ON ABSOLUTELY CONTINUOUS COMPONENTS AND RENEWAL THEORY[1]

BY CHARLES STONE

University of California, Los Angeles

1. Introduction. In this paper we continue the work of [6] and [7] by studying renewal theory in the case of an absolutely continuous component.

Let μ be a probability measure on the Borel subsets of the reals R, let $\mu^{(n)}$ denote the n-fold convolution of μ with itself and set $\nu = \sum_0^\infty \mu^{(n)}$. We assume throughout this paper that *some* $\mu^{(n)}$ *has a non-trivial absolutely continuous component.* If ν assigns measure to bounded intervals, this is equivalent to the assumption that ν has a non-trivial absolutely continuous component. Our main goal is to write $\nu = \nu' + \nu''$, where ν' is absolutely continuous and its density has the proper tail behavior for a renewal density, and ν'' is a finite measure and has the same tail behavior as μ.

To this end we can find an n_0 and probability measures φ and φ' such that $\mu^{(n_0)} = (\varphi + \varphi')/2$, φ has compact support, is absolutely continuous and has a twice continuously differentiable density. Set $\nu''' = \sum_0^\infty \mu^{(nn_0)}$. Let $\hat{\mu}$ denote the characteristic function of μ and let the characteristic functions of the other finite measures be denoted similarly. Also set $\hat{\nu} = (1 - \hat{\mu})^{-1}$ and $\hat{\nu}''' = (1 - \hat{\mu}^{n_0})^{-1}$. Now $\hat{\nu}''' = 1 + \hat{\mu}^{n_0}\hat{\nu}'''$ and $\hat{\mu}^{n_0} = (\hat{\varphi} + \hat{\varphi}')/2$, from which it follows that

$$\hat{\nu}''' = (1 - \tfrac{1}{2}\hat{\varphi}')^{-1}(1 + \tfrac{1}{2}\hat{\varphi}\hat{\nu}''').$$

Thus

$$\hat{\nu} = \sum_{n=0}^\infty \hat{\mu}^n = (1 + \hat{\mu} + \cdots + \hat{\mu}^{n_0-1}) \sum_{n=0}^\infty \hat{\mu}^{nn_0}$$
$$= (1 + \hat{\mu} + \cdots + \hat{\mu}^{n_0-1})(1 - \tfrac{1}{2}\hat{\varphi}')^{-1}(1 + \tfrac{1}{2}\hat{\varphi}\hat{\nu}''').$$

Correspondingly, $\nu = \nu' + \nu''$, where

$$\nu'' = (1 + \mu + \cdots + \mu^{(n_0-1)}) * (\textstyle\sum_{n=0}^\infty (\varphi')^n/2^n)$$

and $\nu' = \frac{1}{2}\varphi * \nu'' * \nu'''$, $*$ denoting convolution. Note that ν'' has total measure $2n_0$ and essentially the same tail behavior as μ(to be made more precise below), and that ν' is absolutely continuous and has a continuous density p.

Suppose that μ has a finite positive first moment $\mu_1 = \int x\mu(dx)$. Then $\mu^{(n_0)}$ has first moment $n_0\mu_1$ and the renewal theorem of Blackwell [1] asserts that if I is a bounded interval with length $|I|$, then $\nu'''(I)$ is finite,

$$\lim_{x\to-\infty} \nu'''(x + I) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \nu'''(x + I) = |I|/n_0\mu_1 .$$

Since φ has compact support and a continuous density, it follows easily that

$$\lim_{x\to-\infty} p(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} p(x) = 1/\mu_1 .$$

Received 22 June 1965.

[1] The preparation of this paper was sponsored in part by N.S.F. Grant GP-1606.

Therefore if A is a Borel subset of $[0, \infty)$ with finite Lebesgue measure $|A|$, then

$$\lim_{x\to-\infty} \nu(x - A) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \nu(x + A) = |A|/\mu_1.$$

Suppose that μ doesn't have finite non-zero mean, but that ν assigns finite measure to some interval of positive length and hence to all bounded intervals. Then by a renewal theorem of Feller and Orey [3] for all finite intervals I, $\lim_{|x|\to\infty} \nu'''(x + I) = 0$. It follows easily that $\lim_{|x|\to\infty} p(x) = 0$. Consequently, if A is a Borel subset of $[0, \infty)$ having finite Lebesgue measure, then

$$\lim_{x\to-\infty} \nu(x - A) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \nu(x + A) = 0.$$

The results of the last two paragraphs were suggested in part by Theorem 3 of Breiman [2].

