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On Absolute Almost Generalized Nörlund Summability of Orthogonal Series

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Abstract. In this paper we present some results on absolute almost generalized Nörlund summability of orthogonal series. The most important corollaries of the main results also are deduced.

1. Introduction

It is a known fact that the absolute summability is a generalization of the concept of the absolute convergence just as the summability is an extension of the concept of the convergence. More about theory of summability the reader can find in Hardy [5]. Lorentz [16], for the first time in 1948, defined almost convergence of a bounded sequence. It is shown that every convergent sequence is almost convergent [17]. The idea of almost convergence led to the definition of almost generalized Nörlund summability method. This notion was introduced by Qureshi [24] which includes almost Nörlund, Riesz, harmonic and Cesàro summability as particular cases.

The absolute Nörlund summability, the absolute generalized Nörlund summability, the absolute Riesz summability, absolute generalized Cesàro summability, absolute Euler summability of an orthogonal series has been studied by many authors. For example, one can see the work of Tandori [27], Leindler [12]–[15], Okuyama and Tsuchikura [20], Okuyama [18]–[22], Szalay [26], Billard [2], Grepaqevskaya [4], Spevakov and Kudrajatsev [25]. In 2002 Okuyama [23] using generalized Nörlund means has proved two theorems which give sufficient conditions in terms of the coefficients of an orthogonal series under which it is absolute generalized Nörlund summable almost everywhere.

After several articles in which we have dealt with absolute summability of an orthogonal series we proved some general results which include all of the theorems that had been proved previously by Okuyama [23], Okuyama and Tsuchikura [20], and gave some new consequences, as well [6]–[11]. This notion motivated us to consider

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not simply absolute almost generalized Nörlund summability of an orthogonal series but its absolute almost generalized Nörlund summability of order k, $1 \leq k \leq 2$. Closing the introduction we suggest the reader that a systematic work regarding to the results of this type can find in the book of Alexits [1], and Okuyama [21].

2. Notations, notions and known results

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums $\{s_n\}$. Then, let p denotes the sequence $\{p_n\}$. For two given sequences p and q,

$$\begin{aligned} P_n &:= p_0 + p_1 + \cdots + p_n, \quad (P_{-1} = p_{-1} = 0), \\ Q_n &:= q_0 + q_1 + \cdots + q_n, \quad (Q_{-1} = q_{-1} = 0), \end{aligned}$$

the convolution $(p * q)_n$ is defined by

$$(p * q)_n = \sum_{v=0}^{n} p_v q_{n-v} = \sum_{v=0}^{n} p_{n-v} q_v.$$

It is obvious that

$$P_n := (p * 1)_n = \sum_{v=0}^{n} p_v \quad \text{and} \quad Q_n := (1 * q)_n = \sum_{v=0}^{n} q_v.$$

When $(p*q)_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{v=0}^{n} p_{n-v} q_v s_v.$$

The infinite series $\sum_{n=0}^{\infty} a_n$ is absolutely $(N, p, q)-$summable if the series

$$\sum_{n=1}^{\infty} |t_n^{p,q} - t_{n-1}^{p,q}|$$

converges, and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |N, p, q|.$$

Note that $|N, p, q|-$summability is introduced by Tanaka [28].

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval (a, b). We assume that $f(x)$ belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x), \tag{2.1}$$

where $a_n = \int_a^b f(x)\varphi_n(x)dx$, $(n = 0, 1, 2, \dots)$.

Notations

$$R_n := (p * q)_n, \; R_n^j := \sum_{v=j}^{n} p_{n-v} q_v$$

and

$$R_{n-1}^n = 0, \; R_n^0 = R_n$$

are needed below.

Regarding to $|N, p, q|-$ summability of the orthogonal series (2.1) the following two theorems are proved.

Theorem 2.1([23]). *If the series*

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is $|N, p, q|$ summable almost everywhere.

