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ON ADMISSIBILITY AND OPTIMALITY OF TREATMENT–CONTROL DESIGNS

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The relationship between admissible incomplete block designs for confidence intervals with maximal coverage probability for treatment–control contrasts and optimal designs for estimation is investigated. For certain types of designs, admissible designs are shown to be precisely those with the number of replications of the control less than or equal to that of an optimal design. Moreover, admissible designs are the Bayes optimal designs for a class of priors.

1. Introduction. There are two widely studied approaches to arriving at optimal block designs for comparing a set of test treatments with a control. One approach, introduced by Bechhofer and Tamhane (1981), is to find a design that maximizes the coverage probability of simultaneous confidence intervals for the control–test treatment contrasts. The other is a Kiefer-style approach of identifying a design that gives the best estimators for these contrasts; the most widely accepted criterion being A-optimality which minimizes the sum of the variances of the estimators.

While the relationship between the two approaches has been discussed by several authors [cf. Hedayat, Jacroux and Majumdar (1988), and the discussions], including Spurrier (1988), who felt that the optimal designs from one approach would perform well under the other, we know of no attempt to investigate rigorously the connection between the two. In this paper we attempt to explore some aspects of the relationship. We show that for a certain type of design that has been studied in the literature, the admissible designs for the confidence interval approach are precisely those that enjoy a certain relation with an optimal design for estimation. Moreover, the set of admissible designs is the set of Bayes optimal designs (for estimation) under squared error loss, for a certain class of priors.

We assume that v test treatments labelled $1, \dots, v$ have to be compared with the control labelled 0 in b blocks of size k $(2 \le k \le v)$ each. The observation y_{ijp}, in plot p of block j, that receives treatment i will be assumed to be normal and follow an additive, homoscedastic linear model:

$$(1.1)\quad E(y_{ijp}) = \mu + \alpha_i + \beta_j, \qquad V(y_{ijp}) = \sigma^2, \qquad y_{ijp}\text{'s uncorrelated,}$$

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where α_i and β_j are the treatment and block effects, respectively. Let us use $\mathscr{D}(v+1, b, k)$ to denote the set of all connected designs. For $d \in \mathscr{D}(v+1, b, k)$, let n_{dij} be the number of occurrences of treatment i $(0 \le i \le v)$ in block j $(1 \le j \le b)$ and $r_{di} = \sum_{j=1}^{b} n_{dij}$.

A design d is called a balanced treatment incomplete block (BTIB) design [Bechhofer and Tamhane (1981), henceforth abbreviated as BT] if

$$\sum_{j=1}^{b} n_{d0j} n_{dij} = \lambda_{d0}, \qquad \sum_{j=1}^{b} n_{dij} n_{di'j} = \lambda_{d1}$$

for all i, i', $i \neq i'$ for some λ_{d0} and λ_{d1}. In the incomplete block context, BTIB designs are the same as the designs with supplemented balance of Pearce (1960). Let

$$\mathscr{D}_2(v+1, b, k) = \{d: d \in \mathscr{D}(v+1, b, k),\ d \text{ is a BTIB design}\}.$$

For $d \in \mathscr{D}_2(v+1, b, k)$,

$$\operatorname{Var}\left(\hat{\alpha}_{di} - \hat{\alpha}_{d0}\right) = \sigma^2 \tau_d^2, \qquad \operatorname{Corr}\left(\hat{\alpha}_{di} - \hat{\alpha}_{d0}, \hat{\alpha}_{di'} - \hat{\alpha}_{d0}\right) = \rho_d,$$

where

$$\tau_d^2 = \frac{k(\lambda_{d0} + \lambda_{d1})}{\lambda_{d0}(\lambda_{d0} + v\lambda_{d1})} \tag{1.2}$$

and

$$\rho_d = \frac{\lambda_{d1}}{\lambda_{d0} + \lambda_{d1}},$$

where $\hat{\alpha}_{di} - \hat{\alpha}_{d0}$ is the BLUE of $\alpha_i - \alpha_0$.

For $d \in \mathscr{D}_2(v+1, b, k)$ the coverage probability for simultaneous confidence intervals of the $(\alpha_i - \alpha_0)$'s is a function of τ_d^2, ρ_d and δ/σ, where δ is the specified "yardstick" (equation 5.1 of BT) given by

$$\Pr\left(\alpha_0 - \alpha_i \ge \hat{\alpha}_{d0} - \hat{\alpha}_{di} - \delta,\ i = 1, \dots, v\right).$$

In general, it is difficult to determine an optimal design by this approach. However, Bechhofer and Tamhane [(1981), Definition 5.2] proposed an admissibility criterion which is very useful. In the original definition of admissibility, v and k were fixed, but b was allowed to vary. In order to compare with A-optimal designs in $\mathscr{D}(v+1, b, k)$ we have to adapt the definition to classes of designs with v, b, k fixed. This is done next.

