[8] H. P. F. Swinnerton-Dyer, *Two special cubic surfaces*, Mathematika 9 (1962), pp. 54–56.
[9] H. Wada, *A table of fundamental units of purely cubic field*, Proc. Japan Acad. 46 (10) (1970), pp. 1135–1140.
[10] B. L. van der Waerden, *Noch eine Bemerkung zu der Arbeit „Zur Arithmetik der Polynome" von U. Wegner in Math. Ann. 105, S. 628–631*, Math. Ann. 109 (1934), pp. 679–680.

DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY

Received on 5. 10. 1971 (229)

ACTA ARITHMETICA
XXII (1973)

On absolute (j, ε)-normality in the rational fractions with applications to normal numbers

by

R. G. Stoneham (New York, N.Y.)

1. Introduction. In [1, Th. 6, p. 233], we established the (j, ε)-normality [1, Def., p. 222] of a broad class of rational fractions $Z/m < 1$ in lowest terms of type A [1, Th. 4, p. 227, and Def., Type A, p. 229] when represented in bases g such that $(g, m) = 1$.

We shall now present results based on a relaxation of the requirement $(g, m) = 1$ and consider the consequences for the (j, ε)-normal properties of the representations of Z/m in bases g such that $(g, m) > 1$ where g contains some but not all prime factors of m.

Essentially, the above implies that we shall now permit the representations to have non-periodic parts for such g and, of course, the definition of (j, ε)-normality [1, Lemma, and Def., p. 222) does not preclude this occurrence.

Let $m = 2^b \prod_{i=1}^{r} p_i^{b_i}$ and assume in contrast to the basic requirement for Type A, i.e. $b_i > z_i + s_i$ for *at least one* odd prime p_i that one or more of the p_i are such that $b_i > z_i + s_i$, hence, Z/m is surely of Type A and (j, ε)-normal on all g such that $2 \leqslant g < m/D$ where $(g, m) = 1$. Since we obtain non-periodic parts for those g which contain some but not all prime factors of m, we may write

$$Z/m = ZI(u)/g^u M = Q/g^u + R/g^u M \tag{1.0}$$

where $ZI(u)/M = Q + R/M$ with $I(u)$ some positive integer, and $Q \geqslant 0$ is the set of u digits in the non-periodic part. We shall call $R/M < 1$ in lowest terms the "associated" fraction when Z/m is represented in a base such that $(g, M) = 1$ since M contains all the residual prime factors of m not contained in g.

Now if the associated fraction R/M is still of Type A, then Z/m is (j, ε)-normal in all such additional bases g, i.e. those that contain some but not all prime factors of m. The essential point is to select those prime factors in the choice of g which leaves behind in the associated fraction

R/M prime factors such that $b_i > z_i + s_i$ for at least one of the remaining factors. This can, of course, always be done if the structure of m is such that more than one of the p_i is such that $b_i > z_i + s_i$. If only one of the p_i out of the r distinct odd primes is such that $b_i > z_i + s_i$, then we can have (j, ε)-normality in all other g which are multiples of every prime which is not the particular prime such that $b_i > z_i + s_i$, i.e. the associated fraction R/M can always be made Type A; hence, (j, ε)-normal.

It is in the above sense that we may extend the set of bases g contained in $2 \leqslant g < m/D$ when we permit non-periodic parts in the expansions such that the associated fractions are of Type A. We have now proved the following theorem:

THEOREM 1. *If the rational fraction* $Z/m = Z/2^b \prod_{i=1}^{r} p_i^{b_i}$ *has one or more of the odd primes* p_i *such that* $b_i > z_i + s_i$, *then* Z/m *is* (j, ε)*-normal when represented in all additional bases* g *contained in* $2 \leqslant g < m/D$ *which are multiples of those prime factors that are selected in such a way as to leave the remaining associated fraction* R/M *of Type* A.

