, where there are two different ways of executing α, both with (exponentially distributed duration with) rate λ should model the same behavior as $(\alpha, 2 \cdot \lambda).P$, and not as $(\alpha, \lambda).P$, as it would be the case if one would look at the term as a standard process algebra one. Several, significantly different, approaches have been proposed for addressing the issue of *transition multiplicity* raised by the race condition principle ranging, e.g. from *multi relations* [19], to *proved transition systems* [24,13], to LTS with *numbered transitions* [16], to *unique rate names* [10,6]. A different approach has been taken in [15] for IML, a language for Interactive Markov Chains, IMCs, where actions are de-coupled from rates and interaction transitions, labelled with actions, are kept separated from Markovian ones, labelled by rates. Multi-relations are used for Markovian transitions. It should also be noted that some of the most successful approaches, e.g. [19,15] suffer from technical imprecision in that they define the relevant transition multi-relation as the *least* multi-relation satisfying a set of Structured Operational Semantics (SOS) axioms and rules. Unfortunately, such a least multi-relation turns out to be a relation, thus failing to formally representing transition multiplicity. In [20] a variant of LTSs, namely *Rated Transition Systems* ($RdTS$) has been proposed as a model for the definition of the semantics of Markovian process calculi by relying on the general framework of SGSOS. Moreover, in [20] conditions are put forward for guaranteeing associativity of the parallel composition operator in the SGSOS framework. It is then proved that one cannot guarantee associativity of the parallel composition operator up to stochastic bisimilarity when the synchronisation paradigm of CCS is used in combination with the synchronisation rate computation based on *apparent rates* [19]. This implies for instance that parallel composition of the Stochastic π-calculus is not associative.

In the present paper, we use *Rate Transition Systems* (RTS) a variant of $RdTS$ where the transition relation $\rightarrowtail$ associates to a given process P and a given transition label α a *next state* function, say $\mathscr{P}$, mapping each term into a non-negative real number. The transition $P \stackrel{\alpha}{\rightarrowtail} \mathscr{P}$ has the following meaning: if $\mathscr{P}(Q) = v,\ (with\ v \neq 0)$, then Q is reachable from P by executing α, the duration of such execution being exponentially distributed with rate v; if $\mathscr{P}(Q) = 0$, then Q is not reachable from P via α. The approach is somewhat reminiscent of that of Deng et al. [12] where probabilistic process algebra terms are associated to a discrete probability distribution over such terms. RTSs are similar to Continuous Time Markov Decision Processes (CTMDPs) as defined, e.g., in [18,2] or Continuous Time Probabilistic Automata (CPA) (see [21,22,5]), as we shall discuss in more detail in Sect. 2. A distinguishing feature of our approach is compositionality, which, as in [20], is a direct consequence of a structured approach to semantics; furthermore, in our approach, next state functions are composed and manipulated using operators which are in one to one correspondence with those of process calculi. A pleasant side-effect of the resulting framework is a simple and elegant solution to the transition multiplicity problem. Furthermore, RTSs make it relatively easy to define *associative* parallel composition operators for calculi based on the CCS interaction paradigm. Finally, the possibility of defining different stochastic process languages within a single, uniform framework facilitates reasoning about them; their similarities and their major differences. In this paper we will consider only a small number of stochastic process calculi, due to space limitations. Moreover, we will focus only on the fragment of each calculus which is relevant for the stochastic extension. For the sake of conciseness, we will introduce the operators in an incremental fashion, pointing out the relative differences, avoiding repeating the relevant definitions for each language. We will not deal with behavioural relations and we will focus only on language definition: that is why in the title we mention process *languages* and not *calculi*. The reader interested in process equivalencies is referred to [8,9] for some initial results. The rest of the paper is organised as follows: in Sect. 2 some preliminary notions and definitions are recalled. Sect. 3 introduces the RTS semantics for a simple language for CTMCs. Sect. 4 shows the RTS semantics of significant fragments of major Markovian Process Calculi. Emphasis is put on calculi based on the multi-party CSP interaction paradigm, like $TIPP$ and $PEPA$. A brief discussion of other calculi, based on the binary, CCS, interaction paradigm is also provided. The RTS semantics of a language based on the Interactive Markov Chain principle of separating actions from rates is presented in Sect.5. Irrelevance of self-loops for transient analysis of CTMCs is proved in the appendix; for all other proofs concerning results presented in this paper the interested reader is referred to [11], where the $EMPA$ calculus is dealt with as well; results are proved by induction either on the structure of terms or on the length of the derivation in the relevant semantics deduction system. Basic knowledge of prominent stochastic process calculi is assumed in the rest of the paper.

2 Preliminaries

We let $\mathbb{N}_{\geq 0}$ ($\mathbb{R}_{\geq 0}$, respectively) denote the set $\{n \in \mathbb{N} \mid n \geq 0\}$ ($\{x \in \mathbb{R} \mid x \geq 0\}$, respectively) and, similarly, $\mathbb{N}_{>0}$ ($\mathbb{R}_{>0}$, respectively) denote the set $\{n \in \mathbb{N} \mid n > 0\}$ ($\{x \in \mathbb{R} \mid x > 0\}$, respectively). For set S we let 2^S denote the power-set of S and 2^S_{fin} the set of *finite* subsets of S. In function definitions as well as application *Currying* will be used whenever convenient.

Definition 1 (Negative Exponential Distributions). *A random variable X has a* negative exponential distribution *with* rate λ *if and only if* $\mathbb{P}\{X \leq t\} = 1 - e^{\lambda \cdot t}$ *for* $t > 0$ *and* 0 *otherwise.* •

The expected value of an exponentially distributed r.v. with rate λ is λ^{-1} while its variance is λ^{-2}. The *min* of exponentially distributed independent r.v. $X_1, \ldots, X_n$ with rates $\lambda_1, \ldots, \lambda_n$ respectively is an exponentially distributed r.v. with rate $\lambda_1 + \ldots + \lambda_n$ while the probability that X_j is the *min* is $\frac{\lambda_j}{\lambda_1 + \ldots + \lambda_n}$. The *max* of exponentially distributed r.v. is not exponentially distributed. For the purpose of the present paper, CTMCs are defined as follows:

Definition 2 (Continuous Time Markov Chains). *A* Continuous Time Markov Chain (CTMC) *is a tuple* $(S, \mathbf{R})$ *where* S *is a countable non-empty set of* states, *and* $\mathbf{R} : S \to S \to I\!R_{\geq 0}$ *is the* rate matrix, *where for all* $s \in S$ *there exists* $K_s < \infty$ *such that* $\sum_{s' \in S} \mathbf{R}\, s\, s' = K_s$. •

We will often use the matrix notation $\mathbf{R}[s, s']$ for $\mathbf{R}\, s\, s'$. $\mathbf{R}[s, s'] > 0$ means that a transition from s to s' can be taken. The sojourn time at state s before taking a transition is an exponentially distributed r.v. with rate $\sum_{s' \in S} \mathbf{R}[s, s']$ and the probability that the transition from s to s' is taken is $\mathbf{R}[s, s'] / \sum_{s'' \in S} \mathbf{R}[s, s'']$.

Notice that the above definition allows $\mathbf{R}[s, s] > 0$, i.e. self-loops are allowed, which is not the case in traditional definitions of CTMCs. The following proposition, proved in Appendix A, shows that, as long as traditional measures of CTMCs like transient (and consequently steady state) probabilities are concerned, this more liberal definition does not affect the meaning of the CTMC and, in fact, self-loops can be removed (i.e. $\mathbf{R}[s, s]$ set to zero) or added without affecting transient and steady state probability analysis results.

