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ON ADDITIONAL MOTION INVARIANTS OF CLASSICAL HAMILTONIAN WAVE SYSTEMS

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It is shown that the existence of an analytic invariant in addition to the natural ones (momentum, energy and, in some cases, "number of particles") leads to the existence of infinitely many such invariants. Nevertheless, the existence of the additional motion invariant does not guarantee complete integrability. Complete integrability follows from the existence of an additional invariant only if the dispersion law is non-degenerative with respect to decays. If the dispersion law is degenerative, the "number of" motion invariants is insufficient for complete integrability and the S-matrix is factorized via decay processes "one into two" with real intermediate particles. In this paper we present also our results concerning enumeration of degenerative dispersion laws.

1. Introduction

In theoretical physics, exactly solvable, integrable dynamical systems are very important because a considerable part of science consists in the study of systems close to exactly solvable ones. There has been growing a special interest in the exactly solvable systems after the discovery [1] of the inverse scattering transform (IST), allowing us to find the exact solutions of the Korteweg–de Vries equation and it was realized that this equation is the completely integrable Hamiltonian system in Liouville's sense [11]. In the process of the IST development the variety of essentially nonlinear systems, allowing in this or another sense the exact solution, has been found. To the present moment the one-dimensional exactly solvable solutions have been studied in detail. Among these, the Korteweg–de Vries equation [1, 13], the nonlinear Schrödinger equation [7, 8, 14], the sine-Gordon equation [15], the principal chiral field equation [16] and the equation of the resonance interaction of three wave packets [17, 18, 22] are the most famous ones. For details and a systematic account of the inverse scattering transform in its different variants see ref. [19]. To get to know the stages of the method development, see refs. [6–13].

For some of the systems found, the complete integrability was proved in Liouville's sense [11, 14, 15]; for these systems via the inverse scattering transform one is sure to construct the action-angle variables (see [2, 47], for example).

In all such models, solitons interact trivially, i.e. the pair collisions result in a phase shift only and change neither the form nor the velocity, and collision effects involving three or more solitons are absent.

In other models the non-trivial interaction of solitons is possible i.e. their fusions and decays [16–18]. For such systems the complete integrability has not been proved and it does not seem to exist. This follows from the results of the present paper also. We show that such type of division takes place for two-dimensional systems too. Among these, the Kadomtzev–Petviashvilli equations [9, 20], the resonance interaction equation of the three wave packets [9, 21, 25] and the Davey–Stewartson equations [23, 24, 26] are the best known. It is important to note that all the systems mentioned above possess one remarkable property – they have infinite sets of motion invariants, not following from the natural symmetries. An infinite set of motion invariants is, of course, not enough for integrability (it is also necessary to prove that the set of invariants is complete), but the very existence of integrals testifies to the deep inner symmetry of the system. Moreover, the existence of just one additional integral is very important. Experience shows that, usually, the existence of an infinite number of integrals follows from the existence of one such integral. Historically, that was how the matters stood with the KdV equation [1, 4], the integrability proof of the concrete systems has begun more than once from finding in them one or several "unnecessary" motion invariants. The methods [3–6] of the explicit calculation of the equations, having additional integrals or symmetries of a certain form – usually local ones – have been developing for a long time (apropos of this see ref. [3] and references therein).

The present paper differs strongly from the above-mentioned works in the sense of idea and is closest in its spirit to the works by Poincaré [27, 28], who showed in the last century that the existence of just one additional motion integral in the Hamiltonian system is an exceptional fact.

In a sense, our paper follows Poincaré's approach, applied to the infinite-dimensional case. We shall consider wave fields obeying the Hamiltonian system of equations in the homogeneous space of d dimensions of the form

$$\mathrm{i}\dot{a}_k = \frac{\delta H}{\delta \overset{*}{a}_k}, \quad k = (k_1, \ldots, k_d), \tag{1.1}$$

with the Hamiltonian H analytic in the field variables:

$$\begin{gathered} H = H_0 + H_1 + \cdots; \quad H_0 = \int \omega_k |a_k|^2 \,\mathrm{d}k, \\ H_{\mathrm{int}} = \frac{1}{3!} \sum_{ss_1s_2} \int V^{ss_1s_2}_{kk_1k_2} a^s_k a^{s_1}_{k_1} a^{s_2}_{k_2} \delta(sk + s_1k_1 + s_2k_2)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2, \\ s, s_i = \pm 1, \quad a^1_k = a_k, \quad a^{-1}_k = \overset{*}{a}_k. \end{gathered} \tag{1.2}$$

The system with Hamiltonian H_0 is trivially completely integrable. We shall be interested in the problem of existence of an additional motion invariant of the system (1.1) which is analytic in the field variables a^s_k, $s = \pm 1$ and has the form

$$\begin{gathered} F = F_0 + F_1 + \cdots; \quad F_0 = \int f_k |a_k|^2 \,\mathrm{d}k, \\ F_1 = \frac{1}{3!} \sum_{ss_1s_2} \int \Pi^{ss_1s_2}_{kk_1k_2} a^s_k a^{s_1}_{k_1} a^{s_2}_{k_2} \delta(sk + s_1k_1 + s_2k_2)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2, \end{gathered} \tag{1.3}$$

with continuous coefficients Π. F is the integral of the system with Hamiltonian H at any f_k. As we see from (1.2)–(1.3), the smallness of the wave amplitude a_k is a parameter, analogous to a small parameter in the Poincaré theorem.

The system considered possesses a continuous number of degrees of freedom which makes the direct use of Poincaré's results impossible. We show that the existence of integral (1.3) imposes rather strict limitations on the coefficients of the Hamiltonian H. These limitations prove to be formulated conventionly, introducing a new quantity – a classical scattering matrix, given by the set of amplitudes $W^{n,m}_{k_1,\dots,k_n,k_{n+1},\dots,k_{n+m}}$, each of them is determined on the resonance surface:

$$\begin{aligned} &k_1 + \cdots + k_n = k_{n+1} + \cdots + k_{n+m}, \\ &\omega_{k_1} + \cdots + \omega_{k_n} = \omega_{k_{n+1}} + \omega_{k_{n+m}}. \end{aligned} \tag{1.4}$$

The main result may be formulated as follows. In order that I is an additional integral of the motion, it is necessary that for all possible integers n and m either the following identity is fulfilled on the surface (1.4):

$$f_{k_1} + \cdots + f_{k_n} = f_{k_{n+1}} + \cdots + f_{k_{n+m}}, \tag{1.5}$$

or the amplitude of the process $W^{n,m}$ is equal to zero on (1.4), with the manifold of measure zero excluded.

Now it is clear that the problem of several variables function theory arises: under what conditions the system of equations (1.4)–(1.5) has a nontrivial solution. We did not suceed in finding a complete solution of that difficult problem, but we have obtained a lot of rather advanced results (see section 3 of the present paper). In section 2 we present the statement of the problem, the definition of the classical scattering matrix and the proof of the main result, formulated above.

In the finite-dimensional case the Poincaré methods represent a rather effective way of proving the non-integrability of the concrete systems. For systems with the continuous number of degrees of freedom the matter stands much simpler than in the finite-dimensional case. We hope to show that the procedure developed by us is well worthy of becoming a working apparatus in the cases when it is necessary to make an examination whether the given concrete wave system, found in the applications, may be exactly solvable or not. In section 4 we discuss a few examples of checking of the concrete nonlinear wave systems, representing a certain physical interest in the existence in them of the additional integrals of the form (1.3). Section 5 is devoted to a deeper study of the limitations on classical scattering matrix structure, following from the existence of an additional motion invariant. We show that under certain, very natural, assumptions the infinite series for the scattering matrix can be summed up explicitly and this summation results in a simple nonlinear integral equation. Finally, section 6 is devoted to the following problem: in what sense does the existence of an additional motion invariant of the form (1.3) result in the existence of an infinite set of such invariants and in what sense does it imply the complete integrability of the system (1.1)? The latter notion must be defined more precisely. In the case when a wave field in a physical space vanishes sufficiently rapidly at infinity (that corresponds to smooth functions a_k) all the systems of form (1.1) are completely integrable. The only question of interest is the one about the complete integrability of systems at periodic in space boundary conditions when the amplitudes a_k are represented as a set of δ-functions.

In conclusion, let us mention that some statements of the present paper were formulated by us in the past while using another language (applying the kinetic equations for waves) in ref. [26]. Some results of the present paper were formulated, without full proofs, in paper [30] of one of the authors (V.E. Zakharov). A brief statement of the results of the present paper was given in ref. [33].

The diagram technique developed in the present paper is close to the one given in ref. [48].

2. The formal classical scattering matrix in the non-soliton sector

Following [30], let us consider the homogeneous medium of d dimensions, where waves of only one type with the dispersion law ω_k, $k=(k_1,\dots,k_d)$ can propagate. The Hamiltonian of such a medium can be represented as follows:

$$H[a_k(t)] = H_0 + H_{\text{int}}, \tag{2.1}$$

where $a_k(t)$ is the complex wave amplitude with wave vector k,

$$H_0 = \int \omega_k |a_k|^2 \, \mathrm{d}k$$

and H_{int} is the interaction Hamiltonian, representing a power expansion in a_k, $\overset{*}{a}_k$. The equation for a_k has the form

$$\mathrm{i}\dot{a}_k = \omega_k a_k + \frac{\delta H_{\text{int}}}{\delta \overset{*}{a}_k}. \tag{2.2}$$

Following a similar approach as used in quantum scattering theory, let us consider the system with interaction, adiabatically decreasing as $t \to \pm\infty$,

$$H = H_0 + H_{\text{int}} \mathrm{e}^{-\varepsilon|t|}, \quad \varepsilon > 0. \tag{2.3}$$

For the system (2.2) the global solvability theorem may not be fulfilled and asymptotic states as $t \to \pm\infty$ may not exist. However, for the system with the Hamiltonian (2.3) at finite ε and sufficiently small a_k they exist, i.e. the solution of the equation (2.2) turns asymptotically into the solution of the linear equation:

$$a_k(t) \to a_k^{\pm}(t) = c_k^{\pm}\, \mathrm{e}^{-\mathrm{i}\omega_k t}. \tag{2.4}$$

Besides, the asymptotic states may contain solitons, which certainly cannot exist at finite ε. So our consideration should be restricted to the class of initial states without solitons and with smooth c_k^-. We shall call this class as the non-soliton sector.

Though the consideration is restricted to a special class of initial states, the result will be very useful because the obtained structure of formal series for the S-matrix provides us with the structure of motion invariants (see section 6 of this paper) and normal form (if it exist) of the Hamiltonian (E.I. Schulman, to be published).

The functions c_k are not independent and there exists a nonlinear operator $S_\varepsilon[c^-]$, transforming one into the other. To study this operator we use as usual the interaction representation:

$$a_k^s(t) = b_k^s(t)\, \mathrm{e}^{-\mathrm{i}s\omega_k t}. \tag{2.5}$$

Here $s=\pm 1$, $a_k^1(t)=a_k(t)$, $a_k^{-1}(t)=\overset{*}{a}_k(t)$. The equations of motion now take the form

$$\mathrm{i} s \dot{b}_k^s=\frac{\delta \tilde{H}_{\mathrm{int}}}{\delta b_k^{-s}} \mathrm{e}^{-\varepsilon|t|}. \tag{2.6}$$

In (2.6) H_{int} is the interaction Hamiltonian expressed in terms of the variables b_k^s. Eq. (2.6) is equivalent to an integral equation:

$$b_k^s(t)=c_k^{-s}-\frac{\mathrm{i} s}{2} \int_{-\infty}^t \mathrm{d} t_1 \frac{\delta \tilde{H}_{\mathrm{int}}}{\delta b_k^{-s}(t_1)} \mathrm{e}^{-\varepsilon|t_1|}. \tag{2.7}$$

Eq. (2.7) gives a map $c_k^{-s} \rightarrow b_k^s(t)$ which can be written in the form

$$b_k^s=S_\varepsilon^s(-\infty, t)[c_k^{-s}]. \tag{2.8}$$

Letting, $t \rightarrow+\infty$, one finds

$$c_k^{+}=S_\varepsilon[c_k^{-s}], \tag{2.9}$$

where $S_\varepsilon=S_\varepsilon(-\infty, \infty)$.

At finite ε and sufficiently small a_k the operators $S_\varepsilon(-\infty, t)$ and S_ε can be obtained in the form of convergent series by the iteration of eq. (2.7). Let $\varepsilon \rightarrow 0$ in each term of the series. As we shall see the expression obtained is finite in a sense of generalized functions. We shall call the series obtained for the operator $S_\varepsilon(-\infty, t)$ as $\varepsilon \rightarrow 0$ the classical transition matrix. The corresponding series for S_ε will be called the formal classical scattering matrix. Let us designate

$$\begin{aligned} & S(-\infty, t)=\lim_{\varepsilon \rightarrow 0} S_\varepsilon(-\infty, t), \\ & S=\lim_{\varepsilon \rightarrow 0} S_\varepsilon(-\infty, \infty), \end{aligned} \tag{2.10}$$

where the limits are to be understood in the above-mentioned sense.