If we now assume that $b_i > z_i + s_i$ for every p_i in m, then every possible associated fraction R/M is of Type A. We shall call such a fraction Z/m a "complete" rational fraction of Type A. The associated fraction R/M which here is necessarily complete and of Type A generates the periodic portion alone in the representation of Z/m and consists of $\omega(M) = \mathrm{ord}_M g$ digits. A useful bound on the number of digits u in the non-periodic part is given by

$$0 \leqslant u \leqslant \mathrm{Max}\,(b, b_1, b_2, \ldots, b_r) = B. \tag{1.1}$$

If we allow expansions under these conditions in bases g such that $(g, m) \geqslant 1$, we find an interesting property for complete rational fractions of Type A that we shall call "absolute" (j, ε)-normality, i.e. we find that we have (j, ε)-normality for each expansion of Z/m in every consecutive positive integer base of a bounded set of $g \geqslant 2$. This, apparently, is the analog in the rationals for the notion of an absolutely normal number introduced by E. Borel [3] in 1909, i.e. an irrational which is normal when represented in every positive integer base $g \geqslant 2$.

In [1, Th. 6, p. 233], we proved (j, ε)-normality for Type A when represented in any base g such that $(g, m) = 1$ where $2 \leqslant g < m/D$. There is no change in the upper bound m/D but if we assume that Z/m is complete, then we have absolute (j, ε)-normality on all consecutive positive integers contained in $2 \leqslant g < 2^h \prod_{i=1}^{r} p_i$ where $h = 0$ if $b = 0$ and $h = 1$ if $b > 0$. For those $g > 2^h \prod_{i=1}^{r} p_i$, we necessarily delete those g as acceptable bases which contain all prime factors of m since these lead to terminating expansions. Thus, we have absolute (j, ε)-normality for those g in $2 \leqslant g < 2^h \prod_{i=1}^{r} p_i$ and what we shall call "almost" absolute (j, ε)-normality on the rest of the range $2^h \prod_{i=1}^{r} p_i < g < m/D$ where the exceptional set of bases contain all the prime factors of m. The following theorem is easily demonstrated based on [1, Th. 6, p. 233].

THEOREM 2. *A complete rational fraction* $Z/m < 1$ *in lowest terms of Type* A *is absolutely* (j, ε)*-normal in all* g *such that* $2 \leqslant g < 2^h \prod_{i=1}^{r} p_i < m/D$ *where* $h = 0$ *if* $b = 0$, *and* $h = 1$ *if* $b > 0$ *in* $m = 2^b \prod_{i=1}^{r} p_i^{b_i}$. *On those* g *such that* $2^h \prod_{i=1}^{r} p_i \leqslant g < m/D$, *we have almost absolute* (j, ε)*-normality where the exceptional* g *are those* g *which contain all prime factors of* m.

Proof. We have (j, ε)-normality on all g which contain some but not all prime factors of m since the associated fraction $R/M < 1$ in lowest terms is necessarily complete, and therefore of Type A. Using the basic definition of (j, ε)-normality in [1, p. 222] and the (j, ε)-normality of Type A in [1, Th. 6, p. 233], the conclusion follows.

Every g contained in $2 \leqslant g < 2^h \prod_{i=1}^{r} p_i$ will either have some but not all prime factors of m or none. Hence, Z/m will be (j, ε)-normal when represented in every consecutive positive integer in this range. For those g such that $2^h \prod_{i=1}^{r} p_i \leqslant g < m/D$, we may have some g that contain all prime factors of m. These will constitute an exceptional set since representation in these g will produce terminating forms. On the other hand, Z/m is (j, ε)-normal in every other g in this range, i.e. there are g that will contain some but not all prime factors of m and others none at all. Therefore, we have almost absolute (j, ε)-normality on the rest of the range $2^h \prod_{i=1}^{r} p_i \leqslant g < m/D$. Q.E.D.

In Section 2, we will prove that we may extend the set of bases to which the normal number construction in [2] is valid. We show that for any choice of m in the sequence of fractions Z_n/m^n for $n = 1, 2, \ldots$ used in the construction that we obtain normality of $x(g, m)$ for all consecutive g contained in $2 \leqslant g < 2^h \prod_{i=1}^{r} p_i$ and all $g > 2^h \prod_{i=1}^{r} p_i$ which contain some but not all prime factors of m. The exceptional set in which $x(g, m)$ is non-normal are those $g \geqslant 2^h \prod_{i=1}^{r} p_i$ that contain all prime factors of m. In bases g such that $(g, m) > 1$ that contain some but not all prime factors of m, we show that the presence of non-periodic parts in the limit does not affect the normality or transcendence of the construction.