Proposition 1. *The transient behaviour of CTMC* $C = (S, \mathbf{R})$ *with* $\mathbf{R}[\bar{s}, \bar{s}] > 0$ *for some* $\bar{s} \in S$ *coincides with that of CTMC* $\tilde{C} = (S, \tilde{\mathbf{R}})$, *such that* $\tilde{\mathbf{R}}[s, s'] =_{\text{def}} 0$, *if* $s = s'$, *and* $\tilde{\mathbf{R}}[s, s'] =_{\text{def}} \mathbf{R}[s, s']$ *otherwise.* □

As a consequence of the above result, the infinitesimal generator matrix representation of CTMCs, traditionally used for CTMCs without self-loops, can be safely used also for those with such loops.

For countable non-empty set S, we consider the set $S \to \mathbb{R}_{\geq 0}$ of total functions from S to $\mathbb{R}_{\geq 0}$. We let $\mathscr{P}, \mathscr{Q}, \mathscr{R}, \ldots$ range over $S \to \mathbb{R}_{\geq 0}$. We let $[]$ denote the 0 constant function in $S \to \mathbb{R}_{\geq 0}$, i.e. $[]\, s =_{\text{def}} 0$ for all $s \in S$; moreover given $s_1, \ldots, s_n \in S$ and, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{>0}$ we let $[s_1 \mapsto \lambda_1, \ldots, s_n \mapsto \lambda_n]$ denote the function in $S \to \mathbb{R}_{\geq 0}$ which maps s_1 to $\lambda_1, \ldots, s_n$ to λ_n and any $s \in S \setminus \{s_1, \ldots, s_n\}$ to 0. The following definition characterises *Rate Transition Systems* [8,9].

Definition 3 (Rate Transition Systems). *A* Rate Transition System (RTS) *is a tuple* $(S, A, \rightarrowtail)$ *where* S *is a countable non-empty set of* states, A *is a countable non-empty set of* labels *and* $\rightarrowtail \subseteq S \times A \times (S \to I\!R_{\geq 0})$ *is the* transition relation.

In the sequel RTSs will be denoted by $\mathcal{R}$, $\mathcal{R}_1$, $\mathcal{R}'$,... As usual, we let $s \overset{\alpha}{\rightarrowtail} \mathscr{P}$ denote $(s, \alpha, \mathscr{P}) \in \rightarrowtail$. Intuitively, $s_1 \overset{\alpha}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\, s_2) = \lambda \neq 0$ means that s_2 is reachable from s_1 via the execution of α and that the duration of such an execution is characterised by a random variable whose distribution function is negative exponential with rate λ. On the other hand, $(\mathscr{P}\, s_2) = 0$ means that s_2 is not reachable from s_1 via α.

Definition 4 (Σ_S). Σ_S *denotes the subset of* $S \to I\!R_{\geq 0}$ *including only all functions expressed using the* $[\ldots]$ *notation, i.e.* $\mathscr{P} \in \Sigma_S$ *if and only if* $\mathscr{P} = []$ *or there exist* $n > 0$, $s_1, \ldots, s_n \in S$ *and* $\lambda_1, \ldots, \lambda_n \in I\!R_{>0}$ *such that* $\mathscr{P} = [s_1 \mapsto \lambda_1, \ldots, s_n \mapsto \lambda_n]$ •

We equip Σ_S with a few useful operations, i.e. $+ : \Sigma_S \times \Sigma_S \to (S \to \mathbb{R}_{\geq 0})$ with $(\mathscr{P} + \mathscr{Q})\, s =_{\text{def}} (\mathscr{P}\, s) + (\mathscr{Q}\, s)$ and $\bigoplus : \Sigma_S \to 2^S \to \mathbb{R}_{\geq 0}$ with $\bigoplus \mathscr{P}\, C =_{\text{def}} \sum_{s \in C} (\mathscr{P} s)$, for $C \subseteq S$, and we use the shorthand $\oplus \mathscr{P}$ for $\bigoplus \mathscr{P}\, S$. The proposition below trivially follows from the relevant definitions:

Proposition 2. *(i) All functions in Σ_S yield zero almost everywhere, i.e. for all $\mathscr{P} \in \Sigma_S$ the set $\{s \in S \mid (\mathscr{P}\, s) \neq 0\}$ is finite; (ii) Σ_S is* closed *under $+$, i.e. $+ : \Sigma_S \to \Sigma_S \to \Sigma_S$.* □

Proposition 2(i) above guarantees that $\bigoplus$ is well defined.

Definition 5. *Let $\mathcal{R} = (S, A, \rightarrowtail)$ be an RTS, then: (i) $\mathcal{R}$ is* total *if for all $s \in S$ and $\alpha \in A$ there exists $\mathscr{P} \in (S \to \mathbb{R}_{\geq 0})$ such that $s \overset{\alpha}{\rightarrowtail} \mathscr{P}$; (ii) $\mathcal{R}$ is* functional[2] *if for all $s \in S$, $\alpha \in A$, and $\mathscr{P}, \mathscr{Q} \in (S \to \mathbb{R}_{\geq 0})$ we have: $s \overset{\alpha}{\rightarrowtail} \mathscr{P}, s \overset{\alpha}{\rightarrowtail} \mathscr{Q} \Longrightarrow \mathscr{P} = \mathscr{Q}$; (iii) $\mathcal{R}$ is* well formed *if $\rightarrowtail \subseteq S \times A \times \Sigma_S$.* •

Discussion

It is worth noting that RTSs are a slight generalization of Continuous Time Markov Decision Processes (CTMDPs) as defined by Hermanns and Johr [18] and Continuous Time Probabilistic Automata, as defined in [22]. In [18,22], in fact, the transition relation is a subset of $S \times A \times (S \to \mathbb{R}_{\geq 0})$, i.e. it is *not* required to be a *function* in $S \times A \to (S \to \mathbb{R}_{\geq 0})$, but sets S and A are required to be *finite* and in [18] an *initial state* is assumed as well. There is also a direct relationship between RTSs and Continuous Time Probabilistic Automata proposed by Knast in [21], although the latter are studied in a language theoretic framework: the element $a_{i,j}(x)$ of the infinitesimal matrix used in [21] coincides with $(\mathscr{P}\, j)$ for $i \overset{x}{\rightarrowtail} \mathscr{P}$. Finally, the Continuous Time Probabilistic Automata used by Dang Van Hung and Zhou Chaochen in [5] are based on standard automata, where transitions are elements of $S \times S$ and have a rate and no label associated. In [20] *Rated Transition Systems* (RdTSs) are proposed by Klin and Sassone. RdTSs coincide with the class of *functional RTS*: the transition relation is required to be a *function* in $S \times A \times S \to \mathbb{R}_{\geq 0} = S \times A \to (S \to \mathbb{R}_{\geq 0})$. In [2], Baier et al. define CTMDPs as tuples $(S, A, \rightarrowtail)$ where S and A are *finite* sets and $\rightarrowtail$ is a function in $S \times A \times S \to \mathbb{R}_{\geq 0}$, while we allow also infinite sets and relations over $S \times A \times (S \to \mathbb{R}_{\geq 0})$. Finally, we point out that RTSs can be used also to model (passive) action *weights*, e.g. in $EMPA$ or $PEPA$ as well as *interactive* transitions of Interactive Markov Chains in a natural way.

In the rest of the present paper we will consider only *well-formed* RTSs, since they are powerful enough to provide a semantic model for the stochastic process calculi we are interested in.

