60. *On Affine Surfaces whose Cubic Forms are Parallel Relative to the Affine Metric*[†)]

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Let M^n be a nondegenerate affine hypersurface in affine space $\boldsymbol{R}^{n+1}$ and denote by ∇, h and $\hat{\nabla}$ the induced connection, the affine metric, and the Levi-Civita connection for h, respectively. (We follow the terminology of [4].) Let $C=\nabla h$ be the cubic form.

It is a classical theorem that if $C=0$, then M^n is a quadratic hypersurface. In [5], it is shown that for $n=2$ the condition $\nabla C=0$, $C\neq 0$ characterizes, up to an equiaffine congruence, a Cayley surface, namely, the graph of the cubic polynomial $z=xy-y^3/3$. For an arbitrary dimension, [1] has shown that the tensor ∇C is totally symmetric (i.e. symmetric in all its indices) if and only if $\hat{\nabla} C$ is totally symmetric, and this symmetry condition implies that M^n is an affine hypersphere. It is also shown that the condition $\nabla C=0$, $C\neq 0$ implies that M^n is an improper affine hypersphere such that h is hyperbolic metric and the Pick invariant J is 0. As for the case $n=2$, affine spheres M^2 whose affine metric h is flat have been completely determined in [3], although the case where h is elliptic was already done in [2].

In this note, we study affine surfaces with the property $\hat{\nabla} C=0$, $C\neq 0$, and prove the following classification.

Theorem. *If a nondegenerate affine surface in* $\boldsymbol{R}^3$ *satisfies* $\hat{\nabla} C=0$, $C\neq 0$, *then it is equiaffinely congruent to a piece of one of the following surfaces*:

1) *the graph of* $z=1/xy$ (h: *elliptic*);
2) *the graph of* $z=1/(x^2+y^2)$ (h: *hyperbolic and* $J\neq 0$);
3) *Cayley surface* (h: *hyperbolic and* $J=0$).

The proof is given along the following lines. First, from the results quoted from [1] we see that the surface is an affine sphere. Next, we show that the assumption of the theorem implies that the connection $\hat{\nabla}$ is flat by using the argument similar to that in [5]. Now the result in [3] leads to our classification by using a concrete procedure to show that the graph of $z=xy+\varphi(y)$, where φ is an arbitrary cubic polynomial, is equiaffinely congruent to the Cayley surface.

Proof of the theorem. *Step* 1. We show that $\hat{\nabla} C=0$ implies that M^2 is an affine sphere. Indeed, from [1] we know that ∇C is totally symmetric, and this implies our assertion.

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Step 2. We show that $\hat{\nabla}C=0$, $C\neq 0$, implies that $\hat{\nabla}$ is flat.

We can follow the arguments in the proof of Lemma 3 in [5] with a slight modification. In the case where h is elliptic or where h is hyperbolic and $J\neq 0$, we have the same arguments to conclude that the holonomy group of $\hat{\nabla}$ is a finite group and hence the curvature tensor $\hat{R}$ of $\hat{\nabla}$ is identically 0, that is, h is flat.

In the case where h is hyperbolic and $J=0$, we know (proof of Lemma 3, [5]) that we can locally find vector fields X and Y such that

(1) $\quad h(X,X)=0,\quad h(X,Y)=1,\quad \text{and}\quad h(Y,Y)=0$

(2) $\quad C(X,U,V)=0\quad$ for any vector fields U and V

(3) $\quad C(Y,Y,Y)=1.$

Now applying covariant differentiation $\hat{\nabla}_X$ to (2) and (3) we obtain

(4) $$\hat{\nabla}_X X=\lambda X \quad\text{and}\quad \hat{\nabla}_X Y=\mu X.$$

Applying $\hat{\nabla}_X$ to $h(X,Y)=1$, and using (4), we obtain $\hat{\nabla}_X X=0$. Also, applying $\hat{\nabla}_X$ to $h(Y,Y)=0$ and using (4), we obtain $\hat{\nabla}_X Y=0$. Thus

(5) $$\hat{\nabla}_X X=0 \quad\text{and}\quad \hat{\nabla}_X Y=0.$$

Similar to (4) we get

(6) $$\hat{\nabla}_Y X=\nu X \quad\text{and}\quad \hat{\nabla}_Y Y=\tau X.$$

Applying $\hat{\nabla}_Y$ to $h(X,Y)=1$ and $h(Y,Y)=0$ and using (6) we obtain

(7) $$\hat{\nabla}_Y X=0 \quad\text{and}\quad \hat{\nabla}_Y Y=0.$$

From (5) and (7) we see that $\hat{\nabla}$ is flat.

Step 3. We continue the case where h is hyperbolic and $J=0$ to show that ∇ is also flat (so M^2 is an improper affine sphere) and that ∇C is also 0. From what we know, we also get $[X,Y]=\hat{\nabla}_X Y-\hat{\nabla}_Y X=0$. Thus we may find a local coordinate system $\{x,y\}$ such that $X=\partial/\partial x$ and $Y=\partial/\partial y$. This means that $\{x,y\}$ are flat null coordinates for $\hat{\nabla}$. Writing x^1, x^2 for x, y, we see that the components of the cubic form C are all zero except C_{222}. Since $\hat{\nabla}C=0$, we see

$$0=(\hat{\nabla}_Y C)(Y,Y,Y)=YC_{222}=\partial C_{222}/\partial y,$$

and similarly $\partial C_{222}/\partial x=0$. Thus C_{222} is a constant.