To conclude this section, we emphasize an important point. These normal numbers are, in a sense, "base dependent", i.e. each constructed normal number for given fixed choices of the basic parameters Z_i, m, and the repetition sequence a_i as represented in a given acceptable base is a distinct irrational on the real line. Their position varies with the choice of g for fixed Z_i, m, and a_i. This is due to the fact that we use in the construction only finite portions of the infinite periodic expansions of the Z_i/m^i.

Finally, there may be a temptation here to view these results as a kind of "absolute normality" in the sense of Borel [3] who defined normal numbers and gave an existential proof in 1909 that "almost all real numbers are absolutely normal" with the non-normal irrationals of measure zero. However, the differences here are distinct. Borel showed, essentially, that *there exists* a fixed irrational on the real line which is normal when represented in every positive integer base $g \geqslant 2$. We have irrationals here which are normal when represented in every positive integer of a bounded set and, even though, we can fix the choice of the parameters Z_i, m, and a_i; we obtain a sequence of distinct normal numbers $x(g_i, m)$ for each acceptable $g_1, g_2, \ldots$ above and below the bound $2^h \prod_{i=1}^{r} p_i$. Apparently, to date, no simple arithmetic construction of an absolutely normal number has been given, nor does the existential result of Borel help in any way to prove the difficult proposition that a given irrational like "e" or π is normal to any base.

2. Normal number construction. Basically, we shall follow the proofs in [2] and attend to those aspects affected by the presence of the non-periodic parts in the arguments.

Consider the rational fractions $Z_i/m^i = Q_i/g^{iu} + R_i/g^{iu}M^i$ where we shall denote by $T_iE_i(a_i)E_i$, the non-periodic part T_i consisting of ui digits and a_i repetitions of complete periods E_i of the associated fractions R_i/M^i as represented in bases g such that $(g, M) = 1$ which contain some but not all prime factors of m. We use a construction similar to [2, p. 242, (2.1)]

$$(2.0)\quad x(g, m, n) = .T_1E_1(a_1)E_1 \ldots T_{n-1}E_{n-1}(a_{n-1})E_{n-1}T_nE_n(k)E_nB_r$$

where B_r is the first r digits into the $(k+1)$st repetition of the E_nth period.

Let $N(B_j, E_i)$ denote the number of occurrences of B_j in E_i extending at most $j-1$ places into either a next E_i or at the end of a repeated sequence $E_i(a_i)E_iT_{i+1}$ into the juncture E_iT_{i+1}. Also, let $N(B_j, T_i)$ denote the number of occurrences of the block B_j contained in the non-periodic part and extending at most $j-1$ places into the first E_i. Therefore, we have [2, (2.5)]

$$(2.1)\qquad |N(t, B_j, x)/t - I| \leqslant 2n(j-1)/t = R_n$$

where

$$(2.2)\qquad I = \Big(\sum_{i=1}^{n} N(B_j, T_i) + \sum_{i=1}^{n-1} a_i N(B_j, E_i) + kN(B_j, E_n) + N(B_j, r)\Big)/t$$

and $N(t, B_j, x)$ denotes the number of occurrences of B_j in the first t digits given by

$$(2.3)\qquad t = \sum_{i=1}^{n} iu + \sum_{i=1}^{n-1} a_i\omega(M^i) + k\omega(M^n) + r$$

where $u = 0$ and $M = m$ if $(g, m) = 1$. Otherwise, for a fixed choice of g which contains some but not all prime factors of m, u is fixed and M contains the prime factors not in g.