As $\varepsilon \rightarrow 0$ the series for $S_\varepsilon(-\infty, t)$ and S_ε are generally divergent and formal. Consider the structure of the classical scattering matrix in the simplest case of cubic over the field variables interaction Hamiltonian H_{int}:

$$H_{\mathrm{int}}=\frac{1}{3!} \sum_{s s_1 s_2} \int V_{k k_1 k_2}^{s s_1 s_2} a_k^s a_{k_1}^{s_1} a_{k_2}^{s_2} \delta(s k+s_1 k_1+s_2 k_2) \mathrm{d} k \,\mathrm{d} k_1 \,\mathrm{d} k_2. \tag{2.11}$$

From the fact that the Hamiltonian is real it follows that

$$\overset{*}{V}{}_{k\ k_1\ k_2}^{-s-s_1-s_2}=V_{k k_1 k_2}^{s s_1 s_2}. \tag{2.12}$$

Besides, the coefficient functions V do possess an evident symmetry:

$$V_{k k_1 k_2}^{s s_1 s_2}=V_{k k_2 k_1}^{s s_2 s_1}=V_{k_1 k k_2}^{s_1 s s_2}. \tag{2.13}$$

In the interaction representation the integral equation (2.7) takes the form

$$\mathrm{i}s\left(b_k^s(t) - c_k^{-s}\right) = \tfrac{1}{2}\sum_{s_1 s_2}\int_{-\infty}^{t} \mathrm{d}t_1 \int \mathrm{d}k_1\,\mathrm{d}k_2\, V^{-ss_1s_2}_{kk_1k_2}(t_1)\, b_{k_1}^{s_1}(t_1)\, b_{k_2}^{s_2}(t_1)\,\delta(-sk + s_1k_1 + s_2k_2), \tag{2.14}$$

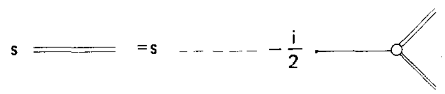
$$V^{ss_1s_2}_{kk_1k_2}(t) = V^{ss_1s_2}_{kk_1k_2}\exp\left(\mathrm{i}\Delta^{ss_1s_2}_{kk_1k_2}t - \varepsilon|t|\right), \tag{2.15}$$

$$\Delta^{ss_1s_2}_{kk_1k_2} = s\omega_k + s_1\omega_{k_1} + s_2\omega_{k_2}. \tag{2.16}$$

For example in the second order (2.14) gives for $t < 0$,

$$(b_k^s)_{\mathrm{sec}} = \frac{s}{2}\sum_{s_i,\, s'}\Bigg[s' \int \Bigg\{ V^{-ss_1s'}_{kk_1k'} V^{-s's_2s_3}_{k'k_2k_3}\,\delta(sk - s_1k_1 - s'k')\,\delta(s'k' - s_2k_2 - s_3k_3)\, c^{-s_1}_{k_1} c^{-s_2}_{k_2} c^{-s_3}_{k_3}$$
$$\times \exp\left(\mathrm{i}\left(\Delta^{-ss_1s_2s_3}_{kk_1k_2k_3} - 2\mathrm{i}\varepsilon\right)t\right)\frac{1}{\left(\Delta^{-ss_1s_2s_3}_{kk_1k_2k_3} - 2\mathrm{i}\varepsilon\right)\left(\Delta^{-s's_2s_3}_{k'k_2k_3} - \mathrm{i}\varepsilon\right)}\Bigg\}\,\mathrm{d}k'\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\Bigg].$$

(For the definition of $\Delta^{s \cdots s_q}_{k \cdots k_q}$, see (2.22).) Eq. (2.14) can be symbolically represented in the graphical form:



(2.17)

where ═══ indicates the two-component unknown value b_k^s, $s = \pm 1$, ---- designates c_k^{-s}, ·— corresponds to the factor $\exp\{-\mathrm{i}\Delta^{-ss_1s_2}_{kk_1k_2}t - \varepsilon|t|\}$, ○ indicates $V^{-ss_1s_2}_{kk_1k_2}\delta(-sk + s_1k_1 + s_2k_2)$ and the summation over s_1 and s_2 is assumed. Using (2.17) it may be possible to attribute certain graphical expression (diagram) to each term of the series arising when iterating eq. (2.14). These graphical expressions are connected graphs, having no loops or, in other words, "trees". Each graph consists of two types of elements: lines and vertices. Lines are divided further into inner and external ones. One of the external lines is distinguished (we shall call it a "root"), the other ones can be called "leaves". Each tree, correspondent to the nth iteration, contains exactly n vertices and $n+2$ leaves. Inner lines are usually called "branches". It corresponds to each of both the external and internal lines a certain value of the wave vector k_i and the index s_i. The "external" values of k and s correspond to the root. The integration is over all k_i except $k_i = k$, the summation is over all s_i except $s_i = s$. To each leaf with the wave vector k_q and index s_q there corresponds a factor $c^{-s_q}_{k_q}$.

The graph corresponding to the Nth iteration contains N integrations over the time variables $t_1, \ldots, t_N$. Each time variable t_i in the diagrams for the transition matrix corresponds to its own branch. The external time t corresponds to the root. Distinguishing the root leads to the partial ordering of the graph elements. From each vertex in which three lines meet there is a unique way to go to the root. We shall call the line leading to the root as a going out one. Let the corresponding wave vector and index be k_α and s_α. The two other lines are entering. Let the corresponding wave vectors be k_β, k_γ and indices s_β, s_γ. It is important that both entering lines are corresponded with one and the same time variable t_q. The vertex factor corresponding to this is

$$V^{-s_\alpha s_\beta s_\gamma}_{k_\alpha k_\beta k_\gamma}\,\mathrm{e}^{\mathrm{i}\Delta^{-s_\alpha s_\beta s_\gamma}_{k_\alpha k_\beta k_\gamma}t_q - \varepsilon|t_q|}\,\delta\left(-s_\alpha k_\alpha + s_\beta k_\beta + s_\gamma k_\gamma\right). \tag{2.18}$$

Let us cut the graph across the line going out of the vertex. Now the part of the graph which is cut off from the root is to be integrated over the variable t_q in the limits $-\infty < t_q \le t_p$. In fact this way of ordering of time variables agrees with the chronologic ordering used in quantum field theory. To end the diagram technique description, let us notice that the set of diagrams which correspond to the nth iteration consists of all possible trees, containing n-vertices and root fixed. Before each diagram there is a numerical factor $1/P$. The number P is equal to the number of symmetry group elements for the diagram considered, i.e. it is equal to the number of rotations at different vertices which leave the diagram unchanged, the identity transformation included. At finite $\varepsilon > 0$, the actual calculation of diagrams is a rather difficult task. However, it becomes much simpler when ε tends to zero. We shall call the integration over the time variable t_1 closest to the root the outer integration. All the other integrations will be called the inner integrations. A very important fact is that when integrating over any inner variable t_q one may replace

$$e^{-\varepsilon|t_q|} \to e^{\varepsilon t_q}. \tag{2.19}$$

We shall prove this statement here. The analogous statement is proved in quantum field theory (see [35], for example). What is important is to notice that, using (2.19), all the integrations over inner times can be carried out explicitly. After this, the diagram technique is greatly simplified. Consider an inner branch with the wave vector k_p and the index s_p such that when cutting it we can separate from the root a tree, having m leaves ($m \ge 2$). Let these leaves have wave vectors k_i and indices s_i, $i = 1, \dots, m$. To the vertex from which this tree "grows" enters the other side lines (branches or leaves) with the wave vectors and indices k_q, k_r and s_q, s_r, say. Then the expression corresponding to this vertex is as follows (the line with k_q, s_q is the going out one):

$$V^{-s_q s_r s_p}_{k_q k_r k_p} \delta(-s_q k_q + s_p k_p + s_r k_r), \tag{2.20}$$

while corresponding to the branch with the wave vector k_p and the index s_p the expression is

$$G_m = \lim_{\varepsilon \to 0} \frac{\exp\{i\Delta_m t + m\varepsilon t\}}{i(\Delta_m - im\varepsilon)} = \frac{\exp\{i\Delta_m t\}}{i(\Delta_m - i0)}, \tag{2.21}$$

$$\Delta_m = \Delta^{-s_p, s_1 \cdots s_m}_{k_p, k_1 \cdots k_m} = -s_p \omega_{k_p} + \sum_{i=1}^{m} s_i \omega_{k_i}. \tag{2.22}$$

Consider now the last (outer) integration over t_1. We have

$$S_{N\varepsilon}(-\infty, t) = W_N \int_{-\infty}^{t} e^{-\varepsilon|t_1| + i\Delta_N t_1} dt_1. \tag{2.23}$$

Here

$$W_N = W^{-s, s_1 \cdots s_N}_{k, k_1 \cdots k_N} \delta(-sk + s_1 k_1 + \cdots + s_N k_N) \tag{2.24}$$

is some expression which tends to a constant in the limit $\varepsilon \to 0$. At finite t we have from (2.23)

$$S_N(-\infty, t) = \lim_{\varepsilon \to 0} S_{N\varepsilon}(-\infty, t) = \frac{W_N e^{i\Delta_N t}}{i(\Delta_N - i0)}. \tag{2.25}$$

As $t \to +\infty$ we have

$$S_N = \lim_{\varepsilon \to 0} S_{N\varepsilon}(-\infty, \infty) = 2\pi\delta(\Delta_N) W_N. \tag{2.26}$$

So the expressions for the $S_N(-\infty, t)$ and S_N have the singularity on a manifold defined by the equations

$$\begin{aligned} P_N &= -sk + s_1 k_1 + \cdots + s_N k_N = 0, \\ \Delta_N &= -s\omega_k + s_1 \omega_{k_1} + \cdots + s_n \omega_{k_N} = 0. \end{aligned} \tag{2.27}$$

Eq. (2.27) depending on the choice of the $s, s_1, \ldots, s_N$ splits into a set of relations

$$\begin{aligned} &k + k_1 + \cdots + k_n = k_{n+1} + \cdots + k_{n+m}, \\ &\omega_k + \omega_{k_1} + \cdots + \omega_{k_n} = \omega_{k_{n+1}} + \cdots + \omega_{k_{n+m}}, \\ &m + n = N. \end{aligned} \tag{2.28}$$

Eq. (2.28) determines a manifold which we shall call the resonant manifold $\Gamma^{n+1,m}$. We designate the corresponding entity W_N via

$$W^{n+1,m}_{k, k_1, \ldots, k_n, k_{n+1}, \ldots, k_{n+m}} = W^{n+1,m}.$$

It is important to notice that $W^{n+1,m}$ is regular on the manifold (2.28) in the points of a general position. However $W^{n+1,m}$ has singularities on the submanifolds of lower dimension on which it turns into zero at least one of the entities Δ_m corresponding to the one of the inner lines of any diagram constituting the $W^{n+1,m}$. As can be seen from (2.21) these singularities may be of two types in agreement with the two terms in (2.21). The first item in (2.21) is distributed over all $\Gamma^{n+1,m}$ while the second one is localized on a manifold (to be more precise, on a set of manifolds)

$$\begin{aligned} &-s_p \omega_{k_p} + s_1 \omega_{k_1} + \cdots + s_m \omega_{k_m} = 0, \\ &-s_p k_p + s_1 \omega_{k_1} + \cdots + s_m \omega_{k_m} = 0. \end{aligned} \tag{2.29}$$

Manifolds (2.29) can be called the youngest resonant manifolds in comparison with (2.28). Eqs. (2.29) together with (2.28) determine a set of submanifolds of $\Gamma^{n+1,m}$ having the unity codimension. The division of two items in (2.21) has a certain physical meaning. One can say that the first item describes processes which go via virtual waves while the second item describes processes going via real intermediate particles. The elements of classical S-matrix with interactions going via real waves can be called as singular ones. They are decomposing on the singularity powers depending on the number of inner lines in which the Green function G_m denominator changes to zero and on the corresponding codimension of the younger resonant manifold. For any concrete dispersion law there is an element of the scattering matrix possessing the maximal singularity.

Let us now set some additional symmetry property of the amplitudes of the classical scattering matrix, i.e. consider the equation

$$\mathrm{i} s \dot{a}_k^s = \omega_k a_k^s + \frac{\delta H_{\mathrm{int}*}}{\delta a_k^{-s}}, \tag{2.30}$$

where $H_{\text{int}*}$ may be obtained from H_{int} in (2.2) by the substitution of complex conjugated Hamiltonian coefficients for example, into (2.7), $V^{ss_1s_2}_{kk_1k_2} \to V^{-s-s_1-s_2}_{kk_1k_2}$. As before we shall call the interaction to be adiabatically set in and out. Then as $t \to \pm\infty$ the solution of (2.30) and of (2.2) as well will degenerate into those of the linear equation. Let us consider the solution of eq. (2.30), turning into $\overset{*}{c}{}^{+}_{k} \exp\{-\mathrm{i}\omega_k t\}$ as $t \to -\infty$:

$$a_k \to c^{-}_{*k} \mathrm{e}^{-\mathrm{i}\omega_k t} = \overset{*}{c}{}^{+}_{k} \, \mathrm{e}^{-\mathrm{i}\omega_k t}.$$

As in (2.2), eq. (2.30) possesses a classical scattering matrix, $c^{+}_{*k} = S_*[c^{-}_{*k}]$. One should note now that eq. (2.30) is derived from (2.2) by complex conjugation and the change of the time sign. So, on account of the simple solution of the Cauchy problem for (2.2) and also for (2.20), $S_*[\overset{*}{c}{}^{+}_{k}] = \overset{*}{c}{}^{-}_{k}$.

Using the definition of the classical scattering matrix (2.9), we get

$$S_*\left[\overset{*}{S}\left[c^{-}_{k}\right]\right] = \overset{*}{c}{}^{-}_{k}. \tag{2.31}$$

The identity (2.31) is analogous to the unitarity condition for the scattering matrix in quantum mechanics. The nonlinear operator S_* can be easily calculated. It coincides with the operator S, where the Hamiltonian coefficient function V is substituted by its complex conjugate in each vertex of a diagram. It is convenient for us to introduce the operator R by the following formula:

$$S = 1 + R. \tag{2.32}$$

Then from (2.31) we obtain the following condition for R:

$$R_*\left[\overset{*}{c}{}^{-}_{k}\right] + \overset{*}{R}\left[c^{-}_{k}\right] + R_*\left[\overset{*}{R}\left[c^{-}_{k}\right]\right] = 0. \tag{2.33}$$

One can also verify simply that

$$\overset{*}{W}_{m,n+1} = -\frac{m}{n+1} W_{n+1,m}. \tag{2.34}$$

It follows from (2.34) that in particular the amplitude is asymmetric relative to the permutation of m-indices, so that the diagram "root" does not really prove to be a marked line. From physical considerations it is clear that the classical scattering matrix constructed by us coincides with the quantum scattering matrix, where radiation corrections are certainly not to be taken into account but only diagrams of the "tree type" are preserved.

3. Degenerative dispersion laws and theorems about them

In order to clarify the restrictions which the existence of the additional motion invariants imposes on the system, one can use the classical scattering matrix introduced in the previous section. Let the system of the form (2.2) have the additional integral I, analytical over a_k, $\overset{*}{a}_k$ and containing the quadratic part, i.e.

$$I = \int f_k |a_k|^2 \, \mathrm{d}k + \cdots, \tag{3.1}$$

where the three dots mean terms of higher order in a_k. We suppose, that I differs from the momentum and energy integrals, i.e $f_k \neq A\omega_k + (v, k) + \text{const}$. It follows from the conservation of the integral (3.1) that in the non-soliton sector we get

$$\int f_k c_k^- \overset{*}{c}{}_k^- \,\mathrm{d}k + \cdots = \int f_k c_k^+ \overset{*}{c}{}_k^+ \,\mathrm{d}k + \cdots . \tag{3.2}$$

Using (2.5) and (2.32) we get

$$\int f_k \Big[c_k^- \overset{*}{R}_k[c^-] + \overset{*}{c}{}_k^- R_k[c^-] \Big] \mathrm{d}k + \int f_k R_k[c^-] \overset{*}{R}_k[c^-] \,\mathrm{d}k + \cdots = 0. \tag{3.3}$$

The dots in (3.2), (3.3) mean terms of higher orders in $C_k^{\pm}$ and also what we obtain from them after substitution of (2.5). As c_k^- is an arbitrary function, then terms of each order must turn to zero, each taken separately.