For $d_1, d_2 \in \mathscr{D}_2(v+1, b, k)$, d_2 is *inadmissible* with respect to d_1 $(d_1 \succ d_2)$ if, for every δ/σ, the coverage probability given by d_2 is no larger than that given by d_1 and is smaller for some δ/σ. A characterization of $d_1 \succ d_2$ for $d_1, d_2 \in \mathscr{D}_2(v+1, b, k)$ (BT, Theorem 5.1) is

$$\tau_{d_1}^2 \le \tau_{d_2}^2, \qquad \rho_{d_1} \ge \rho_{d_2} \quad \text{with at least one inequality strict.} \tag{1.3}$$

If $d_1 \succ d_2$, then d_2 need not be considered in the search for optimal designs in $\mathscr{D}_2(v+1, b, k)$. Tables of admissible designs for $k = 3$, $3 \le v \le 10$, are given in Notz and Tamhane (1983) and for $k = 4, 5$, $k \le v \le 10$; $v = k = 6$ are given in Ture (1982, 1985).

For the estimation problem, A-optimal designs in $\mathscr{D}(v+1,b,k)$, for many (v,b,k)'s were given by Majumdar and Notz [(1983), Theorem 2.2]. An A-optimal design is one that minimizes $\sum_{i=1}^{v}\mathrm{Var}(\hat{\alpha}_{d0}-\hat{\alpha}_{di})$. To describe these designs we use the notation, due to Stufken (1987), BTIB$(v,b,k;t,s)$ to denote a design d in $\mathscr{D}_2(v+1,b,k)$ with the properties $n_{dij}\in\{0,1\}$ for $i=1,\dots,v$, $j=1,\dots,b$, $n_{do1}=\cdots=n_{dos}=t+1$, $n_{dos+1}=\cdots=n_{dob}=t$. Here $t\in\{0,1,\dots,k-1\}$ and $s\in\{0,1,\dots,b-1\}$. Using $\lfloor x\rfloor$ to denote the largest integer less than or equal to x, we define

$$
\begin{aligned}
h(r) &= \lfloor r/b\rfloor^2(b+b\lfloor r/b\rfloor-r)+(r-b\lfloor r/b\rfloor)(\lfloor r/b\rfloor+1)^2,\\
g(r) &= v/(r-h(r)/k)\\
&\qquad +(v-1)^2/(b(k-1)-r(k-1)/k-(r-h(r)/k)/v).
\end{aligned}
$$

Let R be an integer in the interval $[1, bk/2]$, defined by

$$
g(R)=\min\{g(r)\colon r=1,\dots,\lfloor bk/2\rfloor\}. \tag{1.4}
$$

Let d^* denote a BTIB$(v,b,k;t,s)$ with $r_{d^*0}=R$.

Majumdar and Notz showed that, whenever it exists, d^* is A-optimal in $\mathscr{D}(v+1,b,k)$. Thus R can be viewed as the "optimal replication of the control."

Let

$$
\begin{aligned}
&\mathscr{D}_1(v+1,b,k)\\
&\qquad=\{d\colon d\in\mathscr{D}_2(v+1,b,k),\ n_{dij}\in\{0,1\},\ i=1,\dots,v,\ j=1,\dots,b\},\\
&\mathscr{D}_0(v+1,b,k)\\
&\qquad=\{d\colon d\in\mathscr{D}_2(v+1,b,k),\ d \text{ is BTIB}(v,b,k;t,s) \text{ for some } t,s\}.
\end{aligned}
$$

Clearly,

$$
\mathscr{D}(v+1,b,k)\supset\mathscr{D}_2(v+1,b,k)\supset\mathscr{D}_1(v+1,b,k)\supset\mathscr{D}_0(v+1,b,k)\ni d^*.
$$

We derive several inadmissibility results for designs in $\mathscr{D}_1(v+1,b,k)$ in Section 2, and an admissibility result for designs in $\mathscr{D}_0(v+1,b,k)$ in Section 3. In particular, we show that whenever the design d^* exists, $d\in\mathscr{D}_0(v+1,b,k)$ is admissible if and only if $r_{do}\le R$. Admissibility, A-optimality and Bayes A-optimality are linked in Section 3.