Definition 6 (Derivatives). *Let $\mathcal{R} = (S, A, \rightarrowtail)$ be an RTS; for sets $S' \subseteq S$ and $A' \subseteq A$, the set of derivatives of S'* through *A', denoted $Der(S', A')$, is the smallest set such that: (i) $S' \subseteq Der(S', A')$, and (ii) if $s \in Der(S', A')$ and there exists $\alpha \in A'$ and $\mathscr{Q} \in \Sigma_S$ such that $s \overset{\alpha}{\rightarrowtail} \mathscr{Q}$ then $\{s' \in S \mid \mathscr{Q}(s') \neq 0\} \subseteq Der(S', A')$.* •

[2] *Fully-stochastic* according to the terminology used in [9].

Definition 7 (Derived CTMC). *Let $\mathcal{R} = (S, A, \rightarrowtail)$ be a* functional *RTS; for $S' \subseteq S$, the CTMC of S', when one considers only labels in* finite *set $A' \subseteq A$ is defined as $CTMC[S', A'] =_{\text{def}} (Der(S', A'), \mathbf{R})$ where, for all $s_1, s_2 \in Der(S', A')$, $\mathbf{R}[s_1, s_2] =_{\text{def}} \sum_{\alpha \in A', s_1 \overset{\alpha}{\rightarrowtail} \mathscr{P}} \mathscr{P}(s_2)$.* •

We write $Der(s, A')$ and $CTMC[s, A']$ when $S' = \{s\}$.

The semantics of stochastic process calculi are often defined in the literature by means of Structured Operational Semantics (SOS) which characterize transition systems or *multi*-transition systems, i.e. transition systems where the transition relation is instead a *multi*-relation. Such (multi-)transitions are usually labelled by *rates* $\lambda \in \mathbb{R}_{>0}$, but sometimes they are also labelled with *actions* drawn from some set A. In such LTSs there may be two or more transitions with (the same action label and) different rates from a state to another one; in case of multi-transition systems such distinct transitions may even have the same rate. Henceforth we let $\mathbf{rt}(s_1, s_2)$ and $\mathbf{rt}_a(s_1, s_2)$ denote the *cumulative* rate over *all* transitions from s_1 to s_2 and the *cumulative* rate over *all a-labelled* transitions from s_1 to s_2 as defined below, where we use $\{\!| \_ |\!\}$ as a notation for multi-sets, and $\xrightarrow{\lambda}$ ($\xrightarrow{a,\lambda}$, respectively) for a generic transition (a-labelled transition, respectively):

Definition 8. *The* cumulative rates $\mathbf{rt}(s_1, s_2)$ *and* $\mathbf{rt}_a(s_1, s_2)$ *are defined as follows:* $\mathbf{rt}(s_1, s_2) =_{\text{def}} \sum \{\!| \lambda | s_1 \xrightarrow{\lambda} s_2 |\!\}$ *and* $\mathbf{rt}_a(s_1, s_2) =_{\text{def}} \sum \{\!| \lambda | s_1 \xrightarrow{a,\lambda} s_2 |\!\}$, *with* $\sum \{\!| |\!\} =_{\text{def}} 0$. •

3 A Language for CTMCs

In this section we define a simple language for CTMCs, in a similar way as in [16]. The set $\mathcal{P}_{CTMC}$ of CTMC terms includes *inaction*, *rate-prefix*-, *choice*-, and *constant*-terms as defined by the following grammar:

$$P ::= \mathbf{nil} \;\Big|\; \lambda.P \;\Big|\; P + P \;\Big|\; X$$

where $\lambda \in \mathbb{R}_{>0}$ and X is a constant defined by means of an equation of the form $X \overset{\Delta}{=} P$ where constants $X, X_1, X', \ldots$ may occur only guarded, i.e. under the scope of a prefix $\lambda.\_$, in *defining body* P.

In order to give an RTS semantics to the calculus we first of all choose the set $\mathcal{A}_{CTMC} =_{\text{def}} \{\surd\}$ as labels set; transitions have no action labels in standard CTMCs. The transition relation $\rightarrowtail$, is defined in Fig. 1, where $\alpha = \surd$ is assumed.

Intuitively, from Fig. 1 it is clear that there is no transition from **nil** to any other state, while there is a single transition from $\lambda.P$ to P and λ is the rate associated to such a transition. The rule for choice postulates that if there is a transition from P to a state, say R, with rate $(\mathscr{P}R)$ and a transition from Q to the same state R, with rate $(\mathscr{Q}R)$, then there is a transition from $P + Q$ to R with rate $(\mathscr{P}R) + (\mathscr{Q}R)$. Notice that, for term $P + Q$, if there is a transition *only* from P to R (i.e. not from Q to R) then $(\mathscr{Q}R) = 0$. Similarly, $(\mathscr{P}R) = 0$ if

$$\text{(NIL)}\ \frac{}{\mathbf{nil} \overset{\alpha}{\rightarrowtail} []} \quad \text{(PRF)}\ \frac{}{\lambda.P \overset{\alpha}{\rightarrowtail} [P \mapsto \lambda]} \quad \text{(CHO)}\ \frac{P \overset{\alpha}{\rightarrowtail} \mathscr{P}, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}}{P+Q \overset{\alpha}{\rightarrowtail} \mathscr{P}+\mathscr{Q}} \quad \text{(CNT)}\ \frac{P \overset{\alpha}{\rightarrowtail} \mathscr{P}, X \overset{\Delta}{=} P}{X \overset{\alpha}{\rightarrowtail} \mathscr{P}}$$

Fig. 1. Semantics Rules for the CTMC Language

there is only a transition from Q to R. If, instead, there is *both* a transition from P to R (i.e. $(\mathscr{P}R) > 0$) *and* a transition from Q to R (i.e. $(\mathscr{Q}R) > 0$), then the cumulative rate $(\mathscr{P}R) + (\mathscr{Q}R)$ will be associated directly to the transition from $P + Q$ to R. The use of RTSs, in particular in the rule for choice, incorporates the *race condition* principle and solves the related *transition multiplicity* issue in a simple and elegant way. In fact, from Fig. 1, for $R_1 \neq R_2$ we get $\lambda.R_1 + \mu.R_2 \overset{\surd}{\rightarrowtail} [R_1 \mapsto \lambda, R_2 \mapsto \mu]$ where $\oplus[R_1 \mapsto \lambda, R_2 \mapsto \mu] = \lambda + \mu$ is the exit rate of state $\lambda.R_1 + \mu.R_2$ while $\lambda/(\lambda + \mu)$ and $\mu/(\lambda + \mu)$ are the probabilities of moving to R_1 and R_2, respectively. If $R_1 = R_2 = R$ then we get $\lambda.R + \mu.R \overset{\surd}{\rightarrowtail} [R \mapsto \lambda + \mu]$ and if, moreover, $\lambda = \mu$, we get $\lambda.R + \lambda.R \overset{\surd}{\rightarrowtail} [R \mapsto 2\lambda]$. The following proposition ensures that the semantics are closed w.r.t. $\Sigma_{\mathcal{P}_{CTMC}}$.