In (2.1), the $2n(j-1)/t$ accounts for anomalous blocks [2, p. 243] across $(n-1)E_iT_{i+1}$ junctures and possibly from E_n to B_r. Also, we may have counts across each T_iE_i for $i = 1, 2, \ldots, n$. If now $R_n = 2n(j-1)/t$, it is clear that the argument in the proof of Lemma 1 [2, p. 243] which shows that $\lim_{n\to\infty} R_n = 0$ remains unaffected by a factor of 2 in $2n(j-1)/t$ where now we use

$$t = un(n+1)/2 + \sum_{i=1}^{n-1} a_i\omega(M^i) + k\omega(M^n) + r > \sum_{i=1}^{n-1} a_i.$$

Since $t \to \infty$ as $n \to \infty$, we must evaluate

$$(2.4)\qquad \lim_{n\to\infty} N(t, B_j, x)/t = \lim_{n\to\infty} I.$$

As before, we distinguish 2 cases for k, i.e. $1 \leqslant k < a_n$ and $k = a_n$. Now we may write I in (2.2) for case 1 as

$$(2.5)\qquad I = \Big(\sum_{i=1}^{n} N(B_j, T_i)/P_n + \Big(\sum_{i=1}^{n-1} a_i N(B_j, E_i) + kN(B_j, E_n)\Big)/P_n + N(B_j, r)/P_n\Big)/T$$

where now from (2.3)

$$(2.6)\qquad T = t/P_n = 1 + un(n+1)/2P_n + r/P_n$$

and

$$(2.7)\qquad P_n = \sum_{i=1}^{n-1} a_i\omega(M^i) + k\omega(M^n).$$

For case 2, we have

$$(2.8)\quad I' = \Big(\sum_{i=1}^{n} N(B_j, T_i)/P'_n + \sum_{i=1}^{n} a_i N(B_j, E_i)/P'_n + N(B_j, r)/P'_n\Big)/T'$$

where

$$(2.9)\quad T' = 1 + un(n+1)/2P'_n + r/P'_n$$

and

$$(2.10)\quad P'_n = \sum_{i=1}^{n} a_i \omega(M^i).$$

By the same arguments in [2, (2.14), p. 245, etc.] for the periodic parts, we still have $\lim_{n\to\infty} N(B_j, r)/P_n = 0$ and $\lim_{n\to\infty} r/P_n = 0$. Similarly, we have $\lim_{n\to\infty} N(B_j, r)/P'_n = 0$ and $\lim_{n\to\infty} r/P'_n = 0$.

On the counts $N(B_j, T_i)$ in the non-periodic parts, we have in the possible iu digits of T_i counts for single digits $N(B_1, T_i) \leqslant iu$, $N(B_2, T_i) \leqslant iu-1$ for pairs, etc. In general, we have

$$N(B_j, T_i) \leqslant iu-(j-1) \leqslant iu \leqslant iB$$

where $j \geqslant 1$ and $B = \operatorname{Max}(b, b_1, b_2, \ldots, b_r)$. Therefore, we obtain for the non-periodic parts in (2.5) and (2.8)

$$(2.11)\quad \sum_{i=1}^{n} N(B_j, T_i)/P'_n < \sum_{i=1}^{n} N(B_j, T_i)/P_n \leqslant \sum_{i=1}^{n} iB/P_n = Bn(n+1)/2P_n$$

since $P'_n > P_n$. Now $\lim_{n\to\infty} Bn(n+1)/2P_n = 0$ for a fixed B since

$$(2.12)\quad \lim_{n\to\infty} n(n+1)/P_n < \lim_{n\to\infty} \frac{n(n+1)/M^{n-1-c}}{C_0/M^{n-1-c}+C_1} = 0$$

where we have used

$$P_n > \sum_{i=1}^{n-1} a_i \omega(M^i) \geqslant C_0 + C_1 M^{n-1-c}$$

based on the inequality in [2, (2.31), p. 247, $s = n-1$, $k = c$].

[Note: In [2, (2.31), p. 247], we should have used, say, c instead of k so as to distinguish the k notation for cases 1 and 2 above, with reference to the non-related integer c such that $\omega(M) = \omega(M^2) = \ldots = \omega(M^c)$ in the inequality (2.31).]