2. Inadmissible designs. In this section we will identify some designs in $\mathscr{D}_1(v+1,b,k)$ that are inadmissible. First, some notation. For $j=1,\dots,b$ and $d\in\mathscr{D}(v+1,b,k)$, let

$$
m_{dj}=k-n_{doj}=\sum_{i=1}^{v}n_{dij}.
$$

THEOREM 2.1. *Let $d_1,d_2\in\mathscr{D}_1(v+1,b,k)$ such that*

$$
\tau_{d_1}^2\le\tau_{d_2}^2 \tag{2.1}
$$

and

$$\sum_{j=1}^{b} m_{d_1 j}^2 \Big/ \sum_{j=1}^{b} m_{d_1 j} \geq \sum_{j=1}^{b} m_{d_2 j}^2 \Big/ \sum_{j=1}^{b} m_{d_2 j} \tag{2.2}$$

with at least one inequality (2.1) *or* (2.2) *strict. Then* $d_1 \succ d_2$; *hence,* d_2 *is inadmissible.*

PROOF. For any $d \in \mathscr{D}_1(v+1, b, k)$,

$$\begin{aligned}\lambda_{d0} + \lambda_{d1} &= \lambda_{d0} + (v-1)\lambda_{d1} - (v-2)\lambda_{d1} \\ &= r_{d1}(k-1) - (v-2)\lambda_{d1}\end{aligned}$$

since $r_{d1} = \cdots = r_{dv}$ for $d \in \mathscr{D}_1(v+1, b, k)$. Also $\lambda_{d1}\binom{v}{2} = \sum_{j=1}^{b}\binom{m_{dj}}{2}$. Hence,

$$\rho_d = \frac{\sum m_{dj}^2 - \sum m_{dj}}{(k-1)(v-1)\sum m_{dj} - (v-2)\left(\sum m_{dj}^2 - \sum m_{dj}\right)},$$

which is increasing in $\sum m_{dj}^2 / \sum m_{dj}$. Hence the theorem. □

Note that (2.1) means that d_1 is better than d_2 by the criterion of A-optimality. We may say that d_1 is "A-better" than d_2.

COROLLARY 2.1. *Suppose* d_1 *is a BTIB*$(v, b, k; t, s)$ *design and* d_2 *is a BTIB*$(v, b, k; x, z)$ *design such that* (2.1) *holds and* $r_{d_2 0} > r_{d_1 0}$. *Then* $d_1 \succ d_2$.

PROOF. Let us write $g = k - t - 1$ and $h = k - x - 1$. Note that $r_{d_1 0} = bt + s$, $r_{d_2 0} = bx + z$, $t = \lfloor r_{d_1 0}/b \rfloor$ and $x = \lfloor r_{d_2 0}/b \rfloor$. Clearly, $r_{d_2 0} > r_{d_1 0}$ implies $g \geq h$. From Theorem 2.1 it follows that we have to show

$$\begin{aligned}&\left[sg^2 + (b-s)(g+1)^2\right]\left[sg + (b-s)(g+1)\right]^{-1} \\ &\qquad \geq \left[zh^2 + (b-z)(h+1)^2\right]\left[zh + (b-z)(h+1)\right]^{-1},\end{aligned}$$

that is,

$$\begin{aligned}&szgh(g-h) + s(b-z)g(h+1)(g-h-1) \\ &\quad + (b-s)z(g+1)h(g+1-h) \\ &\quad + (b-s)(b-z)(g+1)(h+1)(g-h) \geq 0.\end{aligned} \tag{2.3}$$

If $g \geq h+1$, (2.3) is obviously true. If $g = h$, then (2.3) reduces to $g(g+1)b(z-s) \geq 0$, which is true since $r_{d_1 0} < r_{d_2 0}$. It is easy to see that the inequality (2.3) is strict unless $g = h = 0$. In this case, the relations $g = h = 0$ and $z > s$ imply that inequality (2.1) is strict. □

Using the A-optimal design d^* and the quantity R defined in Section 1 [see (1.4)], we get the following corollary.