Proposition 3. *For all* $P \in \mathcal{P}_{CTMC}$ *and* $\mathscr{P} \in \mathcal{P}_{CTMC} \to \mathbb{R}_{\geq 0}$, *if* $P \rightarrowtail \mathscr{P}$ *can be derived from the rules of Fig. 1, then* $\mathscr{P} \in \Sigma_{\mathcal{P}_{CTMC}}$. □

Definition 9 (Formal semantics of the Language for CTMCs). *The formal semantics of the calculus for CTMCs is the RTS* $\mathcal{R}_{CTMC} =_{\text{def}} (\mathcal{P}_{CTMC}, \mathcal{A}_{CTMC}, \rightarrowtail)$ *where* $\rightarrowtail \subseteq \mathcal{P}_{CTMC} \times \mathcal{A}_{CTMC} \times \Sigma_{\mathcal{P}_{CTMC}}$ *is the least relation satisfying the rules of Fig. 1.* •

The following theorem characterises the structure of $\mathcal{R}_{CTMC}$.

Theorem 1. $\mathcal{R}_{CTMC}$ *is total and functional.* □

The CTMC associated to a given term $P \in \mathcal{P}_{CTMC}$, $CTMC[P, \{\surd\}]$ is generated according to Def. 7. As a corollary of Theorem 1 we get that whenever $P \overset{\surd}{\rightarrowtail} \mathscr{P}$, the exit rate of P is given by $\oplus\mathscr{P}$ and $\mathscr{P}$ is the row of the rate matrix corresponding to P.

4 Fully Markovian Stochastic Process Calculi

We first introduce some additional notation. Let S and A be countable nonempty sets. We define function $\chi : S \to S \to \mathbb{R}_{\geq 0}$ as $\chi\, s =_{\text{def}} [s \mapsto 1]$. Let, moreover, $\_ \otimes_{\_} \_ : 2^A_{fin} \to S \to S \to S$ be a total function and let us define, with a little bit of overloading, function $\_ \otimes_{\_} \_ : 2^A_{fin} \to (S \to \mathbb{R}_{\geq 0}) \to (S \to \mathbb{R}_{\geq 0}) \to (S \to \mathbb{R}_{\geq 0})$ as follows:

$$(\mathscr{P} \otimes_L \mathscr{Q})\, s =_{\text{def}} \begin{cases} (\mathscr{P}\, s_1) \cdot (\mathscr{Q}\, s_2)\ , & \text{if } \exists s_1, s_2 \in S.\ s = s_1 \otimes_L s_2 \\ 0 & , \text{otherwise} \end{cases}$$

$$\text{(PRF1)}\ \frac{}{(\alpha,\lambda).P \overset{\alpha}{\rightarrowtail} [P \mapsto \lambda]} \qquad \text{(PRF2)}\ \frac{\alpha \neq \beta}{(\alpha,\lambda).P \overset{\beta}{\rightarrowtail} []}$$

$$\text{(PAR1)}\ \frac{\alpha \notin L, P \overset{\alpha}{\rightarrowtail} \mathscr{P}, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}}{P||_L Q \overset{\alpha}{\rightarrowtail} (\mathscr{P}||_L(\chi\, Q)) + ((\chi\, P)||_L \mathscr{Q})} \qquad \text{(PAR2)}\ \frac{\alpha \in L, P \overset{\alpha}{\rightarrowtail} \mathscr{P}, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}}{P||_L Q \overset{\alpha}{\rightarrowtail} \mathscr{P}||_L \mathscr{Q}}$$

Fig. 2. Additional Semantics Rules for $TIPP_k$

We also define function $\_ \cdot \_/\_ : (S \to \mathbb{R}_{\geq 0}) \to \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \to S \to \mathbb{R}_{\geq 0}$ as follows:

$$(\mathscr{P} \cdot \frac{x}{y})\, s \ =_{\text{def}} \begin{cases} (\mathscr{P}\, s) \cdot \frac{x}{y}\ , \text{ if } y \neq 0 \\ \\ 0 \qquad\qquad , \text{ otherwise} \end{cases}$$

The proposition below trivially follows from the relevant definitions:

Proposition 4. *Σ_S is* closed *under the operations $\chi, (\_ \otimes_\_ \_)$, and $\_ \cdot \_/\_$, i.e. $\chi : S \to \Sigma_S$, $(\_ \otimes_\_ \_) : 2^A_{fin} \to \Sigma_S \to \Sigma_S \to \Sigma_S$, and $\_ \cdot \_/\_ : \Sigma_S \to I\!R_{\geq 0} \to I\!R_{\geq 0} \to \Sigma_S$.* □

4.1 $\boldsymbol{TIPP_k}$

Here we consider a kernel language $TIPP_k$ of the version[3] of $TIPP$ presented in [17]. Let $\mathcal{A}_{TIPP_k}$ be a countable set of *actions*, $\tau \notin \mathcal{A}_{TIPP_k}$, and $\mathcal{A}^{\tau}_{TIPP_k} =_{\text{def}} \mathcal{A}_{TIPP_k} \cup \{\tau\}$, with τ representing the internal action. The set $\mathcal{P}_{TIPP_k}$ of $TIPP$ terms we consider includes *inaction*, *choice*-, and *constant*-terms, defined as in Sect. 3; moreover, $\mathcal{P}_{TIPP_k}$ includes *action-prefix* (which replaces rate-prefix) and *parallel composition*, as defined by the following grammar[4]:

$$P ::= (\alpha, \lambda).P \ \Big|\ P||_L P$$

where $\alpha \in \mathcal{A}^{\tau}_{TIPP_k}$, $\lambda \in \mathbb{R}_{>0}$, and finite *synchronisation* set $L \in 2^{\mathcal{A}_{TIPP_k}}_{fin}$. Constants $X, X_1, X', \ldots$ may only occur guarded, i.e. under the scope of a prefix $(\alpha, \lambda).\_$, in defining bodies.

The transition relation $\rightarrowtail$ for $TIPP_k$ is characterised by the set of rules RLS_{TIPP_k} defined below:

Definition 10 ($\boldsymbol{RLS_{TIPP_k}}$). *Set* RLS_{TIPP_k} *is the least set of semantics rules including the rules in Fig. 2 plus rules (NIL), (CHO), (CNT) of Fig. 1, where terms P, Q, X are assumed to range over $\mathcal{P}_{TIPP_k}$ and $\alpha, \beta \in \mathcal{A}^{\tau}_{TIPP_k}$.* •

[3] In [17] the synchronisation rate is defined as the product of those of the synchronising actions, as opposed to the original definition of $TIPP$, given in [13], where, instead, such rate is the *max* of the component rates.

[4] In $TIPP$ the notation **stop**, `i`, [] and $P|[L]|P$ is used instead of **nil**, τ, $+$, and $P||_L P$. Here we prefer to use a standard notation for the sake of uniformity.

In the rules, the generic functions χ and $\otimes$ on S are instantiated with specific functions for $\mathcal{P}_{TIPP_k}$. In particular the specific function $||$ is used in place of the generic function $\otimes$; the specific function $\_||\_\_ : 2^{\mathcal{A}_{TIPP_k}}_{fin} \to \mathcal{P}_{TIPP_k} \to \mathcal{P}_{TIPP_k} \to \mathcal{P}_{TIPP_k}$ is just the syntactical constructor for parallel composition on $TIPP$ terms. Rule (PAR1) ensures that all interesting continuations of $P\,||_L\,Q$ are of the form $R\,||_L\,Q$ where $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\,R) > 0$, for some $\mathscr{P}$ and $\alpha \notin L$, or of the form $P\,||_L\,R$ where $Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}$ and $(\mathscr{Q}\,R) > 0$, for some $\mathscr{Q}$ and $\alpha \notin L$. Rule (PAR2), instead, formalizes the *rate multiplication* principle of $TIPP$: if $\alpha \in L$, $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$, $Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}$, $(\mathscr{P}\,R_P) = \lambda_P > 0$, and $(\mathscr{Q}\,R_Q) = \lambda_Q > 0$, then $P\,||_L\,Q$ evolves, via α, to $R_P\,||_L\,R_Q$ with rate $\lambda_P \cdot \lambda_Q$.

