Hokkaido Mathematical Journal Vol. 14 (1985), p. 277-351

On affine symmetric spaces and the automorphism groups of product manifolds

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(Received January 16, 1985)

Introduction

Let R be a manifold, and let E and F be two differential systems on R, i. e., subbundles of the tangent bundle $T(R)$ of R. Then the pair (E, F) is called a product structure on R, if it satisfies the following conditions:

(P. 1) $T(R)=E+F$ (direct sum),

(P. 2) Both E and F are completely integrable.

A manifold R equipped with a product structure (E, F) is called a product manifold. Let R(resp. R') be a product manifold, and (E, F) (resp. (E', F')) its product structure. By an isomorphism of R onto R' we mean a diffeomorphism ϕ of R onto R' such that the differential ϕ_* of ϕ sends E to E' and F to F'. Clearly the product $M\times N$ of two manifolds M and N, and hence its open submanifolds Ω become naturally product manifolds in our sense.

The main purpose of the present paper is to study the automorphism groups Aut(Ω) of product manifolds Ω together with some related problems, based on the results in our previous works [8], [12] and our recent work [13]. (For several years we have worked on the geometrizations of systems of ordinary differential equations, and the results, obtained, will be published in the near future as a series of papers under the title: On pseudo-product structures and the geometrizations of systems of ordinary differential equations, which we quote by [13])

First of all we shall explain the main theorem in the present paper.

Let $\mathfrak{H}$ be a simple graded Lie algebra of the first kind, by which we mean a graded Lie algebra (over $\boldsymbol{R}$), $\mathfrak{g}=\sum_p \mathfrak{h}_p$, satisfying the following conditions: 1) dim $\mathfrak{g}<\infty$, and $\mathfrak{g}$ is simple, 2) $\mathfrak{h}_{-1}\neq\{0\}$, and $\mathfrak{h}_p=\{0\}$ if $p\leqq-2$ or $p\geqq 2$. (Note that dim $\mathfrak{h}_{-1}=$dim $\mathfrak{h}_1$.) If we set $\mathfrak{h}=\mathfrak{h}_0$ and $\mathfrak{m}=\mathfrak{h}_{-1}+\mathfrak{h}_1$ we see that the system $\mathfrak{S}=\{\mathfrak{g};\mathfrak{h}, \mathfrak{m}\}$ gives an (affine) symmetric triple, that is, it satisfies the following conditions: 1) $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ (direct sum), 2) $[\mathfrak{h}, \mathfrak{m}]\subset\mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}]\subset\mathfrak{h}$. Clearly the symmetric triple $\mathfrak{S}$ is of simple and reducible type, that is, $\mathfrak{g}$ is simple, and the linear isotropy representation of $\mathfrak{h}$ on $\mathfrak{m}$ is reducible. In his paper [3], Berger classified the symmetric triples of simple type, especially showing that any symmetric triple of simple and reducible type can be obtained in this manner.

Now, there is naturally associated to $\mathfrak{H}$ a (non-compact) affine symmetric homogeneous space G/H such that the symmetric triple associated with G/H is given by $\mathfrak{S}$ and such that G/H is standard in a suitable sense (see 2. 4). The space G/H will be called the standard affine symmetric space associated with $\mathfrak{H}$. It should be noticed that the space G/H is endowed with a product structure. In fact, the subspaces $\mathfrak{h}_{-1}$ and $\mathfrak{h}_1$ of $\mathfrak{m}$ naturally give rise to invariant differential systems E and F on G/H respectively, and the pair (E, F) gives an invariant product structure on the space. Giving attention only to the product structure, we denote by $\mathrm{Aut}(G/H)$ the automorphism group of the product manifold G/H.

These being prepared, the main theorem (Theorem 2. 8) in the present paper may be stated as follows: Assume that $\mathfrak{H}$ is of the classical type. If $\mathfrak{H}$ is isomorphic with a definite Möbius (graded Lie) algebra, then the automorphism group $\mathrm{Aut}(G/H)$ is naturally isomorphic with the diffeomorphism group of a sphere. Otherwise, the automorphism group Aut (G/H) is naturally isomorphic with the group G.

Here the definite Möbius algebras mean the simple graded Lie algebras of the first kind which play an important role in the definite conformal geometry and which may be regarded as the symbols of definite conformal structures. (For the precise definition, see 2. 3.)

We shall now make some remarks on the main theorem.

(1) Clearly a product structure is of infinite type and not elliptic in the sense that the equation of local infinitesimal automorphisms of the structure is of infinite type and not elliptic. Nevertheless the main theorem implies the finiteness for the automorphism group $\mathrm{Aut}(G/H)$ of the product manifold G/H, provided $\mathfrak{H}$ is not isomorphic with a definite Möbius algebra. It should be here recalled that the finiteness for the automorphism group of a geometric structure (such as a Riemannian structure, a complex structure and so on) which we have known up to now, is all based on the ellipticity for the geometric structure.

(2) We first remark that the Killing form of the Lie algebra $\mathfrak{g}$ naturally gives rise to an invariant indefinite Riemannian metric g on G/H, and it satisfies the following: 1) $g(E_x, E_x)=g(F_x, F_x)=\{0\}$ at each $x\in G/H$, 2) Both E and F are parallel with respect to the Levi-Civita connection ∇ associated with g. Thus the space G/H equipped with the product structure (E, F) and the metric g turns out to be a space analogous to a hermitian symmetric space of simple and non-compact type (cf. a parahermitian

symmetric space in the sense of [5]). As an important consequence of the main theorem we now remark that the metric g is completely determined (up to constant factors $\neq 0$) by the product structure (E, F), provided $\mathfrak{H}$ is not isomorphic with a definite Möbius algebra (cf. the Kählerian metric of a hermitian symmetric space of simple and non-compact type, and the Bergmann metric of a bounded domain). In connection with this fact, see the problem at the end of § 2.

(3) Through the proof of the main theorem we shall find an analogy between the study of the product manifolds G/H and the study of Siegel domains (of the first and the second kinds) due to Pyatetski-Shapiro [7] as developed by [10]. At the same time we shall find some essential differences between the two studies. For example, consider the finiteness for the automorphism groups.

(4) Finally we remark that the geometry of the product manifold G/H is closely related to the geometry of a certain involutive system of partial differential equations of finite type, provided $\mathfrak{H}$ is of the second class (cf. Remark at the end of 2.2).

Now, we proceed to the descriptions of the various sections.

§ 1 is preliminary to the subsequent sections.

After general remarks on terminologies and notations, we recall several known facts on simple graded Lie algebras ([12]), and prove some facts (Lemma 1.11 and its corollaries) on simple graded Lie algebras of the first kind. We also recall some fundamental facts (Facts A and B) in the equivalence problems associated with simple graded Lie algebras of the first kind ([8] and [12]).

We then introduce the notion of a pseudo-product manifold ([13]), which plays an important role in the present paper. By a pseudo-product structure on a manifold R we mean a pair (E, F) of differential systems E and F on R satisfying the following condtions: 1) $E\cap F=0$, the zero cross section of $T(R)$, 2) Both E and F are completely integrable. A manifold R equipped with a pseudo-product structure (E, F) is called a pseudo-product manifold. Now let R be a submanifold of a product manifold R', and (E', F') the product structure of R'. Then R becomes a pseudo-product manifold simply by setting $E_x=E'_x\cap T(R)_x$ and $F_x=F'_x\cap T(R)_x$ at each point x of R, provided both dim E_x and dim F_x are constant.

Furthermore we introduce the notion of a pseudo-product FGLA, which may be regarded as the symbol of a pseudo-product manifold at a point ([13]). Finally we introduce the notion of a pseudo-product manifold of type $\mathfrak{L}$, where $\mathfrak{L}$ is a pseudo-product FGLA of the second kind, and state some fundamental facts (Facts C and D) in the equivalence problem for pseudo-product manifolds of type $\mathfrak{L}$, under the condition that the prolongation $\mathfrak{G}$ of $\mathfrak{L}$ is simple ([13]).

In § 2, we first prove some propositions on the automorphism groups $\mathrm{Aut}(\Omega)$ of product manifolds Ω, and then state the main theorem together with some related facts.

Let P^n be the n-dimensional projective space over $\boldsymbol{R}$, and $G^r(P^n)$ the Grassmann manifold of r-dimensional projective subspaces of P^n, where $1 \leqq r \leqq n-1$. Let us consider the relation R in the product manifold $P^n \times G^r(P^n)$ defined by

$$R = \{(p, \alpha) \in P^n \times G^r(P^n) \mid p \subset \alpha\}.$$

As is easily seen, R is a compact submanifold of the product manifold, and hence the complement Ω of R is an open submanifold of the product manifold. Then it can be shown that the automorphism group $\mathrm{Aut}(\Omega)$ of the product manifold Ω is naturally isomorphic with the projective transformation group G of P^n (Proposition 2.1), indicating that the projective geometry has a close relationship with the geometry of product manifolds.

Proposition 2.1, this apparently simple fact, is just the starting-point of our study, and it is generalized (or partially generalized) in two manners: One is from the view-point of the manifold theory (Propositions 2.5 and 2.6); The other is from the view-point of the Lie group theory, which is nothing but our subject.

§ 3～§ 8 are devoted to the proof of the main theorem.

We first notice that the simple graded Lie algebras of the first kind (of the classical type) are divided into two classes, called of the first class and of the second class (see § 5). In our notations, the simple graded Lie algebras of the second class consists of $\mathfrak{G}(n, k; K)$ $(k > n \geqq 1, K = \boldsymbol{R}$ or $\boldsymbol{C}$ or $\boldsymbol{Q})$, a class of Grassmann graded Lie algebras, and $\mathfrak{SO}(n, n; K)$ (n odd, $n \geqq 5$, $K = \boldsymbol{R}$ or $\boldsymbol{C}$), where $\boldsymbol{Q}$ denotes the skew field of quaternions. Thus the standard affine symmetric spaces G/H are divided into two classes, which just correspond to the symmetric Siegel domains of the first kind and those of the second kind (but not of the first kind). It should be here noted that to every simple graded Lie algebra $\mathfrak{H}$ of the second class there is suitably associated a simple graded Lie algebra $\mathfrak{G}$ of the second kind such that the underlying Lie algebras of $\mathfrak{H}$ and $\mathfrak{G}$ coincide and such that $\mathfrak{G}$ is pseudo-product in our terminology (see § 3 and § 7).

Now, the proof of the main theorem is treated in different manners, according as $\mathfrak{H}$ is of the first class or of the second class. Let us explain our basic idea for the proof in the case where $\mathfrak{H}$ is of the second class. This is to construct two connected, compact manifolds M and N on which the group

G acts transitively and effectively, and successively to carry out the following procedures (cf. the proofs of Propositions 2. 1, 2. 5 and 2. 6) :

(I) The group G acts on the product manifold $M \times N$ through the diagonal map of G to $G \times G$. Then we show that the action of G on $M \times N$ have a single open orbit Ω and a single minimal dimensional orbit R. Hence Ω is an open, dense subset of $M \times N$, and the submanifold R of $M \times N$ is in the boundary $\partial\Omega$ of Ω.

(II) We show that G/H and Ω are isomorphic as homogeneous product manifolds, and that every automorphism ϕ of the product manifold Ω is naturally extended to a unique automorphism $\phi_{M \times N}$ of the product manifold $M \times N$. Note that $\phi_{M \times N}$ leaves $\partial\Omega$ invariant.

(III) We show that $\phi_{M \times N}$ leaves R invariant and hence the restriction of $\phi_{M \times N}$ to R gives an automorphism of the pseudo-product manifold R. From the procedures so far we see that there are natural injective homomorphisms i_Ω, i_R and j of G to $\mathrm{Aut}(\Omega)$, of G to $\mathrm{Aut}(R)$ and of $\mathrm{Aut}(\Omega)$ to $\mathrm{Aut}(R)$ respectively. Thus we obtain the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{i_\Omega} & \mathrm{Aut}(\Omega) \\ & \underset{i_R}{\searrow} & \downarrow j \\ & & \mathrm{Aut}(R) \end{array}$$

(IV) We apply Facts C and D to the pseudo-product manifold R, and show that the homomorphism i_R gives an isomorphism of G onto $\mathrm{Aut}(R)$. Consequently we have the natural isomorphisms: $\mathrm{Aut}(\Omega) \cong G$, proving the main theorem for the given $\mathfrak{H}$.

Let us return to the descriptions of the various sections. In § 3 we develop a general theory for the proof of the main theorem in the case where $\mathfrak{H}$ is of the second class. We start from any simple graded Lie algebra $\mathfrak{G}$ of the second kind which is pseudo-product. Then it is shown that there are naturally associated to $\mathfrak{G}$ two simple graded Lie algebras $\mathfrak{A}$ and $\mathfrak{B}$ of the first kind whose underlying Lie algebras coincide with that of $\mathfrak{G}$. Our task here is to realize our basic idea for $\mathfrak{H} = \mathfrak{A}$. Indeed, we consider two conditions, (II. 1) and (II. 2), on $\mathfrak{G}$, and show by the realization of the basic idea that the main theorem is true for $\mathfrak{H} = \mathfrak{A}$ under these conditions. Similarly in § 4 we develop a general theory for the proof in the case where $\mathfrak{H}$ is of the first class. We start from any simple graded Lie algebra $\mathfrak{G}$ of the first kind. As above we consider three conditions, (I. 1), (I. 2) and (I. 3), on $\mathfrak{G}$, and show by an analogous idea that the main theorem is true for $\mathfrak{H} = \mathfrak{G}$ under these conditions.

In § 6 we study certain algebraic varieties of the spaces $M_n(C)$ of square matrices over C, etc. We discuss generators of the ideals of the varieties, and determine the linear transformations leaving the varieties invariant. In § 7 we apply the arguments in § 3 to the simple graded Lie algebra $\mathfrak{G}$ of the second kind associated with each simple graded Lie algebra $\mathfrak{H}$ of the second class. Note that $\mathfrak{G}$ is pseudo-product, and the associated $\mathfrak{A}$ coincides with the given $\mathfrak{H}$. We easily verify condition (II, 1), and verify condition (II. 2) by the use of the results in § 6 and by the method of complexification, thus completing the proof of the main theorem for $\mathfrak{H}=\mathfrak{A}$. Similarly in § 8, we apply the arguments in § 4 to each simple graded Lie algebra $\mathfrak{G}$ of the first class. Unless $\mathfrak{G}$ is a definite Möbius algebra, we easily verify condition (I. 1), and verify conditions (I. 2) and (I. 3) by the similar arguments to the above. Thus we complete the proof of the main theorem for $\mathfrak{H}=\mathfrak{G}$. (The exceptional case can be easily settled.)

Finally I thank to Dr. Yamaguchi who kindly helped me through reading the manuscript.

§ 1. Preliminaries

1. 1. General remarks on terminologies and notations. Throughout the present paper, we shall always assume the differentiability of class C^∞.

(a) Let M be a manifold. $T(M)$ denotes the tangent bundle of M. By a differential system on M we mean a subbundle of $T(M)$.

(b) Let E be a vector bundle over a manifold M. For each $x \in M$, E_x or $E(x)$ denotes the fibre of E at x. $\underline{E}$ denotes the sheaf of germs of local cross sections of E. For each $x \in M$, $\underline{E}(x)$ denotes the stalk of E at x.

(c) Let π be a map of a manifold M to a manifold N. Then M is called a fibred manifold over the base space N with projection π, if π is surjective, and further if, for each $x \in M$, the differential π_{*_x} of π at x is surjective. If M is a fibred manifold over N with projection π, the differential system $E=\pi_*^{-1}(0)=\{X \in T(M) \mid \pi_*(X)=0\}$ on M is called the vertical tangent bundle of this fibred manifold.

(d) Let f be a function on a manifold M. For any $x \in M$ and any integer $k \geqq 0$, $j_x^k(f)$ denotes the k-jet of f at x.

(e) Let M be a manifold, and let r be an integer with $1 \leqq r \leqq \dim M-1$. Then $G^r(T(M))$ denotes the Grassmann bundle of r-dimensional contact elements to M. Let ϖ be the projection of $G^r(T(M))$ onto M. For each $z \in G^r(T(M))$, let C_z denote the subspace of the tangent (vector) space $T(G^r(T(M)))_z$ consisting of all vectors X such that the vectors $\varpi_*(X)$ are in the subspace z of the tangent space $T(M)_{\varpi(z)}$. Then the union

$C=\bigcup_z C_z$ gives a differential system on $G^r(T(M))$, which is called the canonical system on $G^r(T(M))$. Let A be an r-dimensional submanifold of M. For each $x\in A$, the tangent space $T(A)_x$ gives a point of $G^r(T(M))$. This being said, $\hat{A}$ denotes the subset $\{T(A)_x | x\in A\}$ of $G^r(T(M))$, which is an r-dimensional submanifold of $G^r(T(M))$ and is called the lift of A to $G^r(T(M))$. Note that $\hat{A}$ is an integral manifold of the canonical system C.

(f) The rest of the paragraph will be devoted to the definition of a Cartan connection (cf. [8], [11] and [12]).

Let $G/G^{(0)}$ be a homogeneous space of a Lie group G over its closed subgroup $G^{(0)}$. Put $m=\dim G/G^{(0)}$, and let $\mathfrak{g}$ (resp. $\mathfrak{g}^{(0)}$) denote the Lie algebra of G (resp. of $G^{(0)}$). Now let M be an m-dimensional manifold, and let P be a principal fibre bundle over the base space M with structure group $G^{(0)}$. Let ω be a $\mathfrak{g}$-valued 1-form on P. Then the pair (P, ω) is called a Cartan connection of type $G/G^{(0)}$ on M if it satisfies the following conditions:

(C. 1) For each $z\in P$, the assignment $X\rightarrow\omega(X)$ gives an isomorphism of $T(P)_z$ onto $\mathfrak{g}$,

(C. 2) $R_a^*\omega=\mathrm{Ad}(a^{-1})\omega,\ a\in G^{(0)}$,

(C. 3) $\omega(A^*)=A,\ A\in\mathfrak{g}^{(0)}$.

Here, as usual, R_a denotes the right translation on P corresponding to a, and A^* the vertical vector field on P corresponding to A.

Let (P, ω) (resp. (P', ω')) be a Cartan connection of type $G/G^{(0)}$ on a manifold M (resp. on M'). Let π (resp. π') be the projection of P onto M (resp. of P' onto M'). Then a diffeomorphism ϕ of M onto M' is called an isomorphism of (P, ω) onto (P', ω'), if there is a bundle isomorphism $\tilde{\phi}$ of P onto P' such that $\pi'\circ\tilde{\phi}=\phi\circ\pi$ and $\tilde{\phi}^*\omega'=\omega$. Note that $\tilde{\phi}$ is uniquely determined by ϕ, if $G/G^{(0)}$ is connected and if the action of G on $G/G^{(0)}$ is effective.

The Lie group G may be naturally regarded as a principal fibre bundle over the base space $G/G^{(0)}$ with structure group $G^{(0)}$. Let ω be the Maurer-Cartan form of G, i.e., the $\mathfrak{g}$-valued 1-form on G defined by $\omega(X_z)=X$ for all $X\in\mathfrak{g}$ and $z\in G$, where X should be identified with a left invariant vector field on G. Then it is easy to see that the pair (G, ω) gives a Cartan connection of type $G/G^{(0)}$ on $G/G^{(0)}$, which is called the standard Cartan connection of type $G/G^{(0)}$.

1. 2. Simple graded Lie algebras ([12]). First of all we recall the definition of a graded Lie algebra and of a FGLA. Let K be the field $\boldsymbol{R}$ of real numbers or the field C of complex numbers. Let $\mathfrak{g}$ be a Lie algebra over K, and let $(\mathfrak{g}_p)_{p\in Z}$ be a family of subspaces of $\mathfrak{g}$, where Z denotes the additive group of integers. Then the pair $\mathfrak{G}=\{\mathfrak{g}, (\mathfrak{g}_p)\}$ is called a graded Lie algebra, if it satisfies the following conditions: 1) $\mathfrak{g}=\sum_p \mathfrak{g}_p$ (direct sum), 2) dim $\mathfrak{g}_p<\infty$, 3) $[\mathfrak{g}_p, \mathfrak{g}_q]\subset\mathfrak{g}_{p+q}$.

Let $\mathfrak{T}=\{\mathfrak{t}, (\mathfrak{g}_p)\}$ be a graded Lie algebra such that $\mathfrak{g}_p=\{0\}$ for all $p\geqq 0$. (Hereafter such a graded Lie algebra will be written as $\mathfrak{T}=\{\mathfrak{t}, (\mathfrak{g}_p)_{p<0}\}$.) Then $\mathfrak{T}$ is called a fundamental graded Lie algebra or briefly a FGLA, if it satisfies the following conditions:

(FGLA. 1) dim $\mathfrak{t}<\infty$,

(FGLA. 2) $\mathfrak{g}_{-1}\neq\{0\}$, and the Lie algebra $\mathfrak{t}$ is generated by $\mathfrak{g}_{-1}$.

Given a positive integer μ, a FGLA, $\mathfrak{T}$, is called of the μ-th kind, if $\mathfrak{g}_{-\mu}\neq\{0\}$ and $\mathfrak{g}_p=\{0\}$ for all $p<-\mu$. It is also called non-degenerate, if the condition "$X\in\mathfrak{g}_{-1}$ and $[X, \mathfrak{g}_{-1}]=\{0\}$" implies $X=0$.

Now, let $\mathfrak{G}=\{\mathfrak{g}, (\mathfrak{g}_p)\}$ be a graded Lie algebra. If we set $\mathfrak{t}=\sum_{p<0}\mathfrak{g}_p$, we see that $\mathfrak{T}=\{\mathfrak{t}, (\mathfrak{g}_p)_{p<0}\}$ is a (truncated) graded subalgebra of $\mathfrak{G}$. Then $\mathfrak{G}$ is called a simple graded Lie algebra or briefly a SGLA, if it satisfies the following conditions:

(SGLA. 1) dim $\mathfrak{g}<\infty$, and $\mathfrak{g}$ is simple,

(SGLA 2) $\mathfrak{T}$ is a FGLA.

A simple graded Lie algebra $\mathfrak{G}$ is called of the μ-th kind, if the FGLA, $\mathfrak{T}$, is of the μ-th kind.

Let $\mathfrak{G}$ be a simple graded Lie algebra of the μ-th kind.

LEMMA 1. 1. *There is a unique element E in the centre of $\mathfrak{g}_0$ such that $[E, X]=pX$ for all $X\in\mathfrak{g}_p$ and p.*

The element E will be called the characteristic element of $\mathfrak{G}$.

Let $\langle\,,\rangle$ be the Killing form of the Lie algebra $\mathfrak{g}$.

LEMMA 1. 2. (1) $\langle\mathfrak{g}_p, \mathfrak{g}_q\rangle=\{0\}$ *if* $p+q\neq 0$.

(2) *For any integer p, the bilinear function $(X, Y)\rightarrow\langle X, Y\rangle$ on $\mathfrak{g}_p\times\mathfrak{g}_{-p}$ is non-degenerate.*

In particular it follows from this fact that dim $\mathfrak{g}_p=$dim $\mathfrak{g}_{-p}$.

LEMMA 1. 3. (1) *The natural representation of the Lie algebra $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is faithful.*

(2) *Let $p\geqq 0$, and $X\in\mathfrak{g}_p$. If $[X, \mathfrak{g}_{-1}]=\{0\}$, then $X=0$.*

From this fact we know that $\mathfrak{g}_0$ may be naturally regarded as a subalgebra of the derivation algebra Der($\mathfrak{T}$) of $\mathfrak{T}$, and that $\mathfrak{G}$ may be naturally regarded as a graded subalgebra of the prolongation of the pair

$(\mathfrak{T}\ \mathfrak{g}_0)$. (For the definition of the prolongation, see [9] or preferably [10].) We also remark that if $\mu \geqq 2$, $\mathfrak{T}$ is necessarily non-degenerate.

Examples. Let K be the field $\boldsymbol{R}$ or the field $\boldsymbol{C}$ or the skew field $\boldsymbol{Q}$ of quaternions. Let us consider the space $M_n(K)$ of matrices of degree n over K, being an associative algebra over $\boldsymbol{R}$. As usual, $M_n(K)$ may be considered as a Lie algebra over $\boldsymbol{R}$, which we denote by $\mathfrak{gl}(n, K)$. Furthermore the Lie algebra $\mathfrak{gl}(n, K)$ is reductive, and $\mathfrak{sl}(n, K)$ denotes its semi-simple part. (For the details, see § 5.) Now take k positive integers, $n_1, \dots, n_k$, and set $n = n_1 + \dots + n_k$, where $k \geqq 2$. Then every matrix X of $\mathfrak{g} = \mathfrak{sl}(n, K)$ may be expressed as follows: $X = (X_{ij})_{1 \leqq i, j \leqq k}$, where X_{ij} are $n_i \times n_j$-matrices. For any integer p, define a subspace $\mathfrak{g}_p$ of $\mathfrak{g}$ by

$$\mathfrak{g}_p = \{X \in \mathfrak{g} \mid X_{ij} = 0 \text{ if } j - i \neq p\}.$$

Then we easily see that $\mathfrak{G} = \{\mathfrak{g}, (\mathfrak{g}_p)\}$ becomes a simple graded Lie algebra of the $(k-1)$-th kind over $\boldsymbol{R}$, which we denote by $\mathfrak{G}(n_1, \dots, n_k; K)$. Correspondingly the space $\mathfrak{g}_p$ will be denoted by $\mathfrak{g}_p(n_1, \dots, n_k; K)$.

In the following we shall consider a fixed simple graded Lie algebra $\mathfrak{G}$ of the μ-th kind over $\boldsymbol{R}$. Let us consider the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of the simple Lie algebra $\mathfrak{g}$. As usual, the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ may be identified with $\mathfrak{g}$, so that $\mathrm{Ad}(a)X = aX$ for all $a \in \mathrm{Aut}(\mathfrak{g})$ and $X \in \mathfrak{g}$.

For any injeger p, define a subspace $\mathfrak{g}^{(p)}$ of $\mathfrak{g}$ by $\mathfrak{g}^{(p)} = \sum_{i \geqq p} \mathfrak{g}_i$. Then we have $[\mathfrak{g}^{(p)}, \mathfrak{g}^{(q)}] \subset \mathfrak{g}^{(p+q)}$, and hence the family $(\mathfrak{g}^{(p)})$ gives a filtration in $\mathfrak{g}$. Thus we get the filtred Lie algebra $F(\mathfrak{G}) = \{\mathfrak{g}, (\mathfrak{g}^{(p)})\}$. Now consider the automorphism group $\mathrm{Aut}(\mathfrak{G})$ of the graded Lie algebra $\mathfrak{G}$, and the automorphism group $\mathrm{Aut}(F(\mathfrak{G}))$ of the filtred Lie algebra $F(\mathfrak{G})$.

LEMMA 1.4. $\mathrm{Aut}(F(\mathfrak{G}))$ *coincides with the normalizer of the subalgebra* $\mathfrak{g}^{(0)}$ *of* $\mathfrak{g}$ *in* $\mathrm{Aut}(\mathfrak{g})$.

LEMMA 1.5. *The Lie algebra of* $\mathrm{Aut}(\mathfrak{G})$ *is* $\mathfrak{g}_0$, *and the Lie algebra of* $\mathrm{Aut}(F(\mathfrak{G}))$ *is* $\mathfrak{g}^{(0)}$.

Clearly the graded Lie algebra associated with the filtred Lie algebra $F(\mathfrak{G})$ is naturally isomorphic with the given $\mathfrak{G}$. Hence we have a natural homomorphism κ of $\mathrm{Aut}(F(\mathfrak{G}))$ onto $\mathrm{Aut}(\mathfrak{G})$. Let $G^{(1)}$ be the Lie subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by the (nilpotent) subalgebra $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$, and let exp be the expential map of $\mathfrak{g}$ to $\mathrm{Aut}(\mathfrak{g})$.

LEMMA 1.6. *The kernel of the homomorphism* κ *is* $G^{(1)}$, *and* $\mathrm{Aut}(F(\mathfrak{G}))$ *is a semi-direct product of* $\mathrm{Aut}(\mathfrak{G})$ *and* $G^{(1)}$. *Furthermore every element* a *of* $G^{(1)}$ *may be written uniquely in the following form*:

$$a=\exp X_1 \dots \exp X_\mu,$$

where $X_i \in \mathfrak{g}_i$ $(1 \leqq i \leqq \mu)$.

LEMMA 1.7. *The natural representation of* Aut($\mathfrak{G}$) *on* $\mathfrak{g}_{-1}$ *is faithful.*

Now, let G_0 be an open subgroup of Aut($\mathfrak{G}$). We define subgroups $G^{(0)}$ and G of Aut($\mathfrak{g}$) respectively by

$$G^{(0)}=\kappa^{-1}(G_0)=G_0\cdot G^{(1)}, \quad G=\mathrm{Aut}(\mathfrak{g})^0\cdot G_0,$$

where Aut$(\mathfrak{g})^0$ stands for the connected component of the identity of Aut($\mathfrak{g}$).

REMARK. The groups $G^{(0)}$ and G will be called associated with the pair $(\mathfrak{G}, G_0)$. In the special case where $G_0=\mathrm{Aut}(\mathfrak{G})$, the groups G_0, $G^{(0)}$ and G will be called associated with $\mathfrak{G}$. In our previous paper [12], we exclusively considered the groups G_0, $G'(=G^{(0)})$ and G associated with $\mathfrak{G}$, on the basis of which the theory was developed. However, as is easily observed, all the results in that paper hold good even in our generalized or modified situation.

Let us consider the homogeneous space $G/G^{(0)}$.

LEMMA 1.8. $G/G^{(0)}$ *is connected and compact, and the action of* G *on* $G/G^{(0)}$ *is effective.*

We have $\mathfrak{g}=\mathfrak{t}+\mathfrak{g}^{(0)}$ (direct sum). This being said, we define a representation ρ of $G^{(0)}$ on $\mathfrak{t}$ by

$$\rho(a)X \equiv \mathrm{Ad}(a)X \ (\mathrm{mod}\ \mathfrak{g}^{(0)}) \quad \text{for all } a\in G^{(0)} \text{ and } X\in\mathfrak{t},$$

which may be naturally regarded as the linear isotropy representation of $G^{(0)}$ on the tangent space $T(G/G^{(0)})_o(\cong\mathfrak{t})$, o being the origin of $G/G^{(0)}$. Let $G^{(\mu)}$ be the Lie subgroup of $G^{(1)}$ generated by the (abelian) ideal $\mathfrak{g}^{(\mu)}$ of $\mathfrak{g}^{(1)}$.

LEMMA 1.9. *The kernel of the homomorphism* ρ *coincides with* $G^{(\mu)}$.

Finally we recall some definitions in the equivalence problem associated with the simple graded Lie algebra $\mathfrak{G}$. A Cartan connection (P, ω) of type $G/G^{(0)}$ on a manifold M is called a connection of type $\mathfrak{G}$ on M, and this connection is called normal, if the curvature $K=\sum_p K^p$ satisfies the following conditions: 1) $K^p=0$ for $p<0$, 2) $\partial^* K^p=0$ for $p\geqq 0$ (see page 47 in [12]). Clearly the standard connection (G, ω) of type $G/G^{(0)}$ is a normal connection of type $\mathfrak{G}$, which is called the standard connection of type $\mathfrak{G}$.

1.3. Some facts on simple graded Lie algebras of the first kind. Let $\mathfrak{G}$ be a simple graded Lie algebra of the first kind. Let G_0 be an open subgroup of Aut($\mathfrak{G}$) such that $\mathrm{Aut}(\mathfrak{g})^0\cap\mathrm{Aut}(\mathfrak{G})\subset G_0$. Let $G^{(0)}$ and G be the groups associated with the pair $(\mathfrak{G}, G_0)$: $G^{(0)}=G_0\cdot G_1$ and $G=\mathrm{Aut}(\mathfrak{g})^0\cdot G_0$, where G_1 stands for the group $G^{(1)}$, i.e., the Lie subgroup of $\mathrm{Aut}(\mathfrak{g})^0$ generated by the abelian subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$. Similarly let G_{-1} be the Lie subgroup of $\mathrm{Aut}(\mathfrak{g})^0$ generated by the abelian subalgebra $\mathfrak{g}_{-1}$ of $\mathfrak{g}$. Let us consider the subset of G:

$$S = G_{-1} \cdot G_0 \cdot G_1 .$$

LEMMA 1.10. *Every element a of S can be written uniquely in the following form*:

$$a = \exp X_{-1} \cdot b \cdot \exp X_1 ,$$

where $X_{-1} \in \mathfrak{g}_{-1}$, $b \in G_0$ and $X_1 \in \mathfrak{g}_1$.

PROOF. Let $X_{-1} \in \mathfrak{g}_{-1}$, $b \in G_0$ and $X_1 \in \mathfrak{g}_1$. To prove the lemma, it suffices to prove that the condition "$\exp X_{-1} = b \cdot \exp X_1$" implies that $X_{-1} = X_1 = 0$, and $b = e$, the identity of G. Accordingly assume that $\exp X_{-1} = b \cdot \exp X_1$. Then we have $\mathrm{Ad}(\exp X_{-1})E = \mathrm{Ad}(b \cdot \exp X_1)E$, where E is the characteristic element of G. We have $\mathrm{Ad}(\exp X_{-1})E = E + X_{-1}$, and $\mathrm{Ad}(b \cdot \exp X_1)E = E - \mathrm{Ad}(b)X_1$, whence $X_{-1} = -\mathrm{Ad}(b)X_1$. Therefore we obtain $X_{-1} = X_1 = 0$, and hence $b = e$. Thus the lemma follows.

