ON AFFINELY CONNECTED MANIFOLDS WITH HOMOGENEOUS HOLONOMY GROUP $CL(n, Q) \otimes T^1$

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M. Obata has precisely studied (M. Obata, [1], [2], [3]) the manifolds admitting so-called quaternion structures and some remarkable affine connections in such manifolds leaving invariant the quaternion structures. The restricted homogeneous holonomy group of such an affinely connected manifold is the real representation of quaternion linear group $CL(n, Q)$ or one of its subgroups. If the affinely connected manifold is Riemannian and if the quaternion structures are hermitian with respect to the Riemannian metric, then the restricted homogeneous holonomy group of the Riemannian manifold is the real representation of unitary symplectic group $Sp(n)$ or one of its subgroups (M. Berger, [4]; H. Wakakuwa, [5]).

The purpose of this paper is to study the $4n$-dimensional affinely connected manifold whose restricted homogeneous holonomy group is the real representation of $CL(n, Q) \otimes T^1$, where $CL(n, Q)$ is the quaternion linear group operating on a $2n$-dimensional complex linear space and T^1 is the one parameter torus group operating on a complex line.

1. The group $CL(n, Q) \otimes T^1$ and its real representation. Let Q be the quaternion algebra with bases $1, i, j, k$ ($i^2 = j^2 = k^2 = -1$; $ij = -ji = k, jk = -kj = i, ki = -ik = j$) and let L_Q^n be the n-dimensional quaternic linear space composed of all vectors whose n components are elements of Q.

A linear endomorphism of L_Q^n from the left is given by

$$\mathbf{q}' = F\mathbf{q},$$

where $\mathbf{q}$ is a vector in L_Q^n and F is an (n, n)-matrix with elements in Q. If we put

$$\mathbf{q} = \mathbf{z} + j\mathbf{w},\ \mathbf{q}' = \mathbf{z}' + j\mathbf{w}',\ F = P_n + jQ_n,$$

then we get the complex representation of the above endomorphism whose representative matrix is of the form:

$$(1.1) \qquad \begin{pmatrix} P_n & -\overline{Q}_n \\ Q_n & \overline{P}_n \end{pmatrix}$$

where P_n and Q_n are complex (n, n)-matrices and the bar denotes the complex conjugate.

$CL(n, Q)$ is the subgroup of $GL(2n, C)$ composed of all non-singular linear homogeneous transformations whose matrices are of the form (1.1).

Next, T^1 is the one dimensional torus group on a complex line, its transformations being of the form

$$z' = \sigma z,$$

where z, z' and σ are complex numbers and $|\sigma| = 1$. Then the Kroneckerian product $CL(n, Q) \otimes T^1$ is also a subgroup of $GL(2n, C)$ and it is easily seen that the matrix M_{2n} of transformation of $CL(n, Q) \otimes T^1$ has the form

$$M_{2n} = \begin{pmatrix} \sigma P_n & -\sigma \overline{Q}_n \\ \sigma Q_n & \sigma \overline{P}_n \end{pmatrix}, \qquad (|\sigma| = 1). \tag{1.2}$$

Now, if we put

$$J_{2n} = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} \qquad (E_n : \text{unit matrix of degree } n), \tag{1.3}$$

then we have

$$\frac{1}{\sigma} M_{2n} J_{2n} = \frac{1}{\overline{\sigma}} J_{n2} \overline{M}_{2n},$$

that is

$$M_{2n} J_{2n} = \rho J_{2n} \overline{M}_{2n} \tag{1.4}$$

where $\rho = \sigma^2$.

Conversely, if M_{2n} satisfies (1.4) for J_{2n} of (1.3), then we see that M_{2n} must be of the form (1.2), that is, it gives a transformation of $CL(n, Q) \otimes T^1$.

Now, let z be a vector in a $2n$-dimensional complex linear space and consider a linear transformaton

$$\begin{cases} z' = M_{2n} z \\ \overline{z}' = \overline{M}_{2n} \overline{z} \end{cases}$$

where M_{2n} is given by (1.2), the matrix $\mathfrak{M}$ of the above transformation being

$$\mathfrak{M} = \begin{pmatrix} M_{2n} & 0 \\ 0 & \overline{M}_{2n} \end{pmatrix}. \tag{1.5}$$

If we put

$$z = x + iy,\ z' = x' + iy',\ M_{2n} = H_{2n} + iK_{2n},$$

then we get the real representation of $CL(n, Q) \otimes T^1$ operating on the $4n$-dimensional real linear space R_{4n}, where x, y, x' and y' are real vectors and H_{2n} and K_{2n} are real matrices of degree $2n$. For the sake of brevity we denote this real group by G.