The following proposition ensures that the semantics are closed w.r.t. $\Sigma_{\mathcal{P}_{TIPP_k}}$.

Proposition 5. *For all $P \in \mathcal{P}_{TIPP_k}$, $\alpha \in \mathcal{A}^{\tau}_{TIPP_k}$ and $\mathscr{P} \in \mathcal{P}_{TIPP_k} \to \mathbb{R}_{\geq 0}$, if $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ can be derived using only the rules in set* RLS_{TIPP_k} *of Def. 10, then $\mathscr{P} \in \Sigma_{\mathcal{P}_{TIPP_k}}$.* □

Definition 11 (Formal semantics of $TIPP_k$). *The formal semantics of $TIPP_k$ is the RTS $\mathcal{R}_{TIPP_k} =_{\mathrm{def}} (\mathcal{P}_{TIPP_k}, \mathcal{A}^{\tau}_{TIPP_k}, \rightarrowtail)$ where $\rightarrowtail \subseteq \mathcal{P}_{TIPP_k} \times \mathcal{A}^{\tau}_{TIPP_k} \times \Sigma_{\mathcal{P}_{TIPP_k}}$ is the least relation satisfying the rules of set* RLS_{TIPP_k} *(Def. 10).* •

The following theorem characterises the structure of $\mathcal{R}_{TIPP_k}$.

Theorem 2. *$\mathcal{R}_{TIPP_k}$ is total and functional.*

Corollary 1. *For all $P \in \mathcal{P}_{TIPP_k}, \alpha \in \mathcal{A}^{\tau}_{TIPP_k}$ there exists a* unique *$\mathscr{P}$ such that $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$.*

The following theorem establishes the formal correspondence between the RTS semantics of $TIPP_k$ and the semantics definition given in [17].

Theorem 3. *For all $P, Q \in \mathcal{P}_{TIPP_k}$, $\alpha \in \mathcal{A}^{\tau}_{TIPP_k}$, and unique $\mathscr{P} \in \Sigma_{\mathcal{P}_{TIPP_k}}$ such that $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ the following holds: $(\mathscr{P}\,Q) = \mathbf{rt}_{\alpha}(P, Q)$* □

4.2 $PEPA_k$

The RTS semantics of the full $PEPA$ [19] calculus can be found in [9]. Here we confine our presentation to the kernel language $PEPA_k$. Let $\mathcal{A}_{PEPA_k}$, ranged over by $\alpha, \alpha', \ldots$ be a countable set of *actions*. The set $\mathcal{P}_{PEPA_k}$ of $PEPA$ terms we consider includes *choice*- and *constant*-terms, defined as in Sect. 3, and *action-prefix* and *parallel composition*, defined as in Sect. 4.1, but with synchronisation set[5] $L \in 2^{\mathcal{A}_{PEPA_k}}_{fin}$. Constants $X, X_1, X', \ldots$ may occur only guarded, i.e. under the scope of a prefix $(\alpha, \lambda).\_$, in defining bodies.

The transition relation $\rightarrowtail$ for $PEPA_k$ is characterised by the set of rules RLS_{PEPA_k} defined below:

[5] In $PEPA$ the notation $P \bowtie_L P$ is used instead of $P||_L P$. Here we prefer to use a standard notation for the sake of uniformity.

$$\left(\begin{smallmatrix}\text{PAR}\\\text{PEPA}\end{smallmatrix}\right)\ \frac{\alpha{\in}L, P \overset{\alpha}{\rightarrowtail} \mathscr{P}, Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}}{P||_L Q \overset{\alpha}{\rightarrowtail} \mathscr{P}||_L \mathscr{Q} \cdot \frac{min\{\oplus\mathscr{P},\oplus\mathscr{Q}\}}{\oplus\mathscr{P}\cdot\oplus\mathscr{Q}}}$$

Fig. 3. Additional Semantics Rule for $PEPA_k$

Definition 12 ($\boldsymbol{RLS}_{PEPA_k}$). *Set* RLS_{PEPA_k} *is the least set of semantics rules including the rule in Fig.3 plus rules (CHO), (CNT) of Fig. 1 and Rules (PRF1), (PRF2), and (PAR1) of Fig. 2. In all the above rules terms* P, Q, X *are assumed to range over* $\mathcal{P}_{PEPA_k}$ *and* $\alpha \in \mathcal{A}_{PEPA_k}$. •

In the rules, the generic functions χ and $\otimes$ on S are instantiated with specific functions on $\mathcal{P}_{PEPA_k}$. In particular the specific function $||$ is used in place of the generic function $\otimes$; the specific function $\_||\_\_ : 2_{fin}^{\mathcal{A}_{PEPA_k}} \to \mathcal{P}_{PEPA_k} \to \mathcal{P}_{PEPA_k} \to \mathcal{P}_{PEPA_k}$ is just the syntactical constructor for co-operation on $PEPA$ terms. The rule for interleaving ensures that all continuations of $P\,||_L\,Q$ are of the form $R\,||_L\,Q$ where $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\,R) > 0$, for some α and $\mathscr{P}$ or of the form $P\,||_L\,R$ where $Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}$ and $(\mathscr{Q}\,R) > 0$, for some α and $\mathscr{Q}$. The rule for co-operation, instead, implements the *apparent rate* principle of $PEPA$ (see corollary of Theorem 4): if $\alpha \in L$, $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$, $Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}$, $(\mathscr{P}\,R_P) = \lambda_P > 0$, and $(\mathscr{Q}\,R_Q) = \lambda_Q > 0$, then $P\,||_L\,Q$ evolves to $R_P\,||_L\,R_Q$ with rate $\frac{\lambda_P}{\oplus\mathscr{P}} \cdot \frac{\lambda_Q}{\oplus\mathscr{Q}} \cdot min\{\oplus\mathscr{P}, \oplus\mathscr{Q}\}$.

The following proposition ensures that the semantics are closed w.r.t. $\Sigma_{\mathcal{P}_{PEPA_k}}$.

Proposition 6. *For all* $P \in \mathcal{P}_{PEPA_k}$, $\alpha \in \mathcal{A}_{PEPA_k}$ *and* $\mathscr{P} \in \mathcal{P}_{PEPA_k} \to I\!R_{\geq 0}$, *if* $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ *can be derived from the rules of Fig. 3, then* $\mathscr{P} \in \Sigma_{\mathcal{P}_{PEPA_k}}$. □

Definition 13 (Formal semantics of $PEPA_k$). *The formal semantics of* $PEPA_k$ *is the RTS* $\mathcal{R}_{PEPA_k} =_{\text{def}} (\mathcal{P}_{PEPA_k}, \mathcal{A}_{PEPA_k}, \rightarrowtail)$ *where* $\rightarrowtail \subseteq \mathcal{P}_{PEPA_k} \times \mathcal{A}_{PEPA_k} \times \Sigma_{\mathcal{P}_{PEPA_k}}$ *is the least relation satisfying the rules of set* RLS_{PEPA_k} *(Def. 12).* •

Theorem 4. $\mathcal{R}_{PEPA_k}$ *is total and functional.*

As a corollary of Theorem 4 we get that whenever $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ the apparent rate of α in P—namely the exit rate of P relative to α, denoted by $r_\alpha(P)$ in [19]—is given by $\oplus\mathscr{P}$. In [9] it is shown that the RTS semantics of $PEPA$ coincides with the original one.

