ON ACYCLIC SIMPLICIAL COMPLEXES

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The higher dimensional concepts corresponding to trees are developed and studied. In order to enumerate these 2-dimensional structures called 2-trees, a dissimilarity characteristic theory is investigated. By an appropriate application of certain combinatorial techniques, generating functions are obtained for the number of 2-trees. These are specialized to count those 2-trees embeddable in the plane, thus providing a new approach to the old problem of determining the number of triangulations of a polygon.

1. *Pure Complexes.* By an *n-complex,* we mean a finite n-dimensional simplicial complex. One of us [3, p. 462] defined a *pure n-complex* as an n-complex in which every k-simplex with $k < n$ is contained in an n-simplex. A pure n-complex will be called an *n-plex* for brevity.

We will only be concerned with 2-plexes, and for convenience 0-simplexes, 1-simplexes and 2-simplexes are called *points, lines* and *cells* respectively. We now define for 2-plexes the analogues of the concepts of walk, trail, and path in graph theory [5]. In a given 2-plex P a 1, 2-*walk* is an alternating sequence $x_0, \sigma_1, x_1, \sigma_2, x_2, \ldots, x_{n-1}, \sigma_n, x_n$ of lines x_i and cells σ_j such that each line x_i is incident with the cells σ_i and σ_{i+1}. This walk is *closed* if $x_0 = x_n$, and is *open* otherwise. If all of the cells are distinct, it is called a 1, 2-*trail*; if both the lines and the cells are distinct, it is a 1, 2-*path.*

A 2-plex is 1, 2-*connected* if every pair of lines are joined by a 1, 2-path. It is called 1, 2-*acyclic* if it contains no closed 1, 2-trails with more than one cell. For brevity we will say that a 2-plex is *connected* if it is 1, 2-connected; *acyclic* if 1, 2-acyclic. Note that if a 2-plex is connected, so is its underlying graph (in the usual sense), but not conversely. A 2-plex is *simply connected* if it is connected and its first Betti number is zero; intuitively, it has no " holes ".

2. *2-trees.* In the study of 2-plexes, the 2-tree plays the role of the tree in graph theory. In this section we present some of the important properties of 2-trees. Observe that each of these corresponds to a similar property of trees.

A 2-*tree* is a simply connected, acyclic 2-plex. The *eccentricity* of a line x in a 2-tree is the number of cells in a longest 1, 2-path beginning with x. The *centre* of a 2-tree is the subplex induced by all lines of minimum eccentricity. An *end-cell* has two lines incident with no other cells. The next three propositions follow readily from these definitions.

PROPOSITION 1. *The centre of a 2-tree is a line or a cell.*

The proof is analogous to that of the corresponding proposition for trees: the centre of any tree is a point or a line; see König [8, p. 65].

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PROPOSITION 2. *Every 2-tree with at least two cells has at least two end-cells.*

It is well known that every nontrivial tree (with more than one point) has at least two endpoints. The proof depends on the finiteness of a tree and its acyclicity, as does the proof of this proposition.

Let p, q, and r be the number of points, lines, and cells of a 2-plex.

PROPOSITION 3. *For any 2-tree, the following equations hold:*

$$p - q + r = 1, \tag{1}$$

$$q - 2r = 1, \tag{2}$$

$$p - r = 2, \tag{3}$$

$$2p - q = 3. \tag{4}$$

Obviously any two of these four equations imply the other two. The first of these is, of course, a variant of the Euler-Poincaré equation, and reduces to the familiar $p - q = 1$ for trees. The second is easily verified by induction.

As in the case of trees, there are many alternate definitions for 2-trees. Two characterizations are given in the next proposition.

PROPOSITION 4. *The following are equivalent for a 2-plex T:*

(i) *T is a 2-tree;*

(ii) *$p - q + r = 1$ and T is connected and acyclic;*

(iii) *$p - r = 2$ and T is connected and acyclic.*

That (i) implies (ii) and (iii) follows from Proposition 3 and the definition of 2-tree. To demonstrate that (ii) or (iii) implies (i), we show that T is simply connected. This follows at once from the fact that for any 2-plex which is not simply connected, $p - r > 2$ and $p - q + r < 1$.