The matrix $\mathfrak{M}$ of (1.5) gives the matrix of a transformation of G = real representation of $CL(n, Q) \otimes T^1$ with respect to complex bases.

A simple computation gives us

$$\mathfrak{M}\begin{pmatrix} -iE_{2n} & 0 \\ 0 & iE_{2n} \end{pmatrix}\mathfrak{M}^{-1} = \begin{pmatrix} -iE_{2n} & 0 \\ 0 & iE_{2n} \end{pmatrix} \quad (E_{2n}\text{: unit matrix of degree } 2n),$$

and further by (1.4) we have

$$\mathfrak{M}\begin{pmatrix} 0 & J_{2n} \\ 0 & 0 \end{pmatrix} = \rho\begin{pmatrix} 0 & J_{2n} \\ 0 & 0 \end{pmatrix}\mathfrak{M}$$

or

$$(1.6) \qquad \mathfrak{M}\begin{pmatrix} 0 & J_{2n} \\ 0 & 0 \end{pmatrix}\mathfrak{M}^{-1} = \rho\begin{pmatrix} 0 & J_{2n} \\ 0 & 0 \end{pmatrix}, \qquad (|\rho| = 1)$$

and similarly

$$(1.7) \qquad \mathfrak{M}\begin{pmatrix} 0 & 0 \\ iJ_{2n} & 0 \end{pmatrix}\mathfrak{M}^{-1} = \bar{\rho}\begin{pmatrix} 0 & 0 \\ iJ_{2n} & 0 \end{pmatrix}.$$

These tell us that G leaves invariant the matrix of rank $4n$ $\begin{pmatrix} -iE_{2n} & 0 \\ 0 & iE_{2n} \end{pmatrix}$ and transforms the matrices of rank $2n$ $\begin{pmatrix} 0 & J_{2n} \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ iJ_{2n} & 0 \end{pmatrix}$ into the matrices proportional to them, the proportional factors being ρ and $\bar{\rho}$ ($|\rho| = 1$) respectively.

Making use of a complex matrix

$$I_{4n} = \frac{1}{\sqrt{2}}\begin{pmatrix} E_{2n} & E_{2n} \\ -iE_{2n} & iE_{2n} \end{pmatrix}$$

we have

$$(1.8) \qquad M \equiv I_{4n}\,\mathfrak{M}\,I_{4n}^{-1} = \begin{pmatrix} H_{2n} & -K_{2n} \\ K_{2n} & H_{2n} \end{pmatrix},$$

where H_{2n} and K_{2n} are real matrices of degree $2n$, and M gives a transformation of G with respect to real bases. By the transformation (1.8), the matrix

$$I_{4n}\begin{pmatrix} -iE_{2n} & 0 \\ 0 & iE_{2n} \end{pmatrix}I_{4n}^{-1} = \begin{pmatrix} 0 & E_{2n} \\ -E_{2n} & 0 \end{pmatrix}$$

is left invariant and

$$I_{4n}\begin{pmatrix} 0 & J_{2n} \\ 0 & 0 \end{pmatrix}I_{4n}^{-1} = \frac{1}{2}\begin{pmatrix} J_{2n} & -iJ_{2n} \\ -iJ_{2n} & -J_{2n} \end{pmatrix}$$

and

$$I_{4n}\begin{pmatrix} 0 & 0 \\ iJ_{2n} & 0 \end{pmatrix} I_{4n}^{-1} = \frac{1}{2}\begin{pmatrix} iJ_{2n} & -J_{2n} \\ -J_{2n} & -iJ_{2n} \end{pmatrix}$$

are transformed into themselves up to complex factors ρ and $\bar{\rho}$ ($|\rho| = 1$) respectively. That is to say, if we put

$$(1.9) \qquad \overset{(1)}{\mathfrak{F}} = \frac{1}{2}\begin{pmatrix} J_{2n} & -iJ_{2n} \\ -iJ_{2n} & -J_{2n} \end{pmatrix}, \overset{(2)}{\mathfrak{F}} = \frac{1}{2}\begin{pmatrix} iJ_{2n} & -J_{2n} \\ -J_{2n} & -iJ_{2n} \end{pmatrix}, \overset{(3)}{\mathfrak{F}} = \begin{pmatrix} 0 & E_{2n} \\ -E_{2n} & 0 \end{pmatrix},$$