COROLLARY 2.2. *Suppose the class* $\mathscr{D}(v+1,b,k)$ *is such that* d^* *exists. Then any BTIB*$(v,b,k;x,z)$ *design* d_2 *with* $r_{d_2 0} > R$ *is inadmissible.*

EXAMPLE 2.1. Suppose $v=7$, $k=5$ and $b=35$. We will represent designs in $\mathscr{D}(v+1,b,k)$ by $k\times b$ arrays with entries from $\{0,1,\dots,v\}$ with columns as blocks. Let A_i $(i=1,2,4)$ be a $i\times 7$ array with all entries 0 and let A_3 be a 3×21 array with entries 0. Let δ_1 be a BIB(7, 7, 3, 3, 1) design, δ_2 be a BIB(7, 7, 4, 4, 2) design and δ_3 be a BIB(7, 21, 6, 2, 1) design based on treatments $1,2,\dots,v$. The notation BIB(v,b,r,k,λ) stands for a balanced incomplete block design with parameters v, b, r, k and λ. Let $\delta_4 = (1,2,3,4,5,6,7)$ be a 1×7 array. An A-optimal design in $\mathscr{D}(8,35,5)$ is

$$d^* = \begin{pmatrix} A_2 & A_1 & A_1 & A_1 & A_1 \\ \delta_1 & \delta_2 & \delta_2 & \delta_2 & \delta_2 \end{pmatrix}.$$

Hence, by Corollary 2.2, all of the following designs are inadmissible:

$$d_1 = \begin{pmatrix} A_2 & A_2 & A_1 & A_1 & A_1 \\ \delta_1 & \delta_1 & \delta_2 & \delta_2 & \delta_2 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} A_2 & A_2 & A_2 & A_1 & A_1 \\ \delta_1 & \delta_1 & \delta_1 & \delta_2 & \delta_2 \end{pmatrix},$$

$$d_3 = \begin{pmatrix} A_2 & A_2 & A_2 & A_2 & A_1 \\ \delta_1 & \delta_1 & \delta_1 & \delta_1 & \delta_2 \end{pmatrix},$$

$$d_4 = \begin{pmatrix} A_2 & A_2 & A_2 & A_2 & A_2 \\ \delta_1 & \delta_1 & \delta_1 & \delta_1 & \delta_1 \end{pmatrix},$$

$$d_5 = \begin{pmatrix} A_3 & A_4 & A_4 \\ \delta_3 & \delta_4 & \delta_4 \end{pmatrix}.$$

For d_6,

$$d_6 = \begin{pmatrix} A_4 & A_3 & A_1 \\ \delta_4 & \delta_3 & \delta_2 \end{pmatrix},$$

Corollary 2.2 is not applicable, but a direct application of Theorem 2.1 shows that d_6 is inadmissible.

DEFINITION 2.1. A block j of a design d is called trivial if $n_{dij} = k$ for some $i \in \{0,1,\dots,k\}$.

If $d \in \mathscr{D}_2(v+1,b,k)$ has some trivial blocks, then a design $d^0 \in \mathscr{D}(v+1,b,k)$ obtained from d by replacing each trivial block by any nontrivial block has a bigger (in the sense of nonnegative definite partial order, or Löwner order) Fisher information matrix than d; hence, d^0 gives better inferences on the $(\alpha_i - \alpha_0)$'s than d. For a given d, however, there may exist no d^0 in $\mathscr{D}_2(v+1,b,k)$.

COROLLARY 2.3. *Suppose the BTIB design d_1 is A-best in $\mathscr{D}_1(v+1,b,k)$. Suppose $d_2 \in \mathscr{D}_1(v+1,b,k)$ is a design with no trivial block such that*

$$\sum_{j=1}^{b} m_{d_1 j}^2 \bigg/ \sum_{j=1}^{b} m_{d_1 j} > k + 1 - bk/\left(bk - r_{d_2 0}\right).$$

Then d_2 is inadmissible.

PROOF. This follows from Theorem 2.1 by replacing the right side of inequality (2.2) by the supremum over the $m_{d_2 j}$'s subject to $\sum_{j=1}^{b} m_{d_2 j} = bk - r_{d_2 0}$, $r_{d_2 0}$ fixed. A conservative evaluation of the supremum is obtained when $m_{d_2 j} \in \{1, k\}$. $\square$

EXAMPLE 2.1 (continued). Setting $d_1 = d^*$ in Corollary 2.3, we see that any $d \in \mathscr{D}_1(8,35,5)$ with no trivial block and $r_{d_2 0} \geq 94$ is inadmissible.

EXAMPLE 2.2. If a BTIB$(v,b,k;t,0)$ design is A-best in $\mathscr{D}_1(v+1,b,k)$, then it follows from Corollary 2.3 that any BTIB design d in $\mathscr{D}_1(v+1,b,k)$ with no trivial block and $r_{d0} > bkt/(1+t)$ is inadmissible.