Similarly, it follows that $\lim_{n\to\infty} un(n+1)/2P_n$ or $un(n+1)/2P'_n = 0$ in (2.6) and (2.9) for any fixed u. All remaining ratios in (2.5) and (2.8) approach zero by previous arguments since they involve only periodic parts. Thus for cases 1 and 2 as in [2, (2.16)–(2.18), p. 245]

$$(2.13)\quad \lim_{n\to\infty} I = \lim_{n\to\infty} N(B_j, E_n)/\omega(M^n)$$

where $M = m$ if $(g, m) = 1$. Therefore, we obtain

$$(2.14)\quad \lim_{n\to\infty} N(t, B_j, x)/t = \lim_{n\to\infty} N(B_j, E_n)/\omega(M^n).$$

The rest of the proof follows precisely [2, p. 245–246, from (2.18) to (2.22)] since all associated rational fractions represented in other bases such that $(g, m) > 1$ where the g may contain some but not all prime factors of m are of Type A for n sufficiently large. It is also clear that regardless of the prime structure of m in Z_n/m^n there is some N such that for all $n > N$, we have $nb_i > z_i + s_i$ for any odd prime in m since for a given p_i; z_i and s_i are fixed. Therefore, all the fractions Z_n/m^n for $n > N$ are complete, and consequently all the associated fractions R_n/M^n are also complete.

Thus, we have normality as before such that $(g, m) = 1$ where $T_i = 0$, but now, in addition, we have shown normality in all g which contain some but not all prime factors of m. Furthermore, by the same argument as in the proof of Theorem 2, we have normality of $x(g, m)$ in every positive integer g in $2 \leqslant g < 2^h \prod_{i=1}^{r} p_i$ since these g will not contain all prime factors of m and every $g > 2^h \prod_{i=1}^{r} p_i$ which contain some but not all prime factors of m. The exceptional set on which $x(g, m)$ is non-normal are those $g \geqslant 2^h \prod_{i=1}^{r} p_i$ that contain all prime factors of m.

In order to derive the form similar to [2, (2.0)], we must remove a set of E_n beyond the a_nth E_n in

$$T_n E_n(a_n) E_n = .00\ldots00 T_n E_n(a_n) E_n - .00\ldots00 E_n E_n \ldots.$$

This implies that we must difference the given fraction Z_n/m^n and its associated fraction $Z_n I^n(u)/M^n$ in their appropriate place positions. We find

$$(2.15)\quad T_n E_n(a_n) E_n = Z_n/m^n g^{S'(n-1,M)} - Z_n I^n(u)/M^n g^{S'(n,M)}$$

where for $n = 1, 2, \ldots$ we have, using the definition of $S(n, m)$ in [2, (2.27)], the related quantity

$$(2.16)\quad S'(n, M) = nu + \sum_{i=1}^{n} a_i \omega(M^i) = nu + S(n, M).$$

Differencing the terms which have the same power of g and using the fact that $1/m = I(u)/g^u M \Rightarrow g^{nu}/m^n = I^n(u)/M^n$, we obtain

$$x(g, m) = \sum_{n=0}^{\infty} (Z_{n+1} - mZ_n g^{nu})/m^{n+1} g^{S'(n,M)}. \tag{2.17}$$

In (2.17), depending on g and m, we define $Z_0 = 0$, $a_0 = 0$, $S'(0, M) = 0$; and $M = m$, $u = 0$ if $(g, m) = 1$. Clearly (2.17) reduces to [2, p. 242, (2.0)] if $(g, m) = 1$. Assuming the same definitions of the basic parameters Z_i, m, $\omega(m)$, the a_i, and $S(n, m)$ that enter the construction of $x(g, m)$ as stated in [2, Th. 1, p. 242], we now have proved the following generalization of $x(g, m)$ to $(g, m) > 1$ where those g such that $(g, m) > 1$ contain some but not all prime factors of m.