Theorem 3.1. Let the system (2.2) have an additional integral I of the form (3.1). Then for each scattering process of $n+1$ waves into m waves on the corresponding resonance surface

$$\begin{aligned} &k + k_1 + \cdots + k_n = k_{n+1} + \cdots + k_{n+m}, \\ &\omega_k + \omega_{k_1} + \cdots + \omega_{k_n} = \omega_{k_{n+1}} + \cdots + \omega_{k_{n+m}}, \end{aligned} \tag{3.4}$$

one of the following two conditions is fulfilled: either (1) kernels of the corresponding term in the scattering matrix are equal to zero in the points of a general position on $\Gamma^{n+1,m}$; or (2) the function $f(k)$ satisfies on (3.4) the following condition:

$$f_k + f_{k_1} + \cdots + f_{k_n} = f_{k_{n+1}} + \cdots + f_{k_{n+m}}. \tag{3.5}$$

Proof. Associated with the surface (3.4) the terms in (3.3) contain a combination of fields

$$\overset{*}{c}{}_k^- \overset{*}{c}{}_{k_1}^- \cdots \overset{*}{c}{}_{k_n}^- c_{k_{n+1}}^- \cdots c_{k_{n+m}}^-. \tag{3.6}$$

In this case only the terms generated from the first one in (3.3) are distributed on the whole surface (3.4). Indeed, the terms of order $n+m+1$ generated from the second one and the terms of the highest orders in (3.3) are sure to contain at least one additional δ-function; e.g. at second order, the second integral in (3.3) gives $\frac{1}{4}\sum_{ss_1 \cdots s_4} \int \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4 \int \mathrm{d}k [V^{ss_1s_2}_{kk_1k_2} V^{-ss_3s_4}_{kk_3k_4} \delta(sk + s_1k_1 + s_2k_2)\delta(s\omega_k + s_1\omega_{k_1} + s_2\omega_{k_2})\delta(sk - s_3k_3 - s_4k_4)\delta(s\omega_k - s_3\omega_{k_3} - s_4\omega_{k_4})]$. After integration over k this term gets a δ-functional factor containing one vector and two scalar δ-functions (of k and of ω), that is one δ-function more than the δ-functional factor $\delta(s_1k_1 + s_2k_2 + s_3k_3 + s_4k_4)\delta(s_1\omega_{k_1} + s_2\omega_{k_2} + s_3\omega_{k_3} + s_4\omega_{k_4})$ which arises in the second order term generated by the first integral in (3.3). Besides, terms designated by $\cdots$ in (3.3) arise from expressions $\int \prod^{s_1 \cdots s_q}_{k_1 \cdots k_q} \delta(\sum_1^q s_i k_i)\delta(\sum_1^q s_i \omega_{k_i})[c_{k_1}^{+s_1} \cdots c_{k_q}^{+s_q} - c_{k_1}^{-s_1} \cdots c_{k_q}^{-s_q}]\,\mathrm{d}k_1 \cdots \mathrm{d}k_q$. From (2.26) and (2.32) it is seen that the expression in square brackets contains at least one additional δ-function. Thus, in the points of common position it is necessary to take into account the first integral in (3.3) only.

After symmetrization over k, $k_1, \dots, k_n$ and $k_{n+1}, \dots, k_{n+m}$ with the help of (2.34) we obtain for terms of order (3.6),

$$\begin{aligned}&\int [f_k + f_{k_1} + \cdots + f_{k_n} - f_{k_{n+1}} - \cdots - f_{k_{n+m}}] W^{n+1,m}_{kk_1 \cdots k_n k_{n+1} \cdots k_{n+m}} \\ &\overset{*}{c}{}^{-}_{k} \overset{*}{c}{}^{-}_{k_1} \cdots \overset{*}{c}{}^{-}_{k_n} c^{-}_{k_{n+1}} \cdots c^{-}_{k_{n+m}} \delta(k + k_1 + \cdots + k_n - k_{n+1} - \cdots - k_{n+m}) \\ &\delta\left(\Delta^{-1 \cdots -1 1 \cdots 1}_{k \cdots k_n k_{n+1} \cdots k_{n+m}}\right) \mathrm{d}k \, \mathrm{d}k_1 \cdots \mathrm{d}k_{n+m} = 0.\end{aligned} \tag{3.7}$$

On account of arbitrariness of c_k^- the theorem statement follows immediately from (3.7). ■

The analysis of submanifolds of less dimensionality distinguished by more singular parts of the S-matrix is more complicated and will be performed in section 5. Condition (3.5) may be considered as an equation of the function $f(k)$. We are only interested in the nontrivial solution of this equation, when $f(k) \neq \alpha\omega(k) + (v, k)$, $m \neq n$ and $f \neq \beta\omega(k) + (v, k) + \text{const}$, $m = n$. Here α, β are any constants and v is any constant vector.

Let P be a point of the manifold $\Gamma^{n+1,m}$. The dispersion law $\omega(k)$ is called degenerative at the point P relative to the process $n+1 \to m$ if there exists a finite domain around P where eq. (3.5) has nontrivial solutions. If eq. (3.5) can be solved nontrivially in the domain Ω of manifold $\Gamma^{n+1,m}$, the dispersion law is called degenerative in this region. According to theorem (3.1) in the domain $\Gamma^{n+1,m} \setminus \Omega$ the less singular elements of the scattering amplitude turn into zero. If the domain Ω coincides with all $\Gamma^{n+1,m}$, the dispersion law is called completely degenerate. If a domain Ω exists, but it does not coincide with $\Gamma^{n+1,m}$, the dispersion law is called partially degenerative relative to the process "n into m". Degenerative and even partially degenerative dispersion laws represent an exceptional phenomenon. Let us consider the simplest possible nonlinear process of decaying of the one wave into two. The corresponding manifold $\Gamma^{1,2}$ is determined in the space (k_1, k_2) by the equation

$$\omega(k_1 + k_2) = \omega(k_1) + \omega(k_2). \tag{3.8}$$

If there exists such a manifold, the dispersion law is called decaying (the case $\omega = \omega(|k|)$, $\omega(0) = 0$, $\omega'' > 0$ may serve as an example). For most points in k_1, k_2 space, the manifold (3.8) has dimension $2d - 1$. Eq. (3.5), which now has the form

$$f(k_1 + k_2) = f(k_1) + f(k_2), \tag{3.9}$$

defines a function of d variables. Generally speaking, at $d \geq 2$ (3.8) and (3.9) are incompatible. Nevertheless, at $d = 2$ there do exist the degenerative dispersion laws relative to process (3.8). Let us write the components of vector k as (p, q) and let $\omega(p, q)$ be defined parametrically according to the formulae

$$p = \xi_1 - \xi_2, \quad q = a(\xi_1) - a(\xi_2), \quad \omega = b(\xi_1) - b(\xi_2), \tag{3.10}$$

where $a(\xi)$ and $b(\xi)$ are any functions of one variable. Let us consider a three-dimensional manifold $\tilde{\Gamma}^{1,2}$ determined parametrically according to the formulae

$$\begin{aligned}&p_1 = \xi_1 - \xi_3, \quad p_2 = \xi_3 - \xi_2, \\ &q_1 = a(\xi_1) - a(\xi_3), \quad q_2 = a(\xi_3) - a(\xi_2).\end{aligned} \tag{3.11}$$

Now $p = p_1 + p_2 = \xi_1 - \xi_2$, $q = q_1 + q_2 = a(\xi_1) - a(\xi_2)$. Besides, we have

$$\omega(k) = \omega(k_1 + k_2) = b(\xi_1) - b(\xi_3) + b(\xi_3) - b(\xi_2) = \omega(k_1) + \omega(k_2).$$

Thus, the manifold $\tilde{\Gamma}^{1,2}$ belongs to $\Gamma^{1,2}$. Let us now consider the function $f(p,q)$, parametrizing by the formulae

$$p = \xi_1 - \xi_2, \quad q = a(\xi_1) - a(\xi_2), \quad f = c(\xi_1) - c(\xi_2), \tag{3.12}$$

where $c(\xi)$ is an arbitrary function. It is evident that $f(p,q)$ satisfies eq. (3.9) on $\tilde{\Gamma}^{1,2}$, and the dispersion law (3.10) is at least partially degenerative. The question of its complete degenerability must be considered separately. Let $a(\xi) = \xi^2$, $b(\xi) = 4\xi^3$ in (3.10). Then

$$\omega(p,q) = p^3 + \frac{3q^2}{p}. \tag{3.13}$$

This is the well-known dispersion law for the decaying Kadomtzev–Petviashvili equation often denoted KP–I,

$$\frac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) = 3u_{yy}. \tag{3.14}$$

Eq. (3.8) is now reduced to the form

$$(p_1 + p_2)^2 = \left(\frac{q_1}{p_1} - \frac{q_2}{p_2}\right)^2$$

from which it is clear that $\Gamma^{1,2}$ consists of two parts $\Gamma^{1,2}_{\pm}$. Simple analysis shows that $\tilde{\Gamma}^{1,2}$ coincides with $\Gamma^{1,2}_{+}$ given by the formula

$$p_1 + p_2 = \frac{q_1}{p_1} - \frac{q_2}{p_2}. \tag{3.15}$$

The parametrization $a(\xi) = -\xi^2$, $b(\xi) = 4\xi^3$ results also in the dispersion law (3.13). Now $\tilde{\Gamma}^{1,2}$ coincides with $\Gamma^{1,2}_{-}$, when

$$p_1 + p_2 = -\frac{q_1}{p_1} + \frac{q_2}{p_2}. \tag{3.16}$$

Thus, the dispersion law (3.13) is completely degenerative. Now, in (3.10)–(3.12), let $\xi_1 - \xi_2 = \delta \ll 1$. Then, up to the first order in δ, the identities (the prime means the derivative)

$$q/p = a'(\xi_2), \quad \omega/p = b'(\xi_2) \tag{3.17}$$

determine the dispersion law as well. Formulae (3.15) determines parametrically the homogeneous function of the first degree,

$$\omega = P\Phi(Q/P). \tag{3.18}$$

Let us note, however, that the function (3.18), and the function (3.10) together with it, are not analytical at $p = 0$. Thus, the first degree homogeneous dispersion law with any dependence on angles is degenerative. For the dispersion law (3.18) the manifold $\Gamma^{1,2}$ is determined by the following conditions:

$$q_1/p_1 = q_2/p_2 = q/p,$$

which means that the vectors k_1 and k_2 are parallel and in the same direction.

Examples of dispersion laws, degenerative with respect to the process (3.8) and different from the one just discussed, are not known at present. We can say only some quite probable statements about them. It follows from the degenerability of the dispersion law $\omega(k)$ that the manifolds given by eqs. (3.8) and (3.9) possess a common three-dimensional region. In the points of common position in this region one of the coordinates p_1, p_2, q_1, q_2 is the function of the other ones. Let $q_2 = q_2(p_1, p_2, q_1)$. Let us differentiate eqs. (3.8) and (3.9) with respect to independent variables in all possible ways, so that the summarized order of derivatives would not exceed N. It is easy to calculate that a general number of such differentiations equals $\mathscr{P}_N = (N^3 + 6N^2 + 11N)/6$. We shall obtain $2\mathscr{P}_N$ linear algebraic equations on a number of derivatives form the function $q_2(p_1, p_2, q_1)$. Their consistency conditions represent $\mathscr{P}_N$ of nonlinear differential-functional equations, containing derivatives of functions of two variables, $\omega(p, q)$ and $f(p, q)$, taken in "three positions" p_1, q_1; p_2, q_2 and $p_1 + p_2, q_1 + q_2$. The total number of derivatives of order not exceeding N of functions of two variables equals $Q_N = N(N+3)/2$. At $N = 9$, $\mathscr{P}_N = 21$, $Q_N = 54$, $4Q_N = 216$. Thus, $\mathscr{P}_N > 4Q_N$ at $N \geq 9$. This inequality means that at $N = 9$ from the 219 equations we possess one can exclude 216 derivatives of functions ω and f in some two positions (for example in p_2, q_2 and $p_1 + p_2$, $q_1 + q_2$) in a pure algebraic way. As a result, the functions $f(p_1, q_1)$ and $\omega(p_1, q_1)$ prove to satisfy the system of three differential equations of the ninth order, consisting of polynomials in the derivatives of these functions. If these three equations do not prove to coincide with each other, then their solution may depend only on the finite number of functions of one variable. Analogous results hold for dispersion laws, degenerative relative to other nonlinear processess, if any. All that allows us to propose a statement, that at $d = 2$, degenerative dispersion laws are defined by a finite number of functions of one variable. Let us note that the dispersion law (3.10) is defined by two functions, and the dispersion law (3.18) by one function.

Let us now state the problem of degenerative dispersion laws, which are close to those given by the parametrization (3.10). Let us search for the dispersion laws $\omega(p, q)$ and $f(p, q)$, determined parametrically by the formulae

$$\begin{aligned} &p = \xi_1 - \xi_2, \quad q = a(\xi_1) - a(\xi_2), \\ &\omega = b(\xi_1) - b(\xi_2) + \sum_{n=1}^{\infty} \varepsilon^n \omega_n(\xi_1, \xi_2), \\ &f = c(\xi_1) - c(\xi_2) + \sum_{n=1}^{\infty} \varepsilon^n f_n(\xi_1, \xi_2), \end{aligned} \tag{3.19}$$

here ε is a small parameter. It is convenient to set the three-dimensional resonance manifold parametrically in the form

$$\begin{aligned} &p_1 = \xi_1 - \xi_3, \quad q_1 = a(\xi_1 + \eta) - a(\xi_3 + \eta), \\ &p_2 = \xi_3 - \xi_2, \quad q_2 = a(\xi_3 + \nu) - a(\xi_2 + \nu), \end{aligned}$$

requiring additionally

$$q = q_1 + q_2 = a(\xi_1) - a(\xi_2)$$
$$= a(\xi_1 + \eta) - a(\xi_3 + \eta) + a(\xi_3 + \nu) - a(\xi_2 + \nu). \tag{3.20}$$

Now the conditions (3.8), (3.9) together with (3.20) will yield three equations in five parameters $\xi_1, \xi_2, \xi_3, \eta, \nu$. This system of equations must define η and ν in the form of series in ε:

$$\eta = \sum_{n=1}^{\infty} \varepsilon^n \eta_n(\xi_1, \xi_2, \xi_3), \quad \nu = \sum_{n=1}^{\infty} \varepsilon^n \nu_n(\xi_1, \xi_2, \xi_3).$$

We have a linear overdetermined system in the first order in ε:

$$\begin{aligned}
&[a'(\xi_1) - a'(\xi_3)]\eta_1 + [a'(\xi_3) - a'(\xi_2)]\nu_1 = 0,\\
&[b'(\xi_1) - b'(\xi_3)]\eta_1 + [b'(\xi_3) - b'(\xi_2)]\nu_1 = \Omega_1,\\
&[c'(\xi_1) - c'(\xi_3)]\eta_1 + [c'(\xi_3) - c'(\xi_2)]\nu_1 = F_1.
\end{aligned} \tag{3.21}$$

Here

$$\begin{aligned}
&\Omega_1 = \omega_1(\xi_1, \xi_2) - \omega_1(\xi_1, \xi_3) - \omega_1(\xi_3, \xi_2),\\
&F_1 = f_1(\xi_1, \xi_2) - f_1(\xi_1, \xi_3) - f_1(\xi_3, \xi_2).
\end{aligned} \tag{3.22}$$

Consistency condition of the system (3.26) has the form

$$\Omega_1 B = F_1 A, \tag{3.23}$$

where

$$A(\xi_1, \xi_2, \xi_3) = \alpha(\xi_1, \xi_2) + \alpha(\xi_2, \xi_3) + \alpha(\xi_3, \xi_1) = \Delta_{ab}, \tag{3.24}$$
$$B(\xi_1, \xi_2, \xi_3) = \beta(\xi_1, \xi_2) + \beta(\xi_2, \xi_3) + \beta(\xi_3, \xi_1) = \Delta_{ac}, \tag{3.25}$$
$$\alpha(\xi_1, \xi_2) = b'(\xi_1)a'(\xi_2) - b'(\xi_2)a'(\xi_1),$$
$$\beta(\xi_1, \xi_2) = c'(\xi_1)a'(\xi_2) - c'(\xi_2)a'(\xi_1).$$

The functions $A(\xi_1, \xi_2, \xi_3)$ and $B(\xi_1, \xi_2, \xi_3)$ are antisymmetric relative to all argument permutations. Interchanging ξ_2 and ξ_3 in (3.23) and summing up the results, we are convinced that the functions $\omega_1(\xi_1, \xi_2)$ and $f_1(\xi_1, \xi_2)$ are antisymmetric:

$$\omega_1(\xi_1, \xi_2) = -\omega_1(\xi_2, \xi_1); \quad f_1(\xi_1, \xi_2) = -f_1(\xi_2, \xi_1).$$

Thus, we may put

$$\begin{aligned}
&\Omega_1 = \omega_1(\xi_1, \xi_2) + \omega_1(\xi_2, \xi_3) + \omega_1(\xi_3, \xi_1),\\
&F_1 = f_1(\xi_1, \xi_2) + f_1(\xi_2, \xi_3) + f_1(\xi_3, \xi_1).
\end{aligned} \tag{3.26}$$

So, the problem is to solve the functional equation (3.23). It is easy to check, that eq. (3.23) has the following solution:

$$\omega_1(\xi_1, \xi_2) = \frac{b'(\xi_1) - b'(\xi_2)}{a'(\xi_1) - a'(\xi_2)}(l(\xi_1) - l(\xi_2)), \tag{3.27}$$

$$f_1(\xi_1, \xi_2) = \frac{c'(\xi_1) - c'(\xi_2)}{a'(\xi_1) - a'(\xi_2)}(l(\xi_1) - l(\xi_2)). \tag{3.28}$$

Here $l(\xi)$ is any function. This solution does not result in a new dispersion law, but represents a result of function reparametrization in (3.10)–(3.12). Let us put

$$\begin{aligned}&\xi_1 - \xi_2 = \eta_1 - \eta_2; \quad a(\xi_1) - a(\xi_2) = a(\eta_1) - a(\eta_2) + \varepsilon[l(\eta_1) - l(\eta_2)],\\ &b(\xi_1) - b(\xi_2) = b(\eta_1) - b(\eta_2) + \varepsilon\omega(\eta_1, \eta_2).\end{aligned} \tag{3.29}$$

$\omega(\eta_1, \eta_2)$ represents a series in powers of ε, the first term of this series is given by the formulae (3.27), (3.28).