3. Admissible designs. Establishing admissibility is considerably more difficult than demonstrating inadmissibility. We will focus on designs in $\mathscr{D}_0(v+1,b,k)$ and determine when these designs are admissible. Here admissibility is in the entire set $\mathscr{D}_2(v+1,b,k)$ of BTIB designs, that is, we seek $d_2 \in \mathscr{D}_0(v+1,b,k)$ for which there is no $d_1 \in \mathscr{D}_2(v+1,b,k)$ such that $d_1 \succ d_2$. First we need to recall some results on Bayes A-optimal designs, which will be our tool.

Let Y be the $bk \times 1$ vector of observations y_{ijp}, $\theta_i = \alpha_i - \alpha_0$, $i = 1,\dots,v$, $\eta_j = \mu + \alpha_0 + \beta_j$, $j = 1,\dots,b$; $\theta' = (\theta_1,\dots,\theta_v)$, $\eta' = (\eta_1,\dots,\eta_b)$. We assume that

$$Y|\theta,\eta \sim N(X_{1d}\theta + X_2\eta, E); \qquad \begin{pmatrix}\theta\\ \eta\end{pmatrix} \sim N\left(\begin{pmatrix}\mu_\theta\\ \mu_\eta\end{pmatrix}, \begin{pmatrix}B^* & 0\\ 0 & B\end{pmatrix}\right),$$

for some $E, \mu_\theta, \mu_\eta, B^*, B$; X_{1d} and X_2 are obtained from the model for $E(y_{ijp})$ in (1.1). A Bayes A-optimal design is one [see Owen (1970)] that minimizes the posterior expected loss. We will work with the squared error loss $L(\hat{\theta},\theta) = (\hat{\theta}-\theta)'(\hat{\theta}-\theta)$. For the special case, $E = \sigma^2 I$ (homoscedastic errors), $B^{*-1} = 0$ (vague prior on θ) and $B = (\sigma^2/\alpha)I$ for some $\alpha > 0$, the Bayes A-optimal design d_α minimizes $\operatorname{tr} D_d$ over $d \in \mathscr{D}(v+1,b,k)$, where

$$\sigma^2 D_d^{-1} = M_d(\alpha) = \operatorname{diag}(r_{d1},\dots,r_{dv}) - (k+\alpha)^{-1}\overline{N}_d\overline{N}_d', \tag{3.1}$$

where $\overline{N}_d = (n_{dij})_{i=1,\dots,v,\ j=1,\dots,b}$ [see page 221 of Majumdar (1992)]. If $d \in \mathscr{D}_1(v+1,b,k)$ or if $d \in \mathscr{D}_2(v+1,b,k)$ with the property $r_{d1} = \cdots = r_{dv}$, then $M_d(\alpha)$ is completely symmetric (c.s.), but if d is any other BTIB design in $\mathscr{D}_2(v+1,b,k)$, then $M_d(\alpha)$ need not be c.s. Nevertheless the symmetrized version, $\overline{M}_d(\alpha) = \sum \Pi M_d(\alpha)\Pi'/v!$, where the sum is over all permutation matrices Π of order v, is c.s.; hence, $\overline{M}_d(\alpha)^{-1}$ can be easily evaluated. The following lemma is proved in the Appendix.

LEMMA 3.1. (a) *For* $d \in \mathscr{D}_2(v+1, b, k)$ *and* $\alpha > 0$,

$$\operatorname{tr} \overline{M}_d(\alpha)^{-1} \le \tau_d^2 f(\rho_d), \tag{3.2}$$

where

$$\begin{aligned} f(\rho_d)k[1-\rho_d+\bar{\alpha}(1+(v-2)\rho_d)][1+(v-1)\rho_d+\bar{\alpha}(1+(v-2)\rho_d)] \\ = v(k+\alpha)[1+\bar{\alpha}(1+(v-2)\rho_d)][1-\rho_d][1+(v-1)\rho_d] \end{aligned}$$

and $\bar{\alpha} = \alpha/(k-1)$.

(b) *If* $d \in \mathscr{D}_1(v+1, b, k)$, (3.2) *is an equality.*

(c) $f'(\rho_d) < 0$ *for* $\rho_d > 0$,

where f' *denotes the derivative of* f.