We close this section by observing that $PEPA$ *passive actions* [19] can be easily dealt with in the RTS approach. One has to consider total functions in $\mathcal{P}_{PEPA_k} \to (\mathbb{R}_{\geq 0} \cup \{w \cdot \top \mid w \in \mathbb{N}_{>0}\})$ and define $\Sigma^{\top}_{\mathcal{P}_{PEPA_k}}$ by restricting only to functions expressed using the $[\ldots]$ notation; all definitions involving $\Sigma_{\mathcal{P}_{PEPA_k}}$ must be extended to $\Sigma^{\top}_{\mathcal{P}_{PEPA_k}}$ accordingly and taking into account the equations for $\top$ introduced in [19]. The following is an example resulting from the related derivation using the extended definitions:

$$(\alpha, \sqrt{2}).P||_{\{\alpha\}}((\alpha, 2\top).Q + (\alpha, 4\top).R) \overset{\alpha}{\rightarrowtail} [P||_{\{\alpha\}}Q \mapsto \frac{\sqrt{2}}{3}, P||_{\{\alpha\}}R \mapsto \frac{2\cdot\sqrt{2}}{3}]$$

4.3 CCS-Based Stochastic Process Calculi

Our RTS approach has been successfully applied to several CCS-based calculi including Stochastic CCS [20], Stochastic π-calculus [24] and calculi for modeling Service Oriented Computing [8]. The main issue is the treatment of the CCS one-to-one synchronisation paradigm, as opposed to the CSP multicast one adopted by $TIPP$, $PEPA$ and $EMPA$. RTS semantics allows for an adequate and elegant calculation of normalisation factors which make it possible to preserve nice properties of the original calculi, like associativity of parallel composition, which is not possible using other approaches, as discussed in e.g. [20]. Due to space limitations we do not show the RTS semantics of Stochastic CCS and SOC calculi here and we refer to [8,9].

5 A Language of Interactive Markov Chains

In this section we show an RTS semantics of Hermanns' Language of Interactive Markov Chains (IML). The definition of Interactive Markov Chains (IMC) follows [15]:

Definition 14. *An Interactive Markov Chain is a tuple* $(S, A, \rightarrow, \dashrightarrow, s_0)$ *where* S *is a nonempty, finite set of states,* A *a finite set of actions,* $\rightarrow \subseteq S \times A \times S$ *the set of* interactive transitions, $\dashrightarrow \subseteq S \times \mathbb{R}_{>0} \times S$ *the set of* Markov transitions*, and* $s_0 \in S$ *the initial state.* •

Also for IMCs we let the cumulative transition rate from s to s' be denoted by $\mathbf{rt}(s, s')$. For the sake of simplicity and due to space limitations, in this section we consider a kernel subset IML_k of the language IML defined in [15], which is anyway sufficient for showing how $RTSs$ can be used as a semantic model for IML. Let $\mathcal{A}_{IML_k}$ be a countable set of *actions*. The set $\mathcal{P}_{IML_k}$ of IML_k terms we consider includes *inaction*, *rate-prefix*-, *choice*-, and *constant*-terms, defined as in Sect. 3, and *action-prefix*- and *parallel composition*-terms, defined as in Sect. 4.1, but with $\alpha \in \mathcal{A}_{IML_k}$ and $L \in 2_{fin}^{\mathcal{A}_{IML_k}}$ as synchronisation set[6]. Constants $X, X_1, X', \ldots$ may occur only guarded, i.e. under the scope of a prefix $\lambda.\_$ or $\alpha.\_$, in defining bodies.

In order to give interactive transitions a "first-class objects" status, we consider a slight extension of RTS. We point out here that, technically, such an extension is not necessary, as we shall briefly discuss later on. We use it only because it makes our framework closer to the original model of IMCs. The extension of interest, namely RTS^{ι}, differs from RTS only because, instead of using functions in $\mathcal{P}_{IML_k} \rightarrow \mathbb{R}_{\geq 0}$, we consider those in $\mathcal{P}_{IML_k} \rightarrow \mathbb{R}^{\iota}_{\geq 0}$, where $\mathbb{R}^{\iota}_{\geq 0}$

[6] In IML_k the notation $\mathbf{0}$ and $P|[L]|P$ is used instead of **nil** and $P||_L P$. Here we prefer to use a standard notation for the sake of uniformity.

$+^\iota$	0	ι	v_2
0	0	ι	v_2
ι	ι	ι	ι
v_1	v_1	ι	$v_1 + v_2$

$\cdot^\iota$	0	ι	v_2
0	0	0	0
ι	0	ι	ι
v_1	0	ι	$v_1 \cdot v_2$

Fig. 4. Definition of $+^\iota$ and $\cdot^\iota$

denotes $\mathbb{R}_{\geq 0} \cup \{\iota\}$, with ι a distinguished value such that $\iota \notin \mathbb{R}_{\geq 0}$. Markov transitions are modeled as in Sect. 3, using the special element $\surd \notin \mathcal{A}_{IML_k}$ as a label and defining the label set of the relevant RTS^ι as $\mathcal{A}^{\surd}_{IML_k} =_{\text{def}} \mathcal{A}_{IML_k} \cup \{\surd\}$, ranged over by $\alpha, \alpha_1, \alpha', \ldots$. We define $\Sigma^\iota_{\mathcal{P}_{IML_k}}$ as expected:

Definition 15 ($\Sigma^\iota_{\mathcal{P}_{IML_k}}$)**.** *$\Sigma^\iota_{\mathcal{P}_{IML_k}}$ denotes the subset of $\mathcal{P}_{IML_k} \to \mathbb{R}^\iota_{\geq 0}$ including only all functions expressed using the $[\ldots]$ notation, i.e. $\mathscr{P} \in \Sigma^\iota_{\mathcal{P}_{IML_k}}$ if and only if $\mathscr{P} = []$ or $\mathscr{P} = [P_1 \mapsto v_1, \ldots, P_n \mapsto v_n]$ for $n \in \mathbb{N}_{>0}$, $P_1, \ldots, P_n \in \mathcal{P}_{IML_k}$ and $v_1, \ldots, v_n \in \mathbb{R}_{>0} \cup \{\iota\}$, with $([]\,P) =_{\text{def}} 0$ and $[P_1 \mapsto v_1, \ldots, P_n \mapsto v_n]\,P$ yielding v_j if $P = P_j$ for $1 \leq j \leq n$ and 0 otherwise.* •

We extend operations $+$ and $\cdot$ to $+^\iota, \cdot^\iota : \mathbb{R}^\iota_{\geq 0} \to \mathbb{R}^\iota_{\geq 0} \to \mathbb{R}^\iota_{\geq 0}$ as in Fig. 4, where we assume that $v_1, v_2 \notin \{0, \iota\}$. We lift $+^\iota : \mathbb{R}^\iota_{\geq 0} \to \mathbb{R}^\iota_{\geq 0} \to \mathbb{R}^\iota_{\geq 0}$ to $+^\iota : \Sigma^\iota_{\mathcal{P}_{IML_k}} \to \Sigma^\iota_{\mathcal{P}_{IML_k}} \to \mathcal{P}_{IML_k} \to \mathbb{R}^\iota_{\geq 0}$; we moreover define $\_ ||_L \_ : \Sigma^\iota_{\mathcal{P}_{IML_k}} \to \Sigma^\iota_{\mathcal{P}_{IML_k}} \to \mathcal{P}_{IML_k} \to \mathbb{R}^\iota_{\geq 0}$ by instantiating $\otimes_L$ on the syntactical constructor for parallel composition on IML_k terms and using $\cdot^\iota$ instead of $\cdot$. In the sequel we refrain from using the superscript ι in $+^\iota$ and $\cdot^\iota$ when it is clear from the context that we are using the extended operators. The following proposition trivially follows from the relevant definitions.