One more trivial solution of eq. (3.29) is

$$\omega_1 = p(\xi_1) - p(\xi_2), \quad f_1 = q(\xi_1) - q(\xi_2)$$

($p(\xi)$ and $q(\xi)$ are any functions, representing the variations of $b(\xi)$ and $c(\xi)$).

It is important to note that eq. (3.23) possesses still one more trivial solution. Let us assume

$$\begin{aligned}&\omega_1(\xi_1, \xi_2) = \alpha(\xi_1, \xi_2)S(\xi_1, \xi_2),\\ &f_1(\xi_1, \xi_2) = \beta(\xi_1, \xi_2)S(\xi_1, \xi_2).\end{aligned} \tag{3.30}$$

After substitution of (3.30) into (3.23) we shall be convinced that $S(\xi_1, \xi_2)$ satisfies the unlooked-for simple equation

$$S(\xi_1, \xi_2)[a'(\xi_1) - a'(\xi_2)] + S(\xi_2, \xi_3)[a'(\xi_2) - a'(\xi_3)] + S(\xi_3, \xi_1)[a'(\xi_3) - a'(\xi_1)] = 0, \tag{3.31}$$

$$S(\xi_1, \xi_2) = \frac{\tau(\xi_1) - \tau(\xi_2)}{a'(\xi_1) - a'(\xi_2)}. \tag{3.32}$$

Here $\tau(\xi)$ is an arbitrary function again. The solution (3.32) is also a trivial one and results from the reparametrization of dispersion law of the form

$$p = \xi_1 - \xi_2 + \varepsilon[\tau(\xi_1) - \tau(\xi_2)], \quad q = a(\xi_1) - a(\xi_2), \quad \omega = b(\xi_1) - b(\xi_2),$$

which is to the first order in ε equivalent to (3.10) with a modified function $a(\xi)$. To obtain the given $a(\xi)$, one needs to make a change of variables of the form

$$\xi_1 = \eta_1 + \varepsilon a'(\eta_2)\frac{\tau(\eta_1) - \tau(\eta_2)}{a'(\xi_1) - a'(\xi_2)}; \quad \xi_2 = \eta_2 + \varepsilon a'(\eta_1)\frac{\tau(\eta_1) - \tau(\eta_2)}{a'(\eta_1) - a'(\eta_2)}.$$

Substituting new variables into the expression for ω and expanding in ε we arrive at expression (3.19) with the linear term being of the form (3.30), (3.32).

We shall consider (3.23) as a system of linear algebraic equations relative to the unknown functions $\omega(\xi_1, \xi_2)$ and $f(\xi_1, \xi_2)$ Let the variable ξ_3 take two arbitrary values $\xi_3 = \alpha_1$ and $\xi_3 = \alpha_2$. Let us note that

$$A_{1,2} = A_{1,2}(\xi_1, \xi_2) = A|_{\xi_3=\alpha_{1,2}}; \quad B_{1,2} = B_{1,2}(\xi_1, \xi_2) = B|_{\xi_3=\alpha_{1,2}}, \tag{3.33}$$

$$f(\xi, \alpha_i) = g_i(\xi), \quad \omega(\xi, \alpha_i) = h_i(\xi), \quad i = 1,2. \tag{3.34}$$

We can see from (3.34) that in the most general case the solution of eq. (3.23) may depend on not more than four functions of one variable $g_{1,2}(\xi)$ and $h_{1,2}(\xi)$.

The solution constructed by us depends upon the very four functions $l(\xi)$, $p(\xi)$, $q(\xi)$ and $\tau(\xi)$. Solving evidently eq. (3.23) at $\xi_3 = \alpha_{1,2}$ and making elementary analysis of the solution, we are convinced that we have constructed a general solution of the functional equation (3.23). The result obtained can be considered as local uniqueness theorem for degenerative dispersion laws of the form (3.10). This theorem was presented in [30] without full proof. Unfortunately, we have not yet a global uniqueness theorem and it is possible for degenerative dispersion laws to exist not close to (3.10).

Let $\omega(p, q)$ be a differentiable function and $\omega(0,0) = 0$. Let $\omega(p, q)$ satisfy one more condition

$$\frac{|\omega(p,q)|}{R} \underset{R\to 0}{\longrightarrow} 0, \quad R = |p^2 + q^2|^{1/2}. \tag{3.35}$$

Then the dispersion law $\omega(p, q)$ is decaying. There exists a manifold $\Gamma^{1,2}$, because it contains a two-dimensional plane $p_2 = q_2 = 0$ and some vicinity of this plane given by the following equation:

$$\frac{\partial \omega}{\partial p}(p_1, q_1) \cdot p_2 + \frac{\partial \omega}{\partial q}(p_1, q_1) \cdot q_2 = 0. \tag{3.36}$$

Putting $p_2 = q_2 = 0$ in (3.9), we get $f(0,0) = 0$ and, moreover,

$$\lim_{R\to 0} [f(R, \vartheta)/R] = f_0(\vartheta) < \infty \quad \text{at all } \vartheta.$$

Here $\vartheta = \operatorname{arctg}(q_2/p_2)$.

Thus, in the vicinity of zero $f(p, q)$ may tend asymptotically to the homogeneous function of the first order. But then at $p, q \to 0$, $\omega(p, q)$ must also tend to the homogeneous function of the first order, which is excluded by the condition (3.35). Thus, $f_0(\vartheta)$ and the function f also submit to the condition (3.35). Now in the vicinity of $p_2 = q_2 = 0$ we have from (3.9) the following:

$$\frac{\partial f}{\partial p}(p_1, q_1) \cdot p_2 + \frac{\partial f}{\partial q}(p_1, q_1) \cdot q_2 = 0.$$

From here it follows that the Jacobian between the functions f and ω is equal to zero, and that there exists functional dependence between them:

$$f(p, q) = F[\omega(p, q)].$$

Now we have from eqs. (3.8) and (3.9),

$$F[\omega(p_1,q_1)+\omega(p_2,q_2)] = F[\omega(p_1,q_1)] + F[\omega(p_2,q_2)],$$

from where we get $F(\xi)=\lambda\xi$, λ is a constant. The important consequence of the result obtained is the following:

Theorem 3.2. A dispersion law $\omega(p,q)$ satisfying the condition $\omega(0,0)=0$ and analytic in the vicinity of $p=q=0$ is nondegenerative on $\Gamma^{1,2}$ in the vicinity of the plane $p_2=q_2=0$.

In fact, the proof of this theorem is given above. Only, one should notice that linear terms in the expansion $\omega(p,q)$ in the vicinity of point $p=q=0$ may be excluded. After that the analytical function satisfies the condition (3.35).

Theorem 3.2 is rather important from the viewpoint of applications of the theory developed here to the nonintegrability proof of the concrete wave systems. Two-dimensionality of the coordinate space is essential for degenerability. Suppose $d=3$. Now in common position, the manifold $\Gamma^{1,2}$ has dimensionality 5. Let us designate the wave number, corresponding to a new space via "r" and consider the dispersion law, transforming into (3.10) at $r=0$. Then theorem 3.3 holds.

Theorem 3.3. Let the degenerative dispersion law $\omega(p,q,r)$ be parametrized in the vicinity of $r=0$ as follows:

$$\begin{aligned} &p=\xi_1-\xi_2; \quad q=a(\xi_1)-a(\xi_2),\\ &\omega(p,q,r)=b(\xi_1)-b(\xi_2)+r\sum_{n=0}^{\infty} r^n\omega_n(\xi_1,\xi_2) \end{aligned} \tag{3.37}$$

and the manifold $\Gamma^{1,2}$ has dimensionality 5. Then $\omega_0=\text{const}$, $\omega_n=0$, $n>0$.

Proof. The resonance manifold $\Gamma^{1,2}$ for the dispersion law (3.37) may be given in the form

$$\begin{aligned} &a(\xi_1)-a(\xi_2)=a(\xi_1+\eta)-a(\xi_3+\eta)+a(\xi_3+\nu)-a(\xi_2+\nu),\\ &\sum_{k=0}^{\infty}(r_1+r_2)^{k+1}\omega_k(\xi_1,\xi_2) = -b(\xi_1)+b(\xi_2)+b(\xi_1+\eta)-b(\xi_3+\eta)+b(\xi_3+\nu)-b(\xi_2+\nu)\\ &\qquad\qquad + \sum_{h=0}^{\infty}\left[r_1^{n+1}\omega_n(\xi_1+\eta,\xi_3+\eta)+r_2^{n+1}\omega_n(\xi_3+\nu,\xi_2+\nu)\right]. \end{aligned} \tag{3.38}$$

Let us choose ξ_1, ξ_2, ξ_3, r_1, and r_2 as independent variables and then consider ν and η as their functions, analytical in r_1 and r_2.

The degenerability condition can be written in its usual form:

$$f(p,q,r_1+r_2)=f(p_1,q_1,r_1)+f(p_2,q_2,r_2). \tag{3.39}$$

Its solution may be found in the form

$$f(p,q,r)=c(\xi_1)-c(\xi_2)+r\sum_0^\infty r^n f_n(\xi_1,\xi_2),$$
$$\eta=\sum_{m+n=1}^{\infty}\eta_{mn}r_1^m r_2^n;\quad \nu=\sum_{m+n=1}^{\infty}\nu_{mn}r_1^m r_2^n. \tag{3.40}$$

Considering the terms linear in r_1 and r_2 in (3.38) and (3.39) and noticing $\eta_0=\eta_{10}r_1+\eta_{01}r_2$, $\nu_0=\nu_{10}r_1+\nu_{01}r_2$ we obtain

$$\eta_0[a'(\xi_1)-a'(\xi_3)]+\nu_0[a'(\xi_3)-a'(\xi_2)]=0,$$
$$(r_1+r_2)\omega_0(\xi_1+\xi_2)=r_1\omega_0(\xi_1,\xi_3)+r_2\omega_0(\xi_3,\xi_2)+\eta_0[b'(\xi_1)-b'(\xi_3)]+\nu_0[b'(\xi_3)-b'(\xi_2)], \tag{3.41}$$
$$(r_1+r_2)f_0(\xi_1,\xi_2)=r_1f_0(\xi_1,\xi_3)+r_2f_0(\xi_3,\xi_2)+\eta_0[c'(\xi_1)-c'(\xi_3)]+\nu_0[c'(\xi_3)-c'(\xi_2)].$$

Setting equal the coefficients in (3.41) at r_1, r_2 separately, we obtain an overdetermined system of equations for η_{10}, ν_{10} and η_{01}, ν_{01}. Their consistency conditions are

$$[\omega_0(\xi_1,\xi_2)-\omega_0(\xi_1,\xi_3)]B=[f_0(\xi_1,\xi_2)-f_0(\xi_1,\xi_3)]A, \tag{3.42}$$
$$[\omega_0(\xi_1,\xi_2)-\omega_0(\xi_3,\xi_2)]B=[f_0(\xi_1,\xi_2)-f_0(\xi_3,\xi_2)]A. \tag{3.43}$$

Here A and B are given by the formulae (3.29) and (3.30).

In contrast to eq. (3.28), eqs. (3.42) and (3.43) do not possess nontrivial solutions. To be convinced about it, let us differentiate (3.42) with respect to ξ_3 and then apply the operator $\partial^3/\partial\xi_3^3-\partial^3/\partial\xi_3^2\partial\xi_2$ for the same equation and further put $\xi_3=\xi_2$. We obtain the system of the two homogeneous equations for $\partial\omega_0/\partial\xi_2, \partial f_0/\partial\xi_2$, having nonzero determinant. So, $\partial\omega_0/\partial\xi_2=0$, $\partial f_0/\partial\xi_2=0$. Similarly, we get $\partial\omega_0/\partial\xi_1=0$, $\partial f_0/\partial\xi_1=0$ from (3.43). Thus, the only solution of eqs. (3.42) is $\omega_0=\text{const}$, $f_0=\text{const}$, $\nu_0=\eta_0=0$. We shall show further proof via induction. Let ν_k, η_k be the sums of the sequence terms in (3.40), for which $m+n=k$. Let $\nu_q=\eta_q=0$ at $q<k$. Collecting in (3.38) and (3.39) terms of degree k, we have

$$(r_1+r_2)^k\omega_{k-1}(\xi_1,\xi_2)=r_1^k\omega_{k-1}(\xi_1,\xi_3)+r_2^k\omega_{k-1}(\xi_3,\xi_2)$$
$$+[b'(\xi_1)-b'(\xi_3)]\eta_k+[b'(\xi_3)-b'(\xi_2)]\nu_k=0,$$
$$[a'(\xi_1)-a'(\xi_3)]\eta_k+[a'(\xi_3)-a'(\xi_2)]\nu_k=0,$$

together with an analogous equation for f. Taking mixed derivative in r_1, r_2 of the kth order $\partial^k/\partial r_1^{k-1}\partial r_2$ we get

$$k!\omega_k(\xi_1,\xi_2)=[b'(\xi_1)-b'(\xi_3)]\frac{\partial\eta_k}{\partial r_1^{k-1}\partial r_2}+[b'(\xi_3)-b'(\xi_2)]\frac{\partial\nu_k}{\partial r_1^{k-1}\partial r_2},$$
$$k!f_k(\xi_1,\xi_2)=[c'(\xi_1)-c'(\xi_3)]\frac{\partial\eta_k}{\partial r_1^{k-1}\partial r_2}+[b'(\xi_3)-b'(\xi_2)]\frac{\partial\nu_k}{\partial r_1^{k-1}\partial r_2}.$$

Consistency of these equations with (3.44) results in an equation of the form (3.23):

$$\omega_k(\xi_1, \xi_2)\,\Delta_{ac} = f_k(\xi_1, \xi_2)\,\Delta_{ba},$$

which is not fulfilled, as Δ_{ac}/Δ_{ba} is a function of ξ_1, ξ_2, ξ_3.