Let d_2 be a BTIB$(v, b, k; t, s)$ design with $r_{d_2 0} \le R$. Suppose d_2 is inadmissible, that is, there is a $d_1 \in \mathscr{D}_2(v+1, b, k)$ such that (1.3) holds. It follows from Theorem 4.3 of Majumdar (1992) that d_2 is a Bayes A-optimal design for some $\alpha > 0$. Hence, from the statement and proof of Corollary 3.1 and Theorem 3.1 of Majumdar (1992) it follows that, for this α,

$$\operatorname{tr} M_{d_2}(\alpha)^{-1} = \operatorname{tr} \overline{M}_{d_2}(\alpha)^{-1} \le \operatorname{tr} \overline{M}_{d_1}(\alpha)^{-1}.$$

This together with parts (a) and (b) of Lemma 3.1 implies

$$\tau_{d_2}^2 f(\rho_{d_2}) \le \tau_{d_1}^2 f(\rho_{d_1}). \tag{3.3}$$

On the other hand, (1.3) together with part (c) of Lemma 3.1 implies

$$\tau_{d_2}^2 f(\rho_{d_2}) > \tau_{d_1}^2 f(\rho_{d_1}),$$

which contradicts (3.3). Hence d_2 is admissible.

We have proved the following theorem:

THEOREM 3.1. *A BTIB$(v, b, k; t, s)$ design d with $r_{d0} \le R$ is admissible in the class $\mathscr{D}_2(v+1, b, k)$ of BTIB designs.*

Note that we do not need existence of the design d^* for Theorem 3.1. If it does, then we can combine Theorem 3.1 and Corollary 2.2 to get the following corollary:

COROLLARY 3.1. *If d^* exists, then a BTIB$(v, b, k; t, s)$ design d is admissible in $\mathscr{D}_2(v+1, b, k)$ if and only if $r_{d0} \le R$.*

REMARK 3.1. Corollary 3.1 can be extended to certain classes of designs for which d^* does not exist. This extension is based on the fact that the function g defined in the introduction has the property: for $r_1 < r_2 \le R$, $g(r_1) > g(r_2)$ and for $R \le r_1 < r_2$, $g(r_1) < g(r_2)$ [see Theorem 2.2 of Cheng, Majumdar, Stufken and Ture (1988)]. If $R_l < R$ is such that a $d^{**} \in \mathscr{D}_0(v+1, b, k)$ exists with $r_{d^{**}0} = R_l$ and for all $r_0 \in [R_l + 1, R_0]$, where $R_0 \geq R$ is such that $g(R_0) \leq g(R_l) < g(R_0 + 1)$, there is no design $d \in \mathscr{D}_0(v+1, b, k)$ with $r_{d0} = r_0$, then d^{**} is A-best in $\mathscr{D}_0(v+1, b, k)$. Using Corollary 2.2 and Theorem 3.1 it follows that $d \in \mathscr{D}_0(v+1, b, k)$ is admissible if and only if $r_{d0} \leq R$.

There are many design classes where the situation described in Remark 3.1 occurs, as shown in the next example.

EXAMPLE 3.1. Suppose $k = 3$ and $v = 5$. Then $R \in \{\lfloor 0.949b \rfloor, \lfloor 0.949b \rfloor + 1\}$. Consider design classes $\mathscr{D}(6, b, 3)$ with $b = 10a$, for integers a in the range $1 \leq a \leq 10$. [It can be shown that $\mathscr{D}_0(6, b, 3)$ contains a design with $r_{do} \leq b$ only if $b \equiv 0 \pmod{10}$.] Here $b - 5 \leq R < b$. For $a = 9, 10$, d^* exists; hence, Corollary 3.1 applies. For $a = 5, 6, 7, 8$, d^* does not exist, but the conditions of Remark 3.1 hold with $R_l = b - 5$ and $d^{**} = \mathrm{BTIB}(5, 10a, 3; 0, b - 5)$, which exists. Thus, for $5 \leq a \leq 10$, a design $d \in \mathscr{D}_0(6, 10a, 3)$ is admissible if and only if $r_{d0} \leq R$. For $1 \leq a \leq 4$, neither Corollary 3.1 nor Remark 3.1 applies; any $d \in \mathscr{D}_0(6, b, 3)$ is admissible if $r_{d0} \leq R$, by Theorem 3.1, and inadmissible if $r_{d0} > b = 10a$, by Corollary 2.1, since a $\mathrm{BTIB}(5, 10a, 3; 1, 0)$ exists. The only unresolved case in $\mathscr{D}_0(6, 10a, 3)$ is $d_0 = \mathrm{BTIB}(5, 10a, 3; 1, 0)$ for which $r_{d_0 0} = b$. It follows from Theorem 2.2 of Cheng, Majumdar, Stufken and Ture (1988) that, for $1 \leq a \leq 4$, there is no design $d_1 \in \mathscr{D}_2(6, 10a, 3)$, with $r_{d_1 0} \leq b - 5$ or $r_{d_1 0} \geq b$ such that $d_1 \succ d_0$ holds. If $d_1 \in \mathscr{D}_2(6, 10a, 3)$ satisfies $b - 5 < r_{d_1 0} < b$, the impossibility of $d_1 \succ d_0$ is likely to involve lengthy computations which will not be attempted here. It is our belief that d_0 is admissible in $\mathscr{D}(6, 10a, 3)$. Admissible designs in classes $\mathscr{D}_0(6, b, 3)$ with $b > 100$ can be characterized similarly.