Throughout the remainder of the paper we assume that μ has finite positive first moment μ_1 and finite moment of integer order $m \geqq 2$. Set

$$\begin{aligned} r(x) &= \textstyle\int_x^\infty (y - x)\mu(dy), && x \geqq 0,\\ &= \textstyle\int_{-\infty}^x (s - y)\mu(dy), && x < 0, \end{aligned}$$

and

$$\begin{aligned} s(x) &= \textstyle\int_x^\infty r(y)\, dy, && x \geqq 0,\\ &= \textstyle\int_{-\infty}^x r(y)\, dy, && x < 0. \end{aligned}$$

Then $r(x) = o(x^{-(m-1)})$ and $s(x) = o(x^{-(m-2)})$ as $|x| \to \infty$, $\int_{-\infty}^\infty |x|^{m-2} r(x)\, dx < \infty$ and, for $m \geqq 3$, $\int_{-\infty}^\infty |x|^{m-3} s(x)\, dx < \infty$. In Section 2 we will prove that under these assumptions

$$\begin{aligned} (1)\qquad p(x) &= [r(x)/\mu_1^2] + o(x^{-m}), && x \to -\infty,\\ &= (1/\mu_1) + [r(x)/\mu_1^2] + o(x^{-m}), && x \to +\infty. \end{aligned}$$

From (1) it follows that

$$\begin{aligned} p(x) &= o(x^{-(m-1)}), && x \to -\infty,\\ &= (1/\mu_1) + o(x^{-(m-1)}), && x \to +\infty, \end{aligned}$$

and

$$\int_{-\infty}^0 |x|^{m-2} p(x)\, dx + \int_0^\infty x^{m-2} |p(x) - 1/\mu_1|\, dx < \infty.$$

Now ν'' has finite moment of order m, and hence

$$\int_{-\infty}^0 |x|^{m-2} \nu(dx) + \int_0^\infty x^{m-2} |\nu(dx) - dx/\mu_1| < \infty.$$

(The $m = 2$ case of the last result was obtained by Smith [4].) Let A be a bounded Borel set of Lebesgue measure $|A|$. From (1) it follows that

$$\begin{aligned} \nu(x + A) &= [r(x)|A|/\mu_1^2] + o(x^{-m}), && x \to -\infty,\\ &= (|A|/\mu_1) + (r(x)|A|/\mu_1^2) + o(x^{-m}), && x \to +\infty. \end{aligned}$$

Therefore

$$\nu(x + A) = o(x^{-(m-1)}), \qquad x \to -\infty,$$

and

$$= (|A|/\mu_1) + o(x^{-(m-1)}), \qquad x \to +\infty,$$

$$\int_{-\infty}^{0} |x|^{m-2}\nu(x + A) + \int_{0}^{\infty} x^{m-2}|\nu(x + A) - |A|/\mu_1| \, dx < \infty.$$

Slightly weaker results hold if A is semi-infinite, but of finite Lebesgue measure.

According to Smith [5] (see also [6]), $H(x) = \nu((-\infty, x])$ is finite under our assumptions and

$$\lim_{x\to\infty} [H(x) - (x/\mu_1) - (\mu_2/2\mu_1^2)] = 0,$$

where μ_2 denotes the second moment of μ. From this result, together with (1), we get that

$$H(x) = (s(x)/\mu_1^2) + o(x^{-(m-1)}), \qquad x \to -\infty,$$

$$= (x/\mu_1) + (\mu_2/2\mu_1^2) - (s(x)/\mu_1^2) + o(x^{-(m-1)}), \; x \to +\infty.$$

The bound on the error here is curiously, better than the bound $o(x^{-(m-1)} \log |x|)$ obtained in a similar result in [6] under the weaker assumption that μ be strongly non-lattice.