Let $a \in S$ and $Y_1 \in \mathfrak{g}_1$. The notations being as in Lemma 1.10, we have

$$\mathrm{Ad}(a) Y_1 = \mathrm{Ad}(\exp X_{-1}) \mathrm{Ad}(b) Y_1 \equiv \mathrm{Ad}(b) Y_1 \pmod{\mathfrak{g}_{-1} + \mathfrak{g}_0}.$$

Now, for any $a \in G$, we define an endomorphism $\xi(a)$ of $\mathfrak{g}_1$ by

$$\mathrm{Ad}(a) Y_1 \equiv \xi(a) Y_1 \pmod{\mathfrak{g}_{-1} + \mathfrak{g}_0} \quad \text{for all } Y_1 \in \mathfrak{g}_1,$$

and set

$$S' = \{ a \in G \mid \xi(a) \in GL(\mathfrak{g}_1) \},$$

where $GL(\mathfrak{g}_1)$ denotes the general linear group of $\mathfrak{g}_1$. We have $S \subset S'$, as we have just seen.

LEMMA 1.11. $S = S'$.

PROOF. Take any $a \in S'$, and set $\xi = \xi(a)$. Since $\xi \in GL(\mathfrak{g}_1)$, there is $X_1 \in \mathfrak{g}_1$ such that

$$\mathrm{Ad}(a)E \equiv -\xi X_1 \pmod{\mathfrak{g}_{-1} + \mathfrak{g}_0}.$$

Put $a' = a \cdot \exp(-X_1)$. Then we have $\mathrm{Ad}(a')E = \mathrm{Ad}(a)(E + X_1) \equiv -\xi X_1 + \xi X_1 \pmod{\mathfrak{g}_{-1} + \mathfrak{g}_0}$, and hence $\mathrm{Ad}(a')E \equiv 0 \pmod{\mathfrak{g}_{-1} + \mathfrak{g}_0}$. Therefore there is $X_0 \in \mathfrak{g}_0$ such that

$$\mathrm{Ad}(a')E \equiv X_0 \pmod{\mathfrak{g}_{-1}}.$$

We assert that $X_0=E$. Indeed, let $Y_1\in\mathfrak{g}_1$. Then we have $[E, Y_1]=Y_1$, and hence $[\mathrm{Ad}(a')E, \mathrm{Ad}(a')Y_1]=\mathrm{Ad}(a')Y_1$. We have $\mathrm{Ad}(a')Y_1=\mathrm{Ad}(a)Y_1 \equiv \xi Y_1 \pmod{\mathfrak{g}_{-1}+\mathfrak{g}_0}$. From these facts it follows that $[X_0, \xi Y_1]=\xi Y_1=[E, \xi Y_1]$. Since $\xi\mathfrak{g}_1=\mathfrak{g}_1$, and since the natural representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful (cf. Lemmas 1.2 and 1.7), it follows that $X_0=E$, proving our assertion.

Therefore we have $\mathrm{Ad}(a')E\equiv E \pmod{\mathfrak{g}_{-1}}$, and hence there is $X_{-1}\in\mathfrak{g}_{-1}$ such that

$$\mathrm{Ad}(a')E=E+X_{-1}.$$

If we put $b=\exp(-X_{-1})\cdot a'$, we easily have $\mathrm{Ad}(b)E=E$, which means that $b\in\mathrm{Aut}(\mathfrak{G})\cap G=G_0$. We have thus shown that $a=a'\cdot\exp X_1=\exp X_{-1}\cdot b\cdot\exp X_1\in S$, and hence $S'\subset S$, proving the lemma.

COROLLARY 1. *S is an open, dense subset of G.*

Let π be the projection of G onto $G/G^{(0)}$. We define a map ι of $\mathfrak{g}_{-1}$ to $G/G^{(0)}$ by

$$\iota(X)=\pi(\exp X) \text{ for all } X\in\mathfrak{g}_{-1}.$$

Then we see from Lemma 1.10 that ι is an (open) imbedding. Clearly we have $S=\pi^{-1}(\iota(\mathfrak{g}_{-1}))$. Therefore from Corollary 1 we get

COROLLARY 2. *The image $\iota(\mathfrak{g}_{-1})$ of $\mathfrak{g}_{-1}$ by ι is an open, dense subset of $G/G^{(0)}$.*

1.4. $\tilde{G}_0$-structures ([8] and [12]). Let $\mathfrak{G}$ be a simple graded Lie algebra of the first kind, and let G_0, $G^{(0)}$ and G be the associated groups. Let us consider the homogeneous space $G/G^{(0)}$, and let ρ be the associated linear isotropy representation of the group $G^{(0)}$ on the space $\mathfrak{g}_{-1}(=\mathfrak{t})$. We denote by $\tilde{G}_0$ the image of $G^{(0)}$ by the homomorphism ρ of $G^{(0)}$ to $GL(\mathfrak{g}_{-1})$, which is called the linear isotropy group associated with $G/G^{(0)}$. Since $G^{(0)}=G_0\cdot G_1$, and since the kernel of ρ coincides with $G^{(1)}=G_1$ (Lemma 1.9), we see that ρ gives an isomorphism of G_0 onto $\tilde{G}_0$.

$\tilde{G}_0$ being a Lie subgroup of $GL(\mathfrak{g}_{-1})$, we have the notion of a $\tilde{G}_0$-structure, which we shall clarify from now on.

For this purpose, we first recall the definition of the frame bundle. Put $m=\dim G/G^{(0)}=\dim\mathfrak{g}_{-1}$, and let M be an m-dimensional manifold. For each $x\in M$, we denote by $F(M)_x$ the set of all isomorphisms z of $\mathfrak{g}_{-1}$ to the tangent space $T(M)_x$, and put $F(M)=\bigcup_x F(M)_x$. Then $F(M)$ becomes naturally a principal fibre bundle over the base space M with structure group

$GL(\mathfrak{g}_{-1})$, which is called the frame bundle of M. Let M (resp. M') be an m-dimensional manifold, and $F(M)$ (resp. $F(M')$) its frame bundle. Then a diffeomorphism ϕ of M onto M' naturally induces a bundle isomorphism $\bar{\phi}$ of $F(M)$ onto $F(M')$:

$$\bar{\phi}(z)\cdot X=\phi_*(z\cdot x),\ z\in F(M),\ X\in\mathfrak{g}_{-1}.$$

Let M be an m-dimensional manifold. Then a reduction Q of the frame bundle $F(M)$ to the group $\tilde{G}_0$ (or a $\tilde{G}_0$-subbundle of $F(M)$) is called a $\tilde{G}_0$-structure on M. Let Q (resp. Q') be a $\tilde{G}_0$-structure on a manifold M (resp. on M'). By an isomorphism of Q onto Q' we mean a diffeomorphism ϕ of M onto M' such that the bundle isomorphism $\bar{\phi}$ sends Q to Q'.

We now show that to the homogeneous space $G/G^{(0)}$ there is naturally associated a $\tilde{G}_0$-structure on it. Consider the frame bundle $F(G/G^{(0)})$ of $G/G^{(0)}$. Let z_0 be the point of $F(G/G^{(0)})$ given by

$$z_0\cdot X=\pi_*(X_e),\ X\in\mathfrak{g}_{-1},$$

where π is the projection of G onto $G/G^{(0)}$. For each $a\in G$, let τ_a denote the transformation of $G/G^{(0)}$ induced by a. Then the group G acts on $F(G/G^{(0)})$ through the correspondence $a\rightarrow\bar{\tau}_a$, and let Q be the G-orbit through the point $z_0\in F(G/G^{(0)})$. As is easily verified, we have

$$\bar{\tau}_a(z_0)=z_0\cdot\rho(a),\ a\in G^{(0)}.$$

Using this fact, we easily show that Q is a $\tilde{G}_0$-structure on $G/G^{(0)}$.

Here, we recall the following

LEMMA 1.12 (cf. [6], III, and [8]). *$\mathfrak{G}$ is the prolongation of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ (or precisely of $(\mathfrak{T}, \mathfrak{g}_0)$) if and only if $\mathfrak{G}$ is isomorphic with none of the simple graded Lie algebras $\mathfrak{G}(1, n; K)$ of the first kind, where $n\geqq 1$ and $K=$ **R** or **C**.*

Now we have the notion of a normal connection of type $\mathfrak{G}$, being a connection of type $G/G^{(0)}$. In the following we assume that $\mathfrak{G}$ is the prolongation of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.

FACT A (Theorem 2.7 in [12]). *To every $\tilde{G}_0$-structure Q on a manifold M there is associated a normal connection (P, ω) of type $\mathfrak{G}$ on M in an invariant manner.*

In connection with this fact, see also [8]. Let Q (resp. Q') be a $\tilde{G}_0$-structure on a manifold M (resp. on M'), and let (P, ω) (resp. (P', ω')) be the associated normal connection of type $\mathfrak{G}$ on M (resp. on M'). Then the invariance means that a diffeomorphism ϕ of M onto M' is an isomorphism of Q onto Q', if and only if it is an isomorphism of (P, ω) onto (P', ω'). Therefore it follows that the automorphism group $\mathrm{Aut}(Q)$ of the $\tilde{G}_0$-structure Q coincides with the automorphism group $\mathrm{Aut}(P, \omega)$ of the connection (P, ω), and further that $\mathrm{Aut}(Q)$ becomes naturally a Lie group of dimension at most $\dim \mathfrak{g}$.

Let us now consider the $\tilde{G}_0$-structure Q associated with $G/G^{(0)}$. The following fact also follows from Theorem 2.7 cited above.

FACT B. *The standard connection of type* $\mathfrak{G}$, (G, ω), *may be naturally regarded as the normal connection of type* $\mathfrak{G}$ *associated with the* $\tilde{G}_0$*-structure* Q.

Therefore we have $\mathrm{Aut}(Q)=\mathrm{Aut}(G, \omega)$, from which we can easily derive the following

LEMMA 1.13. *The natural homomorphism,* $a \rightarrow \tau_a$, *of* G *to* $\mathrm{Aut}(Q)$ *gives an isomorphism (onto).*

1.5. Pseudo-product manifolds, and the symbol algebras ([13] and cf. [9]). Let R' be a product manifold, and (E', F') its product structure. Let R be a submanifold of R'. For each $x \in R$, we define subspaces E_x and F_x of $T(R)_x$ respectively by

$$E_x = E'_x \cap T(R), \quad F_x = F'_x \cap T(R)_x,$$

and set $E = \bigcup_x E_x$ and $F = \bigcup_x F_x$. We now assume the following regularity condtion: Both $\dim E_x$ and $\dim F_x$ are constant. Then we see that E and F give differential systems on R and that the pair (E, F) satisfies the following conditions:

(PP. 1) $E \cap F = 0$, the zero cross section of $T(R)$,

(PP. 2) Both E and F are completely integrable.

Now, let R be a manifold, and let E and F be differential systems on it. Then the pair (E, F) is called a pseudo-product structure on R, if it satisfies conditions (PP. 1) and (PP. 2). A manifold R equipped with a pseudo-product structure (E, F) is called a pseudo-product manifold. Let R (resp. R') be a pseudo-product manifold, and (E, F) (resp. (E', F')) its pseudo-product structure. By an isomorphism of R onto R' we mean a diffeomorphism ϕ of R onto R' such that the differential ϕ_* of ϕ sends E to E' and F to F'.

As we have seen above, a submanifold R of a product manifold R' becomes naturally a pseudo-product manifold, provided R satisfies the regularity condition. Conversely it is shown that any pseudo-product manifold R can be locally realized as a submanifold of a product manifold. More precisely, let x be any point of R. Then there are a product manifold R' and an imbedding ι of a neighborhood of x to R' such that the pseudo-product structure (E, F) of R, restricted to the neighborhood, is induced from the product structure (E', F') of R' by the imbedding ι, and such that $2 \dim R = \dim R' + \operatorname{rank} E + \operatorname{rank} F$. Note that such a pair (R', ι) is unique in a suitable sense.

Let R be a pseudo-product manifold, and (E, F) its pseudo-product structure. Set $D = E + F$, which is a differential system on R by (PP. 1). The sheaf $\underline{T(R)}$ of germs of local cross sections of the tangent bundle $T(R)$ is naturally a sheaf of Lie algebras, and the sheaf $\underline{D}$ of germs of local cross sections of the differential system D is a subsheaf of $\underline{T(R)}$. Now assume the following conditions:

(*. 1) D is regular in the sense of [9] and the derived system of D coincides with $T(R)$, or in other words,

$$\underline{T(R)}(x) = [\underline{D}(x), \underline{D}(x)] + \underline{D}(x) \text{ at each } x \in R,$$

(*. 2) $T(R) \supsetneqq D \supsetneqq 0$.

Let $x \in R$. For any negative integer p we define a vector space $\mathfrak{g}_p(x)$ as follows:

$$\mathfrak{g}_p(x) = \{0\} \text{ if } p < -2,$$
$$\mathfrak{g}_{-2}(x) = T(R)(x)/D(x), \quad \mathfrak{g}_{-1}(x) = D(x),$$

and set $\mathfrak{t}(x) = \sum_{p<0} \mathfrak{g}_p(x) = \mathfrak{g}_{-2}(x) + \mathfrak{g}_{-1}(x)$. We now define a bracket operation [,] in $\mathfrak{t}(x)$ by the requirement that $[\mathfrak{g}_{-2}(x), \mathfrak{g}_{-2}(x)] = [\mathfrak{g}_{-2}(x), \mathfrak{g}_{-1}(x)] = \{0\}$, and

$$[X_x, Y_x] = \varpi([X, Y]_x) \text{ for all } X, Y \in \underline{D}(x),$$

where ϖ denotes the projection of $T(R)$ onto $T(R)/D$. Then we easily see that [,] is well defined, and that $\mathfrak{t}(x)$ becomes a Lie algebra with respect to this bracket operation. We also see that $\mathfrak{T}(x) = \{\mathfrak{t}(x), (\mathfrak{g}_p(x))_{p<0}\}$ becomes a FGLA of the second kind, which is called the symbol algebra of the differential system D at the point x ([9]).

Now, $E(x)$ and $F(x)$ are subspaces of $\mathfrak{g}_{-1}(x) = D(x)$. Then the system $\mathfrak{L}(x) = \{\mathfrak{T}(x); E(x), F(x)\}$ will be called the symbol algebra of the pseudo-product manifold R at the point x. Clearly we have $\mathfrak{g}_{-1}(x) = E(x) + F(x)$ (direct sum), and by condtion (PP. 2) we have $[E(x), E(x)] = [F(x), F(x)] = \{0\}$.

1. 6. Pseudo-product FGLA's, and pseudo-product manifolds of type $\mathfrak{L}$ ([13] and cf. [9]). Let $\mathfrak{T}=\{\mathfrak{t}, (\mathfrak{g}_p)_{p<0}\}$ be a FGLA over K, where $K=\boldsymbol{R}$ or $\boldsymbol{C}$, and let $\mathfrak{e}$ and $\mathfrak{f}$ be subspaces of $\mathfrak{g}_{-1}$. Then the system $\mathfrak{L}=\{\mathfrak{T};\ \mathfrak{e}, \mathfrak{f}\}$ is called a pseudo-product FGLA, if it satisfies the following conditions:

(PPF. 1) $\mathfrak{g}_{-1}=\mathfrak{e}+\mathfrak{f}$ (direct sum),

(PPF. 2) $[\mathfrak{e}, \mathfrak{e}]=[\mathfrak{f}, \mathfrak{f}]=\{0\}$.

Let $\mathfrak{L}=\{\mathfrak{T};\ \mathfrak{e}, \mathfrak{f}\}$ and $\mathfrak{L}'=\{\mathfrak{T}';\ \mathfrak{e}', \mathfrak{f}'\}$ be two pseudo-product FGLA's. By an isomorphism of $\mathfrak{L}$ onto $\mathfrak{L}'$ we mean an isomorphism ϕ of $\mathfrak{T}$ onto $\mathfrak{T}'$ such that ϕ sends $\mathfrak{e}$ to $\mathfrak{e}'$ and $\mathfrak{f}$ to $\mathfrak{f}'$.

Let $\mathfrak{L}=\{\mathfrak{T};\ \mathfrak{e}, \mathfrak{f}\}$ be a pseudo-product FGLA. Then $\mathfrak{L}$ is called of the μ-th kind (resp. non-degenerate), if $\mathfrak{T}$ is of the μ-th kind (resp. non-degenerate). Furthermore let $\mathfrak{g}_0$ be the derivation algebra $\mathrm{Der}(\mathfrak{L})$ of $\mathfrak{L}$: $\mathrm{Der}(\mathfrak{L})=\{X\in\mathrm{Der}(\mathfrak{T})\,|\,X\mathfrak{e}\subset\mathfrak{e},\ X\mathfrak{f}\subset\mathfrak{f}\}$. Then the prolongation $\mathfrak{G}$ of $(\mathfrak{T}, \mathfrak{g}_0)$ is called the prolongation of $\mathfrak{L}$.

LEMMA 1. 14. *Let $\mathfrak{G}$ be the prolongation of $\mathfrak{L}$. If $\mathfrak{L}$ is non-degenerate, then the underlying Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ is finite dimensional.*

Let R be a pseudo-product manifold satisfying conditions (*. 1) and (*. 2). Let us consider the symbol algebra $\mathfrak{L}(x)$ of R at each point $x\in R$, which is a pseudo-product FGLA of the second kind. Now let $\mathfrak{L}$ be a fixed pseudo-product FGLA of the second kind over $\boldsymbol{R}$. Then we say that the pseudo-product manifold R is of type $\mathfrak{L}$, if $\mathfrak{L}(x)$ is isomorphic with $\mathfrak{L}$ at each point x.

Hereafter we assume that $\mathfrak{L}$ is non-degenerate and that the prolongation $\mathfrak{G}$ of $\mathfrak{L}$ is simple. Then the automorphism group $\mathrm{Aut}(\mathfrak{L})$ of $\mathfrak{L}$ may be naturally identified with an open subgroup of the automorphism group $\mathrm{Aut}(\mathfrak{G})$ of $\mathfrak{G}$. Set $G_0=\mathrm{Aut}(\mathfrak{L})$, and let $G^{(0)}$ and G be the groups associated with the pair $(\mathfrak{G}, G_0)$.

Now, we have the notion of a normal connection of type $\mathfrak{G}$, being a Cartan connection of type $G/G^{(0)}$.

FACT C. *To every pseudo-product manifold R of type $\mathfrak{L}$ there is associated a connection (P, ω) of type $\mathfrak{G}$ on R in an invariant manner.*

As before it follows from this fact that the automorphism group $\mathrm{Aut}(R)$ of the pseudo-product manifold R coincides with the automorphism group $\mathrm{Aut}(P, \omega)$ of the connection (P, ω), and that $\mathrm{Aut}(R)$ becomes naturally a Lie group of dimension at most $\dim\mathfrak{g}$.

We shall now show that the homogeneous space $G/G^{(0)}$ becomes naturally a pseudo-product manifold of type $\mathfrak{L}$. Let us consider the subspaces $\mathfrak{a}^{(0)}=\mathfrak{e}+\mathfrak{g}^{(0)}$ and $\mathfrak{b}^{(0)}=\mathfrak{f}+\mathfrak{g}^{(0)}$ of $\mathfrak{g}$. Then we can easily verify the following: 1) $\mathrm{Ad}(G^{(0)})\mathfrak{a}^{(0)}=\mathfrak{a}^{(0)}$ and $\mathrm{Ad}(G^{(0)})\mathfrak{b}^{(0)}=\mathfrak{b}^{(0)}$, 2) $\mathfrak{a}^{(0)}\cap\mathfrak{b}^{(0)}=\mathfrak{g}^{(0)}$, 3) Both $\mathfrak{a}^{(0)}$ and $\mathfrak{b}^{(0)}$ are subalgebras of $\mathfrak{g}$. By 1) we see that $\mathfrak{a}^{(0)}$ and $\mathfrak{b}^{(0)}$ naturally induce invariant differential systems E and F on $G/G^{(0)}$ respectively, and by 2) and 3) that the pair (E, F) gives an invariant pseudo-product structure on $G/G^{(0)}$. Since $\mathfrak{g}_{-2}=[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$, it follows easily that the pseudo-product manifold $G/G^{(0)}$, thus obtained, satisfies conditions $(*.1)$ and $(*.2)$. Moreover it is not difficult to verify that the symbol algebra $\mathfrak{L}(o)$ of $G/G^{(0)}$ at the origin o is isomorphic with the given pseudo-product FGLA, $\mathfrak{L}$, indicating that $G/G^{(0)}$ is of type $\mathfrak{L}$.

FACT D. *The standard connection (G, ω) of type $\mathfrak{G}$ may be naturally regarded as the normal connection of type $\mathfrak{G}$ associated with the pseudo-product manifold $G/G^{(0)}$ of type $\mathfrak{L}$.*

Therefore the automorphism group $\mathrm{Aut}(G/G^{(0)})$ of the pseudo-product manifold $G/G^{(0)}$ coincides with the automorphism group $\mathrm{Aut}(G, \omega)$ of the connection (G, ω). For $a\in G$ let τ_a denote the transformation of $G/G^{(0)}$ induced by a. Then we have the following

LEMMA 1.15. *The natural homomorphism, $a\rightarrow\tau_a$, of G to $\mathrm{Aut}(G/G^{(0)})$ gives an isomorphism (onto).*

REMARK. Let (P, ω) be the normal connection of type $\mathfrak{G}$ associated with a pseudo-product manifold R of type $\mathfrak{L}$. We shall explain how the connection is related to the pseudo-product structure (E, F) of R. Set $\mathfrak{G}=\{\mathfrak{g}, (\mathfrak{g}_p)\}$, and set $\mathfrak{e}=\mathfrak{g}^+_{-1}$ and $\mathfrak{f}=\mathfrak{g}^-_{-1}$. According to the decomposition $\mathfrak{g}=\sum_p \mathfrak{g}_p$, the connection form ω is decomposed as follows: $\omega=\sum_p \omega_p$, and according to the decomposition $\mathfrak{g}_{-1}=\mathfrak{g}^+_{-1}+\mathfrak{g}^-_{-1}$, the $\mathfrak{g}_{-1}$-valued 1-form ω_{-1} as follows: $\omega_{-1}=\omega^+_{-1}+\omega^-_{-1}$. Let π be the projection of P onto R. Then the differential system $\pi_*^{-1}(E)$ on P is defined by the equations: $\omega_{-2}=\omega^-_{-1}=0$, and the differential system $\pi_*^{-1}(F)$ on P by the equations: $\omega_{-2}=\omega^+_{-1}=0$. Note that these conditions together with the normality condition characterize the connection (P, ω).

§ 2. The automorphism groups of product manifolds

2.1. The projective geometry. Let (n, r) be a pair of integers with $1\leqq r\leqq n-1$. Let P^n be the n-dimensional projective space over $\boldsymbol{R}$, and $G^r(P^n)$ the Grassmann manifold of r-dimensional projective subspaces of P^n. Let us consider the relation $R=R_{n,r}$ in the product manifold $P^n\times G^r(P^n)$ defined by

$$R=\{(p, \alpha)\in P^n\times G^r(P^n) \mid p\subset\alpha\},$$

where " $p\subset\alpha$ " means that the point p lies on the projective subspace α. It is easy to see that R is a compact submanifold of the product manifold. Let $\Omega=\Omega_{n,r}$ be the complement of R, being an open submanifold of the product manifold. We remark that dim $G^r(P^n)=(r+1)(n-r)$ and dim $R_{n,r}=n+r(n-r)$.

Let G be the projective transformation group of P^n. The group G naturally acts on the manifold $G^r(P^n)$, and hence it acts on the product manifold $P^n\times G^r(P^n)$ through the diagonal map of G to $G\times G$:

$$a\cdot(p, \alpha)=(ap, a\alpha), \ a\in G, \ (p, \alpha)\in P^n\times G^r(P^n).$$

We remark that the actions of G on both P^n and $G^r(P^n)$ are transitive, and that R and Ω are G-orbits.

Let us consider the automorphism group Aut(Ω) of the product manifold Ω. Then we have a natural injective homomorphism i of G to Aut(Ω): For $a\in G$, $i(a)$ is the transformation of Ω induced by a.

We shall prove the following

PROPOSITION 2.1. *The homomorphism i gives an isomorphism of G onto* Aut(Ω).

Let ρ' (resp. ρ'') be the projection of Ω to P^n (resp. of Ω to $G^r(P^n)$), and let (E_Ω, F_Ω) be the product structure of Ω. Then we can easily prove the following

LEMMA 2.2. *Ω is a fibred manifold over the base space P^n (resp. over $G^r(P^n)$) with projection ρ' (resp. with ρ''). Furthermore its fibres are all connected, and its vertical tangent bundle is given by F_Ω (resp. by E_Ω).*

Therefore we see that each fibre of the fibred manifold is a maximal connected integral manifold of F_Ω (resp. of E_Ω), and vice versa. Hence it follows that every $\phi\in$Aut(Ω) naturally induces a diffeomorphism ϕ' (resp. ϕ'') of P^n (resp. of $G^r(P^n)$): $\rho'\circ\phi=\phi'\circ\rho'$ (resp. $\rho''\circ\phi=\phi''\circ\rho''$).

Thus we have the product transformation $\phi'\times\phi''$ of $P^n\times G^r(P^n)$, being an automorphism of the product manifold. Clearly the restriction of $\phi'\times\phi''$ to Ω coincides with the given ϕ, and hence $\phi'\times\phi''$ leaves the boundary $\partial\Omega=R$ of Ω invariant. This clearly means the following: Let $p\in P^n$ and $\alpha\in G^r(P^n)$. Then $p\subset\alpha$, if and only if $\phi'(p)\subset\phi''(\alpha)$. Conseguently we know that ϕ' naturally induces a transformation on the r-dimensional projective subspaces, and hence a transformation on the projective lines. Therefore it follows from a fundamental theorem in the projective geometry that ϕ' is a projective transformation. Hence there is $a\in G$ such that $\phi'(p)=ap$ for all

$p \in P^n$. Clearly we have $\phi''(\alpha)=a\alpha$ for all $\alpha \in G^r(P^n)$. We have thus shown that $\phi(w)=aw$ for all $w \in \Omega$, that is, $\phi=i(a)$, proving the proposition.

As we have mentioned in Introduction, Proposition 2.1 is just the starting-point of our study, which will be generalized in two-manners: One is from the view-point of the manifold theory, and the other from the view-point of the Lie group theory.

2. 2. Generic submanifolds of product manifolds ([13]). As in the preceding paragraph, let (n, r) be a pair of integers with $1 \leqq r \leqq n-1$. Let R be a pseudo-product manifold, and (E, F) its pseudo-product structure. Set $D=E+F$, and, for each point $x \in R$, let γ_x be the torsion of the differential system D at x. Recall that γ_x is the anti-symmetric bilinear map of $D_x \times D_x$ to $T(R)_x/D_x$ defined by

$$\gamma_x(X_x, Y_x)=\varpi([X, Y]_x) \text{ for all } X, Y \in \underline{D}(x),$$

where ϖ denotes the projection of $T(R)$ onto $T(R)/D$.

Now, the pseudo-product manifold R is called a projective system of type (n, r), if it satisfies the following conditions:

(PRS. 1) $\dim R=n+r(n-r)$, rank $E=r$ and rank $F=r(n-r)$,

(PRS. 2) D is non-degenerate, i. e., the torsion γ_x of D at each point x is non-degenerate.

Let $\mathfrak{L}=\{\mathfrak{T}; \mathfrak{e}, \mathfrak{f}\}$ be a pseudo-product FGLA, and set $\mathfrak{T}=\{\mathfrak{t}, (\mathfrak{g}_p)_{p<0}\}$. Then $\mathfrak{L}$ is called a projective FGLA of type (n, r), if it satisfies the following conditions:

(PRF. 1) $\mathfrak{L}$ is of the second kind, and $\dim \mathfrak{g}_{-2}=n-r$, $\dim \mathfrak{e}=m$ and $\dim \mathfrak{f}=r(n-r)$,

(PRF. 2) $\mathfrak{L}$ is non-degenerate.

It is easy to see that there is a unique projective FGLA of type (n, r) up to isomorphism.

LEMMA 2. 3. *Let R be a pseudo-product manifold, and let $\mathfrak{L}$ be a projective* FGLA *of type (n, r). Then R is a projective system of type (n, r), if and only if it is of type $\mathfrak{L}$.*

Let R be a submanifold of the product $M \times N$ of two manifolds M and N. Assume that $\dim M=n$, $\dim N=(r+1)(n-r)$ and $\dim R=n+r(n-r)$. Let (E', F') be the product structure of $M \times N$, and, for each point $x \in R$, set $E_x=E'_x \cap T(R)_x$ and $F_x=F'_x \cap T(R)_x$. Then we easily have

$$\dim E_x \geqq r, \quad \dim F_x \geqq r(n-r).$$

This being remarked, we say that R is generically embedded in the product manifold $M\times N$, if it satisfies the following conditions:

1) $\dim E_x=r$ and $\dim F_x=r(n-r)$ at each point x. Hence (E, F) gives a pseudo-product structure on R.

2) The differential system $D=E+F$ is non-degenerate.

It is clear that, if R is generically embedded in $M\times N$, then R becomes a projective system of type (n, r).

Let us now consider a system $\mathfrak{P}=\{R;\ M, N\}$ of connected manifolds M, N and a submanifold R of $M\times N$. Let Ω be the complement of R in $M\times N$. Let ρ_M^R (resp. ρ_N^R) be the projection of R to M (resp. of R to N), and similarly let ρ_M^Ω (resp. ρ_N^Ω) be the projection of Ω to M (resp. of Ω to N). Now assume the following conditions on the system $\mathfrak{P}$:

i) $\dim M=n$, $\dim N=(r+1)(n-r)$ and $\dim R=n+r(n-r)$,

ii) R is generically embedded in the product manifold $M\times N$, and is closed in it,

iii) The projections ρ_M^R and ρ_N^R are both surjective, and their fibres are all connected,

iv) If $r=n-1$, the fibres of the projections ρ_M^Ω and ρ_N^Ω are all connected.

For example, the system $\mathfrak{P}=\{R_{n,r};\ P^n, G^r(P^n)\}$ satisfies conditions i) $\sim$iv).

We see from conditions i) and ii) that R becomes a projective system of type (n, r), and from ii) that Ω becomes an open submanifold of the product manifold $M\times N$. Let (E_Ω, F_Ω) be the product structure of Ω, and similarly let (E_R, F_R) be the pseudoproduct structure of R.

We easily have the following

LEMMA 2.4. *(1) Ω is a fibred manifold over the base space M (resp. over N) whth projection ρ_M^Ω (resp. with ρ_N^Ω). Furthermore its fibres are all connected, and its vertical tangent bundle is given by F_Ω (resp. by E_Ω).*

(2) R is a fibred manifold over the base space M (resp. over N) with projection ρ_M^R (resp. with ρ_N^R). Furthermore its fibres are all connected, and its vertical tangent bundle is given by F_R (resp. by E_R).

In particular we see from this lemma that both R and Ω become connected.

Now, consider the automorphism group $\mathrm{Aut}(\Omega)$ of the product manifold Ω as well as the automorphism group $\mathrm{Aut}(R)$ of the pseudo-product manifold R.

As before it follows from (1) of Lemma 2.4 that every $\phi\in\mathrm{Aut}(\Omega)$ naturally induces a diffeomorphism ϕ_M (resp. ϕ_N) of M (resp. of N).

Thus we have the product transformation $\phi_{M\times N}=\phi_M\times\phi_N$ of $M\times N$; The restriction of $\phi_{M\times N}$ to Ω coincides with the given ϕ, and hence $\phi_{M\times N}$ leaves R invariant. Let ϕ_R denote the restriction of $\phi_{M\times N}$ to R. Then we see that the assignment $\phi\to\phi_R$ gives an injective homomorphism Λ of $\mathrm{Aut}(\Omega)$ to $\mathrm{Aut}(R)$.

Similarly it follows from (2) of Lemma 2.4 that every $\psi\in\mathrm{Aut}(R)$ naturally induces a diffeomorphism ψ_M (resp. ψ_N) of M (resp. of N). Thus we have the product transformation $\psi_{M\times N}=\psi_M\times\psi_N$; The restriction of $\psi_{M\times N}$ to R coincides with the given ψ, and hence $\psi_{M\times N}$ leaves Ω invariant. Let ψ_Ω denote the restriction of $\psi_{M\times N}$ to Ω. Then we see that the assignment $\psi\to\psi_\Omega$ gives an injective homomorphism Λ' of $\mathrm{Aut}(R)$ to $\mathrm{Aut}(\Omega)$.

Clearly we have $\Lambda'\circ\Lambda=1$ and $\Lambda\circ\Lambda'=1$. Thus we have proved the following

PROPOSITION 2.5. *The homomorphism* Λ *gives an isomorphism of* $\mathrm{Aut}(\Omega)$ *onto* $\mathrm{Aut}(R)$.