then we have

$$(1.10) \qquad \begin{cases} M\overset{(1)}{\mathfrak{F}}M^{-1} = \rho\overset{(1)}{\mathfrak{F}}, \qquad M\overset{(2)}{\mathfrak{F}}M^{-1} = \rho\overset{(2)}{\mathfrak{F}} \; (|\rho| = 1) \\ \qquad M\overset{(3)}{\mathfrak{F}}M^{-1} = \overset{(3)}{\mathfrak{F}} \end{cases}$$

where $\overset{(1)}{\mathfrak{F}}$ and $\overset{(2)}{\mathfrak{F}}$ are of rank $2n$ and $\overset{(3)}{\mathfrak{F}}$ are of rank $4n$.

Conversely, a matrix M which transforms $\overset{(1)}{\mathfrak{F}}, \overset{(2)}{\mathfrak{F}}$ and $\overset{(3)}{\mathfrak{F}}$ of (1.9) by (1.10) gives a transformation of G, which is easily seen by a consideration with respect to the complex bases.

If we put

$$(1.11) \qquad \begin{cases} \overset{(1)}{F} \equiv i\overset{(1)}{\mathfrak{F}} - \overset{(2)}{\mathfrak{F}} = \begin{pmatrix} 0 & J_{2n} \\ J_{2n} & 0 \end{pmatrix}, \\ \overset{(2)}{F} \equiv \overset{(1)}{\mathfrak{F}} - i\overset{(2)}{\mathfrak{F}} = \begin{pmatrix} J_{2n} & 0 \\ 0 & -J_{2n} \end{pmatrix}, \\ \overset{(3)}{F} \equiv \overset{(2)}{\mathfrak{F}} \qquad = \begin{pmatrix} 0 & E_{2n} \\ -E_{2n} & 0 \end{pmatrix}, \end{cases}$$

then we find the quaternic relations among $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$:

$$(1.12) \qquad \begin{cases} \overset{(1)(2)}{FF} = -\overset{(2)(1)}{FF} = \overset{(3)}{F}, \; \overset{(2)(3)}{FF} = -\overset{(3)(2)}{FF} = \overset{(1)}{F}, \; \overset{(3)(1)}{FF} = -\overset{(1)(3)}{FF} = \overset{(2)}{F} \\ \qquad \overset{(1)}{F^2} = \overset{(2)}{F^2} = \overset{(3)}{F^2} = -E_{4n}. \end{cases}$$

From (1.11), we get

$$(1.13) \qquad \overset{(1)}{\mathfrak{F}} = \frac{1}{2}(\overset{(2)}{F} - i\overset{(1)}{F}), \quad \overset{(2)}{\mathfrak{F}} = \frac{1}{2}(i\overset{(2)}{F} - \overset{(1)}{F}), \quad \overset{(3)}{\mathfrak{F}} \equiv \overset{(3)}{F},$$

and putting these $\overset{(1)}{\mathfrak{F}}, \overset{(2)}{\mathfrak{F}}$ and $\overset{(3)}{\mathfrak{F}}$ into (1.10), we have

$$(1.14) \qquad M\overset{(1)}{F}M^{-1} = a\overset{(1)}{F} - b\overset{(2)}{F}, \quad M\overset{(2)}{F}M^{-1} = b\overset{(1)}{F} + a\overset{(2)}{F}, \quad M\overset{(3)}{F}M^{-1} = \overset{(3)}{F},$$

where $\rho = a + ib$ ($a^2 + b^2 = 1$).

That is to say, G transforms $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ of (1.12) by (1.14) and conversely, a transformation which transforms $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ by (1.14) belongs to G.

Summing up the above considerations we have the following two Lemmas

LEMMA 1.1. *With respect to a suitable complex bases the necessary* *and sufficient condition that a complex matrix* $\mathfrak{M}$ *gives a transformation of* $G =$ *real representation of* $CL(n, Q) \otimes T^1$ *is that it leaves invariant the matrix* $\begin{pmatrix} -iE_{2n} & 0 \\ 0 & iE_{2n} \end{pmatrix}$ (E_{2n} : *unit matrix of degree* $2n$) *and transforms the matrices* $\begin{pmatrix} 0 & J_{2n} \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ iJ_{2n} & 0 \end{pmatrix}$, $\left(J_{2n} = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}\right)$, *by* (1.6) *and* (1.7) *where* $\rho^{\frac{1}{2}}$ *is the complex number giving rise the transformation of* T^1.