Really, Δ_{ac} and Δ_{ba} are totally antisymmetric functions, so their ratio is a totally symmetric function of ξ_1, ξ_2 and ξ_3 and is not equal to a constant, as b and c are different functions. Thus theorem 3.3 is proven. ∎

On the grounds of theorem 3.3 one may suggest the hypothesis that $d > 2$ and under a condition of maximal dimensionality for $\Gamma^{1,2}$, there do not exist dispersion laws, degenerative with respect to processes $1 \leftrightarrow 2$. The requirement of maximum dimensionality of $\Gamma^{1,2}$ is essential. For example, at any $d \geq 2$, the linear dispersion law $\omega = |k|\varphi(k/|k|)$ is degenerative. However, the manifold $\Gamma^{2,2}$ is given by the parallelism condition on k_1, k_2 and k and so has dimensionality 4, less than the maximum.

Let us consider the process of scattering of two waves upon each other. The manifold $\Gamma^{2,2}$ is given by the following equations:

$$k + k_1 = k_2 + k_3,\qquad \omega(k) + \omega(k_1) = \omega(k_2) + \omega(k_3). \tag{3.44}$$

The dispersion law $\omega(k)$ is nondegenerative relative to this process, if in some region of the manifold $\Gamma^{2,2}$ the functional equation

$$f(k) + f(k_1) = f(k_2) + f(k_3) \tag{3.45}$$

has nontrivial solution. Apparently, the manifold $\Gamma^{2,2}$ includes two hypersurfaces, set by the conditions

$$k = k_2,\quad k_1 = k_3 \quad \text{or} \quad k = k_3,\quad k_1 = k_2$$

crossing in a straight line $k = k_1 = k_2 = k_3$. On this submanifold $\tilde{\Gamma}^{2,2}$ eq. (3.45) is fulfilled at any $f(k)$. At $d = 1$, $\Gamma^{2,2} = \tilde{\Gamma}^{2,2}$ and any dispersion law is degenerative. At $d \geq 2$, the following holds:

Theorem 3.4. If in the vicinity of point k_0 the dispersion law $\omega(k)$ may be expanded in a Taylor series

$$\omega(k_0 + \kappa) = \omega(k_0) + (\alpha, \kappa) + \sum A_{ij}\kappa_i\kappa_j + \cdots, \tag{3.46}$$

then in the vicinity of $k = k_1 = k_2 = k_3 = k_0$ the dispersion law ω_k is nondegenerative with respect to the process (3.44). Theorem 3.4 is the evident consequence of the following lemma.

Lemma 1. The quadratic dispersion law with any signature is nondegenerative with respect to (3.44) at $d \geq 2$.

Proof. Let us reduce the quadratic form (3.46) to a diagonal form via coordinate system rotation, then $(k = (k^{(1)}, \ldots, k^{(d)}))$.

$$\omega(k) = k^{(1)^2} + \sigma_2 k^{(2)^2} + \cdots + \sigma_d k^{(d)^2},\qquad \sigma_i = \pm 1,\quad i = 2, \ldots, d. \tag{3.47}$$

All signs in (3.47) are independent. For the dispersion law (3.47) the manifold $\Gamma^{2,2}$ has rational parametrization:

$$\begin{aligned}
&k^{(1)}=P_1+\tfrac{1}{2}\mu(1-Q), \quad k_1^{(1)}=P_1-\tfrac{1}{2}\mu(1-Q),\\
&k_2^{(1)}=P_1-\tfrac{1}{2}\mu(1+Q), \quad k_3^{(1)}=P_1+\tfrac{1}{2}\mu(1+Q),\\
&k^{(i)}=P_i+\tfrac{1}{2}\mu(\tau_i+s_i), \quad k_1^{(i)}=P_i-\tfrac{1}{2}\mu(\tau_i+s_i),\\
&k_2^{(i)}=P_i+\tfrac{1}{2}\mu(\tau_i-s_i), \quad k_3^{(i)}=P_i-\tfrac{1}{2}\mu(\tau_i-s_i),\\
&\quad i=2,\dots,d,
\end{aligned} \tag{3.48}$$

where

$$Q=\sum_{n=2}^{d}\sigma_n\tau_n s_n$$

and $P_1,\dots,P_d,\mu,\tau_i,s_i$ are independent coordinates on the resonance surface (3.44). Let us put the parameterization (3.48) into the functional equation (3.45):

$$\begin{aligned}
&f\left(P_1+\tfrac{1}{2}\mu(1-Q),P_2+\tfrac{1}{2}\mu(\tau_2+s_2),\cdots\right)+f\left(P_1-\tfrac{1}{2}\mu(1-Q),P_2-\tfrac{1}{2}\mu(\tau_2+s_2),\cdots\right)\\
&\quad=f\left(P_1-\tfrac{1}{2}\mu(1+Q),P_2+\tfrac{1}{2}\mu(\tau_2-s_2),\cdots\right)+f\left(P_1+\tfrac{1}{2}\mu(1+Q),P_2-\tfrac{1}{2}\mu(\tau_2-s_2),\cdots\right).
\end{aligned} \tag{3.45a}$$

Differentiating (3.45a) in τ_i, s_i, supposing $\tau_i=s_i$, subtracting one from another, differentiating in τ_i and supposing $\mu=0$, we find

$$\partial^2 f(P_1,\dots,P_d)/\partial P_1\,\partial P_i=0, \quad i=2,\dots,d,$$

from where

$$f=F_1(k^{(1)})+\phi(k^{(2)},\dots,k^{(d)}). \tag{3.49}$$

Substituting (3.49) into (3.45a), putting down the equations obtained via differentiation in τ_i,τ_j,s_i,s_j and supposing all τ,s to be equal to zero, after simple transformations we obtain $\partial^2\phi/\partial P_i\,\partial P_j=0$ or

$$f=F_1(k^{(1)})+\cdots+F_d(k^{(d)}). \tag{3.50}$$

Let us substitute (3.50) into (3.45a) and differentiate in P_1. We obtain

$$F_1'\left(P_1+\tfrac{1}{2}\mu(1-Q)\right)+F_1'\left(P_1-\tfrac{1}{2}\mu(1-Q)\right)=F_1'\left(P_1-\tfrac{1}{2}\mu(1+Q)\right)+F_1'\left(P_1+\tfrac{1}{2}\mu(1+Q)\right),$$

differentiating in Q and μ we get two equations on F_1'', for which the consistency condition is written in the form (at $Q=0$)

$$F_1''\left(P_1-\frac{\mu}{2}\right)=F_1''\left(P_1+\frac{\mu}{2}\right).$$

On account of arbitrariness of P_1 and μ we get that $F_1'' = \text{const}$. Exactly in the same way, differentiating (3.45a) in P_i and then in τ_i, s_i, subtracting one from another and supposing $\tau_i = -s_i$, we obtain

$$F_i''(P_i + \mu\tau_i) = F_i''(P_i - \mu\tau_i).$$

From where on account of arbitrariness of P_i, μ, τ_i we get $F_i'' = \text{const}$. Thus, $F_i = C_i k^{(i)^2} + B_i k^{(i)} + D_i$. It is easy to get $C_i = \sigma_i c$ from (3.45a) and that proves nondegenerability. ■

It follows from what has been proved that there do not exist dispersion laws completely degenerative relative to the process (3.44). It is rather doubtful that there exist dispersion laws degenerative relative to this process even partially. Besides theorem 3.4, the statement following below gives grounds for this doubt as well. Let us suppose that the dispersion law $\omega(k)$ is decaying. Then the manifold $\Gamma^{2,2}$ possesses a submanifold $\Gamma_M^{2,2}$ of dimensionality one, given by the system of equations

$$\begin{aligned} &k + k_1 = k_2 + k_3 = q, \\ &\omega(k) + \omega(k_1) = \omega(k_2) + \omega(k_3) = \omega(q). \end{aligned} \tag{3.51}$$

If the dispersion law is degenerative relative to the process "one into two" then on manifold $\Gamma_{\mathrm{M}}^{2,2}$, the function $f(k)$ is sure to satisfy the following equation:

$$f(k) + f(k_1) = f(k_2) + f(k_3) = f(q), \tag{3.52}$$

which, of course, does not mean even partial degenerability of the dispersion law $\omega(k)$. For degenerability to occur, it is necessary to fulfill eq. (3.45) on $\Gamma^{2,2}$ in the vicinity of just one point of the manifold (3.51).

Let us study this possibility in the simplest case of $d = 2$ when the dispersion law is referred to class (3.10) considered by us. Now the manifold $\Gamma_{\mathrm{M}}^{2,2}$ (3.51) is parametrized as follows (at $d = 2$ its dimensionality is equal to four):

$$\begin{aligned} &p = \xi_1 - \xi_2, \quad p_1 = \xi_2 - \xi_3, \quad p_2 = \xi_1 - \xi_4, \quad p_3 = \xi_4 - \xi_3, \\ &q = a(\xi_1) - a(\xi_2), \quad q_1 = a(\xi_2) - a(\xi_3), \quad q_2 = a(\xi_1) - a(\xi_4), \quad q_3 = a(\xi_4) - a(\xi_3). \end{aligned} \tag{3.53}$$

Let us consider on $\Gamma^{2,2}$ the vicinity of point given by the coordinates $\xi_1, \xi_2, \xi_3, \xi_4$. We may fix it, having preserved the expression (3.53) for p_i and defined

$$\begin{aligned} &q = a(\xi_1) - a(\xi_2), \quad q_1 = a(\xi_1 + \nu_1) - a(\xi_3 + \nu_1), \\ &q_2 = a(\xi_1 + \nu_2) - a(\xi_4 + \gamma_2), \quad q_3 = a(\xi_4 + \nu_3) - a(\xi_3 + \nu_3). \end{aligned}$$

Similarly we can define ω_i. Resonance conditions impose two conditions upon ν_i:

$$[a'(\xi_2) - a'(\xi_3)]\nu_1 = [a'(\xi_1) - a'(\xi_3)]\nu_2 + [a'(\xi_4) - a'(\xi_3)]\nu_3,$$

$$[b'(\xi_2) - b'(\xi_3)]\nu_1 = [b'(\xi_1) - b'(\xi_3)]\nu_2 + [b'(\xi_4) - b'(\xi_3)]\nu_3.$$

Degenerability condition yields one more equation:

$$[c'(\xi_2) - c'(\xi_3)]\nu_1 = [c'(\xi_1) - c'(\xi_3)]\nu_2 + [c'(\xi_4) - c'(\xi_3)]\nu_3.$$

If the functions a, b, c are linearly independent, these three equations possess zero solutions only. It follows from here, that the submanifold $\Gamma_{\mathrm{M}}^{2,2}$ cannot be locally dilated, while preserving degenerability.

Let us now consider any process of "m waves into n waves" described by resonance conditions (3.4), and let the dispersion law $\omega(k)$ be decaying and degenerative relative to the process "one into two". In the corresponding manifold $\Gamma^{n,m}$ we can distinguish a set of "minimal" manifolds $\Gamma_{\mathrm{M}}^{n,m}$. To describe these manifolds, let us remember that a set of diagrams of the tree type, possessing a finite number of vertices and internal lines, corresponds to the scattering amplitude $W^{n,m}$. Let us mark via q_i, s_i the wave vectors and their directions corresponding to these and also to the external lines.

Let us suppose that at some vertex the lines with vectors and directions q_i, s_i; q_j, s_j; q_k, s_k are crossed. Then the following identity is fulfilled in it:

$$s_i q_i + s_j q_j + s_k q_k = 0. \tag{3.54}$$

Let us also demand the fulfillment of the condition

$$s_i \omega(q_i) + s_j \omega(q_j) + s_k \omega(q_k) = 0. \tag{3.55}$$

Conditions (3.4) are sure to follow from formulae (3.54) and (3.55) but they define a manifold of less dimensionality (one of the minimal manifolds $\Gamma_{\mathrm{M}}^{n,m}$). If the dispersion law is degenerative relative to the process "one into two", then for each vertex the following condition will be satisfied:

$$s_i f(q_i) + s_j f(q_j) + s_k f(q_k) = 0$$

and, hence, also eq. (3.5). Analogously to the above it may be shown that for degenerative dispersion laws of the form (3.10), it is impossible to enlarge dimensionality of the manifold $\Gamma_{\mathrm{M}}^{n,m}$ while preserving condition (3.5).

Let us now refer to linear dispersion laws, when $\omega(k)$ is a homogeneous function of the first degree. We have already mentioned that such dispersion laws are degenerative relative to the process "one into two" at any d. In this case the resonance manifold describes three collinear vectors. It is apparent that the minimal manifold $\Gamma_{\mathrm{M}}^{n,m}$ also describes sets of collinear vectors k_i. Linear dispersion laws possess one more curious peculiarity with respect to the processes "one into n" with the following resonance equations:

$$\begin{aligned} &k = k_1 + \cdots + k_n, \\ &\omega(k) = \omega(k_1) + \cdots + \omega(k_n). \end{aligned} \tag{3.56}$$

These equations are satisfied now for collinear vectors only, so for linear dispersion laws the resonance manifold $\Gamma^{1,n}$ coincides with the set $\Gamma_{\mathrm{M}}^{1,n}$ and has dimensionality less than in a general decaying case (degenerative included).