Let $\mathfrak{L}$ be a projective FGLA of type (n, r), and $\mathfrak{G}$ its prolongation. Then it can be shown that $\mathfrak{G}$ is isomorphic with the simple graded Lie algebra $\mathfrak{G}(1, r, n-r ; \boldsymbol{R})$ of the second kind. Moreover we know from Lemma 2.3 that the pseudo-product manifold R, being a projective system of type (n, r), is of type $\mathfrak{L}$. Therefore, by virtue of Fact C, there is associated to the pseudo-product manifold R a normal connection (P, ω) of type $\mathfrak{G}$ on it in an invariant manner. In particular it follows that $\mathrm{Aut}(R)$ becomes naturally a Lie group and $\dim \mathrm{Aut}(R)\leqq\dim \mathfrak{sl}(n+1, \boldsymbol{R})=n^2+2n$.

Consequently we have the following

PROPOSITION 2.6 $\mathrm{Aut}(R)$ *as well as* $\mathrm{Aut}(\Omega)$ *becomes naturally a Lie group of dimension at most* n^2+2n, *so that* Λ *gives a Lie group isomorphism of* $\mathrm{Aut}(\Omega)$ *onto* $\mathrm{Aut}(R)$.

REMARK. Let P be as above. Let us consider the Grassmann bundle $G^r(T(M))$ and the canonical system C on it. Let ϖ be the projection of $G^r(T(M))$ onto M. Then it can be shown that there is a unique (open) immersion ϕ of R into $G^r(T(M))$ such that $\varpi\circ\phi=\rho_M^R$ and such that $D=\phi_*^{-1}(C)$, where $D=E_R+F_R$ as before. For simplicity let us assume that ϕ is an (open) imbedding, and identify R with an open submanifold of $G^r(T(M))$.

Now let A be an r-dimensional submanifold of M. Then we say that A is a solution of the pair $\mathscr{R}=(R, E_R)$, if the lift $\hat{A}$ of A to $G^r(T(M))$ is an integral manifold of E_R. In this way $\mathscr{R}$ can be regarded as a differential equation. We can easily show that the equation $\mathscr{R}$ is locally represented by an involutive system of partial differential equations of the second order of the following form:

$$\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}=F^\alpha_{ij}\left((x^k),(y^\beta),\left(\frac{\partial y^\beta}{\partial x^k}\right)\right),\ 1\leqq i,\ j\leqq r,\ 1\leqq\alpha\leqq n-r.$$

In the special case where $r=1$, this system simply means a system of ordinary differential equations of the second order:

$$\frac{d^2y^\alpha}{dx^2}=F^\alpha\left(x, y^1, \dots, y^{n-1}, \frac{dy^1}{dx}, \dots, \frac{dy^{n-1}}{dx}\right),\ 1\leqq\alpha\leqq n-1.$$

By an automorphism of the equation $\mathscr{R}$ we mean a diffeomorphism ϕ of M which naturally induces a transformation on the solutions of $\mathscr{R}$. Then we remark that the automorphism group Aut(R) of the pseudo-product manifold R is naturally isomorphic with the automorphism group Aut$(\mathscr{R})$ of the equation $\mathscr{R}$.

2. 3. Affine symmetric triples, and simple graded Lie algebras of the first kind. This paragraph is preliminary to the subsequent paragraph. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\boldsymbol{R}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, and $\mathfrak{m}$ a subspace $\neq\{0\}$ of $\mathfrak{g}$. Then the system $\mathfrak{S}=\{\mathfrak{g}\,;\,\mathfrak{h},\mathfrak{m}\}$ is called an (affine) symmetric triple, if it satisfies the following conditions:

(AST. 1) $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$(direct sum),

(AST. 2) $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$, and $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h}$.

Let $\mathfrak{S}=\{\mathfrak{g}\,;\,\mathfrak{h},\mathfrak{m}\}$ and $\mathfrak{S}'=\{\mathfrak{g}';\,\mathfrak{h}',\mathfrak{m}'\}$ be two symmetric triples. By an isomorphism of $\mathfrak{S}$ onto $\mathfrak{S}'$ we mean a Lie algebra isomorphism ϕ of $\mathfrak{g}$ onto $\mathfrak{g}'$ which sends $\mathfrak{h}$ to $\mathfrak{h}'$ and $\mathfrak{m}$ to $\mathfrak{m}'$.

Let $\mathfrak{S}=\{\mathfrak{g}\,;\,\mathfrak{h},\mathfrak{m}]$ be a symmetric triple. Then $\mathfrak{S}$ is called of simple type, if the Lie algebra $\mathfrak{g}$ is simple, and it is also called of reducible type, if the linear isotropy representation ρ of $\mathfrak{h}$ on $\mathfrak{m}$ is reducible. (Recall that ρ is defined by $\rho(X)Y=[X, Y]$ for all $X\in\mathfrak{h}$ and $Y\in\mathfrak{m}$.)

In his paper [3], Berger classified the symmetric triples of simple type up to isomorphism.

Now, let $\mathfrak{H}=\{\mathfrak{g},(\mathfrak{h}_p)\}$ be a simple graded Lie algebra of the first kind. We know that $\mathfrak{H}^*=\{\mathfrak{g},(\mathfrak{h}_{-p})\}$ is also a simple graded Lie algebra of the first kind, and is isomorphic with $\mathfrak{H}$ (cf. Lemma 1. 5 in [12]). Furthermore if we set

$$\mathfrak{h}=\mathfrak{h}_0, \text{ and } \mathfrak{m}=\mathfrak{h}_{-1}+\mathfrak{h}_1,$$

we see that the system $\mathfrak{S}=\{\mathfrak{g}\ ;\ \mathfrak{h}, \mathfrak{m}\}$ gives a symmetric triple of simple and reducible type, which will be called associated with $\mathfrak{H}$. Conversely we have the following lemma, which was first proved by Berger [3].

LEMMA 2.7(cf. Lemma 2, Appendix, in [11]). *Any symmetric triple $\mathfrak{S}$ of simple and reducible type is associated with a simple graded Lie algebra $\mathfrak{H}$ of the first kind. Furthermore, such a $\mathfrak{H}$ is unique in the following sense: If $\mathfrak{S}$ is associated with another simple graded Lie algebra $\mathfrak{H}'$ of the first kind, then $\mathfrak{H}'=\mathfrak{H}$ or $\mathfrak{H}'=\mathfrak{H}^*$.*

Accordingly we have known that there is a natural one-to-one correspondence between the symmetric triples of simple and reducible type, and the simple graded Lie algebras of the first kind up to the respective isomorphisms.

We shall now exhibit some examples of simple graded Lie algebras of the first kind.

(1) The Grassmann graded Lie algebras: $\mathfrak{G}(n, k\ ;\ K)$, where $k\geqq n\geqq 1$, and $K=\boldsymbol{R}$ or $\boldsymbol{C}$ or $\boldsymbol{Q}$.

(2) The Möbius graded Lie algebras. Let (n, r) be a pair of integers with $0\leqq 2r\leqq n$. We define a diagonal matrix of degree n, $T_{n,r}$, by

$$T_{n,r}=\begin{pmatrix} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} (-1\ r \text{ times}),$$

and define a symmetric matrix of degree $n+2$, $\tilde{T}_{n,r}$, by

$$\tilde{T}_{n,r}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & T_{n,r} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We then dfine a subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(n+2, \boldsymbol{R})$ by

$$\mathfrak{g}=\{X\in\mathfrak{gl}(n+2, \boldsymbol{R})\,|\,{}^tX\tilde{T}_{n,r}+\tilde{T}_{n,r}X=0\},$$

which is isomorphic with the Lie algebra $\mathfrak{so}(r+1, n-r+1)$. If we put $n_1=n_3=1$ and $n_2=n$, every matrix $X\in\mathfrak{gl}(n+2, \boldsymbol{R})$ can be expressed as follows: $X=(X_{ij})_{1\leqq i, j\leqq 3}$, where X_{ij} are $n_i\times n_j$-matrices. Then the subalgebra $\mathfrak{g}$ is defined by the equations:

$$X_{31}=0,\ {}^tX_{32}+T_{n,r}X_{21}=0,\ X_{11}+X_{33}=0,$$
$${}^tX_{22}T_{n,r}+X_{22}T_{n,r}=0,\ {}^tX_{12}+T_{n,r}X_{23}=0,\ X_{13}=0.$$

For any integer p we now define a subspace $\mathfrak{g}_p$ of $\mathfrak{g}$ by $\mathfrak{g}_p = \mathfrak{g} \cap \mathfrak{g}_p(1, n, 1; \boldsymbol{R}) = \{X \in \mathfrak{g} \mid X_{ij} = 0 \text{ if } j-i \neq p\}$. Then we see that $\mathfrak{G} = \{\mathfrak{g}, (\mathfrak{g}_p)\}$ is a graded subalgebra of $\mathfrak{G}(1, n, 1; \boldsymbol{R})$. If $r=0$, $n \geqq 1$ or $1 \leqq 2r \leqq n$, $n \geqq 3$, $\mathfrak{G}$ turns out to be a simple graded Lie algebra of the first kind, which will be called the Möbius graded Lie algebra of degree n and of index r, and will be denoted by $\mathfrak{M}_r(n)$. In particular, $\mathfrak{M}_0(n)$ will be called the definite Möbius graded Lie algebra of degree n. Note that $\mathfrak{G}(1, 1; \boldsymbol{R}) \cong \mathfrak{M}_0(1)$, $\mathfrak{G}(1, 1; \boldsymbol{C}) \cong \mathfrak{M}_0(2)$ and $\mathfrak{G}(1, 1; \boldsymbol{Q}) \cong \mathfrak{M}_0(4)$.

By using the matrix $\tilde{T}_{n,0}$, we define a subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(n+2, \boldsymbol{C})$ and their subspaces $\mathfrak{g}_p$ in the same manner as above. Then we see that $\mathfrak{G} = \{\mathfrak{g}, (\mathfrak{g}_p)\}$ is a graded subalgebra of $\mathfrak{G}(1, n, 1; \boldsymbol{C})$. If $n \neq 2$, $\mathfrak{G}$ turns out to be a simple graded Lie algebraof the first kind, which will be called the Möbius graded Lie algebra of degree n over $\boldsymbol{C}$, and will be denoted by $\mathfrak{M}(n, \boldsymbol{C})$. Note that $\mathfrak{M}(1, \boldsymbol{C}) \cong \mathfrak{M}_0(2)$.

2. 4. The standard affine symmetric spaces, and the main theorem. Let $\mathfrak{H} = \{\mathfrak{g}, (\mathfrak{h}_p)\}$ be a simple graded Lie algebra of the first kind, and $\mathfrak{S} = \{\mathfrak{g}; \mathfrak{h}, \mathfrak{m}\}$ the associated symmetric triple. We define subgroups H and G of $\mathrm{Aut}(\mathfrak{g})$ respectively by

$$H = \mathrm{Aut}(\mathfrak{H}), \quad G = \mathrm{Aut}(\mathfrak{g})^0 \cdot H.$$

We also consider the subalgebra $\mathfrak{h}^\# = \mathfrak{h} + \mathfrak{h}_1$, of $\mathfrak{g}$, and denote by $H^\#$ the normalizer of $\mathfrak{h}^\#$ in $\mathrm{Aut}(\mathfrak{g})$. The groups H, $H^\#$ and G, thus obtained, are nothing but the groups associated with $\mathfrak{H}$. (Note that $H^\# = \mathrm{Aut}(F(\mathfrak{H})) = H \cdot \exp \mathfrak{h}_1$ (Lemmas 1. 4 and 1. 6).) We know that the homogeneous space $G/H^\#$ is connected and compact, and that the action of G on $G/H^\#$ is effective.

Let us now consider the homogeneous space G/H. It is easy to see that G/H is connected and non-compact, and that the action of G on G/H is effective. We show that G/H becomes naturally an affine symmetric space. Indeed, let α be the involutive automorphism of the Lie algebra $\mathfrak{g}$ associated with the symmetric triple $\mathfrak{S}$:

$$\alpha(X) = X \text{ if } X \in \mathfrak{h}, \text{ and } \alpha(X) = -X \text{ if } X \in \mathfrak{m}.$$

Clearly we have $\alpha \cdot G \cdot \alpha = G$. We define an involutive automorphism θ of the Lie group G by $\theta(a) = \alpha a \alpha$ for all $a \in G$, and a subgroup G_θ of G by $G_\theta = \{a \in G \mid \theta(a) = a\}$. Then we easily have $G_\theta^0 \subset H \subset G_\theta$, where G_θ^0 denotes the connected component of the identity of G_θ. Thus we have seen that G/H becomes an affine symmetric space, and its infinitesimal structure is given by the symmetric triple $\mathfrak{S}$.

We next show that the space G/H is endowed with a product structure. Indeed, we have $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ (direct sum), $\mathrm{Ad}(H)\mathfrak{h}_{-1}=\mathfrak{h}_{-1}$ and $\mathrm{Ad}(H)\mathfrak{h}_1=\mathfrak{h}_1$. Thus the subspaces $\mathfrak{h}_{-1}$ and $\mathfrak{h}_1$ of $\mathfrak{m}$ naturally give rise to invariant differential systems E and F on G/H respectively. Since $\mathfrak{m}=\mathfrak{h}_{-1}+\mathfrak{h}_1$(direct sum), and since both $\mathfrak{h}_{-1}$ and $\mathfrak{h}_1$ are abelian subalgebrat of $\mathfrak{g}$, we see that (E, F) gives an invariant product structure on G/H.

Giving attention only to the product structure, we denote by $\mathrm{Aut}(G/H)$ the automorphism group of the product manifold G/H. Then we have a natural injective homomorphism i of G to $\mathrm{Aut}(G/H)$: For $a\in G$, $i(a)$ is the transformation of G/H induced by a. Now G/H becomes naturally a fibred manifold over the base space $G/H^{\#}$. As is easily observed, its fibres are all connected, and its vertical tangent bundle is given by F. Therefore every $\phi\in\mathrm{Aut}(G/H)$ naturally induces a diffeomorphism $\phi^{\#}$ of $G/H^{\#}$, and the assignment $\phi\rightarrow\phi^{\#}$ gives a homomorphism j of $\mathrm{Aut}(G/H)$ to $\mathrm{Diff}(G/H^{\#})$, the diffeomorphism group of $G/H^{\#}$.

We are now in a position to state the main theorem in the present paper, which completely determines the automorphism group $\mathrm{Aut}(G/H)$.

THEOREM 2.8. *Assume that $\mathfrak{H}$ is of the classical type. If $\mathfrak{H}$ is isomorphic with a definite Möbius (graded Lie) algebra, then the homomorphism j gives an isomorphism of* $\mathrm{Aut}(G/H)$ *onto* $\mathrm{Diff}(G/H^{\#})$. *Otherwise, the homomorphism i gives an isomorphism of G onto* $\mathrm{Aut}(G/H)$.

This theorem will be proved in § 3~§ 8.

We note that, if $\mathfrak{H}\cong\mathfrak{M}_0(n)$, the definite Möbius algebra of degree n, then $G/H^{\#}$ is diffeomorphic with the n-dimensional sphere S^n.

We also note that Theorem 2.8 partially generalizes Proposition 2.1. In fact, consider the product manifold $\Omega_{n,n-1}$, being an open submanifold of the product manifold $P^n\times G^{n-1}(P^n)$. (Note that $G^{n-1}(P^n)$ is the dual space of P^n.) Recall that the projective transformation group G of P^n naturally acts on $P^n\times G^{n-1}(P^n)$, and $\Omega_{n,n-1}$ is an open G-orbit. Let $(z_0, \dots, z_n)$ be the homogeneous coordinate system of P^n. Let p_0 be the point of P^n given by $z_1=\dots=z_n=0$, and α_0 the hyperplane of P^n given by $z_0=0$. Let H be the isotropy group of G at the point $(p_0, \alpha_0)\in\Omega_{n,n-1}$. Then $\Omega_{n,n-1}$ may be represented by the homogeneous space G/H. Moreover we can show that G/H is the standard affine symmetric space associated with $\mathfrak{H}=\mathfrak{G}(1, n; \boldsymbol{R})$, and that G/H and $\Omega_{n,n-1}$ coincides as product manifolds. We have thus confirmed our claim.

2.5. Some remarks on the standard affine symmetric spaces, and a problem. Let G/H be the standard affine symmetric space associated with a simple graded Lie algebra $\mathfrak{H}$ of the first kind. Let $\langle\ ,\ \rangle$ be the Killing form of the Lie algebra $\mathfrak{g}$. By Lemma 1.2 we know that $\langle\mathfrak{h}_{-1}, \mathfrak{h}_{-1}\rangle = \langle\mathfrak{h}_1, \mathfrak{h}_1\rangle = \{0\}$, and the bilinear function $(X, Y) \to \langle X, Y\rangle$ on $\mathfrak{h}_{-1}\times\mathfrak{h}_1$ is non-degenerate. In particular, it follows that the restriction $\langle\ ,\ \rangle|\mathfrak{m}$ of $\langle\ ,\ \rangle$ to $\mathfrak{m}$ is non-degenerate. Clearly we have $\langle \mathrm{Ad}(a)X, \mathrm{Ad}(a)Y\rangle = \langle X, Y\rangle$ for all $a\in H$ and $X, Y\in\mathfrak{m}$. Therefore $\langle\ ,\ \rangle|\mathfrak{m}$ naturally gives rise to an invariant indefinite Riemannian metric g on G/H. Let ∇ be the Levi-Civita connection associated with the metric g. Then it it easy to verify that g satisfies the following conditions:

1) $g(E_x, E_x) = g(F_x, F_x) = \{0\}$ at each point $x\in G/H$,

2) Both E and F are parallel with respect to ∇.

Accordingly, the space G/H equipped with the product structure (E, F) and the metric g turns out to be a space analogous to a hermitian symmetric space of simple and non-compact type.

REMARK. Clearly the product structure (E, F) satisfies rank E = rank F, which means that (E, F) is a paracomplex structure in the sense of [5]. Furthermore the conditions 1) and 2) above mean that the metric g together with the paracomplex structure (E, F) is a parakählerian metric again in the sense of [5].

From the discussions above we see that G/H becomes naturally an oriented manifold, which is even a symplectic manifold. Let ω be the volume element on G/H associated with the metric and the orientation. Let $(x^1, \dots x^m, y^1, \dots, y^m)$ be a coordinate system of G/H which is positive with respect to the orientation and which is compatible with the product structure, i.e., such that x^i and y^i are first integrals of F and E respectively. Then the volume element ω can be expressed as follows:

$$\omega = f dx^1\wedge \dots \wedge dx^m \wedge dy^1 \wedge \dots \wedge dy^m,$$

where f is a positive function on the coordinate neighborhood. Now, we remark that, in terms of the function f, the metric g can be expressed as follows:

$$g = 4\sum_{i,j}\frac{\partial^2 \log f}{\partial x^i \partial y^j} dx^i dy^j.$$

Clearly, invariant volume elements on G/H uniquely exist up to non-zero constant factors. Thus we know from Theorem 2.8 combined with the remark above that the metric g is completely determined by the product structure (E, F) (up to non-zero constant factors), provided $\mathfrak{H}$ is not isomorphic with a definite Möbius algebra.

Finally we propose a problem in connection with Proposition 2. 6 and the observation above.

Let $\mathfrak{P}=\{R; M, N\}$ be a system satisfying conditions i)$\sim$iv) with $r=n-1$. (Note that dim $M=$dim $N=n$, and dim $R=2n-1$.) Let us consider the product manifold Ω, the complement of R in $M\times N$, which is a paracomplex manifold. Assume that R, and hence M and N are compact.

PROBLEM. *Is there associated to the paracomplex manifold Ω a volume element ω on it in an invariant manner so that ω induces a parakählerian metric g (in the same fashion as above)? More weakly, is there associated to Ω any geometric structure or any geometric object on it in an invariant manner?*

Consider the following conditions on the system $\mathfrak{P}$ (cf. Remark at the end of 2. 2):

a) The natural (open) immersion of R to $G^{n-1}(T(M))$ is a diffeomorphism,

b) The natural (open) immersion of R to $G^{n-1}(T(N))$ is a diffeomorphism.

It is known that these conditions considerably restrict the topology of M and of N (see [13]). Consequently we mention that they can be important additional conditions for the problem.

§ 3. Some studies on simple graded Lie algebras of the second kind

3. 1. Pseudo-product SGLA's of the second kind, and the associated SGLA's of the first kind. Following [13], we first introduce the notion for a simple graded Lie algebra (or briefly a SGLA) of being pseudo-product.

Let $\mathfrak{G}=\{\mathfrak{g}, (\mathfrak{g}_p)\}$ be a SGLA of the μ-th kind, where $\mu\geqq 2$. Let us consider the truncated graded subalgebra $\mathfrak{T}=\{\mathfrak{t}, (\mathfrak{g}_p)_{p<0}\}$ of $\mathfrak{G}$, where $\mathfrak{t}=\sum_{p<0}\mathfrak{g}_p$. We recall that $\mathfrak{g}_0$ may be regarded as a subalgebra of the derivation algebra Der$(\mathfrak{T})$ of the FGLA, $\mathfrak{T}$, and that $\mathfrak{G}$ may be regarded as a graded subalgebra of the prolongation of the pair $(\mathfrak{T}, \mathfrak{g}_0)$. We also recall that $\mathfrak{T}$ is non-degenerate.

Now, $\mathfrak{G}$ is called pseudo-product, if there are given subspaces $\mathfrak{e}$ and $\mathfrak{f}$ of $\mathfrak{g}_{-1}$ satisfying the following conditions:

1) $\mathfrak{L}=\{\mathfrak{T}; \mathfrak{e}, \mathfrak{f}\}$ is a pseudo-product FGLA,

2) $\mathfrak{g}_0$ is a subalgebra of the derivation algebra Der$(\mathfrak{L})$ of $\mathfrak{L}$, i.e., $[\mathfrak{g}_0, \mathfrak{e}]\subset\mathfrak{e}$ and $[\mathfrak{g}_0, \mathfrak{f}]\subset\mathfrak{f}$.

LEMMA 3. 1 ([13]). *If $\mathfrak{G}$ is pseudo-product, then it is the prolongation of the pseudo-product* FGLA, $\mathfrak{L}$.

Outline of the proof (cf. the proof of Lemma 3.4 in [11]). Let $\hat{\mathfrak{G}}=\{\hat{\mathfrak{g}}, (\hat{\mathfrak{g}}_p)\}$ be the prolongation of $\mathfrak{L}$ or of the pair $(\mathfrak{T}, \hat{\mathfrak{g}}_0)$, where $\hat{\mathfrak{g}}_0=\mathrm{Der}(\mathfrak{L})$. Since $\mathfrak{g}_0\subset\hat{\mathfrak{g}}_0$, we see that $\mathfrak{G}$ is a graded subalgebra of $\hat{\mathfrak{G}}$. Moreover, since $\mathfrak{L}$ or $\mathfrak{T}$ is non-degenerate, we know from Lemma 1.14 that $\hat{\mathfrak{g}}$ is finite dimensional. Since $\mathfrak{g}$ is simple, it follows from these facts that the radical of $\hat{\mathfrak{g}}$ vanishes, i.e., $\hat{\mathfrak{g}}$ is semi-simple. It is now easy to show that $\hat{\mathfrak{G}}=\mathfrak{G}$.

Hereafter we shall consider a fixed pseudo-product SGLA, $\mathfrak{G}$, of the second kind over $\boldsymbol{R}$. We preserve the notations as above.

We have $\mathfrak{t}=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}=\mathfrak{g}_{-2}+\mathfrak{e}+\mathfrak{f}$. Then we define an endomorphism J of the vector space $\mathfrak{t}$ as follows:

$$JX=0 \text{ if } X\in\mathfrak{g}_{-2}\,;\ JX=X \text{ if } X\in\mathfrak{e}\,;\ JX=-X \text{ if } X\in\mathfrak{f}.$$

Clearly J is in the centre of $\mathfrak{g}_0=\mathrm{Der}(\mathfrak{G})$. By Lemma 1.2 we can easily prove the following

LEMMA 3.2. *If p is even, then $[J, \mathfrak{g}_p]=\{0\}$. If p is odd, then $[J, [J, X]]=X$ for all $X\in\mathfrak{g}_p$.*

REMARK. Let $\mathfrak{G}$ be a SGLA of the second kind. As is easily seen, $\mathfrak{G}$ is pseudo-product, if and only if there is given an element J in the centre of $\mathfrak{g}_0$ such that $[J, \mathfrak{g}_{-2}]=\{0\}$ and $[J, [J, X]]=X$ for all $X\in\mathfrak{g}_{-1}$.

Now, let E be the characteristic element of $\mathfrak{G}$. Using E and J, we define elements $E_{\mathfrak{A}}$ and $E_{\mathfrak{B}}$ in the centre of $\mathfrak{g}_0$ respectively by

$$E_{\mathfrak{A}}=\frac{1}{2}(E-J),\quad E_{\mathfrak{B}}=\frac{1}{2}(E+J),$$

and, for any integer p, define subspaces $\mathfrak{a}_p$ and $\mathfrak{b}_p$ of $\mathfrak{g}$ respectively by

$$\mathfrak{a}_p=\{X\in\mathfrak{g}\mid[E_{\mathfrak{A}}, X]=pX\},\quad \mathfrak{b}_p=\{X\in\mathfrak{g}\mid[E_{\mathfrak{B}}, X]=pX\}.$$

Then we assert that both $\mathfrak{A}=\{\mathfrak{g}, (\mathfrak{a}_p)\}$ and $\mathfrak{B}=\{\mathfrak{g}, (\mathfrak{b}_p)\}$ are SGLA's of the first kind. Indeed, let ε be 1 or -1. We define subspaces $\mathfrak{g}_\varepsilon^+$ and $\mathfrak{g}_\varepsilon^-$ of $\mathfrak{g}_\varepsilon$ respectively by

$$\mathfrak{g}_\varepsilon^+=\{X\in\mathfrak{g}_\varepsilon\mid[J, X]=X\},\quad \mathfrak{g}_\varepsilon^-=\{X\in\mathfrak{g}_\varepsilon\mid[J, X]=-X\}.$$

(Note that $\mathfrak{g}_{-1}^+=\mathfrak{e}$ and $\mathfrak{g}_{-1}^-=\mathfrak{f}$.) By Lemma 3.2 we easily have the following:

$$\mathfrak{a}_{-1}=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}^+,\quad \mathfrak{a}_0=\mathfrak{g}_{-1}^-+\mathfrak{g}_0+\mathfrak{g}_1^+,\quad \mathfrak{a}_1=\mathfrak{g}_1^-+\mathfrak{g}_2\,;$$
$$\mathfrak{b}_{-1}=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}^-,\quad \mathfrak{b}_0=\mathfrak{g}_{-1}^++\mathfrak{g}_0+\mathfrak{g}_1^-,\quad \mathfrak{b}_1=\mathfrak{g}_1^++\mathfrak{g}_2.$$

Hence we obtain $\mathfrak{g}=\sum_p\mathfrak{a}_p=\sum_p\mathfrak{b}_p$, proving our assertion.

LEMMA 3.3. (1) *Both* $\mathfrak{g}_{-1}^{+}+\mathfrak{g}_{1}^{+}$ *and* $\mathfrak{g}_{-1}^{-}+\mathfrak{g}_{1}^{-}$ *are abelian subalgebras of* $\mathfrak{g}$.

(2) $[\mathfrak{g}_{-2\varepsilon}, \mathfrak{g}_{\varepsilon}^{+}]=\mathfrak{g}_{-\varepsilon}^{+}$, $[\mathfrak{g}_{-2\varepsilon}, \mathfrak{g}_{\varepsilon}^{-}]=\mathfrak{g}_{-\varepsilon}^{-}$, $\mathfrak{g}_{2\varepsilon}=[\mathfrak{g}_{\varepsilon}^{+}, \mathfrak{g}_{\varepsilon}^{-}]$,

where ε is 1 or -1.

The proof of this fact is left to the readers as an exercise (cf. Lemma 3.1 in [11]).

Set $G_0=\mathrm{Aut}(\mathfrak{L})$, which may be regarded as an open subgroup of $\mathrm{Aut}(\mathfrak{G})$. Let $G^{(0)}$ and G be the groups associated with the pair $(\mathfrak{G}, G_0)$:

$$G^{(0)}=G_0\cdot G^{(1)},\quad G=\mathrm{Aut}(\mathfrak{g})^0\cdot G_0,$$

where $G^{(1)}$ is the Lie subgroup of $\mathrm{Aut}(\mathfrak{g})^0$ generated by the nilpotent subalgebra $\mathfrak{g}^{(1)}=\mathfrak{g}_1+\mathfrak{g}_2$. We know that the Lie algebra of G_0 and $G^{(0)}$ are $\mathfrak{g}_0$ and $\mathfrak{g}^{(0)}=\mathfrak{g}_0+\mathfrak{g}_1+\mathfrak{g}_2$ respectively. We also know that $G/G^{(0)}$ is connected and compact, and that the action of G on $G/G^{(0)}$ is effective.

We define subgroups A_0 and B_0 of G respectively by

$$A_0=G\cap\mathrm{Aut}(\mathfrak{A}),\quad B_0=G\cap\mathrm{Aut}(\mathfrak{B}),$$

which are open subgroups of $\mathrm{Aut}(\mathfrak{A})$ and $\mathrm{Aut}(\mathfrak{B})$ respectively. Clearly we have

$$G_0=\{a\in G\,|\,\mathrm{Ad}(a)E=E,\ \mathrm{Ad}(a)J=J\},$$
$$A_0=\{a\in G\,|\,\mathrm{Ad}(a)E_{\mathfrak{A}}=E_{\mathfrak{A}}\},\quad B_0=\{a\in G\,|\,\mathrm{Ad}(a)E_{\mathfrak{B}}=E_{\mathfrak{B}}\},$$

whence

$$A_0\cap B_0=G_0.$$

In particular, it follows that

$$G=\mathrm{Aut}(\mathfrak{g})^0\cdot A_0=\mathrm{Aut}(\mathfrak{g})^0\cdot B_0.$$

We now define subgroups $A^{(0)}$ and $B^{(0)}$ of G respectively by

$$A^{(0)}=A_0\cdot A_1,\quad B^{(0)}=B_0\cdot B_1,$$

where A_1 and B_1 are the Lie subgroups of $\mathrm{Aut}(\mathfrak{g})^0$ generated respectively by the abelian subalgebras $\mathfrak{a}_1$ and $\mathfrak{b}_1$ of $\mathfrak{g}$. Then we see from the remark above that $A^{(0)}$ and G are the groups associated with the pair $(\mathfrak{A}, A_0)$, and similarly $B^{(0)}$ and G are the groups associated with the pair $(\mathfrak{B}, B_0)$. We know that the Lie algebras of A_0 and $A^{(0)}$ are $\mathfrak{a}_0$ and $\mathfrak{a}^{(0)}=\mathfrak{a}_0+\mathfrak{a}_1$ respectively, and similarly the Lie algebras of B_0 and $B^{(0)}$ are $\mathfrak{b}_0$ and $\mathfrak{b}^{(0)}=\mathfrak{b}_0+\mathfrak{b}_1$ respectively. We also know that both $G/A^{(0)}$ and $G/B^{(0)}$ are connected and compact, and that the action of G on both $G/A^{(0)}$ and $G/B^{(0)}$ are effective.

LEMMA 3.4. $A^{(0)}\cap B^{(0)}=G^{(0)}$.

PROOF. We have $G_0=A_0\cap B_0\subset A^{(0)}\cap B^{(0)}$. Since $\mathfrak{g}^{(1)}\subset\mathfrak{g}^{(0)}=\mathfrak{a}^{(0)}\cap\mathfrak{b}^{(0)}$, we have $G^{(1)}\subset A^{(0)}\cap B^{(0)}$. Hence it follows that $G^{(0)}\subset A^{(0)}\cap B^{(0)}$. Conversely we show that $A^{(0)}\cap B^{(0)}\subset G_0$. Let $a\in A^{(0)}\cap B^{(0)}$. Then we have $\mathrm{Ad}(a)\mathfrak{a}^{(0)}=\mathfrak{a}^{(0)}$ and $\mathrm{Ad}(a)\mathfrak{b}^{(0)}=\mathfrak{b}^{(0)}$, whence $\mathrm{Ad}(a)\,\mathfrak{g}^{(0)}=\mathfrak{g}^{(0)}$. Consequently we see from Lemma 1.4 that $a\in\mathrm{Aut}(\mathfrak{G})\cdot G^{(1)}$. Now, we have $\mathrm{Ad}(a)E_{\mathfrak{A}}\equiv E_{\mathfrak{A}}$ (mod $\mathfrak{a}_1$) and $\mathrm{Ad}(a)E_{\mathfrak{B}}\equiv E_{\mathfrak{B}}$ (mod $\mathfrak{b}_1$), whence

$$\mathrm{Ad}(a)J\equiv J\,(\mathrm{mod}\ \mathfrak{g}^{(1)}).$$

Since $a\in\mathrm{Aut}(\mathfrak{G})\cdot G^{(1)}$, it can be written as follows: $a=b\cdot c$, where $b\in\mathrm{Aut}(\mathfrak{G})$ and $c\in G^{(1)}$. Then we have

$$\mathrm{Ad}(a)J\equiv\mathrm{Ad}(b)J\quad(\mathrm{mod}\ \mathfrak{g}^{(1)}).$$

Therefore it follows that $\mathrm{Ad}(b)J=J$, i.e., $b\in G_0$. Thus we obtain $a\in G^{(0)}$, and hence $A^{(0)}\cap B^{(0)}\subset G^{(0)}$, proving the lemma.