3. *Dissimilarity characteristic theorem.* Let α be a 1–1 map which sends the points, lines, and cells of one 2-plex onto the points, lines, and cells respectively of another 2-plex. If α preserves the incidence relations, then α is called an *isomorphism.* As usual, an *automorphism* is an isomorphism of a 2-plex with itself. The automorphisms of a 2-plex P form a permutation group denoted $\Gamma(P)$ called the *group of P.* We shall refer to the orbits of this group as *similarity classes.* Thus two points (lines, cells) are *similar* whenever they are in the same similarity class, i.e. there is an automorphism which sends one point (line, cell) onto the other.

A homomorphism of the group of a 2-tree is obtained by restricting the automorphisms to the points of the centre. The image of this homomorphism is called the *group of the centre* of the 2-tree, and according as the centre is a line or a cell (Proposition 1), it is a subgroup of one of the symmetric groups S_2 or S_3.

Corresponding to the three partitions of the number 3, there are three types of cells in a 2-tree:

Type (1), exactly two of its lines are similar;

Type (2), all three lines are similar;

Type (3), no two lines are similar.

For a given 2-tree, we let s_1 and s_2 be the number of similarity classes of type (1) and type (2) respectively. Then we let

$$s = s_1 + 2s_2. \tag{5}$$

The following three observations about the three types of cells are obvious but important.

PROPOSITION 5. *If a 2-tree T contains a cell uvw of type* (1) *in which uv and vw are similar, then line uw is in the centre of T.*

PROPOSITION 6. *If a 2-tree T contains a cell of type* (2), *then that cell is the centre of T and* $s_2 = 1$.

PROPOSITION 7. *If the group of the centre of a 2-tree is the identity, then every cell is of type* (3) *and* $s = 0$.

By the *number of dissimilar points* p^* of a 2-tree we mean the number of similarity classes of points; analogous definitions are made for the number q^* of dissimilar lines and r^* for cells. Otter's dissimilarity characteristic theorem for trees [**9**] provides the key to the enumeration of trees in terms of rooted trees. The next theorem performs the same function for 2-trees. Other dissimilarity characteristic results have been studied by Harary and Norman [**6**].

PROPOSITION 8. (*Dissimilarity Characteristic Theorem.*) *For any 2-tree with* q^* *dissimilar lines,* r^* *dissimilar cells,* s_1 *cells with two similar lines,* s_2 *cells with all three lines similar, and* $s = s_1 + 2s_2$,

$$q^* + s - 2r^* = 1. \tag{6}$$

Otter's proof of his dissimilarity characteristic theorem for trees was based on the fact that for any tree, the identification of its similar lines results in a tree whose number of points and lines leads to his equation. Similarly, given any 2-tree T, one can show that a 2-tree T^* is obtained by identifying similar cells of T. Then one can readily see that T^* has $q^* + s$ lines and r^* cells. On applying Proposition 3 to the 2-tree T^*, the resulting equation is (6).

4. *Enumeration of 2-trees*. In order to describe the counting of 2-trees, we require the usual notation of combinatorial enumeration theory, as given in Pólya [**10**], in [**3, 4, 7**] and especially in [**5**]. This includes the symbols $Z(A)$ for the cycle index of the permutation group A and $Z(A, f(x))$ for the power series obtained when a series $f(x)$ is substituted into $Z(A)$. A familiarity with Pólya's classical enumeration theorem must be assumed. Three particular formulas which occur in 2-tree enumeration involve the substitution of series into the cycle indexes of the symmetric groups S_2 and S_3, and of the alternating group A_2 (actually the identity group of degree 2) in the form $Z(A_2 - S_2) = Z(A_2) - Z(S_2)$. To be explicit, we now list the results of such substitutions:

$$Z(S_2, f(x)) = \tfrac{1}{2}[f^2(x) + f(x^2)], \tag{7}$$

$$Z(S_3, f(x)) = \tfrac{1}{6}[f^3(x) + 3f(x)f(x^2) + 2f(x^3)], \tag{8}$$

$$Z(A_2 - S_2, f(x)) = \tfrac{1}{2}[f^2(x) - f(x^2)]. \tag{9}$$

In equation (9), $Z(A_2 - S_2)$ appears as a special case of a result of Pólya [10] to the effect that $Z(A_n - S_n)$ enumerates those configurations in which the figures are all distinct. Finally we state the useful identity:

$$\sum_{n=0}^{\infty} Z(S_n, f(x)) = \exp \sum_{r=1}^{\infty} \frac{f(x^r)}{r}, \tag{10}$$

which sums the cycle indexes of all the symmetric groups.