The notion of degenerative dispersion law may be generalized for the case when there are several types of waves. Thus, a set of three dispersion laws $\omega_i(k)$, $i = 1,2,3$ is degenerative with respect to the following process:

$$\begin{aligned} &k = k_1 + k_2, \\ &\omega_1(k) = \omega_2(k_1) + \omega_3(k_2), \end{aligned} \tag{3.57}$$

if there exist functions $f_i(k)$, $i=1,2,3$ satisfying the following equation on (3.57):

$$f_1(k)=f_2(k_1)+f_3(k_2). \tag{3.58}$$

At $d=2$ degenerative sets of three dispersion laws exist. They can be defined via parametrization:

$$\begin{aligned} &p=\xi_1-\xi_2, \quad p_1=\xi_1-\xi_3, \quad p_2=\xi_3-\xi_2,\\ &q=a_1(\xi_1)-a_2(\xi_2), \quad q_1=a_1(\xi_1)-a_3(\xi_3), \quad q_2=a_3(\xi_3)-a_2(\xi_2),\\ &\omega=b_1(\xi_1)-b_2(\xi_2), \quad \omega_1=b_1(\xi_1)-b_3(\xi_3), \quad \omega_2=b_3(\xi_3)-b_2(\xi_2). \end{aligned} \tag{3.59}$$

4. The examples of checking of the concrete Hamiltonian systems for integrability

The results obtained in sections 2 and 3 about the requirements which the amplitudes of the classical scattering matrix in the points of a general position of the resonance surfaces must satisfy, allow us to check the Hamiltonian systems for the existence of additional integrals with quadratic main parts. In many cases the proof of the nonexistence of such integrals is almost a trivial procedure. Let the dispersion law be decaying and nondegenerative in the vicinity of point k_1^0, k_2^0 being on surface (3.8). For the proof of nonexistence of the additional integral it is enough to check the following conditions for the first term of the expansion of Hamiltonian:

$$V_{k_1^0+k_2^0,\, k_1^0 k_2^0}\neq 0. \tag{4.1}$$

The situation is especially simplified if the problem considered is isotropic. In that case the dispersion law $\omega(k)$ is a function of the modulus of the wave vector only and, as a rule, satisfies the condition (3.35). On account of condition $k=k_1+k_2$ the matrix element $V_{kk_1k_2}$ is also a function of the moduli $|k|, |k_1|, |k_2|$. On the resonance surface (3.8) we get

$$V_{k_1^0+k_2^0,\, k_1^0,\, k_2^0}=V(\omega_1+\omega_2, \omega_1, \omega_2)=\varphi(\omega_1,\omega_2).$$

For the nonexistence of the additional integral with quadratic main part it is enough to satisfy the condition $\varphi(\omega_1,\omega_2)$ at $\omega_2 \ll \omega_1$. Let us consider, for example, the problem of capillary waves on the fluid surface. The dispersion law $\omega(k)=|k|^{3/2}$ satisfies the condition (3.35). The expression for $V_{kk_1k_2}$ is bulky enough (for example, it is presented in [37]), but at $\omega_2 \ll \omega_1$ it is simplified and has the form $\varphi(\omega_1,\omega_2)=\omega_2^{9/8}\omega_1^{9/4}$ at $\omega_2\ll\omega_1$, so $\varphi(\omega_1,\omega_2)\neq 0$. Thus, the additional integrals of motion are absent in the problem of capillary waves.

As the next simple example let us consider the nondimensional nonlinear Schrödinger equation

$$\mathrm{i}\psi_t+\Delta\psi=q|\psi|^2\psi. \tag{4.2}$$

It has nondegenerative (with respect to the process $2\leftrightarrow 2$) dispersion law k^2 (see theorem 3.6). The other way round the amplitude of such a process over all k-space is identically equal to a constant. Therefore, eq. (4.2) cannot possess any additional integral of motion with quadratic main part.

Let us now consider the equation arising in the theory of the gravitation waves on the surface of deep fluid [37]. In dimensionless coordinates it takes the form

$$i\psi_t - \psi_{xx} + \psi_{yy} = |\psi|^2\psi. \tag{4.3}$$

As it follows from theorem 3.6 (see also [31]) it also has a nondegenerative dispersion law $\omega(k) = -p^2 + q^2$, and the process $2 \leftrightarrow 2$ amplitude $W^{11-1-1}_{kk_1k_2k_3}$ is also constant in all k-space as in (4.1). Hence, (4.3) cannot possess additional integrals of motion with the quadratic main part. However the numerical computations, made in [38], did not show thermalization, and recurrency was observed for some class of initial conditions, computations were made at boundary conditions periodic in x and y. Let us also notice that if we consider real stationary solutions of (4.3), we get the following equation:

$$u_{xx} - u_{yy} + u^3 = 0.$$

This equation was studied numerically in [39] and in those numerical experiments the thermalization was not observed either. Thus, though eq. (4.3) does not possess any additional integral with quadratic main part, it does possess a number of properties characteristic of systems with a great number of integrals. We have not been able to account for the behaviour of eq. (4.3) so far.

Let us enumerate some results of checking of the concrete systems for the integrability.

Let us consider the following system of equations:

$$\begin{aligned} &i\psi_t + L_1\psi + u\psi = 0, \\ &L_2 u = L_3|\psi|^2, \end{aligned} \tag{4.4}$$

where $u(x, t)$ is a real function; $\psi(x, t)$ is a complex function, $x = (x_1, \ldots, x_d)$, $d = 2, 3$ and

$$L_n = \sum_{i,k=1}^{n} c_{ik}^n \frac{\partial^2}{\partial x_i \partial x_k}, \quad n = 1, 2, 3 \tag{4.5}$$

are differential operators of the second order with constant coefficients c_{ik}^n. Equations of such types occur naturally from multiscale expansions particularly in the theory of long waves on the fluid surface of a finite depth at $d = 2$ (the Davey–Stewartson equations [23]), while describing spectrally narrow packets of internal waves in an unbounded stratified fluid with $d = 3$ [40], and in the problem of interaction of high-frequency and low-frequency waves at $d = 2$ [41] as well. Eqs. (4.4) were studied for the existence of the additional integrals in ref. [31]. To study the system (4.4) it is convenient to rewrite it in explicitly Hamiltonian form

$$i\dot{\psi}_k + L_1(k)\psi_k + bi T_{kk_1k_2k_3}\overset{*}{\psi}_{k_1}\psi_{k_2}\psi_{k_3}\delta(k + k_1 - k_2 - k_3)\,dk_1\,dk_2\,dk_3, \tag{4.6}$$

where $L_i(k)$, $i = 1, 2, 3$ are symbols of the operators (4.5) and the vertex

$$2T_{kk_1k_2k_3} = \frac{L_3(k - k_2)}{L_2(k - k_2)} + \frac{L_3(k - k_3)}{L_2(k - k_3)} \tag{4.7}$$

is determined at $k+k_1=k_2+k_3$. The Hamiltonian of eq. (4.6) has the form

$$H=\int L_1(k)|\psi_k|^2\,\mathrm{d}k+\tfrac{1}{2}\int T_{kk_1k_2k_3}\overset{*}{\psi}_k\overset{*}{\psi}_{k_1}\psi_{k_2}\psi_{k_3}\delta(k+k_1-k_2-k_3)\,\mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3. \tag{4.8}$$

We may reduce the quadratic form $L_1(k)$ to the diagonal form (3.47) via the nondegenerate transformation. In this case we shall designate the new coefficients $\tilde{c}^2_{ik}$, $\tilde{c}^3_{ik}$ as α_{ik}, β_{ik} respectively. The dispersion law of eq. (4.5) is degenerative only in the case $\omega=k_1^2$, $k=(k_1,\dots,k_d)$. We shall not investigate this case because of the complicated expunions involved. However, eq. (4.6) with any other dispersion law will possess additional integrals only in the case if the vertex T turns into zero on the resonance surface (3.44). Let us ask if the vertex T turns into zero on the resonance surface (3.44). Observe that if the vertex T equals zero on (3.44) at some L_2, L_3 then on account of its structure it is equal to zero at the substitution L_2, L_3 by places too. At $d=2$ the existence of the additional integrals at $\sigma=1$ is possible in the following cases:

a)

$$\begin{gathered}\beta_{11}=\beta_{22}=\beta,\quad \alpha_{11}=\alpha_{22}=\alpha,\quad \alpha_{12}=0,\\ \mathrm{i}\psi_t+\Delta\psi+u\psi=0,\quad \alpha\Delta u=\left[\beta\left(\partial_{x_1}^2-\partial_{x_2}^2\right)+2\beta_{12}\partial_{x_1}\partial_{x_2}\right]|\psi|^2;\end{gathered} \tag{4.9}$$

b)

$$\begin{gathered}\beta_{12}=0,\quad \beta_{11}=\beta_{12}=\beta,\quad \alpha_{11}=-\alpha_{22}=\alpha,\\ \mathrm{i}\psi_t+\Delta\psi+u\psi=0,\quad \left[\alpha\left(\partial_{x_1}^2-\partial_{x_2}^2\right)+2\alpha_{12}\partial_{x_1}\partial_{x_2}\right]u=\beta\Delta|\psi|^2.\end{gathered} \tag{4.10}$$

Now let $\sigma=-1$. Then, either $\beta_{11}=\beta_{22}=\beta$, $\alpha_{11}=-\alpha_{22}=\alpha$, $\alpha_{12}=0$ and

$$\mathrm{i}\psi_t+\left(\partial_{x_1}^2-\partial_{x_2}^2\right)\psi+u\psi=0,\quad \alpha\left(\partial_{x_1}^2-\partial_{x_2}^2\right)u=\left[\beta\Delta+2\beta_{12}\partial_{x_1}\partial_{x_2}\right]|\psi|^2 \tag{4.11}$$

or $2\beta_{12}=\pm(\beta_1+\beta_2)$. In this case the following cases are possible:

a)

$$\begin{gathered}\left(N\partial_{x_1}\pm\partial_{x_2}\right)\left[\left(\partial_{x_1}\pm\partial_{x_2}\right)u+\left(\partial_{x_1}\mp\partial_{x_2}\right)|\psi|^2\right]=0,\quad \mathrm{i}\psi_t+\left(\partial_{x_1}^2-\partial_{x_2}^2\right)\psi+u\psi=0;\\ N\neq\pm1;\end{gathered} \tag{4.12}$$

b)

$$\mathrm{i}\psi_t+\left(\partial_{x_1}^2-\partial_{x_2}^2\right)\psi+u\psi=0,\quad \left[\alpha\Delta+2\alpha_{12}\partial_{x_1}\partial_{x_2}\right]u=\beta\left(\partial_{x_1}^2-\partial_{x_2}^2\right)|\psi|^2 \tag{4.13}$$

and the solution (4.11) as well. Let us note that the solutions (4.9) and (4.10), (4.11) and (4.13) are changed by the substitution of L_2 and L_3 by places, and (4.12) is reduced to the operators of the first order in ∂_{x_1}, ∂_{x_2}, and the corresponding system in variables $x_1-x_2=\xi$, $x_1+x_2=\eta$ takes the form

$$\mathrm{i}\psi_t+\psi_{\xi\eta}+u\psi=0,\quad u_\xi=|\psi|^2_\eta. \tag{4.14}$$

System (4.14) and the L–A pair for it are contained in [42]. Systems (4.4) with operators L_2 and L_3 of the second order, integrable via the inverse scattering transform, are also enumerated there. Also the systems (4.12) and (4.13) are among them. In the cases (4.9) and (4.11) the calculation of the second order vertex of the perturbation theory allows us to prove the nonexistence of the additional integrals in these cases. However, the corresponding calculations are very tedious and we do not reproduce them here (see [31] for a more detailed information). At $d=3$, an analogous but more tedious analysis shows that the system (4.4) does not possess additional integrals at any values of the coefficients α_{ik}, β_{ik}. We may apply the developed approach for the one-dimensional case as well. However, in this case one should bear in mind that in the one-dimensional case any dispersion law $\omega(k)$ may be considered as degenerative relative to the process "one into two". In the one-dimensional case, eq. (3.8) must describe a curve in the two-dimensional space with coordinates p_1, p_2. Irrespective of the form of the function $\omega(k)$ this curve includes two straight lines, $p_1=0$ and $p_2=0$. This is also referred to the process "two into two", described by eq. (3.44). In a general position these equations have solutions $k=k_2$, $k_1=k_3$ and $k=k_3$, $k_1=k_2$ not depending on the form of the function $\omega(k)$. So, to solve the problem of the existence of the highest integrals with the quadratic major part in the one-dimensional case for the systems with one type of waves it is necessary to refer to the processes of higher order. As in the previous case we may use the first processes "one into two" and "two into two" for the systems with several types of waves. Here we present the results for the system of two joint nonlinear Schrödinger equations [29], the system of equations describing the interaction of the long acoustic and short waves [32]. We shall also give the results of ref. [34], where in comparison with [32] eigennonlinearity and dispersion of long waves were taken into account. The system of joint nonlinear Schrödinger equations occurs in nonlinear optics [43] and has the form

$$\begin{aligned} \mathrm{i}\psi_{1t} &= c_1\psi_{1xx} + 2\alpha|\psi_1|^2\psi_1 + 2\beta|\psi_2|^2\psi_1, \\ \mathrm{i}\psi_{2t} &= c_2\psi_{2xx} + 2\gamma|\psi_2|^2\psi_2 + 2\beta|\psi_1|^2\psi_2. \end{aligned} \tag{4.15}$$

System (4.15) is Hamiltonian. Its Hamiltonian has the form

$$H = \int \left\{ c_1|\psi_{1x}|^2 + c_2|\psi_{2x}|^2 + \alpha|\psi_1|^4 + 2\beta|\psi_1|^2|\psi_2|^2 + \gamma|\psi_2|^4 \right\} \mathrm{d}x. \tag{4.16}$$

At $c_1=c_2$, $\alpha=\beta=\gamma$, the integrability of system (4.15) via the inverse scattering transform was shown in [44]. In order to study the system (4.15) in a general case, in accordance with the results of sections 2 and 3, from the first it is necessary to study a set of dispersion laws:

$$\omega_1(k) = c_1k^2, \quad \omega_2(k) = c_2k^2, \tag{4.17}$$

for degenerability relative to the first nonlinear process, which resonance manifold is not trivial. In our case it is the process

$$\begin{aligned} k + k_1 &= k_2 + k_3, \\ \omega_1(k) + \omega_2(k_1) &= \omega_1(k_2) + \omega_2(k_3). \end{aligned} \tag{4.18}$$

As it was mentioned above we are interested in the cases $\rho = c_1/c_2 \neq \pm 1$. The manifold Γ described by eqs. (4.18) is easily parametrized by the following formulae:

$$k = \frac{\rho - 1}{2} k_1 + \frac{\rho + 1}{2} k_2, \qquad k_3 = \frac{\rho + 1}{2} k_1 + \frac{\rho - 1}{2} k_2. \tag{4.19}$$

Substituting (4.19) into the equation

$$f_1(k) + f_2(k_1) = f_1(k_2) + f_2(k_3), \tag{4.20}$$

differentiating twice in k_1 and supposing $k_1 = k_2 = \xi/\rho$, we get

$$(\rho^2 - 1)(\rho - 1) f_1'''(\xi) = (\rho^2 - 1)(\rho + 1) f_1'''(\xi). \tag{4.21}$$

At $\rho \neq \pm 1$, it follows from (4.21) that $f_1'''(\xi) = 0$,

$$f_1(\xi) = A\xi^2 + B\xi + C, \tag{4.22}$$

Substituting (4.22) into (4.20), we are convinced that

$$f_2(\xi) = \rho A \xi^2 + B\xi + C,$$

which means that at $\rho \neq \pm 1$, the dispersion laws (4.18) are nondegenerative relative to the process (4.18).

As the amplitude of this process is constant in the whole k-space and equal to $2\beta \neq 0$, the system (4.14) does not possess additional integrals at $\rho \neq \pm 1$. At $\rho = \pm 1$ it is necessary to calculate the second order amplitude corresponding to the next nonlinear process. For example, we may calculate the amplitude corresponding to the following process:

$$\omega_1(k) + \omega_1(k_1) + \omega_2(k_2) = \omega_2(k_3) + \omega_1(k_4) + \omega_1(k_5), \qquad k + k_1 + k_2 = k_3 + k_4 + k_5. \tag{4.23}$$

The corresponding manifold in the space $(k_1, \dots, k_5, k)$ being quadratic, allows rational parametrization (see [29]), using which it is easy to show that the quadratic dispersion laws are nondegenerative with respect to the process (4.23).