3.2. The action of the group G on the product manifold $G/A^{(0)}\times G/B^{(0)}$. Set $M=G/A^{(0)}$ and $N=G/B^{(0)}$. Then the group G naturally acts on the product manifold $M\times N$:

$$a\cdot(p,q)=(ap,aq),\quad a\in G,\ (p,q)\in M\times N.$$

Let us consider the natural imbeddings ι_M and ι_N of $\mathfrak{a}_{-1}$ and $\mathfrak{b}_{-1}$ into M and N respectively:

$$\iota_M(X)=\pi_M(\exp X),\quad X\in\mathfrak{a}_{-1};$$
$$\iota_N(Y)=\pi_N(\exp Y),\quad Y\in\mathfrak{b}_{-1},$$

where π_M and π_N denote the projections of G onto M and N respectively (see 1.3). Then the product map $\iota=\iota_M\times\iota_N$ gives an imbedding of $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ into $M\times N$. We know from Corollary 2 to Lemma 1.11 that the images $\iota_M(\mathfrak{a}_{-1})$ and $\iota_N(\mathfrak{b}_{-1})$ of $\mathfrak{a}_{-1}$ and $\mathfrak{b}_{-1}$ by ι_M and ι_N are open, dense subsets of M and N respectively. Consequently it follows that the image $\iota(\mathfrak{a}_{-1}\times\mathfrak{b}_{-1})=\iota_M(\mathfrak{a}_{-1})\times\iota_N(\mathfrak{b}_{-1})$ of $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ by ι is an open, dense subset of $M\times N$.

We have $\mathfrak{a}_{-1}=\mathfrak{g}_{-2}+\mathfrak{g}^{+}_{-1}$ and $\mathfrak{b}_{-2}=\mathfrak{g}_{-2}+\mathfrak{g}^{-}_{-1}$. Thus every point w of $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ may be expressed as follows: $w=(x+y,u+v)$, where $x\in\mathfrak{g}_{-2}$, $y\in\mathfrak{g}^{+}_{-1}$, $u\in\mathfrak{g}_{-2}$ and $v\in\mathfrak{g}^{-}_{-1}$. Hereafter the point w will be also expressed as (x,y,u,v). By the letters x, y, etc. let us now mean the functions $w\to x$, $w\to y$, etc. Then the system of functions, $(x,y,u.\,v)$, may be considered as a coordinate system of $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$, which will be called the canonical coordinate system of $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$.

Let T be the Lie subgroup of G generated by the nilpotent subalgebra $\mathfrak{t}=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}$ of $\mathfrak{g}$, which is "dual" to the subgroup $G^{(1)}$ of G.

LEMMA 3.5. (1) *Let* X_{-2}, $Y_{-2}\in\mathfrak{g}_{-2}$ *and* X_{-1}, $Y_{-1}\in\mathfrak{g}_{-1}$. *Then* $\exp(X_{-2}+X_{-1})\cdot\exp(Y_{-2}+Y_{-1})=\exp(X_{-2}+Y_{-2}+\frac{1}{2}[X_{-1}, Y_{-1}]+X_{-1}+Y_{-1})$.

(2) *Let* $X_{-2}\in\mathfrak{g}_{-2}$, $X^{+}_{-1}\in\mathfrak{g}^{+}_{-1}$ *and* $X^{-}_{-1}\in\mathfrak{g}^{-}_{-1}$. *Then* $\exp(X_{-2}+X^{+}_{-1}+X^{-}_{-1})=\exp(X_{-2}+\frac{1}{2}[X^{-}_{-1}, X^{+}_{-1}]+X^{+}_{-1})\cdot\exp X^{-}_{-1}=$

$\exp(X_{-2}+\frac{1}{2}[X^{+}_{-1}, X^{-}_{-1}]+X^{-}_{-1})\cdot\exp X^{+}_{-1}$.

PROOF. (1) Let X_{-1}, $Y_{-1}\in\mathfrak{g}_{-1}$. Then we can find unique elements Z_{-2} and Z_{-1} of $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ respectively such that $\exp X_{-1}\cdot\exp Y_{-1}=\exp(Z_{-2}+Z_{-1})$ (cf. Lemma 1.6), whence

$$\mathrm{Ad}(\exp X_{-1})\mathrm{Ad}(\exp Y_{-1})E=\mathrm{Ad}(\exp(Z_{-2}+Z_{-1}))E.$$

It follows easily that $Z_{-2}=\frac{1}{2}[X_{-1}, Y_{-1}]$ and $Z_{-1}=X_{-1}+Y_{-1}$.
Hence we obtain

$$\exp X_{-1}\cdot\exp Y_{-1}=\exp(\frac{1}{2}[X_{-1}, Y_{-1}]+X_{-1}+Y_{-1}).$$

Since $\exp\mathfrak{g}_{-2}$ is in the centre of T, (1) follows immediately from this equality.

(2) can be easily obtained from (1), proving the lemma.

Now, we notice that the product $T\cdot G_0$ of T and G_0 gives a (closed) subgroup of G, which is "dual" to the subgroup $G^{(0)}$ of G. Being a subgroup of G, the group $T\cdot G_0$ acts on the product manifold $M\times N$.

LEMMA 3.6 *The group* $T\cdot G_0$ *leaves the subset* $\iota(\mathfrak{a}_{-1}\times\mathfrak{b}_{-1})$ *of* $M\times N$ *invariant.*

Therefore the group $T\cdot G_0$ acts on the product manifold $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ in such a way that the imbedding $\iota:\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}\rightarrow M\times N$ becomes equivariant:

$$\iota(aw)=a\iota(w),\ a\in T\cdot G_0,\ w\in\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}.$$

The following lemma clarifies the action of $T\cdot G_0$ on $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$.

LEMMA 3.7. *For any* $a\in T\cdot G_0$ *and* $(x, y, u, v)\in\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$, *set* $a\cdot(x, y, u, v)=(x', y', u', v')$. *Express* a *as follows*: $a=\exp(X_{-2}+X^{+}_{-1}+X^{-}_{-1})\cdot b$, *where* $X_{-2}\in\mathfrak{g}_{-2}$, $X^{+}_{-1}\in\mathfrak{g}^{+}_{-1}$, $X^{-}_{-1}\in\mathfrak{g}^{-}_{-1}$ *and* $b\in G_0$. *Then the vectors* x', y', *etc. are described as follows*:

$$x' = \mathrm{Ad}(b)x + [X_{-1}^{-}, \mathrm{Ad}(b)y] + X_{-2} + \frac{1}{2}[X_{-1}^{-}, X_{-1}^{+}],$$
$$y' = \mathrm{Ad}(b)y + X_{-1}^{+},$$
$$u' = \mathrm{Ad}(b)u + [X_{-1}^{+}, \mathrm{Ad}(b)v] + X_{-2} + \frac{1}{2}[X_{-1}^{+}, X_{-1}^{-}],$$
$$v' = \mathrm{Ad}(b)v + X_{-1}^{-}.$$

We shall prove Lemmas 3.6 and 3.7 together. For any $a \in T \cdot G_0$ and $(x, y, u, v) \in \mathfrak{a}_{-1} \times \mathfrak{b}_{-1}$, we have

$$a \cdot \iota(x, y, u, v) = (\pi_M(a \cdot \exp(x+y)),\ \pi_N(a \cdot \exp(u+v))).$$

Let us define x', y', etc. to be the right-hand sides of the equalities in Lemma 3.7. Using Lemma 3.5, we calculate $a \cdot \exp(x+y)$. Then we have

$$\begin{aligned} a \cdot \exp(x+y) &= \exp(X_{-2} + X_{-1}^{+} + X_{-1}^{-}) \cdot b \cdot \exp(x+y) \\ &= \exp(X_{-2} + X_{-1}^{+} + X_{-1}^{-}) \cdot \exp(\mathrm{Ad}(b)x + \mathrm{Ad}(b)y) \cdot b \\ &= \exp(x' + y') \cdot \exp X_{-1}^{-} \cdot b. \end{aligned}$$

Similarly, calculating $a \cdot \exp(u+v)$, we obtain

$$a \cdot \exp(u+v) = \exp(u' + v') \cdot \exp X_{-1}^{+} \cdot b.$$

We have thus shown that $a \cdot \iota(x, y, u, v) = \iota(x', y', u', v')$, proving the lemmas.

We define a map Φ of $\mathfrak{a}_{-1} \times \mathfrak{b}_{-1}$ to $\mathfrak{g}_{-2}$ by

$$\Phi(w) = u - x + [v, y] \quad \text{for all } w = (x, y, u, v) \in \mathfrak{a}_{-1} \times \mathfrak{b}_{-1}.$$

Clearly $\mathfrak{a}_{-1} \times \mathfrak{b}_{-1}$ is a fibred mnifold over the base space $\mathfrak{g}_{-2}$ with projection Φ.

By Lemma 3.7 we have the following two lemmas.

LEMMA 3.8. *The group T freely acts on $\mathfrak{a}_{-1} \times \mathfrak{b}_{-1}$, and, for any $w \in \mathfrak{a}_{-1} \times \mathfrak{b}_{-1}$, w and $(0, 0, \Phi(w), 0)$ belong to the same T-orbit.*

LEMMA 3.9. $\Phi(aw) = \mathrm{Ad}(b)\Phi(w)$, $a \in T \cdot G_0$, $w \in \mathfrak{a}_{-1} \times \mathfrak{b}_{-1}$, *where b stands for the G_0-component of a.*

Let us now consider the natural representation of G_0 on $\mathfrak{g}_{-2}$, and denote by $\tilde{G}_0$ the image $\rho_{-2}(G_0)$ of G_0 by ρ_{-2}. In what follows, $T \cdot G_0$ will be exclusively considered as a transformation group on $\mathfrak{a}_{-1} \times \mathfrak{b}_{-1}$.

In view of Lemmas 3.8 and 3.9 we obtain the following

LEMMA 3.10. *If V_1 is a $\tilde{G}_0$-orbit, then the inverse image $\Phi^{-1}(V_1)$ of V_1 by Φ is a $T \cdot G_0$-orbit, and the assignment $V_1 \to \Phi^{-1}(V_1)$ gives a one-to-one correspondence between the $\tilde{G}_0$-orbits and the $T \cdot G_0$-orbits.*

LEMMA 3.11. *Let V_1 be any $\tilde{G}_0$-orbit. If $X \in V_1$, and if t is a non-zero real number, then $tX \in V_1$.*

PROOF. Let λ and μ be any non-zero real numbers. Define a linear transformation $a=a(\lambda, \mu)$ of $\mathfrak{t}$ as follows:

$$aX=\lambda\mu X \text{ if } X\in\mathfrak{g}_{-2};\ aX=\lambda X \text{ if } X\in\mathfrak{g}^{+}_{-1};\ aX=\mu X \text{ if } X\in\mathfrak{g}^{-}_{-1}.$$

Then we easily see that a is an automorphism of the pseudo-product FGLA, $\mathfrak{L}$, i.e., $a\in\mathrm{Aut}(\mathfrak{L})=G_0$. Thus the lemma follows.

Hereafter we shall identify the vector spaces $\mathfrak{a}_{-1}$ and $\mathfrak{b}_{-1}$ with open, dense subsets of the manifolds M and N by the imbeddings ι_M and ι_N respectively, so that the vector space $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ will be identified with an open, dense subset of the product manifold $M\times N$ by the imbedding ι.

Let $X\in\mathfrak{g}$. Let $\tilde{X}$ be the vector field on $M\times N$ induced from the one-parameter group of transformations of $M\times N$: $\phi_t(w)=(\exp tX)\cdot w$, $w\in M\times N$. Denote by δ_X the Lie derivation with respect to $\tilde{X}$. Furthermore, express X as follows: $X=\sum_{p=-2}^{2} X_p$, where $X_p\in\mathfrak{g}_p$, and, for $\varepsilon=1$ or -1, express X_ε as follows: $X_\varepsilon=X^{+}_{\varepsilon}+X^{-}_{\varepsilon}$, where $X^{+}_{\varepsilon}\in\mathfrak{g}^{+}_{\varepsilon}$ and $X^{-}_{\varepsilon}\in\mathfrak{g}^{-}_{\varepsilon}$.

LEMMA 3.12. *Let (x, y, u, v) be the canonical coordinate system of $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$. Then the Lie derivatives $\delta_X x$, $\delta_X y$, etc. of the functions x, y, etc. are described as follows:*

$$\delta_X x=X_{-2}+[X_0, x]+[X^{-}_{-1}, y]+[x, [y, X^{-}_{-1}]]+\frac{1}{2}[x, [x, X_2]],$$

$$\delta_X y=X^{+}_{-1}+[X^{+}_{1}, x]+[X_0, y]+[x, [y, X_2]]+\frac{1}{2}[y, [y, X^{-}_{1}]],$$

$$\delta_X u=X_{-2}+[X_0, u]+[X^{+}_{-1}, v]+[u, [v, X^{+}_{1}]]+\frac{1}{2}[u, [u, X_2]],$$

$$\delta_X v=X^{-}_{-1}+[X^{-}_{1}, u]+[X_0, v]+[u, [v, X_2]]+\frac{1}{2}[v, [v, X^{+}_{1}]].$$

PROOF. Let (x, y, u, v) be any point of $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$. Put $a(t)=\exp tX$, and, assuming that $a(t)\cdot(x, y, u, v)\in\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$, put

$$a(t)\cdot(x, y, u, v)=(x(t), y(t), u(t), v(t)),$$

which clearly means the following two equalities:

$$\pi_M(a(t)\cdot\exp(x+y))=\pi_M(\exp(x(t)+y(t))),$$
$$\pi_N(a(t)\cdot\exp(u+v))=\pi_N(\exp(u(t)+v(t))).$$

Now, the first equality means that there are unique elements $b(t)$ and $Y(t)$ of A_0 and $\mathfrak{a}_1$ respectively such that

$$a(t)\cdot\exp(x+y)=\exp(x(t)+y(t))\cdot b(t)\cdot\exp Y(t),$$

whence

$$\mathrm{Ad}(a(t))\mathrm{Ad}(\exp(x+y))\mathrm{E}_{\mathfrak{A}}=\mathrm{Ad}(\exp(x(t)+y(t)))\mathrm{Ad}(b(t))\mathrm{Ad}(\exp Y(t))\mathrm{E}_{\mathfrak{A}}.$$

Clearly we have

$$x(0)=x,\ y(0)=y,\ x'(0)=\delta_X x,\ y'(0)=\delta_X y,\ b(0)=e,\ Y(0)=0,$$

where $x'(0)$, etc. denote the derivatives of $x(t)$, etc. at $t=0$. Therefore, differentiating the equality above with respect to the variable t at $t=0$, we obtain

$$\mathrm{ad}(X)\mathrm{Ad}(\exp(x+y))\mathrm{E}_{\mathfrak{A}}=\mathrm{ad}(\delta_X x+\delta_X y)\mathrm{E}_{\mathfrak{A}}+\mathrm{Ad}(\exp(x+y))\mathrm{ad}(Y'(0))\mathrm{E}_{\mathfrak{A}},$$

whence

$$\delta_X x+\delta_X y=X_{-2}+X_{-1}^{+}+[X_{-1}^{-}+X_0+X_1^{+},\ x+y]+\frac{1}{2}[x+y,\ [x+y,\ X_1^{-}+X_2]].$$

From this equality combined with Lemma 3.3 we can easily obtain the expressions for $\delta_X x$ and $\delta_X y$. Similarly we obtain the expressions for $\delta_X u$ and $\delta_X v$.

Consider the map Φ of $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ to $\mathfrak{g}_{-2}$, which may be regarded as a $\mathfrak{g}_{-2}$-valued function on $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$. In terms of the canonical coordinate system (x, y, u, v) of $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$, the function Φ may be expressed as follows:

$$\Phi=u-x+[v,\ y].$$

LEMMA 3.13. *For any $X\in\mathfrak{g}$ there is a $\mathfrak{g}_0$-valued function F_X on $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ such that*

$$\delta_X\Phi=[F_X,\ \Phi].$$

PROOF. We have

$$\delta_X\Phi=\delta_X u-\delta_X x+[\delta_X v,\ y]+[v,\ \delta_X y].$$

Using Lemma 3.12, we calculate the right-hand side. Then we obtain

$$\delta_X\Phi=[F_X,\ \Phi]+G_X,$$

where F_X and G_X are respectively given by

$$F_X=X_0-[v,\ X_1^{+}]-[y,\ X_1^{-}]-\frac{1}{2}[\Phi,\ X_2]-[x-[v,\ y],\ X_2]-[[v,\ X_2],\ y],$$

$$G_X=\frac{1}{2}[[v,\ y],\ [[v,\ y],\ X_2]]-[[[v,\ y],\ [v,\ X_2]],\ y].$$

We assert that $G_X=0$. Indeed, take any elements Y and Z of $\mathfrak{g}_1$. Then we have

$$\delta_{[Y,Z]}\Phi=-\delta_Y\delta_Z\Phi+\delta_Z\delta_Y\Phi.$$

Clearly we have $G_Y=G_Z=0$, and hence

$$\delta_Y\Phi=[F_Y,\Phi],\ \delta_Z\Phi=[F_Z,\Phi].$$

Therefore it follows that

$$\delta_{[Y,Z]}\Phi=[-\delta_Y F_Z+\delta_Z F_Y+[F_Y,F_Z],\Phi].$$

Since $\mathfrak{g}_2=[\mathfrak{g}_1,\mathfrak{g}_1]$ by Lemma 3.3, we have therefore shown that, for any $X\in\mathfrak{g}_2$, there is a $\mathfrak{g}_0$-valued function F'_X on $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ such that $\delta_X\Phi=[F'_X,\Phi]$. Consequently we get

$$[\mathrm{F}_X,\Phi]+G_X=[F'_X,\Phi],\ X\in\mathfrak{g}_2.$$

Here, we note that G_X is a function of the variables y and v only. Hence, putting $\Phi=0$, i.e., $u=x-[v,y]$ in the equality above, we obtain $G_X=0$ for all $X\in\mathfrak{g}_2$. Therefore we have $G_X=G_{X_2}=0$ for all $X\in\mathfrak{g}$, proving our assertion.

LEMMA 3.14. *Let Ω_1' be any $T\cdot G_0$-orbit, and let Ω_1 be the G-orbit containing Ω_1'. Then* $\dim\Omega_1=\dim\Omega_1'$.

PROOF. By Lemma 3.10 we know that there is a unique $\tilde{G}_0$-orbit V_1 such that $\Omega_1'=\Phi^{-1}(V_1)$. Now, take any point w of Ω_1'. By Lemma 3.13 we have $\Phi_*(\tilde{X}_w)=(\delta_X\Phi)(w)=[F_X(w),\Phi(w)]$ for all $X\in\mathfrak{g}$. We have $T(\Omega_1)_w=\{\tilde{X}_w|X\in\mathfrak{g}\}$ and $T(V_1)_{\Phi(w)}=[\mathfrak{g}_0,\Phi(w)]$. Therefore it follows that

$$\Phi_{*w}(T(\Omega_1)_w)\subset T(V_1)_{\Phi(w)}.$$

Since $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ is a fibred manifold over the base space $\mathfrak{g}_{-2}$ whth projection Φ, we have

$$\mathrm{Ker}\ \Phi_{*w}\subset T(\Omega_1')_w\subset T(\Omega_1)_w,$$
$$\Phi_{*w}(T(\Omega_1')_w)=T(V_1)_{\Phi(w)}.$$

From these facts it follows immediately that $T(\Omega_1)_w=T(\Omega_1')_w$. Hence we obtain $\dim\Omega_1=\dim\Omega_1'$, proving the lemma.

LEMMA 3.15. *For any point $w\in M\times N$, there is an element a of G such that $aw\in\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$. In other words, any G-orbit Ω_1 intersects the subset*

$\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ *of* $M\times N$.

PROOF. Put $w=(p, q)$. Then there is $a\in G$ such that $ap=o_M$, and hence $aw=(o_M, aq)$, where o_M denotes the origin of $M=G/A^{(0)}$. Thus we may assume that w is of the form (o_M, q). Since $o_M\in\mathfrak{a}_{-1}$, and since $\mathfrak{a}_{-1}$ is an open set of M, there is a neighborhood U of e such that $U\cdot o_M\subset\mathfrak{a}_{-1}$. Since $\mathfrak{b}_{-1}$ is a dense subset of N, we have $U\cdot q\cap\mathfrak{b}_{-1}\neq\phi$. Therefore we can find $a\in G$ such that $a\cdot o_M\in\mathfrak{a}_{-1}$ and $aq\in\mathfrak{b}_{-1}$, and hence $aw=(ao_M, aq)\in\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$, which proves the lemma.

3.3. Condition (II. 1). From now on we assume the following condition:

(II. 1) There is an element g of G such that $g^2=e$ and such that $\mathrm{Ad}(g)E=-E$ and $\mathrm{Ad}(g)J=J$.

Clearly this condition implies that $\mathrm{Ad}(g)E_{\mathfrak{A}}=-E_{\mathfrak{B}}$. Let A_{-1} and B_{-1} be the Lie subgroups of G generated by the abelian subalgebras $\mathfrak{a}_{-1}$ and $\mathfrak{b}_{-1}$ of $\mathfrak{g}$ respectively. Then we easily have the following

LEMMA 3.16. (1) $\mathrm{Ad}(g)\mathfrak{g}_{-2}=\mathfrak{g}_2$, $\mathrm{Ad}(g)\mathfrak{g}^+_{-1}=\mathfrak{g}^+_1$, $\mathrm{Ad}(g)\mathfrak{g}^-_{-1}=\mathfrak{g}^-_1$, $\mathrm{Ad}(g)\mathfrak{g}_0=\mathfrak{g}_0$.

(2) $\mathrm{Ad}(g)\mathfrak{b}_{-1}=\mathfrak{a}_1$, $\mathrm{Ad}(g)\mathfrak{b}_0=\mathfrak{a}_0$, $\mathrm{Ad}(g)\mathfrak{b}_1=\mathfrak{a}_{-1}$.

(3) $gB_0g=A_0$, $gB^{(0)}g=A_{-1}\cdot A_0$, $gA^{(0)}g=B_{-1}\cdot B_0$.

In particular it follows from this fact that $\dim\mathfrak{g}^+_{-1}=\dim\mathfrak{g}^-_{-1}=\dim\mathfrak{g}^+_1=\dim\mathfrak{g}^-_1$ and $\dim\mathfrak{a}_{-1}=\dim\mathfrak{a}_1=\dim\mathfrak{b}_{-1}=\dim\mathfrak{b}_1$.

We shall study open orbits under the action of G on $M\times N$.

Let o_M and o_N be the origins of $M=G/A^{(0)}$ and $N=G/B^{(0)}$ respectively. Let $\Omega_{\mathfrak{A}}$ be the G-orbit through the point (o_M, go_N). Clearly the isotropy group of G at (o_M, go_N) is given by $A^{(0)}\cap gB^{(0)}g$. By Lemmas 1.10 and 3.16 we have $A^{(0)}\cap gB^{(0)}g=A_0\cdot A_1\cap A_{-1}\cdot A_0=A_0$. Hence $\Omega_{\mathfrak{A}}$ may be represented by the homogeneous space G/A_0. We have $\dim G/A_0=\dim(\mathfrak{a}_{-1}+\mathfrak{a}_1)=\dim(M\times N)$, meaning that $\Omega_{\mathfrak{A}}$ is an open orbit. Similarly, let $\Omega_{\mathfrak{B}}$ be the G-orbit through the point (go_M, o_N). As above it is shown that the isotropy group of G at (go_M, o_N) is given by $gA^{(0)}g\cap B^{(0)}=B_0$, and hence $\Omega_{\mathfrak{B}}$ may be represented by the homogeneous space G/B_0. It is also shown that $\Omega_{\mathfrak{B}}$ is an open orbit.

We want to show that $\Omega_{\mathfrak{A}}$ is dense in $M\times N$. For this purpose we define a map ϖ of $G\times G$ to $M\times N$ by

$$\varpi(a, b)=(\pi_M(a), \pi_N(bg))=(ao_M, bgo_N),$$

where $(a, b)\in G\times G$. Clearly $G\times G$ is a fibred manifold over the base space $M\times N$ with projection ϖ. We also define a map η of $G\times G$ to G by

$$\eta(a, b)=b^{-1}a \quad \text{for all } (a, b)\in G\times G.$$

Clearly $G\times G$ is a fibred manifold over the base space G with projection η. Moreover we consider the subset of G:

$$S=A_{-1}\cdot A_0\cdot A_1.$$

In view of the fact that $gB^{(0)}g=A_{-1}\cdot A_0$, we can easily verify the following

LEMMA 3.17. $\varpi^{-1}(\Omega_{\mathfrak{A}})=\eta^{-1}(S)$.

By Corollary 1 to Lemma 1.11 we know that S is dense in G. It follows that $\varpi^{-1}(\Omega_{\mathfrak{A}})=\eta^{-1}(S)$ is dense in $G\times G$, and in turn that $\Omega_{\mathfrak{A}}$ is dense in $M\times N$. Consequently we find that $\Omega_{\mathfrak{A}}$ is a single open G-orbit. Especially we have $\Omega_{\mathfrak{B}}=\Omega_{\mathfrak{A}}$.

We have therefore proved the following

LEMMA 3.18. *The action of G on $M\times N$ has a single open orbit Ω. Furthermore,* 1) $(o_M, go_N)\in\Omega$, *and the isotropy group of G at (o_M, go_N) is given by A_0*; 2) $(go_M, o_N)\in\Omega$, *and the isotropy group of G at (go_M, o_N) is given by B_0. Hence Ω may be represented by the homogeneous space G/A_0 or G/B_0.*

REMARK. The open G-orbit Ω is not contained in the open, dense subset $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ of $M\times N$. Indeed, suppose that $(o_M, go_N)\in\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$, i.e., $go_N\in\mathfrak{b}_{-1}$. Clearly this means that $g\in B_{-1}\cdot B_0\cdot B_1$. On the other hand, we have $\mathrm{Ad}(g)\mathfrak{b}_1=\mathfrak{a}_{-1}\subset\mathfrak{b}_{-1}+\mathfrak{b}_0$ by Lemma 3.16. Therefore it follows from Lemma 1.11 that $g\notin B_{-1}\cdot B_0\cdot B_1$, which is a contradiction.

We know from Lemmas 3.14 and 3.18 that the intersection $\Omega'=\Omega\cap(\mathfrak{a}_{-1}\times\mathfrak{b}_{-1})$ is the union of all open $T\cdot G_0$-orbits, and from Lemma 3.10 that Ω' may be expressed as follows: $\Omega'=\Phi^{-1}(V)$, where V is the union of all open $\tilde{G}_0$-orbits. Therefore it follows that the boundary $\partial\Omega$ of Ω in $M\times N$ is the union of all singular G-orbits, that the boundary $\partial'\Omega'$ of Ω' in $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ is the union of all singular $T\cdot G_0$-orbits, and that the boundary ∂V of V in $\mathfrak{g}_{-2}$ is the union of all singular $\tilde{G}_0$ orbits. Clearly we have

$$\partial\Omega\cap(\mathfrak{a}_{-1}\times\mathfrak{b}_{-1})=\partial'\Omega'=\Phi^{-1}(\partial V).$$

Let ρ_M (resp. ρ_N) be the projection of Ω to M (resp. of Ω to N).

LEMMA 3.19. *Ω is a fibred manifold over the base space M (resp. over N) with projection ρ_M (resp. with ρ_N), and its fibres are all connected.*

PROOF. Clearly both the maps ρ_M and ρ_N are G-equivariant,and we have $\rho_M((o_M, go_N))=o_M$ and $\rho_N((go_M, o_N))=o_N$. By Lemma 3. 18 we know that the isotropy group of G at (o_M, go_N) (resp. at (go_M, o_N)) is A_0 (resp. B_0), and hence Ω may be represented by the homogeneous space G/A_0 (resp. G/B_0). Furthermore we know that G/A_0 (resp. G/B_0) is naturally a fibred manifold over the base space $G/A^{(0)}$ (resp. over $G/B^{(0)}$), and its fibres are all connected (see 2. 4). Now the lemma follows from these facts.

We shall finally study minimal dimensional orbits under the action of G on $M\times N$.

LEMMA 3. 20. *The action of G on $M\times N$ has a single minimal dimensional orbit R. Furthermore, $(o_M, o_N)\in R$, and the isotropy group of G at (o_M, o_N) is given by $G^{(0)}$. Hence R may be represented by the homogeneous space $G/G^{(0)}$.*

PROOF. Let R be the G-orbit through the point (o_M, o_N). Clearly the isotropy group of G at (o_M, o_N) is given by $A^{(0)}\cap B^{(0)}$. By Lemma 3. 4 we have $A^{(0)}\cap B^{(0)}=G^{(0)}$. Hence R may be represented by the homogeneous space $G/G^{(0)}$. Now, consider the subset $\Phi^{-1}(0)$ of $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$, which is the $T\cdot G_0$-orbit through the zero of the vector space or the point (o_M, o_N). We assert that $\Phi^{-1}(0)$ is a single minimal dimensional $T\cdot G_0$-orbit. Indeed, let V_1 be any $\tilde{G}_0$-orbit. Clearly we have $\dim\Phi^{-1}(V_1)\geqq\dim\Phi^{-1}(0)$. Suppose that $\dim\Phi^{-1}(V_1)=\dim\Phi^{-1}(0)$. Then we have $\dim V_1=0$. Since the natural representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-2}$ is irreducible (Lemma 5, Appendix in [11]), this means that $V_1=\{0\}$, proving our assertion. Therefore it follows from Lemmas 3. 14 and 3. 15 that R is a single minimal dimensional G-orbit. We have thus proved the lemma.

From the proof above we know that $R\cap(\mathfrak{a}_{-1}\times\mathfrak{b}_{-1})=\Phi^{-1}(0)$.

3. 4. Condition (II. 2). Let us consider the single open orbit Ω under the action of G on $M\times N$ (Lemma 3. 18). We know that the isotropy group of G at the point $(o_M, go_N)\in\Omega$ is $A^{(0)}\cap gB^{(0)}g=A_0$, and hence Ω may be represented by the homogeneous space G/A_0. Now, let $\bar{G}/\bar{A}_0$ be the standard affine symmetric space associated with $\mathfrak{A}$. Recall that the groups $\bar{A}_0$ and $\bar{G}$ are defined respectively as follows: $\bar{A}_0=\mathrm{Aut}(\mathfrak{A})$ and $\bar{G}=\mathrm{Aut}(\mathfrak{g})^0\cdot\bar{A}_0$. Clearly we have $\bar{G}=G\cdot\bar{A}_0$ and $A_0=\bar{A}_0\cap G$. Hence $\bar{G}/\bar{A}_0=G/A_0$. Thus we have the following identifications:

$$\Omega=G/A_0=\bar{G}/\bar{A}_0.$$

We claim that the product structure (E_Ω, F_Ω) of Ω as an open submanifold of $M\times N$ coincides with the invariant product structure of $\bar{G}/\bar{A}_0$ which is induced from the abelian subalgebras $\mathfrak{a}_{-1}$ and $\mathfrak{a}_1$ of $\mathfrak{g}$. In fact, this can be easily verified by using the following fact:

$$(\mathfrak{a}_{-1}+\mathfrak{a}_1)\cap \mathrm{Ad}(g)\mathfrak{b}^{(0)}=\mathfrak{a}_{-1},\ (\mathfrak{a}_{-1}+\mathfrak{a}_1)\cap \mathfrak{a}^{(0)}=\mathfrak{a}_1.$$

Let us now consider the single minimal dimensional orbit R under the action of G on $M\times N$ (Lemma 3.20). We know that the isotropy group of G at the point $(o_M, o_N)\in R$ is $A^{(0)}\cap B^{(0)}=G^{(0)}$, and hence R may be represented by the homogeneous space $G/G^{(0)}$. We claim that the pseudo-product structure (E_R, F_R) of R as a submanifold of $M\times N$ coincides with the invariant pseudo-product structure of $G/G^{(0)}$ which is induced from the subalgebras $\mathfrak{a}^{(0)}$ and $\mathfrak{b}^{(0)}$ of $\mathfrak{g}$. In fact, this can be easily verified by using the followin facts:

$$\mathfrak{t}\cap \mathfrak{b}^{(0)}=\mathfrak{g}^{+}_{-1},\ \mathfrak{t}\cap \mathfrak{a}^{(0)}=\mathfrak{g}^{-}_{-1}.$$

We shall study the automorphism group $\mathrm{Aut}(\Omega)$ of the product manifold Ω via the automorphism group $\mathrm{Aut}(R)$ of the pseudo-product manifold R.

First of all we have natural injective homomorphisms i_Ω and i_R of G to $\mathrm{Aut}(\Omega)$ and of G to $\mathrm{Aut}(R)$ respectively:

$$i_\Omega(a)=a_\Omega,\ i_R(a)=a_R,\ a\in G,$$

where a_Ω and a_R are respectively the transformations of Ω and R induced *by* a.

Next we see from Lemma 3.19 that every $\phi\in\mathrm{Aut}(\Omega)$ naturally induces diffeomorphisms ϕ_M and ϕ_N of M and N respectively: $\rho_M\circ\phi=\phi_M\circ\rho_M$ and $\rho_N\circ\phi=\phi_N\circ\rho_N$. Thus we have the product transformation $\phi_{M\times N}=\phi_M\times\phi_N$ of $M\times N$. Clearly the restriction of $\phi_{M\times N}$ to Ω coincides with the given ϕ, and the assignment $\phi\rightarrow\phi_{M\times N}$ gives an injective homomorphism of $\mathrm{Aut}(\Omega)$ to $\mathrm{Aut}(M\times N)$, the automorphism group of the product manifold $M\times N$.