Suppose $m = 2$ and, additionally, the right tail of μ decreases exponentially fast; i.e., for some $r > 0$, $\mu((x, \infty)) = o(e^{-rx})$ as $x \to \infty$. Then, as will be shown in the next section, $p(x) - \mu_1^{-1}$ and the right tail of ν'' decrease exponentially fast. Similar results hold for the left tail.

It is curious that we can treat the two tails separately under the assumption of exponentially decreasing tails, but not under the assumption of finite moments of order m.

2. Statement and proof of results. Those of the above results that require further proof are summarized in the following

THEOREM. *Let μ be a probability measure on R having finite first moment $\mu_1 > 0$ and finite moment of integer order $m \geqq 2$, and such that some $\mu^{(n)}$ has a non-trivial absolutely continuous component. Set $\nu = \sum_0^\infty \mu^{(n)}$ and let ν' and ν'' be the measures defined above. Then $\nu = \nu' + \nu''$, ν'' is a finite measure having finite moment of order m, and ν' is absolutely continuous and has a continuous density $p(x)$ such that* (1) *holds.*

If, additionally, the left (or right) tail of μ decreases exponentially fast, then $p(x)$ (or $p(x) - \mu_1^{-1}$) and the left (or right) tail of ν'' decrease exponentially fast as $x \to -\infty$ (or $x \to +\infty$).

PROOF. We need only justify the results on p, those on ν'' being obvious. We first prove (1). The probability measure $\chi = (1/2n_0)\varphi * \nu''$ has finite moment of order m and, for $0 \leqq k \leqq m$ the kth derivative of its characteristic function $\hat{\chi}$ is continuous and of the form $o(\theta^{-2})$ as $|\theta| \to \infty$. As in [6], we have an inversion formula

$$p(x) = (1/2\mu_1) + (n_0/2\pi) \int_{-\infty}^{\infty} \Re[e^{-ix\theta}\hat{\chi}(\theta)[1/1 - \hat{\mu}^{n_0}(\theta)]] \, d\theta$$

$$= (1/2\mu_1) + (1/2\pi) \int_{-\infty}^{\infty} \Re[e^{-ix\theta}\hat{\chi}(\theta)[1/1 - \hat{\mu}(\theta)]] \, d\theta + o(x^{-m}), \quad |x| \to \infty;$$

the second equality follows by integrating by parts m times the difference of the two integrals after algebraically cancelling out the $1 - \hat{\mu}(\theta)$ term in the numerator and denominator of

$$[n_0 - (1 + \hat{\mu}(\theta) + \cdots + \hat{\mu}^{n_0-1}(\theta))]/[(1 - \hat{\mu}(\theta))(1 + \hat{\mu}(\theta) + \cdots + \hat{\mu}^{n_0-1}(\theta))].$$

Since

$$\begin{aligned}(1/2\pi) \int_{-\infty}^{\infty} \Re(e^{-ix\theta}\hat{\chi}(\theta)/-i\theta)\, d\theta &= \chi((-\infty, x)) - \tfrac{1}{2} \\ &= \pm\tfrac{1}{2} + o(x^{-m}), \qquad x \to \pm\infty,\end{aligned}$$

it follows that

$$\begin{aligned}(1/2\pi) \int_{-\infty}^{\infty} e^{-ix\theta}\hat{\chi}(\theta)(1/(1 - \hat{\mu}(\theta)) - 1/-i\mu_1\theta)\, d\theta \\ &= p(x) + o(x^{-m}), \qquad x \to -\infty, \\ &= p(x) - \mu_1^{-1} + o(x^{-m}), \quad x \to +\infty.\end{aligned}$$

We can write the last integral as the sum of two integrals, corresponding to the decomposition

$$\begin{aligned}[1/(1 - \hat{\mu}(\theta))] - (1/-i\mu_1\theta) &= (\hat{\mu}(\theta) - 1 - i\mu_1\theta)/-\mu_1^2\theta^2 \\ &\quad -(\hat{\mu}(\theta) - 1 - i\mu_1\theta)^2/\mu_1^2\theta^2(1 - \hat{\mu}(\theta)).\end{aligned}$$