REMARK 3.2. If a BIB design in all the $v + 1$ treatments exists in $\mathscr{D}(v+1, b, k)$, then it is a $\mathrm{BTIB}(v, b, k; 0, bk/(v+1))$ design d, with $r_{d0} < R$. Hence it is admissible.

If d is a BTIB design, then, for $i \neq i'$, $\mathrm{Var}(\hat{\alpha}_i - \hat{\alpha}_{i'}) = 2\sigma^2 \tau_d^2 (1 - \rho_d) = 2\sigma^2 k/\gamma_d$, say, where $\gamma_d = \lambda_{d0} + v\lambda_{d1}$. For $d_1, d_2 \in \mathscr{D}_2(v+1, b, k)$ we shall say that $d_2 \prec^* d_1$ if

$$\tau_{d_1}^2 \leq \tau_{d_2}^2 \quad \text{and} \quad \gamma_{d_1} \geq \gamma_{d_2} \quad \text{with at least one inequality strict.} \tag{3.4}$$

A design d_2 will be called $*$-inadmissible if there exists a $d_1 \in \mathscr{D}_2(v+1, b, k)$ for which (3.4) holds.

It is clear that for $d_1, d_2 \in \mathscr{D}_2(v+1, b, k)$, $d_2 \prec d_1$ implies $d_2 \prec^* d_1$. Hence Corollaries 2.1 and 2.2 hold for $*$-admissibility. The following result, which is proved in the Appendix, shows that Theorem 3.1 holds for $*$-admissibility.

THEOREM 3.2. *A BTIB$(v, b, k; t, s)$ design d with $r_{d0} \le R$ is $*$-admissible in the class $\mathscr{D}_2(v+1, b, k)$ of BTIB designs.*

It follows that Corollary 3.1 also holds for $*$-admissibility, that is, if d^* exists, then a BTIB$(v, b, k; t, s)$ design d is $*$-admissible in $\mathscr{D}_2(v+1, b, k)$ if and only if $r_{d0} \le R$.

REMARK 3.3. The criterion $*$-admissibility is desirable from an estimation viewpoint since if (3.4) holds, then d_1 gives better estimators than d_2 for the treatment–control contrasts, as well as the treatment–treatment contrasts (hence all elementary treatment contrasts). The close connection between admissibility and $*$-admissibility could be a possible explanation for the close relationship between the two approaches to obtaining optimal designs.

REMARK 3.4. The Bayes A-optimal designs described in this article are also optimal Γ-minimax designs [see Corollary 3.1 of Majumdar (1992)]. When d^* exists, it follows from Theorem 4.3 of Majumdar (1992) that as α varies in $[0, \infty]$, the subset of $\mathscr{D}_0(v+1, b, k)$ formed by Bayes A-optimal designs consists of designs with $r_{d0} \le R$. We have thus shown that when d^* exists, the admissible designs in $\mathscr{D}_0(v+1, b, k)$ are precisely the set of Bayes A-optimal (as well as optimal Γ-minimax) designs. This parallels well-known results in decision theory on the correspondence between admissible and Bayes decision rules.

APPENDIX

PROOF OF LEMMA 3.1. (a) For $d \in \mathscr{D}_2(v+1, b, k)$, $(k+\alpha)\overline{M}_d(\alpha) = (m_1 - m_2)I + m_2 J$, where I is the $v \times v$ identity matrix, J is a matrix of ones, $m_2 = -\lambda_{d1}$ and $m_1 = (k+\alpha)\bar{r}_d - \xi_d$, where $v\bar{r}_d = bk - r_{d0}$ and $v\xi_d = \sum_{i=1}^{v} \sum_{j=1}^{b} n_{dij}^2$. Using Rao [(1973), page 67],

$$\operatorname{tr} \overline{M}_d(\alpha)^{-1} = \frac{v(k+\alpha)\left[(k+\alpha)\bar{r}_d - \xi_d - (v-2)\lambda_{d1}\right]}{\left[(k+\alpha)\bar{r}_d - \xi_d - (v-1)\lambda_{d1}\right]\left[(k+\alpha)\bar{r}_d - \xi_d + \lambda_{d1}\right]}.$$