The amplitude corresponding to the process (4.23) is rather bulky and we shall not reproduce it here. It is important that the amplitude (4.23) turns into zero only in two cases:

$$\rho = 1, \quad \alpha = \beta \quad \text{and} \quad \rho = -1, \quad \alpha = -\beta.$$

Similarly, we may obtain that at $\rho = 1$, $\beta = \gamma$, and at $\rho = -1$, $\beta = -\gamma$. Thus, besides the case $\rho = 1$, "a candidate" for integrability is the case

$$\rho = -1, \quad \alpha = -\beta = \gamma. \tag{4.24}$$

The system with coefficients (4.24) is really integrable via the inverse scattering transform. Indeed, it was shown in [9] that we may apply the inverse scattering transform for the system

$$\begin{aligned} &\mathrm{i}\Psi_t = \Psi_{xx} + \Psi\chi\Psi, \\ &-\mathrm{i}\chi_t = \chi_{xx} + \chi\Psi\chi, \end{aligned} \tag{4.25}$$

where Ψ and χ are matrices. Let

$$\Psi = (\psi_1, \dots, \psi_n), \quad \chi = \begin{pmatrix} x_1 \\ \vdots \\ \chi v_n \end{pmatrix}$$

and consider the reduction $\chi = A\Psi^+$, where A is a Hermitian matrix. Then (4.26) is equivalent to the system

$$\mathrm{i}\psi_{m_t} = \psi_{m_{xx}} + u\psi_m, \quad m = 1, \dots, n, \tag{4.26}$$

where $u = \Psi A \Psi^+$ is a real function. Matrix A may be reduced to the diagonal form $A \to a_i \delta_{ik}$ via the unitary transformation. So at $n = 2$ after scale transformations we obtain that besides 'vector' case [44], system (4.15) with coefficients (4.24) is also integrable. This case of exact solution of system (4.16) was discovered in ref. [46] independent of ref. [29].

Analogously in ref. [32] the nonintegrability of equations of the resonance interaction of long acoustic and short waves [45],

$$\begin{aligned} &\mathrm{i}\psi_t + \psi_{xx} - u\psi = 0, \\ &u_{tt} - c^2 u_{xx} = 2|\psi|^2_{xx}, \end{aligned} \tag{4.27}$$

and also of the system [34]

$$\begin{aligned} &u_t + \left(u^2 + \alpha|\psi|^2 + u_{xx}\right)_x = 0, \\ &\mathrm{i}\psi_t + \psi_{xx} + u\psi = 0 \end{aligned} \tag{4.28}$$

has been proved.

5. About the singular elements of the scattering matrix

Let us consider the singular elements of the classical scattering matrix. We consider the process "n in m", but we shall use some other notations for the wave vectors. We shall designate the nonlinear part of the amplitude of this process via $S^{n,m}_{k_1 \cdots k_n, \tilde{k}_1 \cdots \tilde{k}_m}$.

This value is nonvanishing on the resonance manifold, which satisfies the following equations:

$$\begin{aligned} &k_1 + \cdots + k_n = \tilde{k}_1 + \cdots + \tilde{k}_m, \\ &\omega_{k_1} + \cdots + \omega_{k_n} = \omega_{\tilde{k}_1} + \cdots + \omega_{\tilde{k}_m}. \end{aligned} \tag{5.1}$$

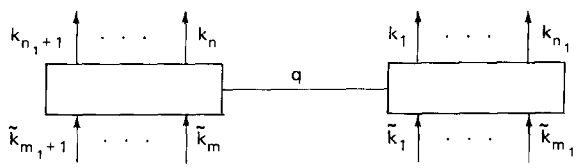
Let us consider some diagram describing the process (5.1), the Green function corresponding to some internal diagram line with the wave vector q is substituted for the δ-function. Let the vector q be directed from the "root" of the diagram and to the right of it (i.e. further from the root) there are external lines with wave vectors $k_1,\dots,k_{n_1},\tilde{k}_1,\dots,\tilde{k}_{m_1}$, $n_1<n$, $m_1<m$. Now the following equations are added to eqs. (5.1):

$$\begin{aligned} &k_1+\cdots+k_{n_1}=\tilde{k}_1+\cdots+\tilde{k}_{m_1}+q,\\ &\omega(k_1)+\cdots+\omega(k_{n_1})=\omega(\tilde{k}_1)+\cdots+\omega(\tilde{k}_{m_1})+\omega(q). \end{aligned} \tag{5.2}$$

Moreover, the following relations obviously take place:

$$\begin{aligned} &k_{n_1+1}+\cdots+k_n=\tilde{k}_{m_1+1}+\cdots+\tilde{k}_m-q,\\ &\omega(k_{n_1+1})+\cdots+\omega(k_n)=\omega(\tilde{k}_{m_1+1})+\cdots+\omega(\tilde{k}_m)-\omega(q). \end{aligned} \tag{5.3}$$

Let us mark via $\tilde{S}^{n,m}_{k_1\cdots k_n,\tilde{k}_1\cdots\tilde{k}_m}$ the singular part of the scattering amplitude "n in m", corresponding to eqs. (5.1)–(5.3). We obtain the expression for $\tilde{S}^{n,m}$, as a result of summing of all the diagrams of the form



so that the representation takes place

$$\tilde{S}^{n,m}_{k_1\cdots k_n,\tilde{k}_1\cdots\tilde{k}_m}=\pi\mathrm{i}\int S^{n_1,m_1+1}_{k_1\cdots k_{n_1},\tilde{k}_1\cdots\tilde{k}_{m_1}q}S^{n-n_1+1,m-m_1}_{k_{n_1+1},\dots,k_n,q;\tilde{k}_{m_1+1},\dots,\tilde{k}_m}\,\mathrm{d}q. \tag{5.4}$$

This formula shows that the singular amplitude $\tilde{S}^{n,m}$ is factorized through the composition of the two nonsingular amplitudes of lower order. It is clear that an analogous statement holds for the amplitude of any degree of singularity, when there are several additional equations of the form (5.2). All of them are factorized in the form of the composition of the finite number of the nonsingular amplitudes of lower orders. In particular, the maximum singular elements of the scattering matrix, defined by the diagrams, where all Green's functions of the internal lines are substituted for the δ-functions, are factorized in the form of the composition of the simplest scattering amplitudes "one into two". These facts have a simple physical meaning. The substitution of one of the internal Green's functions for the δ-function means, that the corresponding wave is the eigenoscillation of the system (a "real particle"), and the process with such a wave occurs stage by stage, combined out of the processes of the lowest order.

Now let the considered dynamical system possess the additional motion integral and let the dispersion law be nondegenerative relative to all nonlinear processes. Then all nonsingular elements of the scattering matrix on the resonance surfaces are vanishing. Singular amplitudes are vanishing too, except 'billiards type' scattering [33], at smooth c_k^- the classical scattering matrix is trivial and the asymptotic states coincide, i.e.

$$c_k^+=c_k^-. \tag{5.5}$$

In particular, this fact holds for the Kadomtsev–Petviashvilli equation KP-II. This circumstance was first mentioned in ref. [30].

We have seen in section 3 that in the two-dimensional case the situation, when the dispersion law is degenerative relative to the lowest-order process "one into two" and nondegenerative relative to all higher-order processes, is typical. All degenerative dispersion laws constructed in section 3 possess this property. In such a situation the classical scattering matrix S is nontrivial, but only its most singular part is nonvanishing, factorizing into the composition of the three wave processes. So the things are with the KP-I equation.

It is very important that in this case one can find the scattering matrix in the explicit form in some sense. Let us note that for the most singular part of the S-matrix one can cancel all inner Green's functions but replace in every vertex

$$V^{-s_p s_q s_r}_{k_p k_q k_r}\delta(-s_p k_p + s_q k_q + s_r k_r) \to \pi \mathrm{i} V^{-s_p s_q s_r}_{k_p k_q k_r}\delta(-s_p k_p + s_q k_q + s_r k_r)\delta(-s_p \omega_{k_p} + s_q \omega_{k_q} + s_r \omega_{k_r}). \tag{5.6}$$

This modified vertex will be denoted symbolically as $\hat{V}$.

Now we must remember that the whole set of diagrams has the factor $2\pi\mathrm{i}$. So one can write symbolically

$$c^+ = c^- + 2\{\hat{V}[c^- c^-] + \cdots\}. \tag{5.7}$$

The estimation in the brackets { } is the whole set of diagrams. Formula (5.7) can be rewritten in the form

$$\frac{c^+ + c^-}{2} = c^- + \hat{V}[c^-, c^-] + \cdots. \tag{5.8}$$

The set in (5.8) is the result of solution of an integral equation

$$c^- = \frac{c^+ + c^-}{2} - \hat{V}\left[\frac{c^+ + c^-}{2}, \frac{c^+ + c^-}{2}\right]. \tag{5.9}$$

Finally, we have

$$\frac{c^+ - c^-}{2} = \hat{V}\left[\frac{c^+ + c^-}{2}, \frac{c^+ + c^-}{2}\right]$$

or more explicitly

$$\begin{aligned} c_k^{+s} - c_k^{-s} = \frac{\pi \mathrm{i}}{2} \int \sum_{s_1 s_2} & V^{-s s_1 s_2}_{k k_1 k_2} \delta(sk - s_1 k_1 - s_2 k_2) \\ & \times \delta(s\omega_k - s_1\omega_{k_1} - s_2\omega_{k_2})\left(c_{k_1}^{+s_1} + c_{k_1}^{-s_1}\right)\left(c_{k_2}^{+s_2} + c_{k_2}^{-s_2}\right) \mathrm{d}k_1\, \mathrm{d}k_2. \end{aligned} \tag{5.10}$$

Formula (5.10) gives a direct connection between asymptotic states in the case of degenerative dispersion law. It was first obtained in ref. [50].

An equation similar to (5.10) occurs in the one-dimensional case if it is one type of waves. In the one-dimensional case any dispersion law is degenerative to the process of two particles scattering. For simplicity consider the Hamiltonian (2.8a). Then we have

$$\frac{c_k^+ + c_k^-}{2} = \pi \int T_{kk_1k_2k_3} \delta(k + k_1 - k_2 - k_3) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times \left(\frac{\overset{*}{c}{}_{k_1}^{+} + \overset{*}{c}{}_{k_1}^{-}}{2} \right) \left(\frac{c_{k_2}^+ + c_{k_2}^-}{2} \right) \left(\frac{c_{k_3}^+ + c_{k_3}^-}{2} \right) dk_1 \, dk_2 \, dk_3. \tag{5.11}$$

It follows from (5.11) that the squared modulus of the classical S-matrix is equal to unity:

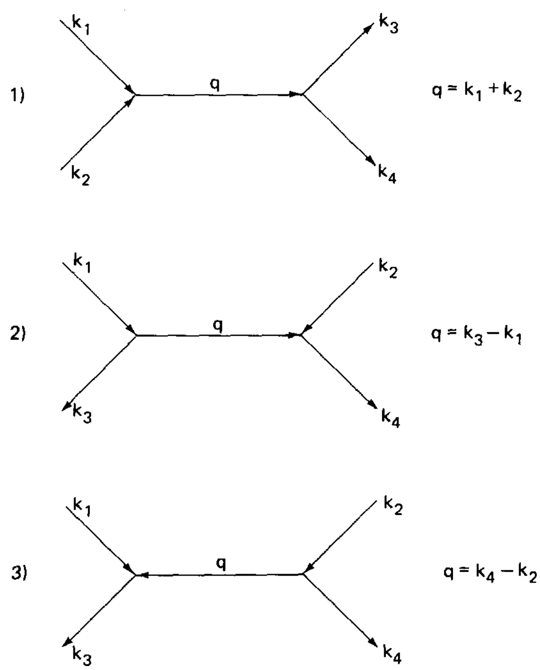
$$|c_k^+|^2 = |c_k^-|^2,$$

but, in general, $\arg c_k^+ \neq \arg c_k^-$. Really, it is well-known that in such one-dimensional systems interaction is reduced to phase shift only.

Now let us return to the two-dimensional case with the decaying degenerative dispersion law and consider the amplitude of the "two into two" process with the resonant conditions

$$\begin{aligned} k_1 + k_2 &= k_3 + k_4, \\ \omega_{k_1} + \omega_{k_2} &= \omega_{k_3} + \omega_{k_4}. \end{aligned} \tag{5.12}$$

This amplitude is described by the following three diagrams:



As we have stated above, the nonsingular part of the amplitude localized on the whole manifold (5.9) must be zero identically. On the other hand, this amplitude turns into infinity near resonant manifolds corresponding to interaction via real waves. (The singular part of the amplitude is localized on these manifolds.) These manifolds are different for the above three diagrams. They are defined by the following formulae:

$$\omega_{k_1+k_2} = \omega_{k_1} + \omega_{k_2} = \omega_{k_3} + \omega_{k_4} \tag{5.13}$$

for diagram 1,

$$\omega_{k_1-k_3} = \omega_{k_1} - \omega_{k_3} = \omega_{k_4} - \omega_{k_2} \tag{5.14}$$

for diagram 2 and

$$\omega_{k_3-k_1} = \omega_{k_3} - \omega_{k_1} = \omega_{k_2} - \omega_{k_4} \tag{5.15}$$

for diagram 3. On account of turning the amplitude of the process (5.9) into zero the singularities localized near manifolds (5.13)–(5.15) must cancel each other. For this cancellation to be true these manifolds must coincide at least partially.

Resonant surface "one into two" for the KP-I equation consists of two connected pieces (see formulae (3.15) and (3.16)). A simple analysis shows that each of the two pieces described by any of the equations (5.13)–(5.15) coincides with the piece described by some other of these three equations. This results in the number of connected manifolds, defined by (5.13)–(5.15), to be equal to three and not six. The statement about the pair compartibility of eqs. (5.13)–(5.15) is a general one for the degenerative dispersion laws and could be used for their enumeration. It is worth noticing that the coincidence of manifolds (5.13)–(5.15) (in the above-mentioned sense) is only necessary but not a sufficient condition for the singularities in (5.9) to cancel each other. Rather strong conditions imposed on the coefficient functions of the three-wave Hamiltonian (2.8) should be satisfied. We check these conditions for the KP-I equation. It is also worth noticing that the checking of cancellation of singularities is a useful and simple way for practical analysis of the existence of additional motion invariants for the concrete systems.

6. About the integrals of motion

One of the important results of the present paper is the statement that the existence of the one additional integral of system (2.2) implies the existence of an infinite set of motion integrals. Let us give the proof of this fact and search for the integral of motion in the form of a formal integropower series,

$$G = \int g_k |a_k|^2 \,\mathrm{d}k + \sum_q \sum_{s,\dots,s_q} \int G_{k\,\dots\,k_q}^{s\,\dots\,s_q} a_k^s \cdots a_{k_q}^{s_q} \delta(P_q)\,\mathrm{d}k\,\mathrm{d}k_1 \cdots \mathrm{d}k_q. \tag{6.1}$$

Here

$$P_q = sk + s_1 k_1 + \cdots + s_q k_q$$

and g_k is some function of the wave numbers. Substituting (6.2) into (2.2), we shall be convinced that the functions $G^{s\ \cdots\ s_q}_{k\ \cdots\ k_q}$ are expressed from the recurrent formulae

$$G^{ss_1s_2}_{kk_1k_2} = \frac{sg_k + s_1 g_{k_1} + s_2 g_{k_2}}{\Delta^{ss_1s_2}_{kk_1k_2}} V^{ss_1s_2}_{kk_1k_2}, \tag{6.2}$$

$$G^{s\ \cdots\ s_q}_{k\ \cdots\ k_q} = \frac{F^{s\ \cdots\ s_q}_{k\ \cdots\ k_q}}{\Delta^{s\ \cdots\ s_q}_{k\ \cdots\ k_q}}. \tag{6.3}$$

In these formulae,

$$\Delta^{s\ \cdots\ s_q}_{k\ \cdots\ k_q} = s\omega_k + s_1\omega_{k_1} + \cdots + s_q\omega_{k_q}$$

and the function $F^{s\ \cdots\ s_q}_{k\ \cdots\ k_q}$ is linearly expressed via $G^{s\ \cdots\ s_{q-1}}_{k\ \cdots\ k_{q-1}}$. It is not necessary for us to write out this dependence.