Let $\phi\in\mathrm{Aut}(\Omega)$. Then it is clear that $\phi_{M\times N}$ leaves the domain Ω invariant, from which follows that $\phi_{M\times N}$ leaves the boundary $\partial\Omega$ of Ω invariant as well. We want to show that $\phi_{M\times N}$ furhter leaves the submanifold R $(\subset\phi\Omega)$ invariant.

Here, we prepare some notations and terminologies for our later arguments. Let U be a finite dimensional vector space over $\boldsymbol{R}$. Then $P(U)$ denotes the algebra of (complex valued) polynomial functions on U. For any non-negative integer m, $P(U)_m$ denotes the subspace of $P(U)$ consisting of all polynomial functions of degree m and the zero polynomial function:

$P(U)=\sum_{m\geqq 0} P(U)_m$ (direct sum). Now, let W be an algebraic variety over $\boldsymbol{R}$ of U. By the (complexified) ideal $I(W)$ of the variety W we mean the ideal of $P(U)$ consisting of all polynomial functions which vanish on W. The ideal $I(W)$ is said to be homogeneous, if $I(W)=\sum_{m\geqq 0} I(W)_m$ with $I(W)_m=I(W)\cap P(U)_m$. Note that $I(W)$ is homogeneous, if and only if $tX\in W$ for all $X\in W$ and $t\in\boldsymbol{R}$. When $I(W)$ is homogeneous, $l(W)$ denotes the smallest non-negative integer l such that $I(W)_l\neq\{0\}$.

Now, consider the union ∂V of all singular $\tilde{G}_0$-orbits, which is a proper algebraic variety over $\boldsymbol{R}$ of $\mathfrak{g}_{-2}$. By Lemma 3.11 we see that the ideal $I(\partial V)$ of the variety ∂V is homogeneous. Note that the integer $l(\partial V)$ is positive.

From now on we assume the following condition:

(II. 2) For every $X\in\partial V$-$\{0\}$ there is a polynomial function $f\in I(\partial V)$ such that $j_X^{l-1}(f)\neq 0$, where $l=l(\partial V)$.

We shall prove the following

Lemma 3.21. *For every $\phi\in\mathrm{Aut}(\Omega)$, $\phi_{M\times N}$ leaves R invariant.*

For this purpose we first prove the following

Lemma 3.22. *Let f be a C^∞ function defined on a neighborhood of* 0 *in $\mathfrak{g}_{-2}$. If f vanishes on a neighborhood of* 0 *in ∂V, then $j_0^{l-1}(f)=0$.*

Proof. For any non-negative integer s, we define a polynomial function $f^{(s)}\in P(\mathfrak{g}_{-2})_s$ by

$$f^{(s)}(X)=\frac{d^s}{dt^s}f(tX)_{t=0} \text{ for all } X\in\mathfrak{g}_{-2}.$$

Since $tX\in\partial V$ for all $X\in\partial V$ and $t\in\boldsymbol{R}$, it follows that $f^{(s)}(X)=0$ for all $X\in\partial V$, i.e., $f^{(s)}\in I(\partial V)_s$. Hence we obtain $f^{(s)}=0$ for all $s\leqq l-1$, meaning that $j_0^{l-1}(f)=0$. This proves the lemma.

As we have remarked, $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ is a fibred menifold over the base space $\mathfrak{g}_{-2}$ with projection Φ. Therefore the next lemma follows immediately from Lemma 3.22 and condition (II. 2).

Lemma 3.23. (1) *Let f be a C^∞ function defined on a neighborhood of $0=(0,0)$ in $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$. It f vanishes on a neighborhood of* 0 *in $\Phi^{-1}(\partial V)$, then $j_0^{l-1}(f)=0$.*

(2) *For every $w\in\Phi^{-1}(\partial V)-\Phi^{-1}(0)$ there is a polynomial function $f\in I(\partial V)$ such that $j_w^{l-1}(f\circ\Phi)\neq 0$.*

Put $o=(o_M, o_N)$, which may be identified with the zero of $\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$.

LEMMA 3.24. Let $\phi\in\mathrm{Aut}(\Omega)$. If $\phi_{M\times N}(o)\in\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$, then $\phi_{M\times N}(o)\in R$.

PROOF. We first recall that $\phi_{M\times N}$ leaves the boundary $\partial\Omega$ of Ω invariant. Therefore, putting $w=\phi_{M\times N}(o)$, we have $w\in\partial\Omega\cap(\mathfrak{a}_{-1}\times\mathfrak{b}_{-1})=\Phi^{-1}(\partial V)$. Suppose that $w\notin\Phi^{-1}(0)$. By (2) of Lemma 3.23 there is a polynomial function $f\in I(\partial V)$ such that $j_w^{l-1}(f\circ\Phi)\neq 0$. Since $\phi_{M\times N}$ maps a neighborhood of o in $\Phi^{-1}(\partial V)$ onto a neighborhood of w in $\Phi^{-1}(\partial V)$, we see that the function $f\circ\Phi\circ\phi_{M\times N}$ vanishes on a neighborhood of o in $\Phi^{-1}(\partial V)$. It follows from (1) of Lemma 3.23 that $j_0^{l-1}(f\circ\Phi\circ\phi_{M\times N})=0$, whence $j_w^{l-1}(f\circ\Phi)=0$. This is a contradiction. Consequently we have shown that $w\in\Phi^{-1}(0)\subset R$, proving the lemma.

Now, Lemma 3.21 can be derived from Lemma 3.24 in the following manner: Let $\phi\in\mathrm{Aut}(\Omega)$ and $w\in R$. We take $a\in G$ such that $w=ao$, and then take $b\in G$ such that $b\cdot\phi_{M\times N}(w)\in\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$ (Lemma 3.15). If we put $\psi=b_\Omega\cdot\phi\cdot a_\Omega$, it is clear that $\psi\in\mathrm{Aut}(\Omega)$, $\psi_{M\times N}(o)\in\mathfrak{a}_{-1}\times\mathfrak{b}_{-1}$, and $\phi_{M\times N}(w)=b^{-1}\cdot\psi_{M\times N}(o)$. Therefore it follows from Lemma 3.24 that $\phi_{M\times N}(w)\in R$, proving Lemma 3.21.

REMARK. The proof above of Lemma 3.21 does not use the property that $\phi_{M\times N}$ is a product transformation. Accordingly we have obtained the stronger result: Let ψ be a diffeomorphism of $M\times N$. If ψ leaves Ω invariant, then it leaves the submanifold R invariant as well.

For $\phi\in\mathrm{Aut}(\Omega)$ we denote by ϕ_R the restriction of $\phi_{M\times N}$ to R, which is an automorphism of the pseudo-product manifold R by Lemma 3.21. Then we see that the assignment $\phi\to\phi_R$ gives an injective homomorphism j of $\mathrm{Aut}(\Omega)$ to $\mathrm{Aut}(R)$, where the injectivity follows from the fact that the projections of R to M and N are both surjective.

In this way we have obtained the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{i_\Omega} & \mathrm{Aut}(\Omega) \\ & {}_{i_R}\searrow & \downarrow j \\ & & \mathrm{Aut}(R). \end{array}$$

Here the homomorphisms i_Ω, i_R and j are all injective.

By Lemma 1.15 we know that the homomorphism i_R gives an isomorphism of G onto $\mathrm{Aut}(R)$. If follows immediately that the homomorphism j gives an isomorphism of $\mathrm{Aut}(\Omega)$ onto $\mathrm{Aut}(R)$, and in turn the homomorphism i_Ω an isomorphism of G onto $\mathrm{Aut}(\Omega)$.

LEMMA 3.25. *$A_0=\mathrm{Aut}(\mathfrak{A})$, and $B_0=\mathrm{Aut}(\mathfrak{B})$.*

PROOF. Consider the standard affine symmetric space $\bar{G}/\bar{A}_0$ associated with $\mathfrak{A}$. Since $\Omega=\bar{G}/\bar{A}_0$ as product manifolds, and since the homomorphism i_Ω gives an isomorphism of G onto $\mathrm{Aut}(\Omega)$, it follows that $\bar{G}=G$. Hence we obtain $A_0=G\cap \mathrm{Aut}(\mathfrak{A})=\mathrm{Aut}(\mathfrak{A})$. Furthermore it follows by Lemma 3.16 that $B_0=gA_0g=g\mathrm{Aut}(\mathfrak{A})g=\mathrm{Aut}(\mathfrak{B})$, proving the lemma.

From this fact we find that G/A_0 and G/B_0 are the standard affine symmetric spaces associated with $\mathfrak{A}$ and $\mathfrak{B}$ respectively.

We have thereby proved the following

PROPOSITION 3.26. *Let $\mathfrak{G}$ be a pseudo-product* SGLA *of the second kind over $\boldsymbol{R}$. Assume that $\mathfrak{G}$ satisfies conditions* (II.1) *and* (II.2). *Then $\Omega=G/A_0$ is the standard affine symmetric space associated with $\mathfrak{A}$, and the homomorphism i_Ω gives an isomorphism of G onto* $\mathrm{Aut}(\Omega)$.

§ 4. Some studies on simple graded Lie algebras of the first kind

4.1. The action of the group G on the product manifold $G/G^{(0)}\times G/G^{(0)}$. As is easily observed, most of the results in the previous section, especially Lemmas 3.5~3.24 there, hold true, even in the degenerate case when $\mathfrak{g}_{-1}=\mathfrak{g}_1=\{0\}$. (Suppose this, which indeed makes sense. Then the graded Lie algebra $\mathfrak{G}$ may be naturally regarded as a SGLA of the first kind, so that $\mathfrak{A}=\mathfrak{B}=\mathfrak{G}$. It follows that $A_0=B_0=G_0$, $A^{(0)}=B^{(0)}=G^{(0)}$, and hence $G/A_0=G/B_0=G/G_0$, $G/A^{(0)}=G/B^{(0)}=G/G^{(0)}$.)

In the present section, we study a SGLA of the first kind, which just corresponds to the study of this degenerate case. The counterparts of the relevant lemmas in that section will be stated without proof in principle.

Let $\mathfrak{G}=\{\mathfrak{g}, (\mathfrak{g}_p)\}$ be a simple graded Lie algebra of the first kind. Let G_0, $G^{(0)}$ and G be the groups associated with $\mathfrak{G}$:

$$G_0=\mathrm{Aut}(\mathfrak{G}),\ G^{(0)}=G_0\cdot G_1,\ G=\mathrm{Aut}(\mathfrak{g})^0\cdot G_0,$$

where G_1 denotes the Lie subgroup of $\mathrm{Aut}(\mathfrak{g})^0$ generated by the abelian subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$.

Set $M=G/G^{(0)}$. Then the group G naturally acts on the product manifold $M\times M$:

$$a\cdot(p, q)=(ap, aq),\ a\in G,\ (p, q)\in M\times M.$$

Let us consider the natural imbedding ι_M of $\mathfrak{g}_{-1}$ into M:

$$\iota_M(X)=\pi_M(\exp X),\ X\in\mathfrak{g}_{-1},$$

where π_M denotes the projection of G onto M. Then the product map $\iota=\iota_M\times\iota_M$ gives an imbedding of $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}$ into $M\times M$. We know that the image $\iota_M(\mathfrak{g}_{-1})$ of $\mathfrak{g}_{-1}$ by ι_M is an open, dense subset of M. It follows that the image $\iota(\mathfrak{g}_{-1}\times\mathfrak{g}_{-1})$ of $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}$ by ι is an open, dense subset of $M\times M$ as well.

Let G_{-1} be the Lie subgroup of G generated by the abelian subalgebra $\mathfrak{g}_{-1}$ of $\mathfrak{g}$. Then the product $G_{-1}\cdot G_0$ of G_{-1} and G_0 gives a (closed) subgroup of G, which acts on $M\times M$ as a subgroup of G. Then it is shown that the group $G_{-1}\cdot G_0$ leaves the subset $\iota(\mathfrak{g}_{-1}\times\mathfrak{g}_{-1})$ of $M\times M$ invariant. Therefore the group $G_{-1}\cdot G_0$ acts on $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}$ in such a way that the imbedding ι becomes equivariant:

$$\iota(aw)=a\iota(w),\quad a\in G_{-1}\cdot G_0,\quad w\in\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}.$$

For any $a\in G_{-1}\cdot G_0$ and $(x, u)\in\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}$, set $a\cdot(x, u)=(x', u')$. Express a as follows: $a=\exp X\cdot b$, where $X\in\mathfrak{g}_{-1}$ and $b\in G_0$. Then we have

$$x'=\mathrm{Ad}(b)x+X,\quad u'=\mathrm{Ad}(b)u+X.$$

We define a map Φ of $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}$ to $\mathfrak{g}_{-1}$ by

$$\Phi(w)=u-x \text{ for all } w=(x, u)\in\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}.$$

Let ρ_{-1} be the natural representation of G_0 on $\mathfrak{g}_{-1}$, which is faithful (Lemma 1.7). Then we denote by $\tilde{G}_0$ the image of G_0 by ρ_{-1}, which is nothing but the linear isotropy group associated with the homogeneous space $M=G/G^{(0)}$. Hereafter $G_{-1}\cdot G_0$ willl be exclusively considered as a transformation group on $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}$.

LEMMA 4.1. *If V_1 is a $\tilde{G}_0$-orbit, then the inverse image $\Phi^{-1}(V_1)$ of V_1 by Φ is a $G_{-1}\cdot G_0$-orbit, and the assignment $V_1\to\Phi^{-1}(V_1)$ gives a one-to-one correspondence between the $\tilde{G}_0$-orbits and the $G_{-1}\cdot G_0$-orbits.*

LEMMA 4.2. *Let V_1 be any $\tilde{G}_0$-orbit. If $X\in V_1$ and if t is a non-zero real number, then $tX\in V_1$.*

PROOF. Let λ be any non-zero real number. Then we define a linear transformation $a=a(\lambda)$ of $\mathfrak{g}$ by

$$aX=\lambda^p X \text{ for all } X\in\mathfrak{g}_p \text{ and all } p.$$

It is easy to see that a is an automorphism of $\mathfrak{G}$, i.e., $a\in\mathrm{Aut}(\mathfrak{G})=G_0$. Accordingly the lemma follows.

In what follows (until the end of 4.3), we shall identify $\mathfrak{g}_{-1}$ with an open, dense subset of M by the imbedding ι_M, so that $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}$ will be identified with an open, dense subset of $M\times M$ by the imbedding ι.

LEMMA 4.3. *Let Ω_1' be any $G_{-1}\cdot G_0$-orbit, and let Ω_1 be the G-orbit containg Ω_1'. Then* $\dim\Omega_1=\dim\Omega_1'$.

LEMMA 4.4. *Any G-orbit, Ω_1, intersects the subset $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}$ of $M\times M$.*

4. 2. Condition (I. 1). Let E be the characteristic element of G. From now on we assume the following condition:

(I. 1) There is an element g of G such that $g^2=e$ and such that $\mathrm{Ad}(g)E=-E$.

Clearly this condition implies the following:

$$\mathrm{Ad}(g)\,\mathfrak{g}_{-1}=\mathfrak{g}_1,\ g\cdot G_0\cdot g=G_0,\ g\cdot G^{(0)}\cdot g=G_{-1}\cdot G_0.$$

Let o_M be the origin of $M=G/G^{(0)}$.

LEMMA 4.5. *The action of G on $M\times M$ has a single open orbit Ω. Furthermore, (o_M, go_M), $(go_M, o_M)\in\Omega$, and the isotropy groups of G at these points are both given by G_0. Hence Ω may be represented in two ways by the homogeneous space G/G_0.*

We know from Lemmas 4.3 and 4.5 that the intersection $\Omega'=\Omega\cap(\mathfrak{g}_{-1}\times\mathfrak{g}_{-1})$ is the union of all open $G_{-1}\cdot G_0$-orbits, and from Lemma 4.1 that Ω' may be expressed as follows: $\Omega'=\Phi^{-1}(V)$, where V is the union of all open $\tilde{G}_0$-orbits. Therefore it follows that the boundary $\partial\Omega$ of Ω in $M\times M$ is the union of all singular G-orbits, that the boundary $\partial'\Omega'$ of Ω' in $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}$ is the union of all singular $G_{-1}\cdot G_0$-orbits, and that the boundary ∂V of V in $\mathfrak{g}_{-1}$ is the union of all singular $\tilde{G}_0$-orbits. We have

$$\partial\Omega\cap(\mathfrak{g}_{-1}\times\mathfrak{g}_{-1})=\partial'\Omega'=\Phi^{-1}(\partial V).$$

Let $\rho_{M,1}$ and $\rho_{M,2}$ be the two kinds of projections of Ω to M:

$$\rho_{M,1}(w)=p,\ \rho_{M,2}(w)=q,\ w=(p,\ q)\in\Omega.$$

LEMMA 4.6. *Ω is a fibred manifold over the base space M with projection $\rho_{M,1}$ or $\rho_{M,2}$, and its fibres are all connected.*

LEMMA 4.7. *The action of G on $M\times M$ has a single minimal dimensional orbit, which is given by the diagonal R of $M\times M$.*

Clearly we have $R\cap(\mathfrak{g}_{-1}\times\mathfrak{g}_{-1})=\Phi^{-1}(0)$.

4. 3. Condition (I. 2). Let us consider the single open orbit Ω under the action of G on $M\times M$ (Lemma 4.5). We know that the isotropy group of G at the point $(o_M, go_M)\in\Omega$ is G_0, and hence Ω may be represented by the homogeneous space G/G_0. We note that G/G_0 is just the standard affine symmetric space associated with $\mathfrak{G}$. We also note that the product structure (E_Ω, F_Ω) of Ω as an open submanifold of $M \times M$ coincides with the invariant product structure of G/G_0.

We shall study the automorphism group $\mathrm{Aut}(\Omega)$ of the product manifold Ω.

First of all we have a natural injective homomorphism i_Ω of G to $\mathrm{Aut}(\Omega)$:

$$i_\Omega(a) = a_\Omega, \quad a \in G,$$

where a_Ω is the transformation of Ω induced by a.

Next we see from Lemma 4.6 that every $\phi \in \mathrm{Aut}(\Omega)$ naturally induces diffeomorphisms $\phi_{M,1}$ and $\phi_{M,2}$ of M : $\rho_{M,i} \circ \phi = \phi_{M,i} \circ \rho_{M,i}$ $(i=1, 2)$. Thus we have the product transformation $\phi_{M\times M} = \phi_{M,1} \times \phi_{M,2}$ of $M \times M$. Clearly the restriction of $\phi_{M\times M}$ to Ω is the given ϕ, and the assignmet $\phi \to \phi_{M\times M}$ gives an injective homomorphism of $\mathrm{Aut}(\Omega)$ to $\mathrm{Aut}(M \times M)$, the automorphism group of the product manifold $M \times M$.

Now, let us consider the union ∂V of all singular $\tilde{G}_0$-orbits, which is a proper algebraic variety of $\mathfrak{g}_{-1}$. Note that the (complexified) ideal $I(\partial V)$ of the variety ∂V is homogeneous (Lemma 4.2), and that the integer $l(\partial V)$ is positive.

From now on we assume the following condition:

(I.2) For every $X \in \partial V - \{0\}$ there is a polynomial function $f \in I(\partial V)$ such that $j_X^{l-1}(f) \neq 0$, where $l = l(\partial V)$.

Lemma 4.8. *For every $\phi \in \mathrm{Aut}(\Omega)$, $\phi_{M\times M}$ leaves the diagonal R of $M \times M$ invariant.*

Therefore it follows that $\phi_{M,1} = \phi_{M,2}$. Denote this by ϕ_M. Then we have

$$\phi_{M\times M} = \phi_M \times \phi_M,$$

and see that the assignment $\phi \to \phi_M$ gives an injective homomorphism j of $\mathrm{Aut}(\Omega)$ to $\mathrm{Diff}(M)$, the diffeomorphism group of M.

Proposition 4.9. *Assume that $\partial V = \{0\}$. Then the homomorphism j gives an isomorphism of* $\mathrm{Aut}(\Omega)$ *onto* $\mathrm{Diff}(M)$.

Proof. We have $\partial\Omega \cap (\mathfrak{g}_{-1} \times \mathfrak{g}_{-1}) = \Phi^{-1}(\partial V) = \Phi^{-1}(0) = R \cap (\mathfrak{g}_{-1} \times \mathfrak{g}_{-1})$, from which follows that $\partial\Omega = R$ (cf. Lemmas 4.4). It is now clear that the proposition is true.

4. 4. Condition (I. 3). We define a subgroup $GL(\partial V)$ of $GL(\mathfrak{g}_{-1})$ by

$$GL(\partial V)=\{a\in GL(\mathfrak{g}_{-1})\mid a(\partial V)=\partial V\}.$$

We have identified $\mathfrak{g}_{-1}$ with an open, dense subset of M by the imbedding ι_M, which we call off from now on.

Let $\phi\in\mathrm{Aut}(\Omega)$. We define an open set $U(\phi)$ of $\mathfrak{g}_{-1}$ by the requirement that

$$\iota_M(U(\phi))=\iota_M(\mathfrak{g}_{-1})\cap\phi_M^{-1}(\iota_M(\mathfrak{g}_{-1})).$$

Clearly we have

$$\phi_M(\iota_M(U(\phi)))=\phi_M(\iota_M(\mathfrak{g}_{-1}))\cap\iota_M(\mathfrak{g}_{-1})=\iota_M(U(\phi^{-1})).$$

This being said, we then define a diffeomorphism ϕ'_M of $U(\phi)$ onto $U(\phi^{-1})$ by the requirement that

$$\phi_M\circ\iota_M=\iota_M\circ\phi'_M.$$

Now, let $X\in U(\phi)$. We define a linear transformation $\partial(\phi, X)$ of $\mathfrak{g}_{-1}$ by

$$\partial(\phi, X)\,Y=\lim_{t\to 0}\frac{1}{t}(\phi'_M(X+tY)-\phi'_M(X)),\quad Y\in\mathfrak{g}_{-1},$$

which may be identified with the differential of ϕ'_M at X.

LEMMA 4. 10. $\partial(\phi, X)\in GL(\partial V)$.

PROOF. Take any $Y\in\partial V$, and let $t\in\boldsymbol{R}$ be such that $X+tY\in U(\phi)$. By Lemma 4. 2 we have $tY\in\partial V$, meaning that $(X, X+tY)\in\Phi^{-1}(\partial V)$. Since $\phi_{M\times M}(\partial\Omega)=\partial\Omega$, and since $\partial\Omega\cap\iota(\mathfrak{g}_{-1}\times\mathfrak{g}_{-1})=\iota(\Phi^{-1}(\partial V))$, it follows that $(\phi'_M(X),\ \phi'_M(X+tY))\in\Phi^{-1}(\partial V)$, that is, $\phi'_M(X+tY)-\phi'_M(X)\in\partial V$. Therefore, again by Lemma 4. 2 we have

$$\frac{1}{t}(\phi'_M(X+tY)-\phi'_M(X))\in\partial V\quad(t\neq 0).$$

It follows immediately that $\partial(\phi, X)\,Y\in\partial V$, and hence $\partial(\phi, X)(\partial V)\subset\partial V$. Similarly we obtain $\partial(\phi^{-1}, Z)(\partial V)\subset\partial V$, where $Z=\phi'_M(X)$. Since $\partial(\phi^{-1}, Z)\circ\partial(\phi, X)=1$, we have thereby shown that $\partial(\phi, X)(\partial V)=\partial V$, which proves the lemma.

Let us now consider the $\tilde{G}_0$-structure Q associated with the homogeneous space $M=G/G^{(0)}$, which is a $\tilde{G}_0$-subbundle of $F(M)$, the frame bundle of M

(see 1. 4). Recall that every diffeomorphism ϕ of M naturally gives rise to a bundle automorphism $\bar{\phi}$ of $F(M)$, and that, by definition, ϕ is an automorphism of Q, if $\bar{\phi}(Q)=Q$.

Our task from now on is to study Aut(Ω) via Aut(Q), the automorphism group of the $\tilde{G}_0$-structure Q.

First of all we have a natural injective homomorphism i_M of G to Aut(Q):

$$i_M(a)=a_M, \quad a\in G,$$

where a_M is the transformation of M induced by a.

Clearly we have $\tilde{G}_0\subset GL(\partial V)$. Hereafter we assume the following condition:

(I. 3) $GL(\partial V)=\tilde{G}_0$.

LEMMA 4. 11. *If* $\phi\in$Aut(Ω), *then* $\phi_M\in$Aut(Q).

PROOF. Consider the point z_0 of Q (see 1. 4). Then we have: For any $z\in Q$ there is $a\in G$ such that $z=\bar{a}_M(z_0)$, and in turn there is $b\in G$ such that $b\cdot\phi_M(a\cdot o_M)=o_M$. If we put $\psi=b_\Omega\cdot\phi\cdot a_\Omega$, we see that $\phi_M(o_M)=o_M$, and $\bar{\psi}_M(z_0)=\bar{b}_M(\bar{\phi}_M(z))$. Therefore to prove the lemma, it suffices to show that $\bar{\phi}_M(z_0)\in Q$ under the condition that $\phi_M(o_M)=o_M$. Accordingly assume this condition. Then we may consider the linear transformation $a=\partial(\phi, 0)$ of $\mathfrak{g}_{-1}$, because 0 of $\mathfrak{g}_{-1}$ is in $U(\phi)$. By Lemma 4. 10 and condition (I. 3) we have $a\in GL(\partial V)=\tilde{G}_0$. For any $Y\in\mathfrak{g}_{-1}$ we clearly have

$$\phi_M(\pi_M(\exp tY))=\pi_M(\exp \phi'_M(tY)),$$

provided $|t|$ is sufficiently small. Since $\phi'_M(tY)=taY+O(t^2)$, it follows that $(\phi_M)_*(z_0\cdot Y)=z_0(aY)$. Hence we obtain $\bar{\phi}_M(z_0)=z_0\cdot a\in Q$, proving the lemma.

By Lemma 4. 11 we see that the homomorphism j, $\phi\to\phi_M$, maps Aut(Ω) to Aut(Q). In this way we have obtained the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{i_\Omega} & \mathrm{Aut}(\Omega) \\ & {\scriptstyle i_M}\searrow & \downarrow{\scriptstyle j} \\ & & \mathrm{Aut}(Q). \end{array}$$

Here the homomorphisms i_Ω, i_M and j are all injective.

Let us consider the definite Möbius algebra of degree 1, $\mathfrak{M}_0(1)$, which satisfies conditions (I. 1)$\sim$(I. 3) as well as the condition that $\partial V=\{0\}$.

LEMMA 4.12. *If* $\mathfrak{G}$ *is not isomorphic with* $\mathfrak{M}_0(1)$, *then* $\mathfrak{G}$ *is the prolongation of the pair* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.

PROOF. Suppose that $\mathfrak{G}$ is not the prolongation of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. By Lemma 1.12, this means that $\mathfrak{G}\cong\mathfrak{G}(1, n; K)$, where $n\geqq 1$, and $K=\boldsymbol{R}$ or $\boldsymbol{C}$. As is easily verified, $\mathfrak{G}(1, n; K)$ does not satisfy condition (I.1) if $n>1$, and $\mathfrak{G}(1,1; \boldsymbol{C})$ satisfies conditions (I.1) and (I.2), but does not satisfy condition (I.3), because $\partial V=\{0\}$. Therefore we have $\mathfrak{G}\cong\mathfrak{G}(1,1: \boldsymbol{R})\cong\mathfrak{M}_0(1)$, which proves the lemma.

Assume the condition in Lemma 4.12. By Lemmas 1.13 and 4.12 we know that the homomorphism i_M gives an isomorphism of G onto Aut(Q). It follows that the homomorphism j gives an isomorphism of Aut(Ω) onto Aut(Q), and in turn the homomorphism i_Ω an isomorphism of G onto Aut(Ω).

We have thereby proved the following

PROPOSITION 4.13. *Let* $\mathfrak{G}$ *be a simple graded Lie algebra of the first kind. Assume that* $\mathfrak{G}$ *satisfies conditions* (I.1)$\sim$(I.3), *and that* $\mathfrak{G}$ *is not isomorphic with* $\mathfrak{M}_0(1)$. *Then* $\Omega=G/G_0$ *is the standard affine symmetric space associated with* $\mathfrak{G}$, *and the homomorphism* i_Ω *gives an isomorphism of* G *onto* Aut(Ω).

§ 5. Simple graded Lie algebras of the first and the second classes

5.1. Spaces of matrices. Let $\boldsymbol{Q}$ be the skew field of quaternions. Let x be any element of $\boldsymbol{Q}$, which may be expressed as follows: $x=x_0+x_1e_1+x_2e_2+x_3e_3$ with $x_0, x_1, x_2, x_3\in\boldsymbol{R}$. Here the elements e_1, e_2 and e_3 which are usually denoted by i, j and k respectively, satisfy the well known relations. Then we define elements ${}^tx$, $\bar{x}$ and x^* of $\boldsymbol{Q}$ respectively as follows:

$$
{}^tx=x_0+x_1e_1-x_2e_2+x_3e_3,\quad \bar{x}=x_0-x_1e_1+x_2e_2-x_3e_3,
$$
$$
x^*={}^t\bar{x}=x_0-x_1e_1-x_2e_2-x_3e_3.
$$

It should be noted that $\bar{x}$ is not the ordinary conjugate of the quaternion x, but is x^*. We have the natural inclusions: $\boldsymbol{R}\subset\boldsymbol{C}\subset\boldsymbol{Q}$. If $x\in\boldsymbol{C}$, we have ${}^tx=x$ and $\bar{x}=x^*$, the latter being the ordinary conjugate of the complex number x. If $x\in\boldsymbol{R}$, we have ${}^tx=\bar{x}=x^*=x$.

Let K be one of the fields: $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$. As before $M_n(K)$ denotes the space of matrices of degree n over K. Let $X=(x_{ij})$ be any matrix of $M_n(K)$. Then we define matrices ${}^tX$, $\bar{X}$ and X^* respectively as follows: ${}^tX=(Y_{ij})$ with $y_{ij}={}^tx_{ji}$, $\bar{X}=(\bar{x}_{ij})$ and $X^*={}^t\bar{X}$.

Let us now give representations of $\boldsymbol{Q}$ and $M_n(\boldsymbol{Q})$ by matrices over $\boldsymbol{C}$.

For this purpose we define a matrix of degree $2n$, $I_{(n)}$, by

$$I_{(n)}=\begin{pmatrix} I_1 & & 0 \\ & \ddots & \\ 0 & & I_1 \end{pmatrix} \quad \text{with } I_1=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and define a subspace over $\boldsymbol{R}$, $\tilde{M}_n$, of $M_{2n}(\boldsymbol{C})$ by

$$\tilde{M}_n=\{ Y\in M_{2n}(\boldsymbol{C}) \mid I_{(n)}Y=\bar{Y}I_{(n)} \}.$$

As is well known, there is a unique isomorphism μ of $\boldsymbol{Q}$ onto $\tilde{M}_1$ as division algebras over $\boldsymbol{R}$ such that $\mu(1)=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mu(e_1)=\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$, $\mu(e_2)=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mu(e_3)=\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$. Clearly we have $\mu({}^t x)={}^t\mu(x)$ $\mu(\bar{x})=\overline{\mu(x)}$ and $\mu(x^*)=\mu(x)^*$ where $x\in\boldsymbol{Q}$.

Every matrix $Y\in\tilde{M}_n$ can be expressed as follows: $Y=(Y_{ij})$, where $Y_{ij}\in\tilde{M}_1$. Consequently we see that the isomorphism μ of $\boldsymbol{Q}$ onto $\tilde{M}_1$ naturally gives rise to an isomorphism of $M_n(\boldsymbol{Q})$ onto $\tilde{M}_n$ as associative algebras over $\boldsymbol{R}$, which we denote by the same letter μ. Clearly we have $\mu({}^tX)={}^t\mu(X)$, $\mu(\bar{X})=\overline{\mu(X)}$ and $\mu(X^*)=\mu(X)^*$, where $X\in M_n(\boldsymbol{Q})$.

For $X\in M_n(\boldsymbol{Q})$, $\det^{\boldsymbol{Q}}(X)$ denotes the determinant of $\mu(X)$, and $\mathrm{Tr}^{\boldsymbol{Q}}(X)$ the trace of $\mu(X)$:

$$\det{}^{\boldsymbol{Q}}(X)=\det(\mu(X)),\quad \mathrm{Tr}^{\boldsymbol{Q}}(X)=\mathrm{Tr}(\mu(X)).$$

Note that both $\det^{\boldsymbol{Q}}(X)$ and $\mathrm{Tr}^{\boldsymbol{Q}}(X)$ are real numbers.

Now, the space $M_n(K)$ naturally becomes a Lie algebra over $\boldsymbol{R}$, which we denote by $\mathfrak{gl}(n, K)$. We define an ideal $\mathfrak{sl}(n, K)$ of $\mathfrak{gl}(n, K)$ by

$$\mathfrak{sl}(n, K)=\{X\in\mathfrak{gl}(n, K) \mid \mathrm{Tr}^K(X)=0\},$$

which is a simple Lie algebra. Here $\mathrm{Tr}^K(X)$ stands for $\mathrm{Tr}(X)$ if $K=\boldsymbol{R}$ or $\boldsymbol{C}$. We denote by $GL(n, K)$ the general linear group of degree n over K, and by $SL(n, K)$ the special linear group of degree n over K:

$$GL(n, K)=\{A\in M_n(K) \mid \det{}^K(A)\neq 0\},$$
$$SL(n, K)=\{A\in M_n(K) \mid \det{}^K(A)=1\},$$

where $\det^K(A)$ stands for $\det(A)$ if $K=\boldsymbol{R}$ or $\boldsymbol{C}$.