THEOREM 3. *Let* $x(g, m) = \sum_{n=0}^{\infty} (Z_{n+1} - mZ_n g^{nu})/m^{n+1} g^{S'(n,M)}$ *where* $Z_n/m^n = Q_n/g^{nu} + R_n/g^{nu}M^n$, *$u$ is the number of digits in the non-periodic part of* Z_1/m, *and* $S'(n, M) = nu + \sum_{i=1}^{n} a_i \omega(M^i)$ *when g contains some but not all prime factors of m. If* $(g, m) = 1$, *then* $u = 0$, $M = m$, *and*

$$S'(n, M) = S(n, m) = \sum_{i=1}^{n} a_i \omega(m^i).$$

Furthermore, $x(g, m)$ is a normal number when represented in every positive integer base contained in $2 \leqslant g < 2^h \prod_{i=1}^{r} p_i$ *and all* $g > 2^h \prod_{i=1}^{r} p_i$ *that contain some but not all prime factors of m. Finally, $x(g, m)$ is non-normal in every positive integer* $g \geqslant 2^h \prod_{i=1}^{r} p_i$ *that contain all prime factors of m.*

3. The transcendence. As in the normality of $x(g, m)$ subject to $(g, m) > 1$ for suitable g, we attend to those aspects of the proof of transcendence and non-Liouville character of $x(g, m)$ which depend upon the presence of the non-periodic parts. Essentially, this amounts to the evaluation of certain limiting forms where now we use (2.16)

$$S'(n, M) = nu + \sum_{i=1}^{n} a_i \omega(M^i) = nu + S(n, M)$$

with $u > 0$ some fixed positive integer. [We find it notationally consistent here to use $S(n, M) = \sum_{i=1}^{n} a_i \omega(M^i)$, even though, earlier we used P'_n for the same quantity in (2.10).] We require that $a_{n+1}\omega(M^{n+1})/S(n, M)$ be bounded as n increases. If $(g, m) = 1$, then $M = m$, $S(n, M) = S(n, m)$ with $u = 0$ which yields the same quantity as in [2, p. 246, (2.27)].

As before [2, p. 247], we let

$$p_s/q_s = \sum_{n=0}^{s} (Z_{n+1} - mZ_n g^{nu})/m^{n+1} g^{S'(n,M)}$$

where $q_s = m^{s+1} g^{S'(s,M)}$ is still the L.C.D. for the same reasons. We identify $q'_s = m^{s+1}$ and $q''_s = g^{S'(s,M)}$ where $q = g$. First, we examine the conditions $\lim_{s\to\infty} \log q'_s/\log q_s = 0$ and $\limsup_{s\to\infty} \log q_{s+1}/\log q_s < \infty$. In several places, we require limits like $\lim_{s\to\infty} (s+c)/S(s, M)$ where $c \geqslant 0$ is some fixed quantity. Since $S(s, M) = \sum_{i=1}^{s} a_i \omega(M^i)$, we have based on [2, p. 247, (2.29)–(2.31)] replacing m by M that

$$S(s, M) \geqslant C_0 + C_1 M^{s-k} \tag{3.0}$$

where k is fixed and M contains the residual prime factors after the choice of g. Therefore, it follows for any fixed c that

$$\lim_{s\to\infty} (s+c)/S(s, M) = 0. \tag{3.1}$$

Hence, we have satisfaction of the first condition

$$\lim_{s\to\infty} \log q'_s/\log q_s = 0$$

as in [2, p. 247, (2.33)]. For the second condition, we have

$$\log q_{s+1}/\log q_s = \log m^{s+2} g^{S'(s+1,M)}/\log m^{s+1} g^{S'(s,M)}. \tag{3.2}$$

[Note: Let (3.2) here stand as a corrigendum of [2, p. 248, (2.36)] wherein the first exponent of g should read $g^{S(s+1,m)}$.]

In (3.2), we find that the $\limsup_{s\to\infty} \log q_{s+1}/\log q_s$ is bounded with the assumption that for some fixed quantity β

$$a_{s+1}\omega(M^{s+1})/S(s, M) < \beta \tag{3.3}$$

where we have used

$$S'(s+1, M) = (s+1)u + S(s, M) + a_{s+1}\omega(M^{s+1}) \tag{3.4}$$

when $u > 0$, $\lim_{s\to\infty} (s+c)/S(s, M) = 0$ for $c = 1, 2$; and also $\lim_{s\to\infty} u/S(s, M) = 0$ for some fixed $u > 0$.

In the demonstration leading to [2, p. 249, (2.46)] that an $x > 1$ exists independent of s, the requirements are all satisfied here which leads to the lower bound $\delta < a_{s+1}\omega(M^{s+1})/S(s, M)$.