It follows from formulae (6.2) and (6.3) that the coefficient functions in the integrals possess singularities on all-possible resonance manifolds of the form

$$\Delta_n = \Delta^{s\ \cdots\ s_n}_{k\ \cdots\ k_n} = 0, \quad P_n = 0. \tag{6.4}$$

Further one might think in the following way. Let the wave field $a(r)$ in a physical space be a rapidly decreasing function. Then its Fourier transform, the field a_k, is a smooth function. That makes it possible to make the regularization in the integrals in the definition of the expression (6.1). That can be achieved in a not unique way. For example, in all denominators one can perform the substitution

$$\Delta^{s\ \cdots\ s_q}_{k\ \cdots\ k_q} \to \Delta^{+\, s\ \cdots\ s_q}_{\ \ k\ \cdots\ k_q} = \Delta^{s\ \cdots\ s_q}_{k\ \cdots\ k_q} + \mathrm{i}0 \tag{6.5}$$

or the substitution

$$\Delta^{s\ \cdots\ s_q}_{k\ \cdots\ k_q} \to \Delta^{-\, s\ \cdots\ s_q}_{\ \ k\ \cdots\ k_q} = \Delta^{s\ \cdots\ s_q}_{k\ \cdots\ k_q} - \mathrm{i}0. \tag{6.6}$$

Generally speaking, in this case we obtain different integrals of motion – let us designate them as $G^{\pm}$. Any linear combination of them may be the integral of motion; particularly, the difference $(1/2\pi\mathrm{i})G^0 = G^+ - G^-$. The integral G^0 not does not have a quadratic part – its expansion in powers of a^s_k starts from the term

$$\sum_{ss_1s_2} \int (sg_k + s_1g_{k_1} + s_2g_{k_2})\,\delta(s\omega_k + s_1\omega_{k_1} + s_2\omega_{k_2}) \\ \times \delta(sk + s_1k_1 + s_2k_2) V^{ss_1s_2}_{kk_1k_2} a^s_k a^{s_1}_{k_1} a^{s_2}_{k_2}\, \mathrm{d}k\, \mathrm{d}k_1\, \mathrm{d}k_2. \tag{6.7}$$

The integral G^0 can be called as an essentially nonlinear one. It is one of the large number of such integrals. The linear equation

$$\dot{a}^s_k + \mathrm{i}s\omega_k a^s_k = 0$$

allows an essentially nonlinear integral of the form

$$I=\int \Phi^{s \cdots s_q}_{k \cdots k_q}\delta\left(\Delta^{s \cdots s_q}_{k \cdots k_q}\right)\delta(P_q)a^s_k \cdots a^{s_q}_{k_q}\,\mathrm{d}k \cdots \mathrm{d}k_q. \tag{6.8}$$

Here q is an arbitrary integer, $\Phi^{s \cdots s_q}_{k \cdots k_q}$ is an arbitrary function. In the nonlinear system (2.2) one can search for the integral in the form of an integropower sequence in a^s_k, the first term of which is the expression (6.8). In this case again there occurs the regularization problem of the denominators of the form $\Delta^{s \cdots s_r}_{k \cdots k_r}$, $r>q$, which again cannot be done uniquely. The different essentially nonlinear integrals obtained will differ from the essentially nonlinear integrals of higher orders. One can attach a simple physical sense to the integrals $G^{\pm}$ occurring as a result of the regularizations (6.5) and (6.6). It is easy to see that

$$G^{\pm}=\int g_k|a^{\pm}_k|^2\,\mathrm{d}k. \tag{6.9}$$

Here $a^{\pm}_k$ are the $t \to \pm\infty$ asymptotic states of the wave field. Formulae (6.9) show that an arbitrary system (2.2) in the rapidly decreasing case is completely integrable. Really, as it is known, the change $a^s_k(t)$ in time is a canonical transformation, so the variables $a^{s^{\pm}}_k(t)=c^{\pm}_k \exp\{-is\omega_k t\}$ are canonical ones. It is evident now that the variables

$$I^{\pm}_k=|a^{\pm}_k|^2 \quad \text{and} \quad \varphi^{\pm}_k=\arg a^{\pm}_k$$

are the action-angle variables for the system (2.2), irrespective of the form of its Hamiltonian. This rather impressive statement is sufficiently based on a rapid decrease of the function $a(r)$ and, respectively, on the smoothness of the function $a(k)$. In the periodic case, when the function $a(k)$ represents a set of δ-functions,

$$a(k)=\sum a_n\delta(k-nk_0), \tag{6.10}$$

where k_0 is the vector of the reverse lattice, n is a multiindex, integrals (6.1) in a general position loose sense (become infinite) and, as a rule, the integrability vanishes. In the periodic case only the integrals do preserve sense, the coefficient functions of which remain finite on all resonance manifolds, i.e. where a reduction of singularities occurs.

To observe the singularities, let us introduce the operators $R^{\pm}$, reverse with respect to the operator of the transition (2,16), taken for simplicity at $t<0$:

$$a^{\pm}_k=R^{\pm}_{\varepsilon}[a_k], \tag{6.11}$$

$$a^{\pm^s}_k=a^s_k+\sum_q\sum_{s \cdots s_q}\int R^{\pm - ss_1 \cdots s_q}_{\varepsilon\; kk_1 \cdots k_q}a^{s_1}_{k_1} \cdots a^{s_q}_{k_q}\delta(P_q)\,\mathrm{d}k_1 \cdots \mathrm{d}k_q.$$

The coefficients $R^{\pm s \cdots s_q}_{\varepsilon\; k \cdots k_q}$ at $\varepsilon \to 0$ do not depend on time. They have singularities on all possible resonance surfaces $\Delta_q=0$. Let us put

$$\lim_{\varepsilon \to 0} R^{\pm ss_1 \cdots s_q}_{\varepsilon\; kk_1 \cdots k_q}=\frac{\tilde{R}^{\pm ss_1 \cdots s_q}_{kk_1 \cdots k_q}}{\Delta_q \pm \mathrm{i}0}. \tag{6.12}$$

The expression $\tilde{R}^{\pm s s_1 \cdots s_q}_{k k_1 \cdots k_q}$ is regular on the resonance manifold $\Delta_q = 0, P_q = 0$, but it can possess singularities on various "junior" resonance manifolds.

Let us consider the operator $R^{\pm}$ and let $t \to -\infty$ in (6.11). In this case $a^s_k \to a^{-s}_k$, and the operator R^+ is to be transformed into a classical scattering matrix. That means that on the resonance surface $\Delta_q = 0$, $P_q = 0$ the numerator in (6.12) coincides with the corresponding element in the scattering matrix:

$$\tilde{R}^{+s \cdots s_q}_{k \cdots k_q} = S^{s \cdots s_q}_{k \cdots k_q}. \tag{6.13}$$

Now let us represent the integral of motion G^+ in the form

$$G^+ = \int g_k a_k \mathring{a}_k \,\mathrm{d}k + \int g_k \mathring{a}_k (a^+_k - a_k) \,\mathrm{d}k + \int g_k a_k (\mathring{a}^+_k - \mathring{a}_k) \,\mathrm{d}k + \int g_k \mathring{a}_k (a^+_k - a_k) \,\mathrm{d}k$$
$$+ \int g_k (a^+_k - a_k)(\mathring{a}^+_k - \mathring{a}_k) \,\mathrm{d}k, \tag{6.14}$$

and substitute (6.11) into (6.14). Let us collect in (6.14) the terms having the singularity on the whole resonance manifold (6.14) and having a complete power q. Such terms are only contained in the second and third terms in (6.14) and after symmetrization are reduced to the form

$$\frac{1}{N} \int \frac{L^{s \cdots s_q}_{k \cdots k_q}}{\Delta^{s \cdots s_q}_{k \cdots k_q} + \mathrm{i}0} \tilde{R}^{+s \cdots s_q}_{k \cdots k_q} a^s_k \cdots a^{s_q}_{k_q} \delta(P_q) \,\mathrm{d}k \,\mathrm{d}k_1 \cdots \mathrm{d}k_q, \tag{6.15}$$

N is some integer.

$$L^{s \cdots s_q}_{k \cdots k_q} = s g_k + s_1 g_{k_1} + \cdots + s_q g_{k_q}. \tag{6.16}$$

Comparing formula (6.14) with (6.3) we are convinced that $F^{s \cdots s_q}_{k \cdots k_q}$ can be represented as follows:

$$F^{s \cdots s_q}_{k \cdots k_q} = \frac{1}{N} L^{s \cdots s_q}_{k \cdots k_q} \tilde{R}^{s s_1 \cdots s_q}_{k k_1 \cdots k_q} + A^{s \cdots s_q}_{k \cdots k_q} \Delta^{s \cdots s_q}_{k \cdots k_q}, \tag{6.17}$$

$s \cdots s_q$ being regular on $\Delta_q = 0$, though probably it has singularities on the "junior" resonance surfaces.

Let the dispersion law $\omega(k)$ be nondegenerative and system (2.2) have an additional integral of motion with continuous coefficients.

As we have already seen, from this follows the triviality of the scattering matrix and the coincidence of asymptotic states $a^{\pm}_k$. Now on the resonance manifold $\Delta_q = 0$, $P_q = 0$ the matrix element $\tilde{R}^{s \cdots s_q}_{k \cdots k_q} = 0$. This means that on the resonance surface $\Delta_q = 0$, $P_q = 0$ the singularity in the motion integral is cancelled. One can see directly from the formula (6.2) that the singularity is cancelled in the junior term of the expansion (6.1) as well. Now, applying induction, we obtain that generally all the singularities are cancelled. Thus, in the considered case one can use an arbitrary function g_k in order to construct the motion invariants of the system (2.2). Roughly speaking, in this case there are as many integrals with continuous coefficients and having a quadratic part as we have them in the linear problem. All these integrals are conserved in the periodic case as well, i.e. in this case the periodic system (2.2) is quite integrable*. Particularly the periodic equation KP-II is integrable. I.M. Krichever** has recently come to this conclusion on the basis of algebrogeometric approach developed by him. We should stress, however, that our results have been obtained on the level of formal series and that convergence of these formal series has not yet been proved.

Now let the dispersion law be degenerative. We restrict ourselves to the consideration of the case when it has the form (3.10) at $d=2$. Now the scattering matrix is different from unity, $S_{k\ldots k_q}^{s\ldots s_q}\neq 0$. However, the nonvanishing scattering matrix is only concentrated on the minimal manifold $\Gamma_{\mathrm{M}}^{n;m}$, when all the scattering occurs with the participation of real intermediate waves only. Now in expression (6.17) $\tilde{R}_{k\ldots k_q}^{s\ldots s_q}\neq 0$, and generally speaking, the integral of the form (6.1) is singular. The only way out of this situation is to require the vanishing of the expression $L_{k\ldots k_q}^{s\ldots s_q}$. It is possible to do that on the manifold $\Gamma_{\mathrm{M}}^{n;m}$, by requiring $g(k)=f(k)$, i.e. the function itself should represent the degenerative dispersion law, permitting the parametrization

$$p=\xi_1-\xi_2;\quad q=a(\xi_1)-a(\xi_2),$$
$$\omega=b(\xi_1)-b(\xi_2),\quad g=c(\xi_1)-c(\xi_2).$$

Here the function $c(\xi)$ is arbitrary. Thus, in the given case, system (2.2) also has an infinite set of integrals of motion with continuous coefficients, but this set is sufficiently narrower than in the previous case; instead of the arbitrary function of two variables at our disposal there is only an arbitrary function of one variable. This is not quite enough for the integrability in the periodic case. So the systems with a degenerative dispersion law under periodic boundary conditions are nonintegrable, though they might possess an infinite set of integrals of motion. It is this very fact that allows one to apply the kinetic equation with the nonvanishing collision term (see [36]) for the statistical description of such systems.

7. Conclusion

Let us summarize. We have tried to show in this paper that the analysis method of the integrability of Hamiltonian dynamic systems, based on the study of sequences of the perturbation theory, proves to be rather effective relative to the nonlinear wave systems. Let us notice that earlier in ref. [49] this method allowed us to prove the nonexistence of a strong recursion operator. One can hope that the method has not exhausted its possibilities yet. A priori, for example, one cannot exclude the probability that some systems of the form (2.2) can have additional integrals, depending evidently on the coordinates and time or just essentially nonlinear integrals only.

If we speak about the systems considered in our paper, i.e. having an additional integral with the quadratic main part, then there remain a number of unsolved problems. First of all, we would like to exhaust the problem of the description of degenerative dispersion laws. In the case when the dispersion law is nondegenerative, it is not quite clear how, in the situation with the periodic boundary conditions, one should construct action-angle variables and what dependence of the Hamiltonian on the action variables

*In this case there exist nonsingular canonical transformations $a_k^s=\mathscr{A}_k^s[\alpha_k^s]$, reducing the system (2.2) to the normal form $\mathrm{i}\dot{\alpha}_k=\Omega_k[|\alpha|^2]\alpha_k$, where $\Omega_k[|\alpha|^2]=\omega_k+\sum_{q=1}^{\infty}\int\Omega_{kk_1\ldots k_q}|\alpha_{k_1}|^2\cdots|\alpha_{k_q}|^2\,\mathrm{d}k_1\cdots\mathrm{d}k_q$ is renormalized frequency. This renormalization is obtained from the requirement of regularity of canonical transformation $\mathscr{A}_k^s$ on all trivial scattering manifolds $\Gamma_{\mathrm{M}}^{n;n}$. (E.I. Schulman, to be published in Teor. Mat. Fiz.)

**Private communication.

for these systems is possible. The initial calculations made in this direction show that though at $s = 1$ the infinities in formula (6.11) vanish, the kernels $R^{\pm\, s s_1 \cdots s_q}_{\;\; k k_1 \cdots k_q}$ conserve on the surfaces of a "trivial scattering"

$$k_1 + \cdots + k_n = \tilde{k}_1 + \cdots + \tilde{k}_n$$

(vectors $\tilde{k}_i$ differ from vectors k_i by permutation only) rather complicated singularities of the type of a jump of derivatives. One should take into account these singularities in the periodic case. However, the case of a degenerative dispersion law is the most interesting one. Despite the fact that the inverse scattering transform can be applied to such systems, such as the KP-I equation, the existing analytical methods do not allow us to construct the solutions of these systems which are not rapidly decreasing and are in a general position. In contrast to the soliton and finite band solutions, which now in the space of all solutions are not dense, such solutions of a general position possess the stochastic property and must be described stochastically. The study of such solutions, which are not necessarily weakly linear, is rather important from the point of view of understanding the turbulence nature in the dynamic systems. A weakly linear solution of a general position in the case of degenerative dispersion laws can be studied via the kinetic equations for waves, the study of which started in ref. [26]. This represents a large field of special interest.

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