A direct computation yields that

$$\int_{-\infty}^{\infty} e^{ix\theta} r(x)\, dx = (\hat{\mu}(\theta) - 1 - i\mu_1\theta)/-\theta^2.$$

It is also easily seen that

$$\int_{-\infty}^{\infty} r(x - y)\chi(dy) = r(x) + o(x^{-m}), \qquad |x| \to \infty.$$

Therefore

$$\begin{aligned}(1/2\pi) \int_{-\infty}^{\infty} e^{-ix\theta}\hat{\chi}(\theta)\, [(\hat{\mu}(\theta) - 1 - i\mu_1\theta)/-\mu_1^2\theta^2]\, d\theta \\ &= [r(x)/\mu_1^2] + o(x^{-m}), \quad |x| \to \infty.\end{aligned}$$

Thus to complete the proof of (1) it suffices to show that

$$\int_{-\infty}^{\infty} e^{-ix\theta}\hat{\chi}(\theta)[(\hat{\mu}(\theta) - 1 - i\mu_1\theta)^2/\theta^2(1 - \hat{\mu}(\theta))]\, d\theta = o(x^{-m}), \qquad |x| \to \infty.$$

This result follows from m integrations by parts and the properties of $\hat{\mu}$ obtained in [6].

Suppose, additionally, that the right tail of μ decreases exponentially fast. Then the right tail of χ decreases exponentially fast and, for some $r > 0$,

$$\begin{aligned}(1/2\pi) \int_{-\infty}^{\infty} \Re(e^{-ix\theta}\hat{\chi}(\theta)/-i\theta)\, d\theta &= \chi((-\infty, x)) - \tfrac{1}{2} \\ &= \tfrac{1}{2} + o(e^{-rx}), \qquad x \to +\infty.\end{aligned}$$

Thus

$$\begin{aligned}p(x) = (1/\mu_1) + (n_0/2\pi) \int_{-\infty}^{\infty} e^{-ix\theta}\hat{\chi}(\theta)\{[1/(1 - \hat{\mu}^{n_0}(\theta)] - (1/-in_0\mu_1\theta)\}\, d\theta \\ + o(e^{-rx}), \quad x \to +\infty.\end{aligned}$$

As in [7], we can apply Cauchy's theorem to get that for some $r > 0$

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{ix\theta}\hat{\chi}(\theta)\{[1/(1 - \hat{\mu}^{n_0}(\theta))] - (1/-in_0\mu_1\theta)\}\, d\theta \\
&\quad = \int_{-\infty}^{\infty} e^{-x(r+i\theta)}\hat{\chi}(\theta - ir)\{[1/(1 - \hat{\mu}^{n_0}(\theta - ir))] - [1/-n_0\mu_1(r + i\theta)]\}\, d\theta \\
&\quad = o(e^{-rx}), \qquad\qquad x \to +\infty.
\end{aligned}$$

Thus $p(x) - \mu_1^{-1}$ approaches zero exponentially fast as $x \to +\infty$. A similar argument works for the left tail.

REFERENCE

[1] BLACKWELL, D. H. (1953). Extension of a renewal theorem. *Pacific J. Math.* **3** 303–309.

[2] BREIMAN, L. (1965). Note concerning some probabilistic aspects of the renewal theorem. Trans. Fourth Prague Conf. on Inf. Theory, Statist. Dec. Functions, and Random Processes, 1965. To appear.

[3] FELLER, W. and OREY, S. (1961). A renewal theorem. *J. Math. Mech.* **10** 619–624.

[4] SMITH, W. L. (1954). Asymptotic renewal theorems. *Proc. Roy. Soc. Edinburgh, Sect. A* **64** 9–48.

[5] SMITH, W. L. (1960). Infinitesimal renewal processes. *Contributions to Probability and Statistics—Essays in Honor of Harold Hotelling*, 396–413, Stanford Univ. Press.

[6] STONE, C. J. (1965). On characteristic functions and renewal theory. *Trans. Amer. Math. Soc.* To appear.

[7] STONE, C. J. (1965). On moment generating functions and renewal theory. *Ann. Math. Statist.* **36** 1298–1301.