Theorem 2.2([23]). *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) w^{(1)}(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |N, p, q|$ almost everywhere, where $w^{(1)}(n)$ is defined by $w^{(1)}(j) := j^{-1} \sum_{n=j}^{\infty} n^2 \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2$.*

Lorentz [16] has given the following definition:

A bounded sequence $\{s_n\}$ is said to be almost convergent to a limit s, if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{v=m}^{n+m} s_v = s \quad \text{uniformly with respect to } m.$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be almost generalized Nörlund (Qureshi [24]) summable to s, if

$$t_{n,m}^{p,q} = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v} q_v s_{v,m}$$

tends to s, for $n \to \infty$, uniformly with respect to m, where

$$s_{v,m} = \frac{1}{v+1} \sum_{k=m}^{v+m} s_k.$$

We say that an infinite series $\sum_{n=0}^{\infty} a_n$ is absolute almost generalized Nörlund summable of order k, $k \geq 1$, if the series

$$\sum_{n=1}^{\infty} n^{k-1} |t_{n,m}^{p,q} - t_{n-1,m}^{p,q}|^k$$

converges uniformly with respect to m, and we write briefly

$$\sum_{n=0}^{\infty} a_n \in |N, p, q; m|_k.$$

Remark 2.3. We note that:

1. The absolute almost generalized Nörlund summability of order k reduces to the absolute almost Nörlund summability of order k ($|N, p; m|_k$−summability), if $q_n = 1$ for all n.
2. The absolute almost generalized Nörlund summability of order k reduces to the absolute almost Riesz summability of order k ($|\overline{N}, q; m|_k$−summability), if $p_n = 1$ for all n.
3. In the special case when $p_n = \binom{n+\alpha-1}{\alpha-1}$, $\alpha > 0$, the absolute almost Nörlund summability of order k reduces to the absolute almost generalized Cesàro summability of order k.
4. If $p_n = 1/(n+1)$ the absolute almost Nörlund summability of order k reduces to the absolute almost harmonic summability of order k.

Throughout this paper K denotes a positive constant that it may depends only on k, not necessarily the same at each occurrence.

The following lemma due to B. Levi (see, for example [1]) is often used in the theory of functions. It will help us to prove main results.

Lemma 2.4. *If $h_n(t) \in L(U)$ are non-negative functions and*

$$\sum_{n=1}^{\infty} \int_U h_n(t)dt < \infty, \tag{2.2}$$

then the series

$$\sum_{n=1}^{\infty} h_n(t)$$

converges (absolutely) almost everywhere on U to a function $h(t) \in L(U)$.

3. Main results

Setting

$$\widehat{R}_n^j := \sum_{v=j}^{n} \frac{p_{n-v}q_v}{v+1}, \quad \widehat{R}_{n-1}^n = 0,$$

we prove the following theorem.

Theorem 3.1. *If for* $1 \leq k \leq 2$ *the series*

$$\sum_{n=1}^{\infty} \left\{ n^{2\left(1-\frac{1}{k}\right)} \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{\frac{k}{2}}$$

converges uniformly with respect to m*, then the orthogonal series*

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

is $|N, p, q; m|_k$*−summable almost everywhere.*

Proof. Let $s_v(x) = \sum_{j=0}^{v} a_j \varphi_j(x)$ be vth partial sums of the series (2.1), and $1 < k < 2$. A straightforward calculation shows that

$$\begin{aligned} s_{v,m}(x) &= \frac{1}{v+1} \sum_{k=m}^{v+m} s_k(x) \\ &= \frac{1}{v+1} \sum_{k=0}^{v} s_{k+m}(x) \\ &= \frac{1}{v+1} \sum_{k=0}^{v} \sum_{j=0}^{k+m} a_j \varphi_j(x) \\ &= s_{m-1}(x) + \sum_{j=0}^{v} \left(1 - \frac{j}{v+1}\right) a_{m+j} \varphi_{m+j}(x). \end{aligned}$$