Proposition 7. *(i) All functions in $\Sigma^\iota_{\mathcal{P}_{IML_k}}$ yield zero almost everywhere, i.e. for all $\mathscr{P} \in \Sigma^\iota_{\mathcal{P}_{IML_k}}$ the set $\{P \in \mathcal{P}_{IML_k} \mid \mathscr{P}\,P \neq 0\}$ is finite; (ii)$\Sigma^\iota_{\mathcal{P}_{IML_k}}$ is closed under the extended operators, namely $+, ||_L : \Sigma^\iota_{\mathcal{P}_{IML_k}} \to \Sigma^\iota_{\mathcal{P}_{IML_k}} \to \Sigma^\iota_{\mathcal{P}_{IML_k}}$.* □

We finally extend the notion of Derived CTMC (see Def. 7) to IMCs in the obvious way:

Definition 16 (Derived IMC). *Let $\mathcal{R} = (S, A, \rightarrowtail)$ be a* functional *RTS^ι; for $s_0 \in S$, the IMC of s_0, when one considers only labels in* finite *set $A' \subseteq A$ is defined as $IMC[\{s_0\}, A'] =_{\text{def}} (Der(\{s_0\}, A'), A', \rightarrow, \dashrightarrow, s_0)$ where for all $s_1, s_2 \in Der(\{s_0\}, A')$, $\alpha \in A'$ such that $s_1 \overset{\alpha}{\rightarrowtail} \mathscr{P}$: (i) $s_1 \overset{\alpha}{\rightarrow} s_2$ iff $(\mathscr{P}\,s_2) = \iota$, and (ii) $s_1 \overset{\lambda}{\dashrightarrow} s_2$ iff $(\mathscr{P}\,s_2) = \lambda > 0$.* •

The transition relation $\rightarrowtail$ for IML_k is characterised by the set of rules RLS_{IML_k} defined below:

$$\frac{}{\lambda.P \overset{\surd}{\rightarrowtail} [P \mapsto \lambda]} \qquad \frac{\alpha \neq \surd}{\lambda.P \overset{\alpha}{\rightarrowtail} []} \qquad \frac{\alpha \neq \surd}{\alpha.P \overset{\alpha}{\rightarrowtail} [P \mapsto \iota]} \qquad \frac{\surd \neq \alpha \neq \beta}{\alpha.P \overset{\beta}{\rightarrowtail} []}$$

Fig. 5. Additional Semantics Rules for the IML_k

Definition 17 ($\boldsymbol{RLS}_{IML_k}$). *Set* RLS_{IML_k} *is the least set of semantics rules including the rules in Fig.5 plus rules (NIL), (CHO), (CNT) of Fig. 1, and rules (PAR1) and (PAR2) of Fig. 2. In all the above rules, terms* P, Q, X *are assumed to range over* $\mathcal{P}_{IML_k}$ *and* $\alpha, \beta \in \mathcal{A}^{\surd}_{IML_k}$. •

The rule for choice allows for the integration of Markov transitions with interaction ones; as usual, if $P \overset{\surd}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\, Q) = \lambda$ then λ is the cumulative rate for reaching Q from P, i.e. $\lambda = \mathbf{rt}(P, Q)$. For instance, for

$$P \overset{\Delta}{=} (\lambda_1.P_1 + \alpha.P_2) + (\alpha.P_2 + \lambda_2.P_1)$$

we have $P \overset{\surd}{\rightarrowtail} [P_1 \mapsto \lambda_1 + \lambda_2]$ and $P \overset{\alpha}{\rightarrowtail} [P_2 \mapsto \iota]$ with, moreover, $P \overset{\alpha'}{\rightarrowtail} []$ for all $\alpha' \notin \{\alpha, \surd\}$. The rule for interleaving ensures that all continuations of $P \,||_L\, Q$ are of the form $R \,||_L\, Q$ where $P \overset{\surd}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\, R) > 0$ or $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ and $(\mathscr{P}\, R) = \iota$ for some $\mathscr{P}$ and α, or of the form $P \,||_L\, R$ where $Q \overset{\surd}{\rightarrowtail} \mathscr{Q}$ and $(\mathscr{Q}\, R) > 0$ or $Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}$ and $(\mathscr{Q}\, R) = \iota$, for $\alpha \notin L$. The rule for synchronisation, instead, applies only in the case of interactive transitions and postulates that the only terms which can be reached from $P \,||_L\, Q$, via $\alpha \in L$ are those of the form $P' \,||_L\, Q'$ with $(\mathscr{P}\, P') = (\mathscr{Q}\, Q') = \iota$, where $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ and $Q \overset{\alpha}{\rightarrowtail} \mathscr{Q}$. It is worth noting that we could have chosen to use standard $\Sigma_{\mathcal{P}_{IML_k}}$ instead of its extension $\Sigma^{\iota}_{\mathcal{P}_{IML_k}}$ by replacing axiom $\alpha.P \overset{\alpha}{\rightarrowtail} [P \mapsto \iota]$ with $\alpha.P \overset{\alpha}{\rightarrowtail} [P \mapsto 1]$. In particular, whenever $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ the number of *different* (interaction) α-transitions from P to Q would be given by $(\mathscr{P}\, Q)$. We preferred the first alternative because we are not interested in counting such transition and we think that keeping different types for the range of the two kinds of transitions makes the framework more clear and closer to the original model of IMCs. We note also a clean separation between internal non-determinism, represented *within* functions, and external non-determinism, represented by different transitions. For instance, assuming P_1, P_2 and P_3 all different terms, the term

$$P \overset{\Delta}{=} \alpha.P_1 + \beta.P_2 + \alpha.P_3$$

has the following transitions: $P \overset{\alpha}{\rightarrowtail} [P_1 \mapsto \iota, P_3 \mapsto \iota]$, $P \overset{\beta}{\rightarrowtail} [P_2 \mapsto \iota]$, and $P \overset{\alpha'}{\rightarrowtail} []$ for all $\alpha' \notin \{\alpha, \beta\}$

Proposition 8. *For all* $P \in \mathcal{P}_{IML_k}$, $\alpha \in \mathcal{A}^{\surd}_{IML_k}$ *and* $\mathscr{P} \in \mathcal{P}_{IML_k} \to I\!R^{\iota}_{\geq 0}$, *if* $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ *can be derived from the rules in set* RLS_{IML_k} *of Def. 17, then* $\mathscr{P} \in \Sigma^{\iota}_{\mathcal{P}_{IMC}}$. □

Proposition 9. *For all* $P \in \mathcal{P}_{IML_k}$, $\alpha \in \mathcal{A}^{\surd}_{IML_k}$ *and* $\mathscr{P} \in \Sigma^{\iota}_{\mathcal{P}_{IMC}}$ *such that* $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ *can be derived from the rules in set* RLS_{IML_k} *of Def. 17, the following holds: (i) if* $\alpha \in \mathcal{A}_{IML_k}$ *and* $\mathscr{P} \neq []$ *then* $(\mathsf{range}\,\mathscr{P}) = \{0, \iota\}$, *(ii) if* $\alpha = \surd$ *then* $\iota \notin (\mathsf{range}\,\mathscr{P})$. □.