Furthermore we define subspaces $S_n(K)$, $S'_n(K)$ and $H'_n(K)$ of $M_n(K)$ respectively as follows:

$$S_n(K)=\{K\in M_n(K) \mid {}^tX=X\},$$
$$S'_n(K)=\{X\in M_n(K) \mid {}^tX=-X\},$$
$$H'_n(K)=\{X\in M_n(K) \mid X^*=-X\}.$$

Note that the product group $GL(n, K)\times GL(n, K)$ naturally acts on the space $M_n(K)$, and the group $GL(n, K)$ on each of the spaces $S_n(K)$, $S'_n(K)$ and $H'_n(K)$.

REMARK. We have $M_{2n}(\boldsymbol{C})=\mu(M_n(\boldsymbol{Q}))+\sqrt{-1}\mu(M_n(\boldsymbol{Q}))$ and $\mu(M_n(\boldsymbol{Q}))\cap\sqrt{-1}\mu(M_n(\boldsymbol{Q}))=\{0\}$. Moreover we have $S_{2n}(\boldsymbol{C})=\mu(S_n(\boldsymbol{Q}))+\sqrt{-1}\mu(S_n(\boldsymbol{Q}))$ and $S'_{2n}(\boldsymbol{C})=\mu(S'_n(\boldsymbol{Q}))+\sqrt{-1}\mu(S'_n(\boldsymbol{Q}))$. Accordingly we find that the complexifications of $M_n(\boldsymbol{Q})$, $S_n(\boldsymbol{Q})$ and $S'_n(\boldsymbol{Q})$ (or of $\mu(M_n(\boldsymbol{Q}))$, $\mu(S_n(\boldsymbol{Q}))$ and $\mu(S'_n(\boldsymbol{Q}))$) may be regarded as $M_{2n}(\boldsymbol{C})$, $S_{2n}(\boldsymbol{C})$ and $S'_{2n}(\boldsymbol{C})$ respectively.

5.2. The graded Lie algebrat $\mathfrak{SO}(n, n; K)$ and $\mathfrak{Sp}(n, K)$. For any positive integer n we define matrices of degree $2n$, J_n and I_n, respectively by

$$J_n=\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix},\quad I_n=\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

where 1_n denotes the unit matrix of degree n. Let K be one of the fields: $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$. Then we define subalgebras $\mathfrak{so}(n, n; K)$, $\mathfrak{su}(n, n; K)$ and $\mathfrak{sp}(n, K)$ of $\mathfrak{sl}(2n, K)$ respectively as follows:

$$\mathfrak{so}(n, n; K)=\{X\in\mathfrak{sl}(2n, K)\mid {}^tXJ_n+J_nX=0\},$$
$$\mathfrak{su}(n, n; K)=\{X\in\mathfrak{sl}(2n, K)\mid X^*J_n+J_nX=0\},$$
$$\mathfrak{sp}(n, K)=\{X\in\mathfrak{sl}(2n, K)\mid {}^tXI_n+I_nX=0\}.$$

Note that these Lie algebras are all simple except for $\mathfrak{so}(1,1;\boldsymbol{R})=\mathfrak{su}(1,1;\boldsymbol{R})$, $\mathfrak{so}(1,1;\boldsymbol{C})$, $\mathfrak{so}(1,1;\boldsymbol{Q})$, $\mathfrak{so}(2,2;\boldsymbol{R})$ and $\mathfrak{so}(2,2;\boldsymbol{C})$.

Every element X of $\mathfrak{sl}(2n, K)$ can be expressed as follows: $X=(X_{ij})_{1\leq i,j\leq 2}$, where X_{ij} are $n\times n$-matrices. Then $\mathfrak{so}(n, n; K)$ is defined by the equations

$${}^tX_{21}+X_{21}=0,\ {}^tX_{11}+X_{22}=0,\ {}^tX_{12}+X_{12}=0,$$

$\mathfrak{su}(n, n; K)$ by the equations

$$X^*_{21}+X_{21}=0,\ X^*_{11}+X_{22}=0,\ X^*_{12}+X_{12}=0,$$

and $\mathfrak{sp}(n, K)$ by the equations

$${}^tX_{21}-X_{21}=0,\ {}^tX_{11}+X_{22}=0,\ {}^tX_{12}-X_{12}=0.$$

Now, let $\mathfrak{g}$ be one of the Lie algebras: $\mathfrak{so}(n, n; K)$, $\mathfrak{su}(n, n; K)$ and $\mathfrak{sp}(n, K)$. For any integer p we define a subspace $\mathfrak{g}_p$ of $\mathfrak{g}$ by $\mathfrak{g}_p=\mathfrak{g}\cap\mathfrak{g}_p(n, n; K)=\{X\in\mathfrak{g}\mid X_{ij}=0 \text{ if } j-i\neq p\}$. Then we see that $\mathfrak{G}=\{\mathfrak{g}, (\mathfrak{g}_p)\}$ becomes a graded subalgebra of $\mathfrak{G}(n, n; K)$, which will be denoted by $\mathfrak{SO}(n, n; K)$

or $\mathfrak{SU}(n, n; K)$ or $\mathfrak{Sp}(n, K)$, according as $\mathfrak{g}$ is $\mathfrak{so}(n, n; K)$ or $\mathfrak{su}(n, n; K)$ or $\mathfrak{sp}(n, K)$.

REMARK. If $\mathfrak{G}$ is simple, its characteristic element E is expressed as follows: $E=\frac{1}{2}\begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}$. Furthermore, according as $\mathfrak{G}$ is $\mathfrak{SO}(n, n; K)$ or $\mathfrak{SU}(n, n; K)$ or $\mathfrak{Sp}(n, K)$, the subspace $\mathfrak{g}_{-1}$ of $\mathfrak{g}$ may be naturally regarded as $S'_n(K)$ or $H'_n(K)$ or $S_n(K)$.

5. 3. Simple graded Lie algebras of the first and the second classes. Hereafter a SGLA of the first kind will always mean that of the classical type. For the classification of SGLA's of the first kind over $\boldsymbol{R}$, we refer to Kobayashi-Nagano [3], I, though it was first done by Berger [3].

A SGLA, $\mathfrak{G}$, of the first kind over $\boldsymbol{R}$ is called of the second class, if it is isomorphic with one of the SGLA's of the first kind in the following table (TABLE 1). Otherwise, $\mathfrak{G}$ is called of the first class.

TABLE 1

$\mathfrak{G}(n, k; \boldsymbol{R})$ $(k>n\geqq 1)$	$\mathfrak{G}(n, k; \boldsymbol{C})$ $(k>n\geqq 1)$	$\mathfrak{G}(n, k; \boldsymbol{Q})$ $(k>n\geqq 1)$
$\mathfrak{SO}(n, n; \boldsymbol{R})$ (n odd, $n\geqq 5$)	$\mathfrak{SO}(n, n; \boldsymbol{C})$ (n odd, $n\geqq 5$)	

Here is a table of the SGLA's of the first class.

TABLE 2

$\mathfrak{G}(n, n; \boldsymbol{R})$ $(n\geqq 3)$	$\mathfrak{G}(n, n; \boldsymbol{C})$ $(n\geqq 3)$	$\mathfrak{G}(n, n; \boldsymbol{Q})$ $(n\geqq 2)$
	$\mathfrak{SU}(n, n; \boldsymbol{C})$ $(n\geqq 3)$	
$\mathfrak{M}_r(n)$ $\left(\begin{matrix} r=0,\ n\geqq 1 \text{ or} \\ 1\leqq 2r\leqq n,\ n\geqq 3 \end{matrix}\right)$	$\mathfrak{M}(n, \boldsymbol{C})$ $(n\geqq 3)$	
$\mathfrak{SO}(n, n; \boldsymbol{R})$ (n even, $n\geqq 6$)	$\mathfrak{SO}(n, n; \boldsymbol{C})$ (n even, $n\geqq 6$)	$\mathfrak{SO}(n, n; \boldsymbol{Q})$ $(n\geqq 3)$
$\mathfrak{Sp}(n, \boldsymbol{R})$ $(n\geqq 3)$	$\mathfrak{Sp}(n, \boldsymbol{C})$ $(n\geqq 3)$	$\mathfrak{Sp}(n, \boldsymbol{Q})$ $(n\geqq 2)$

REMARK 1. $\mathfrak{SU}(n, n; \boldsymbol{R})=\mathfrak{SO}(n, n; \boldsymbol{R})$. $\mathfrak{SU}(n, n; \boldsymbol{Q})\cong\mathfrak{Sp}(n, \boldsymbol{Q})$ (cf. [3], I). $\mathfrak{G}(1,1; \boldsymbol{R})\cong\mathfrak{SU}(1,1; \boldsymbol{C})\cong\mathfrak{Sp}(1, \boldsymbol{R})\cong\mathfrak{M}_0(1)$. $\mathfrak{G}(1,1; \boldsymbol{C})\cong\mathfrak{Sp}(1, \boldsymbol{C})\cong\mathfrak{M}(1, \boldsymbol{C})$. $\mathfrak{G}(1,1; \boldsymbol{Q})\cong\mathfrak{M}_0(4)$. $\mathfrak{G}(2,2; \boldsymbol{R})\cong\mathfrak{M}_2(4)$. $\mathfrak{G}(2,2; \boldsymbol{C})\cong\mathfrak{M}(4, \boldsymbol{C})$. $\mathfrak{SU}(2,2; \boldsymbol{C})\cong\mathfrak{M}_1(4)$, $\mathfrak{SO}(2,2; \boldsymbol{Q})\cong\mathfrak{M}_1(6)$. $\mathfrak{SO}(3,3; \boldsymbol{R})\cong$

$\mathfrak{G}(1,3;\boldsymbol{R})$. $\mathfrak{SO}(3,3;\boldsymbol{C})\cong\mathfrak{G}(1,3;\boldsymbol{C})$. $\mathfrak{SO}(4,4;\boldsymbol{R})\cong\mathfrak{M}_3(6)$. $\mathfrak{SO}(4,4;\boldsymbol{C})\cong\mathfrak{M}(6,\boldsymbol{C})$. $\mathfrak{Sp}(1,\boldsymbol{Q})\cong\mathfrak{M}_0(3)$. $\mathfrak{Sp}(2,\boldsymbol{R})\cong\mathfrak{M}_1(3)$. $\mathfrak{Sp}(2,\boldsymbol{C})\cong\mathfrak{M}(3,\boldsymbol{C})$.

REMARK 2 (cf. Remark in 5.1). The complexifications of $\mathfrak{G}(n,k;\boldsymbol{Q})$ $(k\geqq n)$, $\mathfrak{SO}(n,n;\boldsymbol{Q})$ and $\mathfrak{Sp}(n,\boldsymbol{Q})$ may be regarded as $\mathfrak{G}(2n,2k;\boldsymbol{C})$, $\mathfrak{SO}(2n,2n;\boldsymbol{C})$ and $\mathfrak{Sp}(2n,\boldsymbol{C})$ respectively.

§ 6. Relevant algebraic varieties of the spaces $M_n(\boldsymbol{C})$, $S_n(\boldsymbol{C})$ and $S'_n(\boldsymbol{C})$

6.1. Preliminaries. Let U be a finite dimensional vector space over $\boldsymbol{C}$. Given vectors $X_1,\dots,X_k$ of U, $\mathscr{S}_k(X_1,\dots,X_k)$ benotes the subspace of U spanned (over $\boldsymbol{C}$) by $X_1,\dots,X_k$. $GL_C(U)$ denotes the general linear group of U (as a vector space over $\boldsymbol{C}$).

$P_C(U)$ denotes the algebra of polynomial functions on U (as a vector space over $\boldsymbol{C}$). For any non-negative integer m, $P_C(U)_m$ denotes the subspace of $P_C(U)$ consisting of all polynomial functions of degree m and the zero polynomial function: $P_C(U)=\sum_{m\geqq 0}P_C(U)_m$ (direct sum).

Let W be an algebraic variety over $\boldsymbol{C}$ of U. $I_C(W)$ denotes the ideal of the variety W, that is, the ideal of $P_C(U)$ consisting of all polynomial functions which vanish on W. $GL_C(W)$ denotes the subgroup of $GL_C(U)$ defined by

$$GL_C(W)=\{A\in GL_C(U)\mid AW=W\}.$$

Now, let U be one of the spaces: $M_n(K)$, $S_n(K)$, $S'_n(K)$ and $H'_n(K)$, where $K=\boldsymbol{R}$ or $\boldsymbol{C}$ or $\boldsymbol{Q}$. For any integer $0\leqq r\leqq n$, we define subsets $V_r(U)$ and $W_r(U)$ of U respectively as follows:

$$V_r(U)=\{X\in U\mid \operatorname{rank}(X)=r\},$$
$$W_r(U)=\{X\in U\mid \operatorname{rank}(X)\leqq r\},$$

where rank(X) stands for the rank of the matrix X. (Note that $\operatorname{rank}(X)=\frac{1}{2}\operatorname{rank}(\mu(X))$ if $X\in M_n(\boldsymbol{Q})$.)

Clearly we have $W_r(U)=V_0(U)\cup\dots\cup V_r(U)$. We remark that, if U is $S'_n(\boldsymbol{R})$ or $S'_n(\boldsymbol{C})$, then $V_r(U)=\phi$ for any odd integer $0\leqq r\leqq n$. We also remark that, if U is $M_n(\boldsymbol{C})$ or $S_n(\boldsymbol{C})$ or $S'_n(\boldsymbol{C})$, then $W_r(U)$ is an algebraic variety over $\boldsymbol{C}$ of U, and its ideal $I_C(W_r(U))$ is homogeneous:

$$I_C(W_r(U))=\sum_{m\geqq 0}I_C(W_r(U))_m \text{ with } I_C(W_r(U))_m=I_C(W_r(U))\cap P_C(U)_m.$$

In the present section we study the algebraic varieties: $W_{n-1}(M_n(\boldsymbol{C}))$, $W_{n-2}(M_n(\boldsymbol{C}))$, $W_{n-1}(S_n(\boldsymbol{C}))$, $W_{n-2}(S_n(\boldsymbol{C}))$, and $W_{n-2}(S'_n(\boldsymbol{C}))$ (n even). In the subsequent sections the results here will be applied to the study of the algebraic varieties over $\boldsymbol{R}$, $W_{n-1}(M_n(\boldsymbol{R}))$, $W_{n-1}(M_n(\boldsymbol{Q}))$, $W_{n-2}(S'_n(\boldsymbol{R}))$ (n even), $W_{n-1}(S'_n(\boldsymbol{Q}))$, etc., through the method of complexification.

6.2. The varieties $W_{n-1}(M_n(\boldsymbol{C}))$ and $W_{n-2}(M_n(\boldsymbol{C}))$. For any integer $0 \leqq r \leqq n$, set $V_r = V_r(M_n(\boldsymbol{C}))$ and $W_r = W_r(M_n(\boldsymbol{C}))$. As is well known, $V_0, \ldots, V_n$ are all the orbits under the natural action of $GL(n, \boldsymbol{C}) \times GL(n, \boldsymbol{C})$ on $M_n(\boldsymbol{C})$. In particular V_n is a single open orbit, and hence $W_{n-1} = V_0 \cup \ldots \cup V_{n-1}$ is the union of singular orbits. Clearly the function $\det(X)$ on $M_n(\boldsymbol{C})$ is a homogeneous polynomial function of degree n, and W_{n-1} is the zeros of $\det(X)$.

LEMMA 6.1 (cf. [4]).[1] *The ideal $I_C(W_{n-1})$ of W_{n-1} is generated by* $\det(X)$.

LEMMA 6.2. *Let $0 \leqq r \leqq n$, and let $X \in V_r$. Then*

$$j_X^{n-r-1}(\det) = 0, \text{ and } j_X^{n-r}(\det) \neq 0.$$

PROOF. Clearly we may assume that X is of the form: $\begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}$. Let $X(t)(|t| < \varepsilon)$ be any (C^∞) curve of $M_n(\boldsymbol{C})$ such that $X(0) = X$. Then $X(t)$ may be expressed as $\begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}$, where $A(t) \in M_r(\boldsymbol{C})$. Since $A(0) = 1_r$ and $B(0) = C(0) = D(0) = 0$, it follows that

$$\det(X(t)) = t^{n-r} \det(D'(0)) + O(t^{n-r+1})$$

where $D'(0)$ denotes the derivative of $D(t)$ at $t = 0$. Accordingly the lemma follows.

LEMMA 6.3. *Let F be an endomorphism of $M_n(C)$ as a vector space over $\boldsymbol{C}$. Then the following statements* (1)~(3) *are mutually equivalent*:

(1) $F \in GL_C(W_{n-1})$.

(2) $\det(F(X)) = c \det(X)$, $X \in M_n(\boldsymbol{C})$, *where c is a non-zero constant.*

(3) $F \in GL_C(M_n(\boldsymbol{C}))$, *and* $F(V_r) = V_r$ $(0 \leqq r \leqq n)$.

PROOF. (3)$\Longrightarrow$(1) is clear.

(1)$\Longrightarrow$(2). We have $\det(F(X)) \in I_C(W_{n-1})_n$. Therefore it follows from Lemma 6.1 that $\det(F(X)) = c \det(X)$ for some c. Clearly we have $c \neq 0$.

1) This lemma and also Lemmas 6.10 and 6.15 below can be given elementary proofs, which are left to the readers as exercises.

(2)$\Longrightarrow$(3). We first show that $F\in GL_C(M_n(C))$. Let $X\in M_n(C)$ and $Y\in V_n$. We have $\det(F(Y))=c\det(Y)\neq 0$, and $\det(F(\lambda Y+X))=c\det(\lambda Y+X)$ for all $\lambda\in C$. Consequently it follows that

$$\det(\lambda 1_n+F(Y)^{-1}F(X))=\det(\lambda 1_n+Y^{-1}X),$$

whence $\mathrm{Tr}(F(Y)^{-1}F(X))=\mathrm{Tr}(Y^{-1}X)$. Now, assume that $F(X)=0$, implying that $\mathrm{Tr}(Y^{-1}X)=0$. Since $V_n^{-1}=V_n$, and since V_n is open and dense in $M_n(C)$, we see that $\mathrm{Tr}(ZX)=0$ for all $Z\in M_n(C)$, whence $X=0$. We have thus shown that $F\in GL_C(M_n(C))$. We next show that $F(V_r)=V_r$ $(0\leqq r\leqq n)$. We prove this by induction on the integer r. Accordingly assume that $F(V_s)=V_s$ $(0\leqq s\leqq r-1)$ for some integer $1\leqq r\leqq n-1$. Take any $X\in V_r$. Suppose that $F(X)\in V_{r'}$ for some integer $r+1\leqq r'\leqq n$. By Lemma 6.2 we have $j^{n-r'}_{F(X)}(\det)\neq 0$, whence $j^{n-r'}_X(\det\circ F)\neq 0$. Since $\det\circ F=c\det$, it follows from Lemma 6.2 that $j^{n-r-1}_X(\det)=0$. This is a contradiction, because $n-r'\leqq n-r-1$. Therefore we have $F(X)\in V_r$, and hence $F(V_r)\subset V_r$. Similarly we have $F^{-1}(V_r)\subset V_r$. We have thus shown that $F(V_r)=V_r$, completing the proof of Lemma 6.3.

By the use of Lemma 6.3 we shall prove the following

LEMMA 6.4. *The group $GL_C(W_{n-1})$ consists of all transformations F of $M_n(C)$ of the following form*:

$$F(X)=AXB^{-1} \text{ or } F(X)=A{}^tXB^{-1},\ X\in M_n(C),$$

where $A,\ B\in GL(n, C)$.

Clearly the transformations of the form above form a subgroup Ψ of $GL_C(W_{n-1})$.

Let $\{e_{ij}\}_{1\leqq i,j\leqq n}$ be the canonical basis of $M_n(C)$: The (i, j)-component of e_{ij} is 1, and the other components of e_{ij} are all zero. Clearly we have $e_{ij}\in V_1$. Let T_{ij} be the (holomorphic) tangent space to the submanifold V_1 of $M_n(C)$ at the point e_{ij}, which may be naturally regarded as a subspace of the vector space $M_n(C)$. Then we easily have

$$T_{ij}=\mathscr{S}_{\!p}(e_{i1}, \dots, e_{in}, e_{1j}, \dots, e_{nj}).$$

Let us now show that $GL_C(W_{n-1})\subset\Psi$. Clearly it suffices to deal with the case where $n\geqq 2$. Let $F\in GL_C(W_{n-1})$. Our task from now on is to show that there are $A,\ B\in GL(n, C)$ such that either $AF(X)B=X$ for all X or $AF'(X)B=X$ for all X, where F' is an element of $GL_C(W_{n-1})$ defined by $F'(X)={}^t(F(X))$. By Lemma 6.3 we have $F(V_1)=V_1$ and $F(V_2)=V_2$. Hence it follows that $F(e_{ij})\in V_1$, and that $F(T_{ij})=T_{ij}$, if $F(e_{ij})=e_{ij}$.

Since $F(e_{11})\in V_1$, we can find A, $B\in GL(n, \boldsymbol{C})$ such that $AF(e_{11})B=e_{11}$. Thus we may assume that $F(e_{11})=e_{11}$.

Therefore we have $F(T_{11})=T_{11}$. Since $e_{21}\in T_{11}=\mathscr{S}(e_{11},\dots, e_{1n}, e_{21},\dots, e_{n1})$, it follows that $F(e_{21})\in\mathscr{S}(e_{11},\dots, e_{1n}, e_{21},\dots, e_{n1})$. Since $\mathrm{rank}(F(e_{21}))=\mathrm{rank}(e_{21})=1$, we easily see that either $F(e_{21})\in\mathscr{S}(e_{11},\dots, e_{n1})$ or $F(e_{21})\in\mathscr{S}(e_{11},\dots, e_{1n})$. Clearly we may assume that $F(e_{21})\in\mathscr{S}(e_{11},\dots, e_{n1})$, by replacing F with F' if necessary. This being said, we can find $A\in GL(n, \boldsymbol{C})$ such that $AF(e_{11})=e_{11}$ and $AF(e_{21})=e_{21}$. Thus we may assume that $F(e_{11})=e_{11}$ and $F(e_{21})=e_{21}$.

Therefore we have $F(T_{11}\cap T_{21})=F(T_{11})\cap F(T_{21})=T_{11}\cap T_{21}$. Since $T_{11}\cap T_{21}=\mathscr{S}(e_{11},\dots, e_{n1})$, it follows that $F(e_{i1})\in\mathscr{S}(e_{11},\dots, e_{n1})(3\leqq i\leqq n)$. Consequently we can find $A\in GL(n, \boldsymbol{C})$ such that $AF(e_{i1})=e_{i1}$ $(1\leqq i\leqq n)$. Thus we may assume that $F(e_{i1})=e_{i1}$ $(1\leqq i\leqq n)$.

LEMMA 6.5. $F(e_{ij})\in\mathscr{S}(e_{i1},\dots, e_{in})(1\leqq i\leqq n, 2\leqq j\leqq n)$.

PROOF. Take any integers $1\leqq i\leqq n$ and $2\leqq j\leqq n$. We have $F(T_{i1})=T_{i1}$, $T_{i1}=\mathscr{S}(e_{i1},\dots, e_{in}, e_{11},\dots, e_{n1})$ and $\mathrm{rank}(F(e_{ij}))=1$. From these facts it follows that either $F(e_{ij})\in\mathscr{S}(e_{i1},\dots, e_{in})$ or $F(e_{ij})\in\mathscr{S}(e_{11},\dots, e_{n1})$. Suppose that $F(e_{ij})\in\mathscr{S}(e_{11},\dots, e_{n1})$. Fix an integer k such that $1\leqq k\leqq n$ and $k\neq i$. Then we have $\mathrm{rank}(F(e_{k1}+e_{ij}))=\mathrm{rank}(e_{k1}+e_{ij})=2$. On the other hand, we have $F(e_{k1}+e_{ij})=e_{k1}+F(e_{ij})\in\mathscr{S}(e_{11},\dots, e_{n1})$, whence $\mathrm{rank}(F(e_{k1}+e_{ij}))=1$. This is a contradiction. We have thereby shown that $F(e_{ij})\in\mathscr{S}(e_{i1},\dots, e_{in})$, proving Lemma 6.5.

In particular, by Lemma 6.5 we have $F(e_{1j})\in\mathscr{S}(e_{11},\dots, e_{1n})(2\leqq j\leqq n)$. This being said, we can find $B\in GL(n, \boldsymbol{C})$ such that $F(e_{i1})B=e_{i1}$ and $F(e_{1j})B=e_{1j}$ $(1\leqq i, j\leqq n)$. In this way we may eventually assume that $F(e_{i1})=e_{i1}$ and $F(e_{1j})=e_{1j}$ $(1\leqq i, j\leqq n)$.

In the same manner as in the proof of Lemma 6.5, we can therefore show that $F(e_{ij})\in\mathscr{S}(e_{1j},\dots, e_{nj})$ $(2\leqq i\leqq n, 1\leqq j\leqq n)$. From this fact combined with Lemma 6.5, we easily obtain $F(e_{ij})\in\mathscr{S}(e_{ij})$ $(2\leqq i, j\leqq n)$. Now, take any integers $2\leqq i$, $j\leqq n$. If we put $M_2=\mathscr{S}(e_{11}, e_{i1}, e_{1j}, e_{ij})$, we have $F(M_2)=M_2$, and we have a natural isomorphism, say $X\to X'$, of M_2 onto $M_2(\boldsymbol{C})$. Since $\mathrm{rank}(F(X))=\mathrm{rank}(X)$ for all $X\in M_2$, we have $\mathrm{rank}(F(X)')=\mathrm{rank}(X')$. By Lemma 6.3 (applied for $n=2$), we therefore see that $\det(F(X)')=\det(X')$, from which follows easily that $F(e_{ij})=e_{ij}$. We have thereby shown that $F(X)=X$ for all $X\in M_n(\boldsymbol{C})$, completing the proof of Lemma 6.4.

We shall now study the variety $W_{n-2}=V_0\cup\dots\cup V_{n-2}$. Let X be any matrix of $M_n(\boldsymbol{C})$, and let (i, j) be any pair of integers $1\leqq i$, $j\leqq n$. We denote by $\Delta_{ij}(X)$ the (i, j)-minor of X : $\Delta_{ij}(X)=\det(X_{(i,j)})$, where $X_{(i,j)}$

is the matrix of degree $n-1$ which is obtained from X by extracting the i-th row and the j-th column. Clearly the function $\Delta_{ij}(X)$ on $M_n(\boldsymbol{C})$ is a homogeneous polynomial function of degree $n-1$, and W_{n-2} is the common zeros of the polynomial functions $\Delta_{ij}(X)$.

LEMMA 6.6 (cf. [4])[2] *The ideal* $I_{\boldsymbol{C}}(W_{n-2})$ *of* W_{n-2} *is generated by* $\Delta_{ij}(X)$.

LEMMA 6.7. *Let* $0\leqq r\leqq n-2$, *and let* $X\in V_r$.
(1) $j_X^{n-r-2}(\Delta_{ij})=0$ *for any* (i,j).
(2) $j_X^{n-r-1}(\Delta_{ij})\neq 0$ *for some* (i,j).

PROOF. The notations being as in the proof of Lemma 6.2, we have

$$\begin{aligned}\Delta_{ij}(X(t)) &= t^{n-r-1}\Delta_{i-r,\,j-r}(D'(0))+O(t^{n-r}) \text{ if } r+1\leqq i,\ j\leqq n+1,\\ &= O(t^{n-r}) \text{ otherwise.}\end{aligned}$$

Gonsequently the lemma follows.

LEMMA 6.8. *If* $F\in GL_{\boldsymbol{C}}(W_{n-2})$, *then* $F(V_r)=V_r$ $(0\leqq r\leqq n-2)$.

We can prove this fact by using Lemmas 6.6 and 6.7, and by reasoning in the same manner as in the proof of Lemma 6.3.

Here, we recall that the proof of Lemma 6.4 uses only the fact that every $F\in GL_{\boldsymbol{C}}(W_{n-1})$ satisfies $F(V_1)=V_1$ and $F(V_2)=V_2$. In view of Lemma 6.8 we therefore obtain

LEMMA 6.9. *Assume that* $n\geqq 4$. *Then the group* $GL_{\boldsymbol{C}}(W_{n-2})$ *consists of all transformations* F *of* $M_n(\boldsymbol{C})$ *of the following form*:

$$F(X)=AXB^{-1} \text{ or } F(X)=A^tXB^{-1},\ X\in M_n(\boldsymbol{C}),$$

where $A,\ B\in GL(n,\boldsymbol{C})$.

6.3. The varieties $W_{n-1}(S_n(\boldsymbol{C}))$ and $W_{n-2}(S_n(\boldsymbol{C}))$. For any integer $0\leqq r\leqq n$, set $V_r=V_r(S_n(\boldsymbol{C}))$ and $W_r=W_r(S_n(\boldsymbol{C}))$. As is well known, $V_0,\dots,V_n$ are all the orbits under the natural action of $GL(n,\boldsymbol{C})$ on $S_n(\boldsymbol{C})$. In particular V_n is a single open orbit, and hence $W_{n-1}=V_0\cup\dots\cup V_{n-1}$ is the union of singular orbits. Clearly W_{n-1} is the zeros of the polynomial function $\det(X)$, restricted to $S_n(\boldsymbol{C})$.

2) We shall only use the fact that $I_{\boldsymbol{C}}(W_{n-2})_m=\{0\}$ for $0\leqq m\leqq n-2$, and $I_{\boldsymbol{C}}(W_{n-2})_{n-1}$ is spanned by $\Delta_{ij}(X)$, which can be given an elementary proof. The same remark holds for Lemma 6.13 below.

Lemma 6.10 (cf. [1]). *The ideal $I_C(W_{n-1})$ of W_{n-1} is generated by* $\det(X)$.

Lemma 6.11. *Let F be an endomorphism of $S_n(\boldsymbol{C})$ as a vector space over $\boldsymbol{C}$. Then the following statements* (1)~(3) *are mutually equivalent.*

(1) $F \in GL_C(W_{n-1})$.

(2) $\det(F(X)) = c \det(X)$, $X \in S_n(\boldsymbol{C})$, *where c is a non-zero constant.*

(3) $F \in GL_C(S_n(\boldsymbol{C}))$, *and* $F(V_r) = V_r$ $(0 \leqq r \leqq n)$.

The proof of this fact is quite similar to that of Lemma 6.3. (Note that Lemma 6.2 holds true in our present situation.)

By the use of Lemma 6.11 we shall prove the following

Lemma 6.12. *The group $GL_C(W_{n-1})$ consists of all transformations F of $S_n(\boldsymbol{C})$ of the following form:*

$$F(X) = AX\,{}^tA, \quad X \in S_n(\boldsymbol{C}),$$

where $A \in GL(n, \boldsymbol{C})$.

Clearly the transformations of the form above form a subgroup Ψ of $GL_C(W_{n-1})$.

Using the canonical basis $\{e_{ij}\}$ of $M_n(\boldsymbol{C})$, we put $\varepsilon_{ii} = e_{ii}$ and $\varepsilon_{ij} = e_{ij} + e_{ji}$ $(i \neq j)$. Then we have $\varepsilon_{ij} \in S_n(\boldsymbol{C})$ for all i, j, and $\varepsilon_{ij} (1 \leqq i \leqq j \leqq n)$ form a basis of $S_n(\boldsymbol{C})$. Clearly we have $\varepsilon_{ii} \in V_1$. If we denote by T_{ii} the tangent space to V_1 at ε_{ii}, we have

$$T_{ii} = \mathscr{S}\!\!p(\varepsilon_{i1}, \dots, \varepsilon_{in}).$$

Now, let $i \neq j$. Clearly we have $\varepsilon_{ij} \in V_2$. If we denote by T_{ij} the tangent space to V_2 at ε_{ij}, we have

$$T_{ij} = \mathscr{S}\!\!p(\varepsilon_{i1}, \dots, \varepsilon_{in}, \varepsilon_{j1}, \dots, \varepsilon_{jn}).$$

Let us now show that $GL_C(W_{n-1}) \subset \Psi$. Clearly it suffices to deal with the case where $n \geqq 2$. Let $F \in GL_C(W_{n-1})$. By Lemma 6.11 we have $F(V_1) = V_1$ and $F(V_2) = V_2$. Hence it follows that $F(\varepsilon_{ii}) \in V_1$, $F(\varepsilon_{ij}) \in V_2$ if $i \neq j$, and that $F(T_{ij}) = T_{ij}$ if $F(\varepsilon_{ij}) = \varepsilon_{ij}$.

Since $F(\varepsilon_{11}) \in V_1$, we can find $A \in GL(n, \boldsymbol{C})$ such that $AF(\varepsilon_{11}){}^tA = \varepsilon_{11}$. Thus we may assume that $F(\varepsilon_{11}) = \varepsilon_{11}$.