If G *is the real representation of* $CL(n, Q)$, *then* $\rho = 1$ *and vice versa.*

LEMMA 1.2. *With respect to a suitable real bases, the necessary and sufficient condition that a matrix* M *gives a transformation of* $G =$ *real representation of* $CL(n, Q) \otimes T^1$ *is that it transform the matrices* $\overset{(1)}{\mathfrak{F}}, \overset{(2)}{\mathfrak{F}}$ *and* $\overset{(3)}{\mathfrak{F}}$ *of* (1.9) *by* (1.10) *or it transforms he matrices* $\overset{(1)}{F}, \overset{(2)}{F}$ *and* $\overset{(3)}{F}$ *of* (1.11) *by* (1.14). *If* G *is the real representation of* $CL(n, Q)$, *then* $a = 1$, $b = 0$ *and vice versa.*

It is remarked that the above $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ necessarily satisfy (1.12).

We shall prove furthermore the following Lemma of normalization.

LEMMA 1.3. *Let* $\overset{(1)}{F} = (\overset{(1)}{F^i}_j)$, $\overset{(2)}{F} = (\overset{(2)}{F^i}_j)$, $\overset{(3)}{F} = (\overset{(3)}{F^i}_j)$ *be three matrices satisfying* (1.12), *that is*

$$\text{(A)} \qquad \overset{(1)(2)}{FF} = -\overset{(2)(1)}{FF} = \overset{(3)}{F},\ \overset{(2)(3)}{FF} = -\overset{(3)(2)}{FF} = \overset{(1)}{F},\ \overset{(1)(3)}{FF} = -\overset{(3)(1)}{FF} = \overset{(2)}{F},$$

$$\text{(B)} \qquad \overset{(1)}{F^2} = \overset{(2)}{F^2} = \overset{(3)}{F^2} = -E_{4n}.$$

Then we can choose their components in normal forms (1.11), *that is*

$$\text{(C)} \qquad \overset{(1)}{F} = \begin{pmatrix} 0 & J_{2n} \\ J_{2n} & 0 \end{pmatrix},\ \overset{(2)}{F} = \begin{pmatrix} J_{2n} & 0 \\ 0 & -J_{2n} \end{pmatrix},\ \overset{(3)}{F} = \begin{pmatrix} 0 & E_{2n} \\ -E_{2n} & 0 \end{pmatrix},$$

by performing a suitable change of bases.

PROOF. Let L_{4n} be a $4n$-dimensional real linear space with coordinate system $(u^1, \dots\dots, u^{4n})$ and introduce in L_{4n} an Euclidean metric $G = (G_{ij})$ defined by

$$G_{ij} = \delta_{ij} + \overset{(1)}{F^k}_i \overset{(1)}{F^k}_j + \overset{(2)}{F^k}_i \overset{(2)}{F^k}_j + \overset{(3)}{F^k}_i \overset{(3)}{F^k}_j.$$

Then, it is easily verified that G is positive definite and satisfy

$$\text{(D)} \qquad {}^t\overset{(1)}{F}G\overset{(1)}{F} = G,\ {}^t\overset{(2)}{F}G\overset{(2)}{F} = G,\ {}^t\overset{(3)}{F}G\overset{(3)}{F} = G \qquad ({}^t\overset{(1)}{F} = \text{transpose of } \overset{(1)}{F}, \text{ etc.})$$

that is, G is hermitian with respect to $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$.

It is already known that three matrices $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ satisfying (A), (B) and (D) for a positive definite metric G can be turned into their normal forms (C) by a suitable change of bases (H. Wakakuwa, [5], Theorem 1) but we will sketch the outline of the proof.

At first if we put

$$G_{ik}\overset{(1)}{F}{}^{k}{}_{j} = \overset{(1)}{F}_{ij},\ G_{ik}\overset{(2)}{F}{}^{k}{}_{j} = \overset{(2)}{F}_{ij},\ G_{ik}\overset{(3)}{F}{}^{k}{}_{j} = \overset{(3)}{F}_{ij},$$