Again in the non-Liouville argument, all inequalities and finally the boundedness of y/t in [2, p. 250, (2.56)] for s sufficiently large are satisfied by the limiting form in (3.1) and requiring again that

$$a_{s+1}\omega(M^{s+1})/S(s, M) < \beta.$$

We have obtained the following theorem:

THEOREM 4. *If there exists 2 positive constants δ and β independent of n such that*

$$\delta < a_{n+1}\omega(M^{n+1})/S(n, M) < \beta \tag{3.5}$$

for $n = 1, 2, \ldots$ when $(g, m) > 1$ and $S(n, M) = \sum_{i=1}^{n} a_i \omega(M^i)$ such that g contains some but not all prime factors of m, then $x(g, m)$ in Theorem 3 is a transcendental of the non-Liouville type.

One can easily see that the same boundedness condition as in [2, Th. 2, p. 247] obtains as a requirement for the transcendental non-Liouville character of $x(g, m)$ since (3.5) becomes [2, (2.46)] when $(g, m) = 1$, i.e. $M = m$.

References

[1] R. G. Stoneham, *On (j, ε)-normality in the rational fractions*, Acta Arith. 16 (1970), pp. 221–237.

[2] — *A general arithmetic construction of transcendental non-Liouville normal numbers from rational fractions*, Acta Arith. 16 (1970), pp. 239–253.

[3] E. Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Rend. Circ. Mat. Palermo, 27 (1909), pp. 247–271.

THE CITY COLLEGE
OF THE CITY UNIVERSITY OF NEW YORK

Received on 29. 10. 1971 (150)

ACTA ARITHMETICA
XXII (1973)

Non-divisibility of some multiplicative functions

by

E. J. SCOURFIELD (London)

1. Introduction. Let $f(n)$ be an integer-valued multiplicative function with the property that there exists a polynomial $W(x)$ with integral coefficients such that $f(p) = W(p)$ for all primes p. Further let $N(n \leqslant x: P)$ denote the number of positive integers $n \leqslant x$ with the property P. Our aim in this paper is to find an estimate for

$$N\big(n \leqslant x:\ d \nmid f(n)\big)$$

for any integer $d > 1$. An estimate has been obtained by Narkiewicz in the case when d is squarefree, and we shall be able to derive an explicit formula for his constant A of Theorem II of [5] (see Corollary 1 of Theorem 1 in § 5 below). From Theorem I of [5], it is also easy to deduce an estimate for $N\big(n \leqslant x:\ p^a \nmid f(n)\big)$ for any prime p and any integer $a \geqslant 1$; for

$$N\big(n \leqslant x:\ p^a \nmid f(n)\big) = \sum_{\lambda=0}^{a-1} N\big(n \leqslant x:\ p^\lambda \| f(n)\big)$$

(where the notation $p^\lambda \| f(n)$ means that $p^\lambda | f(n)$ but $p^{\lambda+1} \nmid f(n)$), and an estimate for each term on the right follows from [5]. Thus the result of this paper will be new in the cases when d is neither squarefree nor a prime power.

Let $d = \prod_{i=1}^{r} p_i^{a_i}$, where the p_i are distinct primes and each $a_i \geqslant 1$, and let $S(p, \lambda)$ denote the set $\{n:\ p^\lambda \| f(n)\}$ of positive integers. Then we can state the main result of this paper:

THEOREM 1. *Suppose that $S_i = \bigcup_{\lambda=0}^{a_i-1} S(p_i, \lambda_i) \neq \emptyset$ (the empty set) for $i = 1, 2, \ldots, r$. Then there exist constants B, β, m (dependent on f and d) with $B > 0$, $0 \leqslant \beta \leqslant 1$, and $m \geqslant 0$, where β, m are defined explicitly by (31) and (32), such that as $x \to \infty$,*

(i) *if* $0 < \beta < 1$,

$$N\big(n \leqslant x:\ d \nmid f(n)\big) \sim Bx(\log\log x)^m(\log x)^{\beta-1};$$

(ii) *if* $\beta = 1$,

$$N\big(n \leqslant x:\ d \nmid f(n)\big) \sim Bx, \text{ where } B \leqslant 1;$$