For the almost generalized Nörlund transform $t^{p,q}_{n,m}(x)$ we have that

$$\begin{aligned}
t^{p,q}_{n,m}(x) &= \frac{1}{R_n}\sum_{v=0}^{n} p_{n-v}q_v s_{v,m}(x)\\
&= \frac{1}{R_n}\sum_{v=0}^{n} p_{n-v}q_v \left\{ s_{m-1}(x)+\sum_{j=0}^{v}\left(1-\frac{j}{v+1}\right)a_{m+j}\varphi_{m+j}(x)\right\}\\
&= s_{m-1}(x)+\frac{1}{R_n}\sum_{v=0}^{n} p_{n-v}q_v\sum_{j=0}^{v} a_{m+j}\varphi_{m+j}(x)\\
&\quad -\frac{1}{R_n}\sum_{v=0}^{n}\frac{p_{n-v}q_v}{v+1}\sum_{j=0}^{v} j a_{m+j}\varphi_{m+j}(x)\\
&= s_{m-1}(x)+\frac{1}{R_n}\sum_{j=0}^{n} a_{m+j}\varphi_{m+j}(x)\sum_{v=j}^{n} p_{n-v}q_v\\
&\quad -\frac{1}{R_n}\sum_{j=0}^{n} j a_{m+j}\varphi_{m+j}(x)\sum_{v=j}^{n}\frac{p_{n-v}q_v}{v+1}\\
&= s_{m-1}(x)+\frac{1}{R_n}\sum_{j=0}^{n} R^j_n a_{m+j}\varphi_{m+j}(x)-\frac{1}{R_n}\sum_{j=0}^{n} j\widehat{R}^j_n a_{m+j}\varphi_{m+j}(x)\\
&= s_{m-1}(x)+\frac{1}{R_n}\sum_{j=0}^{n}\left(R^j_n - j\widehat{R}^j_n\right)a_{m+j}\varphi_{m+j}(x).
\end{aligned}$$

Hence, putting

$$\triangle t^{p,q}_{n,m}(x) := t^{p,q}_{n,m}(x) - t^{p,q}_{n-1,m}(x),$$

we obtain

$$\begin{aligned}
\triangle t^{p,q}_{n,m}(x) &= \frac{1}{R_n}\sum_{j=0}^{n}\left(R^j_n - j\widehat{R}^j_n\right)a_{m+j}\varphi_{m+j}(x)\\
&\quad -\frac{1}{R_{n-1}}\sum_{j=0}^{n-1}\left(R^j_{n-1} - j\widehat{R}^j_{n-1}\right)a_{m+j}\varphi_{m+j}(x)\\
&= \frac{1}{R_n}\sum_{j=1}^{n}\left(R^j_n - j\widehat{R}^j_n\right)a_{m+j}\varphi_{m+j}(x)\\
&\quad -\frac{1}{R_{n-1}}\sum_{j=1}^{n}\left(R^j_{n-1} - j\widehat{R}^j_{n-1}\right)a_{m+j}\varphi_{m+j}(x)\\
&= \sum_{j=1}^{n}\left[\frac{R^j_n}{R_n}-\frac{R^j_{n-1}}{R_{n-1}} - j\left(\frac{\widehat{R}^j_n}{R_n}-\frac{\widehat{R}^j_{n-1}}{R_{n-1}}\right)\right]a_{m+j}\varphi_{m+j}(x).
\end{aligned}$$

Using the Hölder's inequality, orthogonality, and the above equality we have that

$$\begin{aligned}\int_a^b |\triangle t_{n,m}^{p,q}(x)|^k dx &\leq (b-a)^{1-\frac{k}{2}} \left(\int_a^b |t_{n,m}^{p,q}(x) - t_{n-1,m}^{p,q}(x)|^2 dx \right)^{\frac{k}{2}} \\ &= K \left\{ \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j \left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{\frac{k}{2}}.\end{aligned}$$

Thus, the series

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\triangle t_{n,m}^{p,q}(x)|^k dx$$

$$\leq K \sum_{n=1}^{\infty} n^{k-1} \left\{ \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j \left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{\frac{k}{2}} \tag{3.1}$$

converges, since the last one converges uniformly with respect to m. Since the functions $|\triangle t_{n,m}^{p,q}(x)|$ are non-negative, then by the Lemma 2.4 the series

$$\sum_{n=1}^{\infty} n^{k-1} |\triangle t_{n,m}^{p,q}(x)|^k$$

converges almost everywhere. For $k = 1$ we use the Schwartz's inequality, until for $k = 2$ we use just the orthogonality. This completes the proof of the theorem. □

For shortening we denote

$$\mathcal{D}_n^j := \frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j \left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right).$$