Since $k\bar{r}_d - \xi_d = \lambda_{d0} + (v-1)\lambda_{d1}$, we can write $\operatorname{tr} \overline{M}_d(\alpha)^{-1} = v(k+\alpha)(A + \alpha\xi_d/k)(B + \alpha\xi_d/k)^{-1}(C + \alpha\xi_d/k)^{-1}$, where $B = (1+\alpha/k)\lambda_{d0} + (v-1)\alpha\lambda_{d1}/k$, $A = B + \lambda_{d1}$ and $C = B + v\lambda_{d1}$. It can be shown that $\partial \operatorname{tr} \overline{M}_d(\alpha)^{-1}/\partial \xi_d < 0$, since $\xi_d > 0$. Hence, for a $d \in \mathscr{D}_2(v+1, b, k)$, minimum ξ_d will give an upper bound to $\operatorname{tr} \overline{M}_d(\alpha)^{-1}$.

Since $v\xi_d = \sum_{i=1}^{v} \sum_{j=1}^{b} n_{dij}^2 \ge v\bar{r}_d$ and $\xi_d = k\bar{r}_d - (\lambda_{d0} + (v-1)\lambda_{d1})$, we get

$$\xi_d \ge \left(\lambda_{d0} + (v-1)\lambda_{d1}\right)/(k-1). \tag{A.1}$$

Substituting the right side of (A.1) in A, B and C, we get, after simplification,

$$\operatorname{tr} \overline{M}_d(\alpha)^{-1} \le \frac{v(k+\alpha)(\lambda_{d0} + \lambda_{d1})(1 + A_1)}{\lambda_{d0}(\lambda_{d0} + v\lambda_{d1})(1 + B_1)(1 + C_1)},$$

where $B_1 = \alpha(\lambda_{d0} + (v-1)\lambda_{d1})/((k-1)\lambda_{d0})$, $A_1 = B_1\lambda_{d0}/(\lambda_{d0} + \lambda_{d1})$ and $C_1 = B_1\lambda_{d0}/(\lambda_{d0} + v\lambda_{d1})$. Further simplification of this expression establishes (a).

(b) This follows from the fact that (A.1) is an equality whenever $d \in \mathscr{D}_1(v+1, b, k)$.

(c) The function $f(\rho_d)$ is the ratio of two polynomials in ρ_d. Lengthy but straightforward computations show that $f'(\rho_d) < 0$ for all ρ_d in $(0,1)$. □

PROOF OF THEOREM 3.2. Suppose $d_2 \in \mathscr{D}_1(v+1, b, k)$, $d_1 \in \mathscr{D}_0(v+1, b, k)$ and $d_2 \prec^* d_1$, that is,

$$\tau_{d_1}^2 \le \tau_{d_2}^2, \tag{A.2}$$

$$\lambda_{d_1 0} + v\lambda_{d_1 1} \ge \lambda_{d_2 0} + v\lambda_{d_2 1} \tag{A.3}$$

with at least one of (A.2) and (A.3) strict.

Case 1: $\lambda_{d_1 0} \le \lambda_{d_2 0}$. Then (A.3) implies $\lambda_{d_1 1} \ge \lambda_{d_2 1}$, from which it follows that $d_2 \prec d_1$. This contradicts Theorem 3.1.

Case 2: $\lambda_{d_1 0} > \lambda_{d_2 0}$.

Case 2.1: $r_{d_1 0} < r_{d_2 0} \le R$. It follows from Theorem 2.2 of Cheng, Majumdar, Stufken and Ture (1988) that

$$\tau_{d_1}^2 \ge g(r_{d_1 0}) > g(r_{d_2 0}) = \tau_{d_2}^2,$$

which contradicts (A.2).

Case 2.2: $r_{d_1 0} \ge r_{d_2 0}$. This implies $r_{d_2 1} \ge \bar{r}_{d_1} = (bk - r_{d_1 0})/v$. Hence,

$$\lambda_{d_1 0} + (v-1)\lambda_{d_1 1} \le (k-1)\bar{r}_{d_1} \le (k-1)r_{d_2 1} = \lambda_{d_2 0} + (v-1)\lambda_{d_2 1}.$$

Since $\lambda_{d_1 0} > \lambda_{d_2 0}$, this implies, $\lambda_{d_1 0} + v\lambda_{d_1 1} < \lambda_{d_2 0} + v\lambda_{d_2 1}$, which contradicts (A.3). □

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