We have $F(T_{11}) = T_{11}$, and $T_{11} = \mathscr{S}\!\!p(\varepsilon_{11}, \dots, \varepsilon_{1n})$, whence $F(\varepsilon_{1i}) \in \mathscr{S}\!\!p(\varepsilon_{11}, \dots, \varepsilon_{1n})$ $(2 \leqq i \leqq n)$. This being said, we can find $A \in GL(n, \boldsymbol{C})$ such that $AF(\varepsilon_{1i}){}^tA = \varepsilon_{1i}$ $(1 \leqq i \leqq n)$. Thus we may assume that $F(\varepsilon_{1i}) = \varepsilon_{1i}$ $(1 \leqq i \leqq n)$.

Take any integer $2 \leqq i \leqq n$. Then we have $F(T_{1i}) = T_{1i}$, $\varepsilon_{ii} \in T_{1i} =$

$\mathscr{S}(\varepsilon_{11}, \ldots, \varepsilon_{1n}, \varepsilon_{i1}, \ldots, \varepsilon_{in})$, and $\operatorname{rank}(F(\varepsilon_{ii}))=1$. From these facts it follows that $F(\varepsilon_{ii})\in\mathscr{S}(\varepsilon_{11}, \varepsilon_{1i}, \varepsilon_{ii})$. If we put $S_2=\mathscr{S}(\varepsilon_{11}, \varepsilon_{1i}, \varepsilon_{ii})$, we have $F(S_2)=S_2$, and we have a natural isomorphism, say $X\to X'$, of S_2 onto $S_2(\boldsymbol{C})$. As before we then obtain $\det(F(X)')=c \det(X')$ for all $X\in S_2$, from which follows that $F(\varepsilon_{ii})=\varepsilon_{ii}$.

Let us consider the case where $n\geqq 3$. Take any integers $2\leqq i<j\leqq n$. Then we have $F(T_{ii}\cap T_{jj})=T_{ii}\cap T_{jj}$, $\varepsilon_{ij}\in T_{ii}\cap T_{jj}=\mathscr{S}(\varepsilon_{ii}, \varepsilon_{jj}, \varepsilon_{ij})$, whence $F(\varepsilon_{ij})\in\mathscr{S}(\varepsilon_{ii}, \varepsilon_{jj}, \varepsilon_{ij})$. If we put $S_2=\mathscr{S}(\varepsilon_{ii}, \varepsilon_{jj}, \varepsilon_{ij})$, we have $F(S_2)=S_2$, and we have a natural isomorphism, say $X\to X'$, of S_2 onto $S_2(\boldsymbol{C})$. As before we then obtain $\det(F(X)')=c \det(X')$ for all $X\in S_2$, from which follows that $F(\varepsilon_{ij})\in\mathscr{S}(\varepsilon_{ij})$. Furthermore, if we put $S_3=\mathscr{S}(\varepsilon_{11}, \varepsilon_{ii}, \varepsilon_{jj}, \varepsilon_{1i}, \varepsilon_{1j}, \varepsilon_{ij})$, we have $F(S_3)=S_3$, and we have a natural isomorphism, say $X\to X'$, of S_3 onto $S_3(\boldsymbol{C})$. As before we then obtain $\det(F(X)')=c \det(X')$ for all $X\in S_3$, from which follows that $F(\varepsilon_{ij})=\varepsilon_{ij}$. We have thus shown that $F(X)=X$ for all $X\in S_n(\boldsymbol{C})$, completing the proof of Lemma 6.12.

We shall now consider the variety $W_{n-2}=V_0\cup\ldots\cup V_{n-2}$, which is the common zeros of the polynomial functions $\Delta_{ij}(X)$, restricted to $S_n(\boldsymbol{C})$.

LEMMA 6.13 (cf. [1]). *The ideal* $I_C(W_{n-2})$ *of* W_{n-2} *is generated by* $\Delta_{ij}(X)$.

LEMMA 6.14. *Assume that* $n\geqq 4$. *Then the group* $GL_C(W_{n-2})$ *consists of all transformations F of* $S_n(\boldsymbol{C})$ *of the following form*:

$$F(X)=AX{}^tA, \quad X\in S_n(\boldsymbol{C}),$$

where $A\in GL(n, \boldsymbol{C})$.

The proof of this fact is quite similar to that of Lemma 6.9. (Note that Lemmas 6.7 and 6.8 hold true in our present situation.)

6.4. The varieties $W_{n-2}(S_n'(\boldsymbol{C}))$ (n even). Set $m=\frac{n}{2}$. For any integer $0\leqq r\leqq m$, set $V_{2r}=V_{2r}(S_n'(\boldsymbol{C}))$ and $W_{2r}=W_{2r}(S_n'(\boldsymbol{C}))$. As is well known, $V_0, V_2, \ldots, V_{2m}$ are all the orbits under the action of $GL(n, \boldsymbol{C})$ on $S_n'(\boldsymbol{C})$. In particular V_{2m} is a single open orbit, and hence $W_{2(m-1)}=V_0\cup V_2\cup\ldots\cup V_{2(m-1)}$ is the union of singular orbits.

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\boldsymbol{C}^n$. For any matrix $X=(x_{ij})$ of $S_n'(\boldsymbol{C})$, we define a 2-vector ω_X on $\boldsymbol{C}^n$ by

$$\omega_X=\frac{1}{2}\sum_{i,j} x_{ij}e_i\wedge e_j,$$

and define a complex number $\det^{\frac{1}{2}}(X)$ by

$$\omega_X \wedge \dots \wedge \omega_X = \det^{\frac{1}{2}}(X) e_1 \wedge \dots \wedge e_n \quad (\omega_X \ m \text{ times}).$$

As is well known, we have

$$(\det^{\frac{1}{2}}(X))^2 = \det(X), \ \det^{\frac{1}{2}}(AX{}^tA) = \det(A) \det^{\frac{1}{2}}(X),$$

where $A \in GL(n, \boldsymbol{C})$.

Clearly the function $\det^{\frac{1}{2}}(X)$ on $S_n'(\boldsymbol{C})$ is a homogeneous polynomial function of degree m, and $W_{2(m-1)}$ is the zeros of $\det^{\frac{1}{2}}(X)$.

LEMMA 6.15 (cf. [2]). *The ideal* $I_{\boldsymbol{C}}(W_{2(m-1)})$ of $W_{2(m-1)}$ *is generated by* $\det^{\frac{1}{2}}(X)$.

LEMMA 6.16. *Let* $0 \leqq r \leqq m$, *and let* $X \in V_{2r}$. *Then*

$$j_X^{m-r-1}(\det^{\frac{1}{2}}) = 0, \ \textit{and} \ j_X^{m-r}(\det^{\frac{1}{2}}) \neq 0.$$

PROOF. Clearly we may assume that X is of the form: $\begin{pmatrix} I_{(r)} & 0 \\ 0 & 0 \end{pmatrix}$. Let $X(t)$ $(|t| < \varepsilon)$ be any curve of $S_n'(\boldsymbol{C})$ such that $X(0) = X$. Then $X(t)$ may be expressed as $\begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}$, where $A(t) \in S_{2r}'(\boldsymbol{C})$. Since $A(0) = I_{(r)}$, and $B(0) = C(0) = D(0) = 0$, it follows that

$$\det^{\frac{1}{2}}(X(t)) = t^{m-r} \det^{\frac{1}{2}}(D'(0)) + O(t^{m-r+1}).$$

Accordingly the lemma follows.

LEMMA 6.17. *Let* F *be an endomorphism of* $S_n'(\boldsymbol{C})$ *as a vector space over* $\boldsymbol{C}$. *Then the following statements* (1)$\sim$(3) *are mutually equivalent*:

(1) $F \in GL_{\boldsymbol{C}}(W_{2(m-1)})$.

(2) $\det^{\frac{1}{2}}(F(X)) = c \det^{\frac{1}{2}}(X)$, $X \in S_n'(\boldsymbol{C})$, *where* c *is a nonzero constant.*

(3) $F \in GL_{\boldsymbol{C}}(S_n'(\boldsymbol{C}))$, *and* $F(V_{2r}) = V_{2r}$ $(0 \leqq r \leqq m)$.

This fact can be proved in the same manner as in the proof of Lemma 6.3, based on Lemmas 6.15 and 6.16. Note that (2) implies $\det(F(X)) = c^2 \det(X)$.

By the use of Lemma 6.17, we shall prove the following

LEMMA 6.18. *Assume that* $n = 2m \geqq 6$. *Then the group* $GL_{\boldsymbol{C}}(W_{2(m-1)})$ *consists of all transformations* F *of* $S_n'(\boldsymbol{C})$ *of the following form*:

$$F(X) = AX{}^tA, \ X \in S_n'(\boldsymbol{C}),$$

where $A\in GL(n, \boldsymbol{C})$.

Clearly the transformations of the form above form a subgroup Ψ of $GL_{\boldsymbol{C}}(W_{2(m-1)})$.

Using the canonical basis $\{e_{ij}\}$ of $M_n(\boldsymbol{C})$, we put $\varepsilon_{ij}=e_{ij}-e_{ji}$. Then we have $\varepsilon_{ij}\in S_n'(\boldsymbol{C})$, and ε_{ij} $(1\leqq i<j\leqq n)$ form a basis of $S_n'(\boldsymbol{C})$. Now, let $i\neq j$. Clearly we have $\varepsilon_{ij}\in V_2$. If we denote by T_{ij} the tangent space to V_2 at ε_{ij}, we have

$$T_{ij}=\mathscr{S}_{\!\!p}(\varepsilon_{i1}, \ldots, \varepsilon_{in}, \varepsilon_{j1}, \ldots, \varepsilon_{jn}).$$

Let us now show that $GL_{\boldsymbol{C}}(W_{2(m-1)})\subset\Psi$. Let $F\in GL_{\boldsymbol{C}}(W_{2(m-1)})$. By Lemma 6.17 we have $F(V_2)=V_2$ and $F(V_4)=V_4$. Hence it follows that $F(\varepsilon_{ij})\in V_2$ if $i\neq j$, and that $F(T_{ij})=T_{ij}$ if $i\neq j$ and if $F(\varepsilon_{ij})=\varepsilon_{ij}$ or more generally $F(\varepsilon_{ij})\in\mathscr{S}_{\!\!p}(\varepsilon_{ij})$.

Since $F(\varepsilon_{12})\in V_2$, we can find $A\in GL(n, \boldsymbol{C})$ such that $AF(\varepsilon_{12}){}^tA=\varepsilon_{12}$. Thus we may assume that $F(\varepsilon_{12})=\varepsilon_{12}$.

We have $F(T_{12})=T_{12}$, and $\varepsilon_{13}\in \mathrm{T}_{12}=\mathscr{S}_{\!\!p}(\varepsilon_{12}, \ldots, \varepsilon_{1n}, \varepsilon_{23}, \ldots, \varepsilon_{2n})$, whence $F(\varepsilon_{13})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \ldots, \varepsilon_{1n}, \varepsilon_{23}, \ldots, \varepsilon_{2n})$. Define a $2\times(n-2)$-matrix (z_{ij}) by $F(\varepsilon_{13})\equiv\sum_{i=1}^{2}\sum_{j=3}^{n} z_{ij}\varepsilon_{ij}$ (mod ε_{12}). Since $F(\varepsilon_{12})=\varepsilon_{12}$ and $\operatorname{rank}(F(\varepsilon_{13}))=2$, we see that the rank of the matrix (z_{ij}) is 1. This being said, we can find $A\in GL(n, \boldsymbol{C})$ such that $AF(\varepsilon_{12}){}^tA=\varepsilon_{12}$ and $AF(\varepsilon_{13}){}^tA=\varepsilon_{13}$. Thus we may assume that $F(\varepsilon_{12})=\varepsilon_{12}$ and $F(\varepsilon_{13})=\varepsilon_{13}$.

LEMMA 6.19. *$F(\varepsilon_{23})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})$, and $F(\varepsilon_{1j})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \ldots, \varepsilon_{1n})$ $(4\leqq j\leqq n)$.*

PROOF. Take any integer $4\leqq j\leqq n$. We have $F(T_{12}\cap T_{13})=T_{12}\cap T_{13}$, and ε_{23}, $\varepsilon_{1j}\in T_{12}\cap T_{13}=\mathscr{S}_{\!\!p}(\varepsilon_{12}, \ldots, \varepsilon_{1n}, \varepsilon_{23})$, whence $F(\varepsilon_{23})$, $F(\varepsilon_{1j})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \ldots, \varepsilon_{1n}, \varepsilon_{23})$. Since $\operatorname{rank}(F(\varepsilon_{23}))=2$, it follows that either $F(\varepsilon_{23})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \ldots, \varepsilon_{1n})$ or $F(\varepsilon_{23})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})$. Similarly, since $\operatorname{rank}(F(\varepsilon_{1j}))=2$, it follows that either $F(\varepsilon_{1j})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \ldots, \varepsilon_{1n})$ or $F(\varepsilon_{1j})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})$. Now, suppose that $F(\varepsilon_{23})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \ldots, \varepsilon_{1n})$. Since $\operatorname{rank}(F(\varepsilon_{1j}+\varepsilon_{23}))=\operatorname{rank}(\varepsilon_{1j}+\varepsilon_{23})=4$, we then see that $F(\varepsilon_{1j})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})$. Hence we obtain $F(\mathscr{S}_{\!\!p}(\varepsilon_{14}, \ldots, \varepsilon_{1n}))\subset\mathscr{S}_{\!\!p}(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})$. Clearly this is a contradiction, if $n\geqq 8$. If $n=6$, we have $F(\mathscr{S}_{\!\!p}(\varepsilon_{14}, \varepsilon_{15}, \varepsilon_{16}))=\mathscr{S}_{\!\!p}(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})$, and hence $\varepsilon_{12}=F^{-1}(\varepsilon_{12})\in\mathscr{S}_{\!\!p}(\varepsilon_{14}, \varepsilon_{15}, \varepsilon_{16})$, which is a contradiction. Consequently we have shown that $F(\varepsilon_{23})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})$. Moreover, we easily see that $F(\varepsilon_{1j})\in\mathscr{S}_{\!\!p}(\varepsilon_{12}, \ldots, \varepsilon_{1n})$ $(4\leqq j\leqq n)$, because $\operatorname{rank}(F(\varepsilon_{1j}+\varepsilon_{23}))=4$.

In view of Lemma 6.19, we can find $A\in GL(n, \boldsymbol{C})$ such that $AF(\varepsilon_{1j}){}^tA=\varepsilon_{1j}$ $(2\leqq j\leqq n)$, and $AF(\varepsilon_{23}){}^tA=\varepsilon_{23}$. Thus we may assume that $F(\varepsilon_{1j})=\varepsilon_{1j}$ $(2\leqq j\leqq n)$, and $F(\varepsilon_{23})=\varepsilon_{23}$.

Take any integer $4 \leqq j \leqq n$. Then we have $F(T_{1j} \cap T_{12} \cap T_{23}) = T_{1j} \cap T_{12} \cap T_{23}$, and $\varepsilon_{2j} \in T_{1j} \cap T_{12} \cap T_{23} = \mathscr{S}(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{2j})$, whence $F(\varepsilon_{2j}) \in \mathscr{S}(\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{2j})$. Since $\mathrm{rank}(F(\varepsilon_{2j})) = 2$, it follows that $F(\varepsilon_{2j}) \in \mathscr{S}(\varepsilon_{12}, \varepsilon_{2j})$. This being said, we can find $A \in GL(n, \boldsymbol{C})$ such that $AF(\varepsilon_{1i})^tA = \varepsilon_{1i}$ $(2 \leqq i \leqq n)$, $AF(\varepsilon_{23})^tA = \varepsilon_{23}$, and $AF(\varepsilon_{2j})^tA \in \mathscr{S}(\varepsilon_{2j})$ $(4 \leqq j \leqq n)$. In this way we may eventually assume that $F(\varepsilon_{1i}) = \varepsilon_{1i}$ $(2 \leqq i \leqq n)$, $F(\varepsilon_{23}) = \varepsilon_{23}$ and $F(\varepsilon_{2j}) \in \mathscr{S}(\varepsilon_{2j})$ $(4 \leqq j \leqq n)$.

Take any integers $3 \leqq i < j \leqq n$. Then we have $F(T_{1i} \cap T_{1j} \cap T_{2i} \cap T_{2j}) = T_{1i} \cap T_{1j} \cap T_{2i} \cap T_{2j}$, and $\varepsilon_{ij} \in T_{1i} \cap T_{1j} \cap T_{2i} \cap T_{2j} = \mathscr{S}(\varepsilon_{ij}, \varepsilon_{12})$, whence $F(\varepsilon_{ij}) \in \mathscr{S}(\varepsilon_{ij}, \varepsilon_{12})$. Since $\mathrm{rank}(F(\varepsilon_{ij})) = 2$, it follows that $F(\varepsilon_{ij}) \in \mathscr{S}(\varepsilon_{ij})$.

We have thus shown that $F(\varepsilon_{ij}) \in \mathscr{S}(\varepsilon_{ij})$ $(2 \leqq i < j \leqq n)$. Fix such integers i and j. If we put $S_4' = \mathscr{S}(\varepsilon_{12}, \varepsilon_{1i}, \varepsilon_{1j}, \varepsilon_{2i}, \varepsilon_{2j}, \varepsilon_{ij})$, we have $F(S_4') = S_4'$, and we have a natural isomorphism, say $X \to X'$, of S_4' onto $S_4'(\boldsymbol{C})$. As before we then obtain $\mathrm{rank}(F(X)') = \mathrm{rank}(X')$ for all $X \in S_4'$. By Lemma 6.17 (applied for $n=4$) we therefore see that $\det^{\frac{1}{2}}(F(X)') = c \det^{\frac{1}{2}}(X')$, from which follows that $F(\varepsilon_{ij}) = \varepsilon_{ij}$. We have thereby shown that $F(X) = X$ for all $X \in S_n'(C)$, completing the proof of Lemma 6.18.

§7. Proof of Theorem 2.8: The case where $\mathfrak{H}$ is a simple graded Lie algebra of the second class.

At the outset we prepare some notations for our later arguments. Let $n_1, \dots, n_k$ be positive integers, and put $n = n_1 + \dots + n_k$. Every matrix A of $GL(n, K)$ may be expressed as follows: $A = (A_{ij})_{1 \leqq i, j \leqq n}$, where A_{ij} are $n_i \times n_j$-matrices. Then $G_0(n_1, \dots, n_k; K)$ denotes the subgroup of $GL(n, K)$ defined by

$$G_0(n_1, \dots, n_k; K) = \{A \in GL(n, K) \mid A_{ij} = 0 \text{ if } i \neq j\}.$$

Given a Lie group L, $C(L)$ denotes the centre of L.

7.1. The case where $\mathfrak{H} = \mathfrak{G}(n, k; K)$ $(k > n \geqq 1$, $K = \boldsymbol{R}$ or $\boldsymbol{C}$ or $\boldsymbol{Q})$. Set $m = k - n$, and let us consider the simple graded Lie algebra $\mathfrak{G} = \mathfrak{G}(n, m, n; K)$ of the second kind. Let $\mathfrak{G} = \{\mathfrak{g}; (\mathfrak{g}_p)\}$. If we put $n_1 = n_3 = n$ and $n_2 = m$, every matrix X of $\mathfrak{g} = \mathfrak{sl}(2n+m, K)$ may be expressed as follows: $X = (X_{ij})_{1 \leqq i, j \leqq 3}$, where X_{ij} are $n_i \times n_j$-matrices. Then we recall that the subspaces $\mathfrak{g}_p$ of $\mathfrak{g}$ are defined by $\mathfrak{g}_p = \{X \in \mathfrak{g} \mid X_{ij} = 0 \text{ if } j - i \neq p\}$.

Now, define subspaces $\mathfrak{e}$ and $\mathfrak{f}$ of $\mathfrak{g}_{-1}$ respectively as follows:

$$\mathfrak{e} = \{X \in \mathfrak{g}_{-1} \mid X_{32} = 0\}, \quad \mathfrak{f} = \{X \in \mathfrak{g}_{-1} \mid X_{21} = 0\}.$$

It is easy to see that $\mathfrak{G}$ is pseudo-product with respect to the subspaces $\mathfrak{e}$ and $\mathfrak{f}$ of $\mathfrak{g}_{-1}$. We shall apply the arguments in § 3 to the pseudo-product SGLA,

$\mathfrak{G}$, of the second class.

We easily see that the elements E and J in the centre of $\mathfrak{g}_0$ are given respectively by

$$E=\begin{pmatrix} 1_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1_n \end{pmatrix}, \quad J=\begin{pmatrix} \alpha 1_n & 0 & 0 \\ 0 & \beta 1_n & 0 \\ 0 & 0 & \alpha 1_n \end{pmatrix},$$

where $\alpha=-m/(2n+m)$, and $\beta=2n/(2n+m)$. It follows that $\mathfrak{A}=\mathfrak{G}(n, m+n; K)=\mathfrak{H}$, and $\mathfrak{B}=\mathfrak{G}(n+m, n; K)$.

Let us now verify that $\mathfrak{G}$ satisfies condition (II. 1) and (II. 2). Set $L=GL(2n+m, K)$, and define a subgroup L_0 of L by

$$L_0=G_0(n, m, n; K)=\{A\in L \mid AEL^{-1}=E,\ AJA^{-1}=J\}.$$

Furthermore set $G'=L/C(L)$, and $G_0'=L_0/C(L)$, which may be naturally identified with subgroups of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{L})$ respectively.

LEMMA 7.1 (1) *If $K=\boldsymbol{R}$ or $\boldsymbol{Q}$, then $G_0=G_0'$ and $G=G'$.*

(2) *If $K=\boldsymbol{C}$, then G_0' and G' are open, normal subgroups of G_0 and G respectively, and $G_0=G_0'\cup\tau\cdot G_0'$, and $G=G'\cup\tau\cdot G'$, where τ denotes the conjugation of $\mathfrak{g}=\mathfrak{sl}(2n+m, \boldsymbol{C})$; $\tau(X)=\bar{X}$, $X\in\mathfrak{g}$.*

Now, define an element g of G' by

$$g=\begin{pmatrix} 0 & 0 & 1_n \\ 0 & 1_m & 0 \\ 1_n & 0 & 0 \end{pmatrix} \qquad (\mathrm{mod}\ C(L)).$$

Then we easily see that $g^2=e$, $\mathrm{Ad}(g)E=-E$ and $\mathrm{Ad}(g)J=J$, showing that $\mathfrak{G}$ satisfies condition (II. 1).

LEMMA 7.2. *The space $\mathfrak{g}_{-2}$ may be naturally identified with $M_n(K)$, and the subgroup $\tilde{G}_0$ of $GL(\mathfrak{g}_{-2})$ consists of all transformations F of $M_n(K)$ of the following form :*

1) *If $K=\boldsymbol{R}$ or $\boldsymbol{Q}$, $F(X)=AXB^{-1}$,*

2) *If $K=\boldsymbol{C}$, $F(X)=AXB^{-1}$ or $F(X)=A\bar{X}B^{-1}$,*

where A, $B\in GL(n, K)$, and $X\in M_n(K)$.

For any integer $0\leqq r\leqq n$, set $V_r=V_r(M_n(K))$. Then we see from Lemma 7.2 that $V_0, \dots, V_n$ are all the $\tilde{G}_0$-orbits, and especially V_n is a single open orbit. Therefore we have $V=V_n$, and $\partial V=V_0\cup\dots\cup V_{n-1}$.

If $K=\boldsymbol{R}$ or $\boldsymbol{C}$, it is clear that ∂V is the zeros of the polynomial function $\det(X)$ on $M_n(K)$. For any pair (i, j) of integers $1\leqq i$, $j\leqq 2n$, we now define a function $\chi_{ij}(X)$ on $M_n(\boldsymbol{Q})$ by

$$\chi_{ij}(X)=\Delta_{ij}(\mu(X)),\ X\in M_n(\boldsymbol{Q}),$$

which is a homogeneous polynomial function of degree $2n-1$. Since rank $(X)=\frac{1}{2}$ rank$(\mu(X))$, we then see that, if $K=\boldsymbol{Q}$, ∂V is the common zeros of the polynomial functions $\chi_{ij}(X)$. Note that ∂V is also the zeros of the polynomial function $\det^{\boldsymbol{Q}}(X)$ on $M_n(\boldsymbol{Q})$.

Now, consider the (complexified) dieal $I(\partial V)$ of the variety ∂V.

LEMMA 7.3. (1) *If* $K=\boldsymbol{R}$, *then* $I(\partial V)$ *is generated by* $\det(X)$.
(2) *If* $K=\boldsymbol{C}$, *then* $I(\partial V)$ *is generated by* $\det(X)$ *and* $\det(\bar{X})$.
(3) *If* $K=\boldsymbol{Q}$, *then* $I(\partial V)$ *is generated by* $\chi_{ij}(X)$.

PROOF. (1) Take any polynomial function $f(X)$ of $I(\partial V)$, which may be naturally regarded as a polynomial function of $P_C(M_n(\boldsymbol{C}))$. Put $\tilde{V}_{n-1}=V_{n-1}(M_n(\boldsymbol{C}))$, and $\tilde{W}_{n-1}=W_{n-1}(M_n(\boldsymbol{C}))$. Since V_{n-1} is a real part of $\tilde{V}_{n-1}$, we see that $f(X)$ is in $I_C(\tilde{W}_{n-1})$. Therefore it follows from Lemma 6.1 that $f(X)$ is in the ideal of $P_C(M_n(\boldsymbol{C}))$ generated by $\det(X)$, and hence $f(X)$ is in the ideal of $P(M_n(\boldsymbol{R}))$ generated by $\det(X)$.

(2) Take any polynomial function $f(X)$ of $I(\partial V)$. We first remark that there is a unique polynomial function $\tilde{f}(X,Y)$ of $P_C(M_n(\boldsymbol{C})\times M_n(\boldsymbol{C}))$ such that $\tilde{f}(X,\bar{X})=f(X)$. Since $\{(X,\bar{X})\,|\,X\in V_{n-1}\}$ is a real part of $V_{n-1}\times V_{n-1}$, it follows that $\tilde{f}(X,Y)$ is in $I_C(\partial V\times\partial V)$. Using Lemma 6.1 twice, we therefore see that $\tilde{f}(X,Y)$ is in the ideal of $P_C(M_n(\boldsymbol{C})\times M_n(\boldsymbol{C}))$ generated by $\det(X)$ and $\det(Y)$, and hence $f(X)$ is in the ideal of $P(M_n(\boldsymbol{C}))$ generated by $\det(X)$ and $\det(\bar{X})$.

(3) Take any polynomial function $f(X)$ of $I(\partial V)$. We know that $\mu(M_n(\boldsymbol{Q}))$ is a real part of $M_{2n}(\boldsymbol{C})$ (see Remark in 5.1). Hence there is a unique polynomial function $\tilde{f}(Y)$ of $P_C(M_{2n}(\boldsymbol{C}))$ such that $f(X)=\tilde{f}(\mu(X))$. Put $\tilde{V}_{2n-2}=V_{2n-2}(M_{2n}(\boldsymbol{C}))$, and $\tilde{W}_{2n-2}=W_{2n-2}(M_{2n}(\boldsymbol{C}))$. Since $\mu(V_{n-1})$ is a real part of $\tilde{V}_{2n-2}$, it follows that $\tilde{f}(Y)$ is in $I_C(\tilde{W}_{2n-2})$. By Lemma 6.6 we therefore see that $\tilde{f}(Y)$ is in the ideal of $P_C(M_{2n}(\boldsymbol{C}))$ generated by $\Delta_{ij}(Y)$, and hence $f(X)$ is in the ideal of $P(M_n(\boldsymbol{Q}))$ generated by $\chi_{ij}(X)$.

We have thus proved Lemma 7.3.

We see from Lemma 7.3 that $l(\partial V)=n$ if $K=\boldsymbol{R}$ or $\boldsymbol{C}$ and $l(\partial V)=2n-1$ if $K=\boldsymbol{Q}$, and therefore from Lemma 6.2 that $\mathfrak{G}$ satisfies condtion (II.2). Consequently we know by Proposition 3.26 that $\Omega=G/A_0$ is the standard affine symmetric space associated with $\mathfrak{A}=\mathfrak{H}$, and that $\mathrm{Aut}(\Omega)$ is naturally isomorphic with G, proving Theorem 2.3 for $\mathfrak{H}$.

7.2. The case where $\mathfrak{H}=\mathfrak{SO}(n+1, n+1; K)$ (n even, $n\geqq 4$, $K=\boldsymbol{R}$ or $\boldsymbol{C}$). We defin a matrix $\tilde{J}_{n+1}$ of degree $2n+2$ by

$$\tilde{J}_{n+1}=\begin{pmatrix} 0 & 0 & 1_n \\ 0 & J_1 & 0 \\ 1_n & 0 & 0 \end{pmatrix},$$

where $J_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as before. We then define a subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(2n+2, K)$ by

$$\mathfrak{g}=\{X\in\mathfrak{gl}(2n+2, K) \mid {}^tX\tilde{J}_{n+1}+\tilde{J}_{n+1}X=0\},$$

which is isomorphic with the simple Lie algebra $\mathfrak{so}(n+1, n+1; K)$. If we put $n_1=n_3=n$ and $n_2=2$, every matrix X of $\mathfrak{gl}(2n+2, K)$ may be expressed as follows: $X=(X_{ij})_{1\leq i, j\leq 3}$, where X_{ij} are $n_i\times n_j$-matrices. Then $\mathfrak{g}$ is defined by the equations

$$\begin{aligned} &{}^tX_{31}+X_{31}=0,\ {}^tX_{32}+J_1X_{21}=0,\ {}^tX_{11}+X_{33}=0, \\ &{}^tX_{22}J_1+J_1X_{22}=0,\ {}^tX_{12}+J_1X_{23}=0,\ {}^tX_{13}+X_{13}=0. \end{aligned}$$

Here, we remark that the equation for X_{22} means X_{22} is of the form: αT_1 with $\alpha\in K$, where $T_1=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

For any integer p, we now define a subspace $\mathfrak{g}_p$ of $\mathfrak{g}$ by $\mathfrak{g}_p=\mathfrak{g}\cap\mathfrak{g}_p(n, 2, n; K)=\{X\in\mathfrak{g} \mid X_{ij}=0 \text{ if } j-i\neq p\}$. Then we see that $\mathfrak{G}=\{\mathfrak{g}, (\mathfrak{g}_p)\}$ is a graded subalgebra of $\mathfrak{G}(n, 2, n; K)$, which is even a simple graded Lie algebra of the second kind. Note that the characteristic element E of $\mathfrak{G}$ is represented by the same matrix as in the preceding paragraph.

Moreover let us define an element J of $\mathfrak{g}$ by

$$J=\begin{pmatrix} 0 & 0 & 0 \\ 0 & T_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that J is in the centre of $\mathfrak{g}_0$, and that $[J, \mathfrak{g}_{-2}]=\{0\}$ and $[J, [J, X]]=X$ for all $X\in\mathfrak{g}_{-1}$, showing that $\mathfrak{G}$ is pseudo-product (see Remark in 3.1). We shall apply the arguments in § 3 to the pseudo-product SGLA, $\mathfrak{G}$, of the second kind.

We assert that $\mathfrak{A}\cong\mathfrak{H}$. Indeed, the characteristic elements $E_{\mathfrak{A}}$ and $E_{\mathfrak{H}}$ of $\mathfrak{A}$ and $\mathfrak{H}$ respectively are represented by the same matrix $\frac{1}{2}\begin{pmatrix} 1_{n+1} & 0 \\ 0 & -1_{n+1} \end{pmatrix}$.

Now, define a matrix P of degree $n+1$ by $P=\begin{pmatrix} 0 & 1_n \\ 1 & 0 \end{pmatrix}$, and a matrix $\tilde{P}$ of degree $2n+2$ by $\tilde{P}=\begin{pmatrix} 1_{n+1} & 0 \\ 0 & P \end{pmatrix}$. Then we see that ${}^t\tilde{P}J_{n+1}\tilde{P}=\begin{pmatrix} 0 & P \\ {}^tP & 0 \end{pmatrix}=\tilde{J}_{n+1}$ and $\tilde{P}E_{\mathfrak{A}}\tilde{P}^{-1}=E_{\mathfrak{H}}$. Hence it follows that $\tilde{P}X\tilde{P}^{-1}\in\mathfrak{so}(n+1, n+1\,;K)$ for all $X\in\mathfrak{g}$, and that the assignment $X\to\tilde{P}X\tilde{P}^{-1}$ gives an isomorphism of $\mathfrak{A}$ onto $\mathfrak{H}$, proving our assertion.

Let us now verify that $\mathfrak{G}$ satisfies conditions (II. 1) and (II. 2). Define a subgroup L of $GL(2n+2, K)$ by

$$L=\{A\in GL(2n+2, K)\,|\,\det(A)=\varepsilon,\ {}^tA\tilde{J}_{n+1}A=\varepsilon\tilde{J}_{n+1} \text{ with } \varepsilon=1 \text{ or } -1\},$$

and a subgroup L_0 of L by

$$L_0=L\cap G_0(n, 2, n\,;K)=\{A\in L\,|\,AEA^{-1}=E,\ AJA^{-1}=J\}.$$

Set $G'=L/C(L)$, and $G_0'=L_0/C(L)$, which may be naturally identified with subgroups of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{L})$ respectively.