In the special case, when $p_v = 1$ for all v, we obtain the equality

$$\begin{aligned}\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} &= \frac{1}{Q_n Q_{n-1}} \left(Q_{n-1} \sum_{v=j}^{n} \frac{q_v}{v+1} - Q_n \sum_{v=j}^{n-1} \frac{q_v}{v+1} \right) \\ &= \frac{1}{Q_n Q_{n-1}} \left[(Q_n - q_n) \sum_{v=j}^{n} \frac{q_v}{v+1} - Q_n \left(\sum_{v=j}^{n} \frac{q_v}{v+1} - \frac{q_n}{n+1} \right) \right] \\ &= \frac{q_n}{Q_n Q_{n-1}} \left(\frac{Q_n}{n+1} - \sum_{v=j}^{n} \frac{q_v}{v+1} \right),\end{aligned}$$

and in a similar way one can find that

$$\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}=-\frac{q_nQ_{j-1}}{Q_nQ_{n-1}}.$$

Thus we have

$$\mathcal{D}_n^j=-\frac{q_n}{Q_nQ_{n-1}}\left[Q_{j-1}+j\left(\frac{Q_n}{n+1}-\sum_{v=j}^{n}\frac{q_v}{v+1}\right)\right],$$

and from theorem 3.1 the following corollary holds true.

Corollary 3.2. *If for* $1\leq k\leq 2$ *the series*

$$\sum_{n=1}^{\infty}\left(\frac{n^{\left(1-\frac{1}{k}\right)}q_n}{Q_nQ_{n-1}}\right)^k\left\{\sum_{j=1}^{n}\left[Q_{j-1}+j\left(\frac{Q_n}{n+1}-\sum_{v=j}^{n}\frac{q_v}{v+1}\right)\right]^2|a_{m+j}|^2\right\}^{\frac{k}{2}}$$

converges uniformly with respect to m*, then the orthogonal series*

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is $|\overline{N},q;m|_k-$*summable almost everywhere.*

Another interesting consequence of the theorem 3.1 is:

Corollary 3.3. *If for* $1\leq k\leq 2$ *the series*

$$\sum_{n=1}^{\infty}\left(\frac{n^{\left(1-\frac{1}{k}\right)}p_n}{P_nP_{n-1}}\right)^k\left\{\sum_{j=1}^{n}p_{n-j}^2\left[\mathcal{H}(j;n)\right]^2|a_{m+j}|^2\right\}^{\frac{k}{2}}$$

converges uniformly with respect to m*, then the orthogonal series*

$$\sum_{n=0}^{\infty}a_n\varphi_n(x)$$

is $|N,p;m|_k-$*summable almost everywhere, where*

$$\mathcal{H}(j;n):=1-\frac{P_{n-1-j}}{p_{n-j}}+j\sum_{v=0}^{n-j}\frac{P_n-(n+1-v)p_n}{(n-v)(n+1-v)p_np_{n-j}}p_v.$$

Proof. If $q_v = 1$ for all v, we have

$$\begin{aligned}
\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} &= \frac{1}{P_nP_{n-1}}\left(P_{n-1}\sum_{v=j}^{n}\frac{p_{n-v}}{v+1} - P_n\sum_{v=j}^{n-1}\frac{p_{n-1-v}}{v+1}\right)\\
&= \frac{1}{P_nP_{n-1}}\left(P_{n-1}\sum_{v=0}^{n-j}\frac{p_v}{n+1-v} - P_n\sum_{v=0}^{n-1-j}\frac{p_v}{n-v}\right)\\
&= \frac{1}{P_nP_{n-1}}\left[(P_n-p_n)\sum_{v=0}^{n-j}\frac{p_v}{n+1-v} - P_n\sum_{v=0}^{n-1-j}\frac{p_v}{n-v}\right]\\
&= \frac{1}{P_nP_{n-1}}\left[P_n\left(\sum_{v=0}^{n-j}\frac{p_v}{n+1-v} - \sum_{v=0}^{n-1-j}\frac{p_v}{n-v}\right) - p_n\sum_{v=0}^{n-j}\frac{p_v}{n+1-v}\right]\\
&= \frac{1}{P_nP_{n-1}}\left[\frac{p_{n-j}P_n}{j+1} - (P_{n-1}+p_n)\sum_{v=0}^{n-1-j}\frac{p_v}{(n-v)(n+1-v)} - \right.\\
&\qquad\left. -p_n\left(\sum_{v=0}^{n-1-j}\frac{p_v}{n+1-v} + \frac{p_{n-j}}{j+1}\right)\right]\\
&= \frac{1}{P_nP_{n-1}}\left[\frac{p_{n-j}(P_n-p_n)}{j+1} - \sum_{v=0}^{n-1-j}\frac{P_n-(n+1-v)p_n}{(n-v)(n+1-v)}p_v\right]\\
&= \frac{1}{P_nP_{n-1}}\left[\frac{p_{n-j}(P_{n-1}-p_n)}{j} - \sum_{v=0}^{n-j}\frac{P_n-(n+1-v)p_n}{(n-v)(n+1-v)}p_v\right]\\
&= \frac{p_np_{n-j}}{P_nP_{n-1}}\left[\frac{P_{n-1}-p_n}{jp_n} - \sum_{v=0}^{n-j}\frac{P_n-(n+1-v)p_n}{(n-v)(n+1-v)p_np_{n-j}}p_v\right],
\end{aligned}$$