Now we proceed to develop the generating functions for 2-trees. Let t_n be the number of 2-trees with n cells. The counting series for 2-trees is denoted by

$$t(x) = \sum_{n=1}^{\infty} t_n x^n. \tag{11}$$

In order to derive formulas for t_n we will make use of the corresponding series for various kinds of rooted 2-trees.

First let $M_1(x)$ and $N_1(x)$ be the series for 2-trees rooted at a symmetric and an unsymmetric end-line respectively. Further, let $M(x)$ and $N(x)$ be the series for 2-trees rooted at any symmetric and any unsymmetric line respectively. The following two equations express $M_1(x)$ and $N_1(x)$ in terms of $M(x)$ and $N(x)$:

$$M_1(x) = x(1 + M(x^2) + 2N(x^2)), \tag{12}$$

$$N_1(x) = xZ(A_2 - S_2,\ 1 + M(x) + 2N(x)). \tag{13}$$

Equation (13) can be stated more explicitly by applying (9). Next we express $M(x)$ in terms of $M_1(x)$ and $N_1(x)$:

$$M(x) = \sum_{n=1}^{\infty} Z(S_n,\ M_1(x) + N_1(x^2)). \tag{14}$$

Using the identity (10), equation (14) may be written:

$$M(x) = -1 + \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} [M_1(x^n) + N_1(x^{2n})]\right\}. \tag{15}$$

Now note that the counting series for 2-trees rooted at an oriented line is simply $M(x) + 2N(x)$. From this observation we have:

$$M(x) + 2N(x) = \sum_{n=1}^{\infty} Z(S_n,\ M_1(x) + 2N_1(x)). \tag{16}$$

Again using the identity (10) we may write (16) as

$$M(x) + 2N(x) = -1 + \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} [M_1(x^n) + 2N_1(x^n)]\right\}. \tag{17}$$

Thus equations (15) and (17) may be used to solve for $N(x)$ in terms of $M_1(x)$ and $N_1(x)$. Now using all four formulas (12), (13), (15) and (17), the coefficients of $M(x)$ and $N(x)$ can be calculated. For the first few terms we have:

$$M(x) = x + x^2 + 2x^3 + 3x^4 + 6x^5 + \ldots, \tag{18}$$

$$N(x) = x^2 + 4x^3 + 18x^4 + 77x^5 + \ldots. \tag{19}$$

The series for 2-trees rooted at a line is denoted $L(x)$ and since $L(x) = M(x) + N(x)$, we have immediately:

$$L(x) = x + 2x^2 + 6x^3 + 21x^4 + 83x^5 + \dots. \tag{20}$$

We denote the series for 2-trees rooted at a cell (triangle) by $\Delta(x)$. It can be shown that

$$\Delta(x) = xZ(S_3,\ 1 + M(x) + 2N(x)) - xN(x)(1 + M(x^2) + 2N(x^2)). \tag{21}$$

Having expressed $\Delta(x)$ in terms of $M(x)$ and $N(x)$, we substitute (18) and (19) in equation (21) to obtain

$$\Delta(x) = x + x^2 + 3x^3 + 10x^4 + 39x^5 + \dots. \tag{22}$$

Following the notation of the Dissimilarity Characteristic Theorem, we denote by $s_1(x)$ the counting series for 2-trees rooted at a cell of type (1). Similarly $s_2(x)$ is the series for 2-trees rooted at a cell of type (2). These two series are readily expressed as functions of $M_1(x)$, $M(x)$ and $N(x)$:

$$s_1(x) = M_1(x)(1 + M(x)) - x(1 + M(x^3)), \tag{23}$$

$$s_2(x) = x(1 + M(x^3) + N(x^3)). \tag{24}$$

Making the appropriate substitutions we obtain

$$s_1(x) = x^2 + 2x^3 + 2x^4 + 7x^5 + \dots, \tag{25}$$

$$s_2(x) = x + x^4 + 2x^7 + 6x^{10} + \dots. \tag{26}$$

In order to obtain the formula for $t(x)$, the series for 2-trees, we use the Dissimilarity Characteristic Theorem in the same manner as was done by Otter [9] for the enumeration of trees. If we sum equation (6) over all 2-trees with a given number n of cells, we find that the total number of such trees is

$$t_n = \Sigma q^* + \Sigma s - 2\Sigma r^*. \tag{27}$$

But then Σq^* is just the coefficient of x^n in $L(x)$, while Σs is the coefficient of x^n in $s_1(x) + 2s_2(x)$, and Σr^* is the coefficient of x^n in $\Delta(x)$. This result is summarized by the following theorem.