then $\overset{(1)}{F}_{ij}$, $\overset{(2)}{F}_{ij}$, $\overset{(3)}{F}_{ij}$ are anti-symmetric in i and j by virtue of (B) and (D). Let $\mathbf{u} = (u^i)$ be an arbitrary non-zero vector in L_{4n}, then $\mathbf{u}, \overset{(1)}{F}\mathbf{u}$, $\overset{(2)}{F}\mathbf{u}$ and $\overset{(3)}{F}\mathbf{u}$ are mutually orthogonal by virtue of (A) and the anti-symmetricity of $\overset{(1)}{F}_{ij}$, $\overset{(2)}{F}_{ij}$ and $\overset{(3)}{F}_{ij}$. And furthermore, let $\mathbf{v} = (v^i)$ be an arbitrary non-zero vector orthogonal to the above four vectors $\mathbf{u}$, $\overset{(1)}{F}\mathbf{u}, \overset{(2)}{F}\mathbf{u}$ and $\overset{(3)}{F}\mathbf{u}$, then the 8 vectors $\mathbf{u}$, $\overset{(1)}{F}\mathbf{u}, \ldots\ldots, \overset{(3)}{F}\mathbf{v}$ are all mutually orthogonal, and so on.

Now, we choose the bases $e_1, \ldots\ldots, e_{4n}$ as follows. Choose an arbitrary vector as e_1 and since the three vectors obtained from e_1 by performing the collineations given by $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ are mutually orthogonal, we choose the three vectors as $-e_{3n+1}$, $-e_{n+1}$ and $-e_{2n+1}$. Then, with respect to such system of bases, we see that

$$\overset{(1)}{F}{}^{3n+1}{}_{1} = \overset{(2)}{F}{}^{n+1}{}_{1} = \overset{(3)}{F}{}^{2n+1}{}_{1} = -1,$$

and the other $\overset{(1)}{F}{}^{i}{}_{1}, \overset{(2)}{F}{}^{i}{}_{1}$ and $\overset{(3)}{F}{}^{i}{}_{1}$ are all zero. In the next place, choose an arbitrary vector orthogonal to e_1, e_{n+1}, e_{2n+1} and e_{3n+1} as e_2. Since the three vectors obtained from e_2 by performing the collineations given by $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ are mutually orthogonal and orthogonal to e_1, e_{n+1}, e_{2n+1} and e_{3n+1}, we choose the three vectors as $-e_{3n+2}$, $-e_{n+2}$ and $-e_{2n+2}$. Then with respect to such system of bases we see that

$$\overset{(1)}{F}{}^{2n+1}{}_{2} = \overset{(2)}{F}{}^{n+1}{}_{2} = \overset{(3)}{F}{}^{2n+1}{}_{2} = -1,$$

and the other $\overset{(1)}{F}{}^{i}{}_{2}$, $\overset{(2)}{F}{}^{i}{}_{2}$ and $\overset{(3)}{F}{}^{i}{}_{2}$ are all zero. Repeating the similar processes we get a system of orthogonal bases and with respect to such system of bases, $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ take the following form:

$$\overset{(1)}{F} = \begin{pmatrix} 0 & 0 & 0 & E_n \\ 0 & & & \\ 0 & & * & \\ -E_n & & & \end{pmatrix},\ \overset{(2)}{F} = \begin{pmatrix} 0 & E_n & 0 & 0 \\ -E_n & & & \\ 0 & & ** & \\ 0 & & & \end{pmatrix},\ \overset{(3)}{F} = \begin{pmatrix} 0 & 0 & E_n & 0 \\ 0 & & & \\ -E_n & & \overset{*}{**} & \\ 0 & & & \end{pmatrix}.$$

We can find the elements of the parts of $*, **$ and $\overset{*}{**}$ by means of (A) and (B) and the verification shows that the forms of $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ are no other than the required normal forms. q. e. d.

Now, we consider the infinitesimal transformations of the group G = real representation of $CL(n, Q) \otimes T^1$, which are the real representations of infinitesimal transformations of $CL(n, Q) \otimes T^1$.

Let σ ($|\sigma| = 1$) be a complex number which gives an infinitesimal transformation of T^1 : $z' = \sigma z_1$ then

$$\sigma = 1 + d\sigma,$$

where $d\sigma$ is a complex number infinitely near to 0 and since $|\sigma| = 1$, it must be necessarily purely imaginary: $d\sigma = i\alpha$. That is,

$$\sigma = 1 + i\alpha \qquad (\alpha\text{: infinitesimal real number}). \tag{1.15}$$