For the difference tensor K: $K(U,V)=\nabla_U V-\hat{\nabla}_U V$, we know

$$h(K(U,V),W)=-\frac{1}{2}C(U,V,W).$$

Using this, we find

(8) $$\nabla_X X=\nabla_X Y=\nabla_Y X=0 \quad\text{and}\quad \nabla_Y Y=-\frac{1}{2}C_{222}X.$$

It follows that the curvature tensor R of ∇ is 0 and so ∇ is also flat. It follows that M^2 is an improper affine sphere. Furthermore, using constancy of C_{222} and (8), we conclude $\nabla C=0$.

Step 4. We have thus shown that M^2 is an affine sphere and h is flat. From the results in [3], M^2 must be either

1) the graph of $z=1/xy$ (if h is ellptic)

or

2) the graph of $z=1/(x^2+y^2)$ (if h is hyperbolic and $J\neq 0$)

or

3*) the graph of $z=xy+\varphi(y)$, where φ is an arbitrary function of y (if h hyperbolic and $J=0$).

The surfaces 1) and 2) have the property that $\hat{\nabla}C=0$, $C\neq 0$. In order to verify this, we may represent the surfaces as in [3] with parameters which become flat coordinates for the affine metric and see that the Christoffel symbols for the induced connection ∇ are constants. Then the Christoffel symbols for the connection $\hat{\nabla}$ being all 0, we see that the components of the cubic form are all constants. This implies that $\hat{\nabla}C=0$ (but of course $C\neq 0$, since the surfaces are not quadrics).

In order to conclude that 3*) above leads to 3) in the theorem under our assumption $\hat{\nabla}C=0$, we can proceed as follows. In Step 3, we have seen that the surface satisfies $\nabla C=0$. Thus if we appeal to the theorem in [5], we conclude that it is a Cayley surface. On the other hand, we may take the following route. For the graph

$$(x, y)\longmapsto(x, y, xy+\varphi(y)) \tag{9}$$

we may compute

$$f_*(\partial/\partial x)=(1, 0, y), \qquad f_*(\partial/\partial y)=(0, 1, x+\varphi'(y))$$
$$(\partial/\partial x)f_*(\partial/\partial x)=(0, 0, 0), \qquad (\partial/\partial y)f_*(\partial/\partial x)=(0, 0, 1)$$
$$(\partial/\partial y)f_*(\partial/\partial y)=(0, 0, \varphi''(y))$$

so that we have

$$\nabla_{\partial/\partial x}(\partial/\partial x)=\nabla_{\partial/\partial x}(\partial/\partial y)=\nabla_{\partial/\partial y}(\partial/\partial x)=\nabla_{\partial/\partial y}(\partial/\partial y)=0$$
$$h(\partial/\partial x, \partial/\partial x)=0, \quad h(\partial/\partial x, \partial/\partial y)=1, \quad h(\partial/\partial y, \partial/\partial y)=\varphi''(y).$$

The affine normal is $(0, 0, 1)$ and the surface is an improper affine sphere. The components of C are 0 except possibly $C(\partial/\partial y, \partial/\partial y, \partial/\partial y)=\varphi^{(3)}$, and the component of ∇C are 0 except possibly $(\nabla_{\partial/\partial y}C)(\partial/\partial y, \partial/\partial y, \partial/\partial y)=\varphi^{(4)}$. Now if the surface (9) satisfies $\hat{\nabla}C=0$, then it also satisfies $\nabla C=0$, thus, $\varphi^{(4)}=0$, that is, $\varphi(y)$ is a cubic polynomial in y. In order to show that the surface is a Cayley surface, it is sufficient to show the following lemma.

Lemma. *The graph of $z=xy+\varphi(y)$, where φ is an arbitrary cubic polynomial in y, can be mapped onto the graph of $z=xy-y^3/3$ by an equiaffine transformation of $\boldsymbol{R}^3$.*

This can be shown by using a change of variables as in Cardano's well-known method of solving a cubic equation. Write $\varphi(y)=ay^3+by^2+cy+d$. We may find suitable constants p, q and r such that

$$\varphi(y)=(a^{1/3}(y+b/3a))^3+py+q.$$

Let

$$\bar{x}=a^{-1/3}x, \quad \bar{y}=a^{1/3}(y+b/3a), \quad \bar{z}=z+(b/3a)x-py-q, \tag{11}$$

which define an equiaffine transformation of $\boldsymbol{R}^3$. Then we see that

$$\bar{x}\bar{y}+\bar{y}^3=xy+(b/3a)x+\varphi(y)-py-q=\bar{z}$$

when $z=xy+\varphi(y)$. In other words, the image of the graph of $z=xy+\varphi(y)$ by the equiaffine transformation (11) is the graph of $z=xy+y^3$. Now we can take an equiaffine transformation $(x, y, z)\mapsto(-3^{-1/3}x, -3^{1/3}y, z)$ to change the surface to the graph of $z=xy-y^3/3$. This completes the proof of the lemma.

References

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[3] M. Magid and P. Ryan: Flat affine spheres in R^3 (to appear).

[4] K. Nomizu: Introduction to Affine Differential Geometry. part I, Lect. Notes, MPI preprint, p. 37.

[5] K. Nomizu and U. Pinkall: Cayley surfaces in affine differential geometry. MPI preprint, p. 37.