PROPOSITION 9. (*Enumeration Theorem for 2-trees.*) *The counting series for 2-trees is*

$$t(x) = L(x) + s_1(x) + 2s_2(x) - 2\Delta(x). \tag{28}$$

Substituting equations (20, 22, 25, 26) into equation (28) gives

$$t(x) = x + x^2 + 2x^3 + 5x^4 + 12x^5 + \dots. \tag{29}$$

Note that this theorem can be used to count 2-trees with specified properties provided that formulas for $L(x)$, $s_1(x)$, $s_2(x)$ and $\Delta(x)$ are found for 2-trees with these properties. The corresponding investigation for trees of various species was made by Harary and Prins [7].

5. *On the number of triangulations of a polygon.* By a *triangulation of a polygon* we mean a graph obtained from a regular n-gon by adding non-intersecting chords until every interior region is a triangle. Obviously n–3 chords are required and n–2 triangles are obtained. Generating functions for the number of different triangula-

tions of the n-gon, i.e. those not isomorphic as graphs, have been found by Brown [1], but our purpose here is to present an entirely different approach toward finding such a generating function. We alter the formulation of the problem into a statement involving 2-dimensional simplicial complexes by observing that triangulations of a polygon correspond precisely with planar 2-trees. We then proceed to enumerate the latter by the same methods exploited for counting 2-trees.

To illustrate the configurations being counted, we show in Figure 1 the unique triangulations of a triangle, a quadrilateral, and a pentagon, and the three different triangulations of a hexagon. Note that these graphs are not taken as rooted or labelled in any way. Observe also the correspondence between these and the planar 2-trees with one, two, three and four cells.

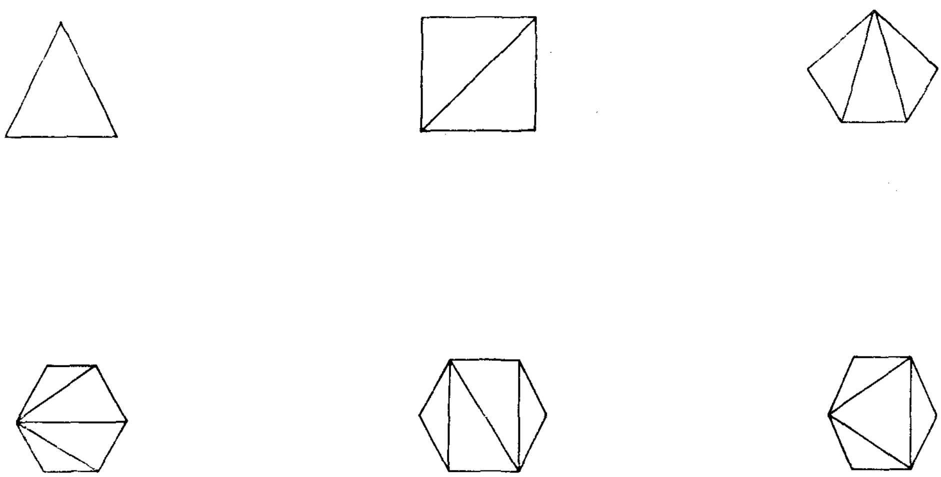

FIG. 1.—The triangulations of the n-gon, $n = 3$ to 6.

The enumeration of planar 2-trees can be accomplished by using almost all of the formulas that have already been developed for 2-trees. Therefore we alter the notation used for 2-trees only slightly by writing a bar to indicate the generating functions for planar 2-trees.

Thus let $\bar{M}_1(x)$ and $\bar{N}_1(x)$ be the series for planar 2-trees rooted at a symmetric and an unsymmetric end-line respectively. Then the following two formulas (compare (12) and (13)) specify the relationship between $\bar{M}_1(x)$ and $\bar{N}_1(x)$.