and in a similar way one can find that

$$\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} = \frac{p_np_{n-j}}{P_nP_{n-1}}\left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\right).$$

So,

$$\mathcal{D}_n^j = \frac{p_np_{n-j}}{P_nP_{n-1}}\left[1 - \frac{P_{n-1-j}}{p_{n-j}} + j\sum_{v=0}^{n-j}\frac{P_n-(n+1-v)p_n}{(n-v)(n+1-v)p_np_{n-j}}p_v\right]$$

for all $q_v = 1$. Now obviously the proof follows immediately from Theorem 3.1. □

Now we shall prove a very general theorem on $|N, p, q; m|_k-$ summability almost everywhere of an orthogonal series. It involves a sequence that satisfies certain conditions. During the proof of this theorem is used a similar method with one of Okuyama [18], and Ul'yanov [29].

Indeed, if we put

$$\mathcal{B}^{(k)}(j) := \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^{\frac{2}{k}} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 \tag{3.2}$$

then the following theorem holds true.

Theorem 3.4. *Let $1 \leq k \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum_{n=1}^{\infty} |a_{m+n}|^2 \Omega^{\frac{2}{k}-1}(n)\mathcal{B}^{(k)}(n)$ converges uniformly with respect to m, then the orthogonal series $\sum_{n=0}^{\infty} a_n\varphi_n(x) \in |N, p, q; m|_k$ almost everywhere, where $\mathcal{B}^{(k)}(n)$ is defined by (3.2).*

Proof. Applying Hölder's inequality to the inequality (3.1) we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\triangle t_{n,m}^{p,q}(x)|^k dx \leq \\
&\leq K \sum_{n=1}^{\infty} n^{k-1} \left\{ \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{\frac{k}{2}} \\
&= K \sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))^{\frac{2-k}{2}}} \times \\
&\qquad \times \left\{ n\Omega^{\frac{2}{k}-1}(n) \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{\frac{k}{2}} \\
&\leq K \left(\sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))} \right)^{\frac{2-k}{2}} \times \\
&\qquad \times \left\{ \sum_{n=1}^{\infty} n\Omega^{\frac{2}{k}-1}(n) \sum_{j=1}^{n} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 |a_{m+j}|^2 \right\}^{\frac{k}{2}} \\
&\leq K \left\{ \sum_{j=1}^{\infty} |a_{m+j}|^2 \sum_{n=j}^{\infty} n\Omega^{\frac{2}{k}-1}(n) \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 \right\}^{\frac{k}{2}} \\
&\leq K \left\{ \sum_{j=1}^{\infty} |a_{m+j}|^2 \left(\frac{\Omega(j)}{j} \right)^{\frac{2}{k}-1} \sum_{n=j}^{\infty} n^{\frac{2}{k}} \left[\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} - j\left(\frac{\widehat{R}_n^j}{R_n} - \frac{\widehat{R}_{n-1}^j}{R_{n-1}} \right) \right]^2 \right\}^{\frac{k}{2}} \\
&= K \left\{ \sum_{j=1}^{\infty} |a_{m+j}|^2 \Omega^{\frac{2}{k}-1}(j) \mathcal{B}^{(k)}(j) \right\}^{\frac{k}{2}}
\end{aligned}$$

which is finite uniformly with respect to m by assumption. For the proof now one can do the same reasoning as in the proof of theorem 3.1. We omit details. □

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