LEMMA 7.4. (1) *If $K=\boldsymbol{R}$, then $G_0=G_0'$ and $G=G'$.*

(2) *If $K=\boldsymbol{C}$, then G_0' and G' are open, normal subgroups of G_0 and G respectively, and $G_0=G_0'\cup\tau\cdot G_0'$ and $G=G'\cup\tau\cdot G'$, where τ is the conjugation of $\mathfrak{g}$.*

Clearly the matrix

$$\begin{pmatrix} 0 & 0 & 1_n \\ 0 & 1_2 & 0 \\ 1_n & 0 & 0 \end{pmatrix}$$

is in the group L, and hence it defines an element g of G'. Then we see that $\mathfrak{G}$ satisfies condition (II. 1) with respect to the element g.

LEMMA 7.5. *The space $\mathfrak{g}_{-2}$ may be naturally identified with $S_n'(K)$, and the group $\tilde{G}_0$ consists of all transformations F of $S_n'(K)$ of the following form:*

1) *If $K=\boldsymbol{R}$, $F(X)=\varepsilon AX{}^tA$,*

2) *If $K=\boldsymbol{C}$, $F(X)=AX{}^tA$ or $F(X)=A\bar{X}{}^tA$,*

where $A\in GL(n, K)$, $\varepsilon=1$ or -1, and $X\in S_n'(K)$.

Set $n=2m$. For any integer $0\leqq r\leqq m$, set $V_{2r}=V_{2r}(S_n'(K))$. Then we see from Lemma 7.5 that V_0, V_2, …, V_{2m} are all the $\tilde{G}_0$-orbits, and especially V_{2m} is a single open orbit. Therefore we have $V=V_{2m}$, and $\partial V=V_0\cup V_2\cup\ldots\cup V_{2(m-1)}$. Clearly ∂V is the zeros of the polynomial function $\det^{\frac{1}{2}}(X)$ on $S_n'(K)$.

LEMMA 7.6. (1) *If* $K=\boldsymbol{R}$, *then* $I(\partial V)$ *is generated by* $\det^{\frac{1}{2}}(X)$.

(2) *If* $K=\boldsymbol{C}$, *then* $I(\partial V)$ *is generated by* $\det^{\frac{1}{2}}(X)$ *and* $\det^{\frac{1}{2}}(\bar{X})$.

This fact follows from Lemma 6.15 (cf. the proofs of (1) and (2) of Lemma 7.3).

We see from Lemma 7.6 that $l(\partial V)=m$, and therefore from Lemma 6.16 that $\mathfrak{G}$ satisfies condition (II. 2). Since $\mathfrak{A}\cong\mathfrak{H}$, we have thus proved Theorem 2.8 for $\mathfrak{H}$.

§ 8. Proof of Theorem 2.8: The case where $\mathfrak{H}$ is a simple graded Lie algebra of the first class

Let $\mathfrak{G}$ be one of the simple graded Lie algebras of the first class in TABLE 2. The main aim of the present section is to apply the arguments in § 4 to $\mathfrak{G}$, and is to prove the following

PROPOSITION 8.1. (1) *If* $\mathfrak{G}$ *is a definite Möbius algebra, then* $\mathfrak{G}$ *satisfies conditions* (I. 1), (I. 2) *and the condition that* $\partial V=\{0\}$.

(2) *Otherwise,* $\mathfrak{G}$ *satisfies conditions* (I. 1)$\sim$(I 3).

Consider the standard affine symmetric space $\Omega=G/G_0$ associated with $\mathfrak{G}$. By Propositions 4.9 and 4.13 we therefore have the following: If $\mathfrak{G}$ is a definite Möbius algebra, then $\mathrm{Aut}(\Omega)$ is naturally isomorphic with $\mathrm{Diff}(M)$; Otherwise, $\mathrm{Aut}(\Omega)$ is naturally isomorphic with G. Thus Proposition 8.1 combined with the results in the preceding section accomplishes the proof of Theorem 2.8.

8.1. The case where $\mathfrak{G}=\mathfrak{G}(n, n; K)$ ($n\geqq 3$ if $K=\boldsymbol{R}$ or $\boldsymbol{C}$; $n\geqq 2$ if $K=\boldsymbol{Q}$). Set $L=GL(2n, K)$, and define its subgroup L_0 by

$$L_0=G_0(n, n; K)=\{A\in L \mid AEA^{-1}=E\}.$$

Set $G'=L/C(L)$, and $G_0'=L_0/C(L)$, which may be naturally identified with subgroups of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{G})$ respectively.

Now, define an element g of G' by

$$g=J_n \pmod{C(L)},$$

and an automorphism θ of the Lie algebra $\mathfrak{g}=\mathfrak{sl}(2n, K)$ by

$$\theta(X)=\mathrm{Ad}(g)(-{}^tX), \quad X\in\mathfrak{g}.$$

It is easy to see that $\theta\in\mathrm{Aut}(\mathfrak{G})$.

LEMMA 8.2. *G_0' and G' are open, normal subgroups of G_0 and G respectively, and*

(1) *If* $K=\boldsymbol{R}$ *or* $\boldsymbol{Q}$, *then* $G_0=G_0'\cup\theta\cdot G_0'$ *and* $G=G'\cup\theta\cdot G'$.

(2) *If* $K=\boldsymbol{C}$, *then* $G_0=G_0'\cup\theta\cdot G_0'\cup\tau\cdot G_0'\cup\theta\tau\cdot G_0'$, *and* $G=G'\cup\theta\cdot G'\cup\tau\cdot G'\cup\theta\tau\cdot G'$, *where* τ *denotes the conjugation of* $\mathfrak{g}=\mathfrak{sl}(2n, \boldsymbol{C})$.

Note that $\theta^2=\tau^2=(\theta\tau)^2=e$.

Clearly the element g of G' satisfies $g^2=e$ and $\mathrm{Ad}(g)E=-E$, showing that $\mathfrak{S}$ satisfies condition (I. 1).

LEMMA 8. 3. *The space* $\mathfrak{g}_{-1}$ *may be naturally identified with* $M_n(K)$, *and the subgroup* $\tilde{G}_0$ *of* $GL(\mathfrak{g}_{-1})$ *consists of all transformations* F *of* $M_n(K)$ *of the following form*:

1) *If* $K=\boldsymbol{R}$ *or* $\boldsymbol{Q}$, $F(X)=AXB^{-1}$ *or* $F(X)=A{}^tXB^{-1}$,

2) *If* $K=\boldsymbol{C}$, $F(X)=AXB^{-1}$ *or* $F(X)=A{}^tXB^{-1}$ *or* $F(X)=A\bar{X}B^{-1}$ *or* $F(X)=AX^*B^{-1}$,

where A, $B\in GL(n, K)$, *and* $X\in M_n(K)$.

For any integer $0\leqq r\leqq n$, set $V_r=V_r(M_n(K))$. As before we see from Lemma 8.3 that $V=V_n$, and $\partial V=V_0\cup\ldots\cup V_{n-1}$. Clearly Lemma 7.3 holds true in our present situation. Accordingly we know that $\mathfrak{S}$ satisfies condition (I. 2).

We shall now show that $\mathfrak{S}$ satisfies condition (I. 3). Take any element F of $GL(\partial V)$. Our task from now on is to show that $F\in\tilde{G}_0$.

The case where $\mathfrak{S}=\mathfrak{S}(n, n;\boldsymbol{R})$. By Lemma 7.3 we have

LEMMA 8. 4. $\det(F(X))=c\det(X)$, $X\in M_n(\boldsymbol{R})$, *where* c *is a constant.*

Since $M_n(\boldsymbol{R})$ is a real part of $M_n(\boldsymbol{C})$, there is a unique $\tilde{F}\in GL_{\boldsymbol{C}}(M_n(\boldsymbol{C}))$ such that $\tilde{F}(X)=F(X)$. By Lemma 8.4 we then have

$$\det(\tilde{F}(Y))=c\det(Y),\quad Y\in M_n(\boldsymbol{C}).$$

Therefore it follows from Lemmas 6.3 and 6.4 that there are $A, B\in GL(n, \boldsymbol{C})$ such that either $\tilde{F}(Y)=AYB^{-1}$ or $\tilde{F}(Y)=A{}^tYB^{-1}$. For our purpose it clearly suffices to deal with the case where $\tilde{F}(Y)=AYB^{-1}$. Put $A_1=\mathrm{Re}\,A$, $A_2=\mathrm{Im}\,A$, $B_1=\mathrm{Re}\,B$ and $B_2=\mathrm{Im}\,B$. Then there is $t\in\boldsymbol{R}$ such that $\det(A_1+tA_2)\neq 0$ and $\det(B_1+tB_2)\neq 0$. Since $F(X)B_1=A_1X$ and $F(X)B_2=A_2X$, it follows that $F(X)(B_1+tB_2)=(A_1+tA_2)X$, and hence $F(X)=(A_1+tA_2)X(B_1+tB_2)^{-1}$. This implies that $F\in\tilde{G}_0$.

The case where $\mathfrak{S}=\mathfrak{S}(n, n;\boldsymbol{C})$. By Lemma 7.3 we have

LEMMA 8. 5. $\det(F(X))=a\det(X)+b\det(\bar{X})$, $X\in M_n(\boldsymbol{C})$, *where* a *and* b *are constants.*

LEMMA 8. 6. *Either* F *or* $F\circ\tau$ *is in* $GL_{\boldsymbol{C}}(M_n(\boldsymbol{C}))$, *where* τ *denotes the conjugation of* $M_n(\boldsymbol{C})$.

PROOF. We first notice that there are unique endomorphisms F_1 and F_2 of $M_n(\boldsymbol{C})$ as a vector space over $\boldsymbol{C}$ such that $F(X)=F_1(X)+F_2(\bar{X})$. Since $\{(X, \bar{X}) \mid X \in M_n(\boldsymbol{C})\}$ is a real part of $M_n(\boldsymbol{C}) \times M_n(\boldsymbol{C})$, it follows from Lemma 8.5 that

$$\det(F_1(X)+F_2(Y))=a \det(X)+b \det(Y), \quad X, \ Y \in M_n(\boldsymbol{C}).$$

Clearly we have either $a \neq 0$ 0 or $b \neq 0$. For our purpose it clearly suffices to show that $F_1 \in GL_C(M_n(\boldsymbol{C}))$, and $F_2=0$, assuming that $a \neq 0$. Accordingly assume this condition. By the equality above we have $\det(F_1(X))= a \det(X)$. Consequently we see from Lemma 6.3 that $F_1 \in GL_C(M_n(\boldsymbol{C}))$, and $F_1(V_n)=V_n$. Now, let $X \in V_n$, and $Y \in M_n(\boldsymbol{C})$. We have $\det(F_1(X))=a \det(X)$, and $\det(F_1(\lambda X)+F_2(Y))=a \det(\lambda X)+b \det(Y)$ for all $\lambda \in \boldsymbol{C}$. It follows that

$$\det(\lambda 1_n+F_1(X)^{-1}F_2(Y))=\lambda^n+a^{-1}b \det(X^{-1}Y),$$

whence $\operatorname{Tr}(F_1(X)^{-1}F_2(Y))=0$. Since $V_n^{-1}=V_n$, and V_n is open and dense in $M_n(\boldsymbol{C})$, we therefore see that $\operatorname{Tr}(ZF_2(Y))=0$ for all $Y, Z \in M_n(\boldsymbol{C})$, whence $F_2=0$. We have thus prove Lemma 8.6.

By Lemmas 8.5 and 8.6 we have: If $F \in GL_C(M_n(\boldsymbol{C}))$, then $\det(F(X))=a \det(X)$; If $F \circ \tau \in GL_C(M_n(\boldsymbol{C}))$, then $\det(F(\bar{X}))=b \det(X)$. Therefore we see from Lemmas 6.3 and 6.4 that $F \in \tilde{G}_0$.

The case where $\mathfrak{G}=\mathfrak{G}(n, n; \boldsymbol{Q})$. By Lemma 7.3 we have

LEMMA 8.7. $\chi_{ij}(F(X))=\sum_{k,l} c_{ij}^{kl}\chi_{kl}(X)$, $X \in M_n(\boldsymbol{Q})$, where c_{ij}^{kl} are constants.

Since $\mu(M_n(\boldsymbol{Q}))$ is a real part of $M_{2n}(\boldsymbol{C})$, there is a unique $\tilde{F} \in GL_C(M_{2n}(\boldsymbol{C}))$ such that $\tilde{F}(\mu(X))=\mu(F(X))$. Since $\chi_{ij}(X)=\Delta_{ij}(\mu(X))$, it follows from Lemma 8.7 that

$$\Delta_{ij}(\tilde{F}(Y))=\sum_{k,l} c_{ij}^{kl}\Delta_{kl}(Y), \quad Y \in M_{2n}(\boldsymbol{C}).$$

Putting $\tilde{W}_{2n-2}=W_{2n-2}(M_{2n}(\boldsymbol{C}))$, we therefore obtain $\tilde{F}(\tilde{W}_{2n-2}) \subset \tilde{W}_{2n-2}$. Considering the inverse F^{-1} of F, we similarly obtain $\tilde{W}_{2n-2} \subset \tilde{F}(\tilde{W}_{2n-2})$. Hence $\tilde{F}(\tilde{W}_{2n-2})=W_{2n-2}$ or $\tilde{F} \in GL_C(\tilde{W}_{2n-2})$. By Lemma 6.9 we can thereby find $A, B \in GL(2n, \boldsymbol{C})$ such that either $\tilde{F}(Y)=AYB^{-1}$ or $\tilde{F}(Y)=A{}^tYB^{-1}$. Consequently it follows that $F \in \tilde{G}_0$ (cf. the case where $\mathfrak{G}=\mathfrak{G}(n, n; \boldsymbol{R})$).

We have thus shown that $GL(\partial V)=\tilde{G}_0$, meaning that $\mathfrak{G}$ satisfies condition (I.3).

8. 2. The case where $\mathfrak{G}=\mathfrak{S}\mathfrak{U}(n, n; C)$ $(n\geqq 3)$. Define a subgroup L of $GL(2n, C)$ by

$$L=\{A\in GL(2n, C) \mid A^*J_nA=\varepsilon J_n \text{ with } \varepsilon=1 \text{ or } -1\},$$

and its subgroup L_0 by

$$L_0=L\cap G_0(n, n; C).$$

Set $G'=L/C(L)$, and $G'_0=L_0/C(L)$, which may be naturally identified with subgroups of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{G})$ respectively.

As before define an element g of G' by

$$g=J_n \pmod{C(L)},$$

and an automorphism θ of the Lie algebra $\mathfrak{g}=\mathfrak{su}(n, n; C)$ by

$$\theta(X)=\bar{X}=\mathrm{Ad}(g)(-{}^tX), \quad X\in\mathfrak{g}.$$

LEMMA 8. 8. G'_0 and G' are open, normal subgroups of G_0 and G respectively, and $G_0=G'_0\cup\theta\cdot G'_0$ and $G=G'\cup\theta\cdot G'$.

Then we see that $\mathfrak{G}$ satisfies condition (I. 1) with respect to the element g of G'.

LEMMA 8. 9. *The space* $\mathfrak{g}_{-1}$ *may be naturally identified with* $H'_n(C)$, *and the group* $\tilde{G}_0$ *consists of all transformations* F *of* $H'_n(C)$ *of the following form*:

$$F(X)=\varepsilon AXA^* \text{ or } F(X)=\varepsilon A\bar{X}A^*,$$

where $A\in GL(n, C)$, $\varepsilon=1$ *or* -1, *and* $X\in H'_n(C)$.

For any integer $0\leqq r\leqq n$, set $V_r=V_r(H'_n(C))$. Then we see from Lemma 8. 9 that each V_r is the union of all $\tilde{G}_0$-orbits of the same dimension, and especially V_n is the union of all open orbits. Therefore we have $V=V_n$, and $\partial V=V_0\cup\ldots\cup V_{n-1}$. Clearly ∂V is the zeros of the polynomial function $\det(X)$ on $H'_n(C)$.

LEMMA 8. 10. $I(\partial V)$ *is generated by* $\det(X)$.

Clearly $H'_n(C)$ is a real part of $M_n(C)$, and correspondingly V_{n-1} is a real part of $\tilde{V}_{n-1}=V_{n-1}(M_n(C))$. Accordingly Lemma 8. 10 follows from Lemma 6. 1 (cf. the proof of (1) of Lemma 7. 3).

We see from Lemma 8. 10 that $l(\partial V)=n$, and therefore from Lemma 6. 2 that $\mathfrak{G}$ satisfies condition (I. 2).

Let us now show that $\mathfrak{G}$ satisfies condition (I. 3). Take any element F of $GL(\partial V)$. By Lemma 8. 10 we have

LEMMA 8.11. $\det(F(X))=c\ \det(X),\ X\in H_n'(\boldsymbol{C}),$ *where c is a constant.*

Since $H_n'(\boldsymbol{C})$ is a real part of $M_n(\boldsymbol{C})$, there is a unique $\tilde{F}\in GL_{\boldsymbol{C}}(M_n(\boldsymbol{C}))$ such that $\tilde{F}(X)=F(X)$. By Lemma 8.11 we then have

$$\det(\tilde{F}(Y))=c\ \det(Y),\quad Y\in M_n(\boldsymbol{C}).$$

Therefore it follows from Lemmas 6.3 and 6.4 that there are $A,\ B\in GL(n,\boldsymbol{C})$ such that either $\tilde{F}(Y)=AYB$ or $\tilde{F}(Y)=A{}^tYB$. For our purpose, it clearly suffices to deal with the case where $\tilde{F}(Y)=AYB$. If $X\in H_n'(\boldsymbol{C})$, we have $X^*=-X$ and $F(X)^*=-F(X)$, whence $B^*XA^*=AXB$. From this fact it follows that $B=\lambda A^*$ with some $\lambda\in\boldsymbol{R}$. Hence we obtain $F(X)=\lambda AXA^*$, implying that $F\in\tilde{G}_0$.

8.3. The case where $\mathfrak{G}=\mathfrak{SO}(n,\ n\,;K)$ (n even, $n\geqq 6$ if $K=\boldsymbol{R}$ or $\boldsymbol{C}$; $n\geqq 3$ if $K=\boldsymbol{Q}$). Define a subgroup L of $SL(2n,K)$ by

$$L=\{A\in SL(2n,\ K)\,|\,{}^tAJ_nA=\varepsilon J_n \text{ with } \varepsilon=1 \text{ or } -1\},$$

and its subgroup L_0 by

$$L_0=L\cap G_0(n,\ n\,;K).$$

Set $G'=L/C(L)$, and $G_0'=L_0/C(L)$, which may be naturally identified with subgroups of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{G})$ respectively.

LEMMA 8.12. (1) *If $K=\boldsymbol{R}$ or $\boldsymbol{Q}$, then $G_0=G_0'$ and $G=G'$.*

(2) *If $K=\boldsymbol{C}$, then G_0' and G' are open, normal subgroups of G_0 and G respectively, and $G_0=G_0'\cup\tau\cdot G_0'$ and $G=G'\cup\tau\cdot G'$, where τ denotes the conjugation of $\mathfrak{g}=\mathfrak{so}(n,\ n\,;\boldsymbol{C})$.*

As before define an element g of G' by

$$g=J_n\ (\mathrm{mod}\ C(L)).$$

Then we see that $\mathfrak{G}$ satisfies condition (I.1) with respect to the element g.

LEMMA 8.13. *The space $\mathfrak{g}_{-1}$ may be naturally identified with $S_n'(K)$, and the group $\tilde{G}_0$ consitst of all transformations F of $S_n'(K)$ of the following form:*

1) *If $K=\boldsymbol{R}$ or $\boldsymbol{Q}$, $F(X)=\varepsilon AX{}^tA$,*

2) *If $K=\boldsymbol{C}$, $F(X)=AX{}^tA$ or $F(X)=A\bar{X}{}^tA$,*

where $A\in GL(n,K)$, $\varepsilon=1$ or -1, and $X\in S_n'(K)$.

We shall show that $\mathfrak{G}$ satisfies condition (I.2). We first consider the case where $K=\boldsymbol{R}$ or $\boldsymbol{C}$. Set $n=2m$. For any integer $0\leqq r\leqq m$, set $V_{2r}=V_{2r}(S_n'(K))$. As before we see from Lemma 8.13 that $V=V_{2m}$, and $\partial V=$

$V_0 \cup V_2 \cup \ldots \cup V_{2(m-1)}$. Clearly Lemma 7.6 holds true in our present situation. Accordingly we know that $\mathfrak{G}$ satisfies condition (I. 2).

We next consider the case where $K=\boldsymbol{Q}$. For any integer $0 \leqq r \leqq n$, set $V_r = V_r(S'_n(\boldsymbol{Q}))$. Then we see from Lemma 8.13 and Remark below that each V_r is the union of all $\tilde{G}_0$-orbits of the same dimension, and especially V_n is the union of all open orbits. Therefore we have $V=V_n$, and $\partial V = V_0 \cup \ldots \cup V_{n-1}$.

REMARK (cf. Sylvester's law of inertia). For every matrix $X \in S'_n(\boldsymbol{Q})$ there is a matrix $A \in GL(n, \boldsymbol{C})$ such that the matrix $AX{}^tA$ may be represented by a diagonal matrix whose diagonal vector is of the form: $(-e_2, \ldots, -e_2, e_2, \ldots, e_2, 0, \ldots, 0)$. Furthermore the multiplicity of $-e_2$ as well as e_2 does not depend on the choice of A, and hence is uniquely determined by X.

We know that $\mu(S'_n(\boldsymbol{Q}))$ is a real part of $S'_{2n}(\boldsymbol{C})$ (see Remark in 5.1). This being said, we define a function $\chi(X)$ on $S'_n(\boldsymbol{Q})$ by

$$\chi(X) = \det{}^{\frac{1}{2}}(\mu(X)), \quad X \in S'_n(\boldsymbol{Q}),$$

which is a homogeneous polynomial function of degree n. Clearly ∂V is the zeros of $\chi(X)$.

LEMMA 8.14. *$I(\partial V)$ is generated by $\chi(X)$.*

This fact follows from Lemma 6.15 (cf. the proof of (1) or (3) of Lemma 7.3). Note that $\mu(V_{n-1})$ is a real part of $\tilde{V}_{2(n-1)} = V_{2(n-1)}(S'_{2n}(\boldsymbol{C}))$.

We see from Lemma 8.14 that $l(\partial V)=n$, and therefore from Lemma 6.16 that $\mathfrak{G}$ satisfies condition (I. 2).

We shall now show that $\mathfrak{G}$ satisfies condition (I. 3). Take any element F of $GL(\partial V)$.

The case where $\mathfrak{G} = \mathfrak{SO}(n, n; \boldsymbol{R})$. By Lemma 7.6 we have

LEMMA 8.15. $\det^{\frac{1}{2}}(F(X)) = c \det^{\frac{1}{2}}(X)$, $X \in S'_n(\boldsymbol{R})$, *where c is a constant.*

Since $S'_n(\boldsymbol{R})$ is a real part of $S'_n(\boldsymbol{C})$, there is a unique $\tilde{F} \in GL_C(S'_n(\boldsymbol{C}))$ such that $\tilde{F}(X) = F(X)$. By Lemma 8.15 we then have

$$\det{}^{\frac{1}{2}}(\tilde{F}(Y)) = c \det{}^{\frac{1}{2}}(Y), \quad Y \in S'_n(\boldsymbol{C}).$$

Therefore we see from Lemmas 6.17 and 6.18 that there is $A \in GL(n, \boldsymbol{C})$ such that $\tilde{F}(Y) = AY{}^tA$. Consequently it follows that there are $A_0, B_0 \in GL(n, \boldsymbol{R})$ such that $F(X) = A_0 X B_0$ (cf. the case of $\mathfrak{G}(n, n; \boldsymbol{R})$), and in turn it follows that $B_0 = \lambda {}^tA_0$ with some $\lambda \in \boldsymbol{R}$ (cf. the case of $\mathfrak{SU}(n, n; \boldsymbol{C})$).

Hence we obtain $F(X)=\lambda A_0 X {}^tA_0$, implying that $F\in\tilde{G}_0$.

The case where $\mathfrak{G}=\mathfrak{SO}(n, n; \boldsymbol{C})$. By Lemma 7.6 we have

LEMMA 8.16. $\det^{\frac{1}{2}}(F(X))=a \det^{\frac{1}{2}}(X)+b \det^{\frac{1}{2}}(\bar{X})$, $X\in S_n'(\boldsymbol{C})$, *where a and b are constants.*

By virtue of Lemmas 6.17 and 8.16 it can be shown that either F or $F\circ\tau$ is in $GL_C(S_n'(\boldsymbol{C}))$, where τ denotes the conjugation of $S_n'(\boldsymbol{C})$ (cf. Lemma 8.6). (Note that Lemma 8.16 implies $\det(F(X))=a^2 \det(X)+2ab \det^{\frac{1}{2}}(X)\det^{\frac{1}{2}}(\bar{X})+b^2 \det(\bar{X})$.) Consequently it follows from Lemmas 6.17 and 6.18 that $F\in\tilde{G}_0$.

The case where $\mathfrak{G}=\mathfrak{SO}(n, n; \boldsymbol{Q})$. By Lemma 8.14 we have

LEMMA 8.17. $\chi(F(X))=c\, \chi(X)$, $X\in S_n'(\boldsymbol{Q})$, *where c is a constant.*

Since $\mu(S_n'(\boldsymbol{Q}))$ is a real part of $S_{2n}'(\boldsymbol{C})$, there is a unique $\tilde{F}\in GL_C(S_{2n}'(\boldsymbol{C}))$ such that $\tilde{F}(\mu(X))=\mu(F(X))$. Since $\chi(X)=\det^{\frac{1}{2}}(\mu(X))$, it follows from Lemma 8.17 that

$$\det^{\frac{1}{2}}(\tilde{F}(Y))=c \det^{\frac{1}{2}}(Y), \quad Y\in S_{2n}'(\boldsymbol{C}).$$

Therefore we see from Lemmas 6.17 and 6.18 that there is $A\in GL(2n, \boldsymbol{C})$ such that $\tilde{F}(Y)=AY{}^tA$. As above, it follows that there are $A_0, B_0\in GL(n, \boldsymbol{Q})$ such that $F(X)=A_0XB_0$, and in turn it follows that $B_0=\lambda\, {}^tA_0$ with some $\lambda\in\boldsymbol{R}$. Hence we obtain $F\in\tilde{G}_0$.

8.4. The case where $\mathfrak{G}=\mathfrak{Sp}(n, K)$ ($n\geqq 3$ if $K=\boldsymbol{R}$ or $\boldsymbol{C}$; $n\geqq 2$ if $K=\boldsymbol{Q}$). The discussions from now on are similar to the cases of $\mathfrak{G}(n, n; K)$ and $\mathfrak{SO}(n, n; K)$.

Define a subgroup L of $GL(2n, K)$ by

$$L=\{A\in GL(2n, K)\,|\,{}^tAI_nA=\varepsilon I_n \text{ with } \varepsilon=1 \text{ or } -1\},$$

and its subgroup L_0 by

$$L_0=L\cap G_0(n, n; K).$$

Set $G'=L/C(L)$, and $G_0'=L_0/C(L)$, which may be naturally identified with subgroups of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{G})$ respectively.

We notice that Lemmas 8.12 and 8.13 hold true in our present situation, where $\mathfrak{so}(n, n; \boldsymbol{C})$ and $S_n'(K)$ sholuld be of course replaced by $\mathfrak{Sp}(n, \boldsymbol{C})$ and $S_n(K)$ respectively.

Now, define an element g of G' by

$$g = I_n \pmod{C(L)}.$$

Then we see that $\mathfrak{G}$ satisfies condition (I. 1) with respect to the element g.

We shall show that $\mathfrak{G}$ satisfies condition (I. 2) and (I. 3). For any integer $0 \leqq r \leqq n$, set $V_r = V_r(S_n(K))$. If $K = \boldsymbol{C}$, we see that $V_0, \dots, V_n$ are all the $\tilde{G}_0$-orbits, and especially V_n is a single open orbit. If $K = \boldsymbol{R}$ or $\boldsymbol{Q}$, we see that each V_r is the union of all $\tilde{G}_0$-orbits of the same dimension, and especially V_n is the union of all open orbits. In any cases we therefore have $V = V_n$, and $\partial V = V_0 \cup \dots \cup V_{n-1}$.

Furthermore if $K = \boldsymbol{R}$ or $\boldsymbol{C}$, ∂V is the zeros of the polynomial function $\det(X)$ on $S_n(K)$; If $K = \boldsymbol{Q}$, ∂V is the common zeros of the polynomial functions $\chi_{ij}(X)$ on $S_n(\boldsymbol{Q})$, which is also the zeros of the polynomial function $\det^{\boldsymbol{Q}}(X)$ on $S_n(\boldsymbol{Q})$.

Here, we notice that Lemma 7. 3 holds true in addition. (This fact follows from Lemmas 6. 10 and 6. 13, together with the fact that $S_n(\boldsymbol{R})$ and $\mu(S_n(\boldsymbol{Q}))$ are real parts of $S_n(\boldsymbol{C})$ and $S_{2n}(\boldsymbol{C})$ respectively.) We also recall that there is an anlogous fact concerning the polynomial function $\det(X)$ on $S_n(\boldsymbol{C})$, which corresponds to Lemma 6. 2. Accordingly we find that $l(\partial V) = n$ if $K = \boldsymbol{R}$ or $\boldsymbol{C}$ and $l(\partial V) = 2n-1$ if $K = \boldsymbol{Q}$, and that $\mathfrak{G}$ satisfies condition (I. 2).

Finally we can show that $\mathfrak{G}$ satisfies condition (I. 3), by the use of Lemmas 6. 11, 6. 12 and 6. 14 combined with the fact corresponding to Lemma 7. 3.

8. 5. The case where $\mathfrak{G} = \mathfrak{M}_r(n)$ $(r = 0,\ n \geqq 1$ or $1 \leqq 2r \leqq n,\ n \geqq 3)$ or $\mathfrak{G} = \mathfrak{M}(n, \boldsymbol{C})$ $(n \geqq 3)$. We first consider the case where $\mathfrak{G} = \mathfrak{M}_r(n)$. Define a subgroup L of $GL(n+2, \boldsymbol{R})$ by

$$L = \{A \in GL(n+2, \boldsymbol{R}) \mid {}^t A \tilde{T}_{n,r} A = \varepsilon \tilde{T}_{n,r} \text{ with } \varepsilon = 1 \text{ or } -1\},$$

and its subgroup L_0 by

$$L_0 = L \cap G_0(1, n, 1 ; \boldsymbol{R}).$$

LEMMA 8. 18. *The groups G_0 and G may be naturally identified with the factor groups $L_0/C(L)$ and $L/C(L)$ respectively.*

Now, define an element g of G by

$$g = \tilde{T}_{n,r} \pmod{C(L)}.$$

Then we see that $\mathfrak{G}$ satisfies condition (I. 1).

LEMMA 8. 19. *The space $\mathfrak{g}_{-1}$ may be naturally identified with $\boldsymbol{R}^n$, and the group $\tilde{G}_0$ consists of all transformations F of $\boldsymbol{R}^n$ of the following form :*

$$F(x)=Ax,\ x\in \boldsymbol{R}^n,$$

where $A\in GL(n, \boldsymbol{R})$, *and* ${}^tA\tilde{T}_{n,r}A=\lambda T_{n,r}$ *with some* $\lambda\in \boldsymbol{R}$.

Let us define a quadratic form $q(x)$ on $\boldsymbol{R}^n$ by

$$q(x)=\sum_{i=1}^{n} \varepsilon_i x_i^2,\ x=(x_1, \dots, x_n)\in \boldsymbol{R}^n,$$

where $\varepsilon_i=-1$ if $1\leqq i\leqq r$, and $\varepsilon_i=1$ otherwise. Let us also define subsets V_0, V_1 and V_2 of $\boldsymbol{R}^n$ respectively as follows:

$$V_0=\{0\},\ V_1=\{x\,|\,x\neq 0,\ q(x)=0\},\ V_2=\{x\,|\,q(x)\neq 0\}.$$

The case where $r=0$: Clearly we have $V_1=\phi$, and we see from Lemma 8.19 that V_2 is an open $\tilde{G}_0$-orbit. Hence $V=V_2$ and $\partial V=V_0=\{0\}$. Clearly G satisfies condition (I.2).

The case where $r\geqq 1$: We see from Lemma 8.19 that V_0, V_1, V_2 are all the $\tilde{G}_0$-orbits, and especially V_2 is a single open orbit. Hence $V=V_2$, and $\partial V=V_0\cup V_1$. Moreover it is easy to see that $\mathfrak{G}$ satisfies conditions (I.2) and (I.3).

It remains to consider the case where $\mathfrak{G}=\mathfrak{M}(n, \boldsymbol{C})$. However this case can be similarly dealt with to the case of an indefinite Möbius algebra, and it is shown that $\mathfrak{G}$ satisfies conditions (I.1)$\sim$(I.3). We only remark the following

LEMMA 8.20. *The space* $\mathfrak{g}_{-1}$ *may be naturally identified with* $\boldsymbol{C}^n$, *and the group* $\tilde{G}_0$ *consists of all transformations F of* $\boldsymbol{C}^n$ *of the following form*:

$$F(x)=Ax \text{ or } F(x)=A\bar{x},\ x\in \boldsymbol{C}^n,$$

where $A\in GL(n, \boldsymbol{C})$, *and* ${}^tAA=\lambda 1_n$ *with some* $\lambda\in \boldsymbol{C}$.

We have thereby completed the proof of Proposition 8.1.

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