$$\bar{M}_1(x) = x(1 + \bar{M}_1(x^2) + 2\bar{N}_1(x^2)), \tag{30}$$

$$\bar{N}_1(x) = xZ(A_2 - S_2,\ 1 + \bar{M}_1(x) + 2\bar{N}_1(x)). \tag{31}$$

These two equations (30) and (31) can be used to obtain the coefficients in the two series $\bar{M}_1(x)$ and $\bar{N}_1(x)$ by exhaustion, as for (12) and (13). However, as Brown [2, p. 752] pointed out, a formula due to Euler (and repeatedly rediscovered) shows that the number of triangulations of an $(n + 2)$-gon which is rooted by orienting one of its boundary edges (and hence the polygon) is

$$f_n = \frac{2(2n-1)!}{(n-1)!(n+1)!} = \frac{1}{n}\binom{2n}{n-1}. \tag{32}$$

Hence it follows that $\bar{M}_1(x) + 2\bar{N}_1(x) = \Sigma f_n x^n$, so that

$$\bar{M}_1(x) + 2\bar{N}_1(x) = \sum_{n=1}^{\infty} \frac{2(2n-1)!}{(n-1)!(n+1)!} x^n. \tag{33}$$

Now from Euler's formula (32) and equation (30) for $\bar{M}_1(x)$ we have

$$\bar{M}_1(x) = x + \sum_{n=1}^{\infty} \frac{2(2n-1)!}{(n-1)!(n+1)!} x^{2n+1}. \tag{34}$$

For the first few terms of $\bar{M}_1(x)$ and $\bar{N}_1(x)$ we have:

$$\bar{M}_1(x) = x + x^3 + 2x^5 + 5x^7 + 14x^9 + \dots, \tag{35}$$

$$\bar{N}_1(x) = x^2 + 2x^3 + 7x^4 + 20x^5 + 66x^6 + 212x^7 + 715x^8 + \dots. \tag{36}$$

The series for planar 2-trees rooted at a line is denoted $\bar{L}(x)$ and can be expressed in terms of $\bar{M}_1(x)$ and $\bar{N}_1(x)$:

$$\bar{L}(x) = Z(S_2, 1 + \bar{M}_1(x) + \bar{N}_1(x)) + Z(S_2, \bar{N}_1(x)) - 1. \tag{37}$$

Substitution of (35) and (36) in equation (37) for $\bar{L}(x)$ yields

$$\bar{L}(x) = x + 2x^2 + 4x^3 + 12x^4 + 34x^5 + 111x^6 + 360x^7 + 1226x^8 + \dots. \tag{38}$$

From this point on, since the equations and procedure are virtually the same for planar 2-trees as for 2-trees, we will simply list the formulas for $\bar{\Delta}(x)$, $\bar{s}_1(x)$, $\bar{s}_2(x)$, and $\bar{t}(x)$. We have

$$\bar{\Delta}(x) = xZ(S_3, 1 + \bar{M}_1(x) + 2\bar{N}_1(x)) - x\bar{N}_1(x)(1 + \bar{M}_1(x^2) + 2\bar{N}_1(x^2)), \tag{39}$$

$$\bar{s}_1(x) = \bar{M}_1(x)(1 + \bar{M}_1(x)) - x(1 + \bar{M}_1(x^3)), \tag{40}$$

$$\bar{s}_2(x) = x\bar{N}_1(x^3) + x(1 + \bar{M}_1(x^3)), \tag{41}$$

and as before

$$\bar{t}(x) = \bar{L}(x) + \bar{s}_1(x) + 2\bar{s}_2(x) - 2\bar{\Delta}(x), \tag{42}$$

which is obtained by barring equation (28), the Enumeration Theorem for 2-trees.

Substituting the calculations (35) and (36) for $\bar{M}_1(x)$ and $\bar{N}_1(x)$ in these formulas gives

$$\bar{t}(x) = x + x^2 + x^3 + 3x^4 + 4x^5 + 12x^6 + 27x^7 + 82x^8 + \dots \tag{43}$$

for the number of planar 2-trees.

In an unpublished manuscript, R. K. Guy has worked further with the results of Brown [1] and (with the help of a computing machine) obtains several additional coefficients for $\bar{t}(x)$:

$$\begin{aligned}\bar{t}(x) = {} & x + x^2 + x^3 + 3x^4 + 4x^5 + 12x^6 + 27x^7 + 82x^8 + 228x^9 + 733x^{10} \\ & + 2282x^{11} + 7528x^{12} + 24{,}834x^{13} + 83{,}898x^{14} + 285{,}357x^{15} \\ & + 1{,}046{,}609x^{16} + 3{,}412{,}420x^{17} + 11{,}944{,}614x^{18} + 42{,}080{,}170x^{19} \\ & + 149{,}197{,}152x^{20} + 531{,}883{,}768x^{21} + 1{,}905{,}930{,}975x^{22} \\ & + 6{,}861{,}221{,}666x^{23} + \dots. \end{aligned} \tag{44}$$

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