Definition 18 (Formal semantics of IML_k**).** *The formal semantics of* IML_k *is the* RTS^{ι} $\mathcal{R}_{IML_k} =_{\mathrm{def}} (\mathcal{P}_{IML_k}, \mathcal{A}^{\surd}_{IML_k}, \rightarrowtail)$ *where* $\rightarrowtail \subseteq \mathcal{P}_{IML_k} \times \mathcal{A}^{\surd}_{IML_k} \times \Sigma^{\iota}_{\mathcal{P}_{IML_k}}$ *is the least relation satisfying the rules in set* RLS_{IML_k} *of Def. 17.* •

Theorem 5. $\mathcal{R}_{IML_k}$ *is total and functional.*

Corollary 2. *For all* $P \in \mathcal{P}_{IML_k}, \alpha \in \mathcal{A}^{\surd}_{IML_k}$ *there exists a* unique $\mathscr{P}$ *such that* $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$.

The following theorem establishes the formal correspondence between the RTS^{ι} semantics of IML_k and the semantics definition given in [15]. Notice that in this case the cumulative rate must be computed over *all* copies of all transitions from P to Q in the *multi*-relation $\dashrightarrow$ defined in [15].

Theorem 6. *For all* $P, Q \in \mathcal{P}_{IML_k}$, $\alpha \in \mathcal{A}_{IML_k}$, *and unique functions* $\mathscr{P}, \mathscr{P}' \in \Sigma_{\mathcal{P}_{IML_k}}$ *such that* $P \overset{\alpha}{\rightarrowtail} \mathscr{P}$ *and* $P \overset{\surd}{\rightarrowtail} \mathscr{P}'$ *the following holds: (i)* $(\mathscr{P}\,Q) = \iota$ *if and only if* $P \overset{\alpha}{\rightarrow} Q$*; (ii)* $(\mathscr{P}'\,Q) = \mathbf{rt}(P, Q)$. □

6 Conclusions

In this paper we introduced *Rate Transition Systems* and we showed how they can be used as a unifying framework for the definition of the semantics of stochastic process algebras. RTSs facilitate the *compositional* definition of such semantics exploiting operators on the next state functions which are the functional counterpart of classical process algebra operators. We applied this framework to representative fragments of major stochastic process calculi including $TIPP$, $PEPA$ and IML and showed how they solve the issue of transition multiplicity in a simple and elegant way[7]. Moreover, we showed how RTSs throw light on differences and similarities of different languages. For each calculus, we also proved the formal correspondence between its RTS semantics and its standard SOS one. It turned out that, in all cases we considered here, it is sufficient to use *functional* RTSs, i.e. RTS where the transition relation is indeed a function. General RTSs are however useful in translations of Interactive Markov Chains to Continuous Time Markov Decision Processes [18], or in the definition of the RTS semantics for the Stochastic π-calculus (see [9]). Future work includes the investigation of the nature and actual usefulness of general $RTSs$, and in particular their explicit representation of non-determinism, also in the context of behavioural relations, along the lines of [22].

[7] The approach has been applied also to $EMPA$ but is not reported here due to space limitations. The details can be found in [11].

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A Proof of Proposition 1

Proposition 1. *The transient behaviour of CTMC* $C = (S, \mathbf{R})$ *with* $\mathbf{R}[\bar{s}, \bar{s}] > 0$ *for some* $\bar{s} \in S$ *coincides with that of CTMC* $\tilde{C} = (S, \tilde{\mathbf{R}})$*, such that*

$$\tilde{\mathbf{R}}[s, s'] \;=_{\mathrm{def}}\; \begin{cases} 0 & \textit{if } s = s' \\ \mathbf{R}[s, s'] & \textit{otherwise} \end{cases}$$

□

Proof. Suppose $\mathbf{R}[\bar{s}, \bar{s}] > 0$ and let $(\pi\, \bar{s}\, t)$ be the probability that C is in state $\bar{s}$ at time t, $\mathbb{P}\{C(t) = \bar{s}\}$. For h small enough, the evolution of C in the period $[t, t+h)$ can be captured using $(\pi\, \bar{s}\, t)$ as shown below, letting $p_{\bar{s}}$ denote the probability that no transition from $\bar{s}$ is taken during the period $[t, t+h)$ and $p_{s,\bar{s}}$ denote the probability that a transition from s to $\bar{s}$ takes place during the period $[t, t+h)$[8]:

[8] Notice that, we do *not* require $s \neq \bar{s}$, as usually found in the literature (see, e.g. [14]).

$\pi\,\bar{s}\,(t+h)$

$=$ $\{$Probability Theory; Definition of $p_{\bar{s}}$ and $p_{s,\bar{s}}$; h small$\}$

$(\pi\,\bar{s}\,t)\cdot\big(1-\sum_{s\in S}\mathbf{R}[\bar{s},s]\cdot h\big)+\sum_{s\in S}(\pi\,s\,t)\cdot\mathbf{R}[s,\bar{s}]\cdot h+o(t)$

$=$ $\{$Algebra$\}$

$(\pi\,\bar{s}\,t)-(\pi\,\bar{s}\,t)\cdot\sum_{s\in S\setminus\{\bar{s}\}}\mathbf{R}[\bar{s},s]\cdot h-(\pi\,\bar{s}\,t)\cdot\mathbf{R}[\bar{s},\bar{s}]\cdot h+$

$\sum_{s\in S\setminus\{\bar{s}\}}(\pi\,s\,t)\cdot\mathbf{R}[s,\bar{s}]\cdot h+(\pi\,\bar{s}\,t)\cdot\mathbf{R}[\bar{s},\bar{s}]\cdot h+o(t)$

$=$ $\{$Algebra$\}$

$(\pi\,\bar{s}\,t)-(\pi\,\bar{s}\,t)\cdot\sum_{s\in S\setminus\{\bar{s}\}}\mathbf{R}[\bar{s},s]\cdot h+\sum_{s\in S\setminus\{\bar{s}\}}(\pi\,s\,t)\cdot\mathbf{R}[s,\bar{s}]\cdot h+o(t)$

Thus the evolution of C in the period $[t,t+h)$ does *not* depend on $\mathbf{R}[\bar{s},\bar{s}]$. And in fact, letting

$$\mathbf{Q}_{\mathbf{R}}[s,s'] =_{\text{def}} \begin{cases}\mathbf{R}[s,s'], \text{if } s\neq s'\\ -\sum_{s''\in S\setminus\{s\}}\mathbf{R}[s,s''], \text{if } s=s'\end{cases}$$

we get $\pi\,\bar{s}\,(t+h)=(\pi\,\bar{s}\,t)+\big(\sum_{s\in S}(\pi\,s\,t)\cdot\mathbf{Q}_{\mathbf{R}}[s,\bar{s}]\big)\cdot h+o(t)$ from which we get

$$\frac{d(\pi\,\bar{s}\,t)}{dt}=lim_{h\to 0}\frac{(\pi\,\bar{s}\,(t+h))-(\pi\,\bar{s}\,t)}{h}=\sum_{s\in S}\mathbf{Q}_{\mathbf{R}}[s,\bar{s}]\cdot(\pi\,s\,t)$$

The vector $((\pi\,s\,t))_{s\in S}$ of the transient probabilities for C is thus characterised as the solution of the equation

$$\left(\frac{d(\pi\,s\,t)}{dt}\right)_{s\in S}=((\pi\,s\,t))_{s\in S}\mathbf{Q}_{\mathbf{R}} \qquad\qquad \text{given } ((\pi\,s\,0))_{s\in S}$$

which clearly coincides with the equation for the transient probabilities of $\tilde{C}$ observing that $\mathbf{Q}_{\mathbf{R}}=\mathbf{Q}_{\tilde{\mathbf{R}}}$.