Since the ρ in (1.10) is given by $\rho = \sigma^2$ from (1.4), we have

$$\rho = \sigma^2 = 1 + 2i\alpha \qquad (\alpha\text{: inf. real}).$$

If we therefore denote an infinitesimal transformation of G by $I + dI$ (I: identity), then we get

$$dI(\overset{(1)}{\mathfrak{F}}) = i\varepsilon \overset{(1)}{\mathfrak{F}},\ dI(\overset{(2)}{\mathfrak{F}}) = -i\varepsilon \overset{(2)}{\mathfrak{F}},\ dI(\overset{(3)}{\mathfrak{F}}) = 0 \tag{1.16}$$

$$(\varepsilon = 2\alpha\text{: inf. real number}),$$

by virtue of (1.10), or

$$dI(\overset{(1)}{F}) = -\varepsilon \overset{(2)}{F},\ dI(\overset{(2)}{F}) = \varepsilon \overset{(1)}{F},\ dI(\overset{(3)}{F}) = 0, \tag{1.17}$$

by virtue of (1.14).

Conversely, an infinitesimal transformation satisfying (1.16) or (1.17) is an infinitesimal transformation of G.

LEMMA 1.4. *The necessary and sufficint condition that an infinitesimal transformation $I + dI$ (I: identity) belongs to the group G which is the real rep. of $CL(n\ Q) \otimes T^1$ is that dI transforms $\overset{(1)}{\mathfrak{F}}, \overset{(2)}{\mathfrak{F}}$ and $\overset{(3)}{\mathfrak{F}}$ by* (1.16); *or what is the same, dI transforms $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ given by* (1.11) *by* (1.17) *with respect to a suitable system of bases. Especially, the infinitesimal transformation in consideration belongs to the group which is the real representation of $CL(n, Q)$ if and only if $\varepsilon = 0$.*

From (1.2), (1.5) and (1.8), we get easily the following Lemma.

LEMMA 1.5. *With respect to a suitable system of bases, the necessary and sufficient condition that an infinitesimal transformation belongs to $G = CL(n, Q) \otimes T^1$ is that it is given by a matrix of the form:*

$$\begin{pmatrix} A & -B & -C-\alpha E & -D \\ B & A & -D & C-\alpha E \\ C+\alpha E & D & A & -B \\ D & -C+\alpha E & B & A \end{pmatrix} \tag{1.18}$$

where $A - E, B, C$ and D are real matrices of degree n whose elements are *infinitesimal real numbers and* α *is an infinitesimal real number.*

2. Fundamental characterizations. In the following, the indices run over 1, 2, 3. …… $4n$, if othewise stated.

THEOREM 2. 1. *Let* A_{4n} *be a* $4n$*-dimensional affinely connected manifold (with torsion or without torsion). If the restricted homogeneous holonomy group* H *of* A_{4n} *be the real representation of* $G = CL(n, Q) \otimes T^1$ *or one of its subgroups, then* A_{4n} *admits three almost complex structures* $\overset{(1)}{F}{}^i{}_j$, $\overset{(2)}{F}{}^i{}_j$ *and* $\overset{(3)}{F}{}^i{}_j$ *satisfying quaternic relations*:

$$\overset{(1)}{F}{}^i{}_k \overset{(2)}{F}{}^k{}_j = - \overset{(2)}{F}{}^i{}_k \overset{(1)}{F}{}^k{}_j = \overset{(3)}{F}{}^i{}_j,\ \overset{(2)}{F}{}^i{}_k \overset{(3)}{F}{}^k{}_j = - \overset{(3)}{F}{}^i{}_k \overset{(2)}{F}{}^k{}_j = \overset{(1)}{F}{}^i{}_j,$$
$$\overset{(3)}{F}{}^i{}_k \overset{(1)}{F}{}^k{}_j = - \overset{(1)}{F}{}^i{}_k \overset{(3)}{F}{}^k{}_j = \overset{(2)}{F}{}^i{}_j \tag{2. 1}$$

and

$$\overset{(1)}{F}{}^i{}_{j/k} = - \varphi_k \overset{(2)}{F}{}^i{}_j,\ \overset{(2)}{F}{}^i{}_{j/k} = \varphi_k \overset{(1)}{F}{}^i{}_j,\ \overset{(3)}{F}{}^i{}_{j/k} = 0, \tag{2. 2}$$

where / denotes covariant differentiation with respect to the affine connection of A_{4n} *and* φ_k *is a covariant vector field.*

Conversely, if A_{4n} *admits three almost complex structures satisfying* (2. 1) *and* (2. 2), *then the restricted homogeneous holonomy group of* A_{4n} *is the real representation of* $CL(n, Q) \otimes T^1$ *or one of its subgroups.*

PROOF. At first assume that A_{4n} be simply connected. If we attach a suitable frame R_0 at a point O of A_{4n}, then the restricted homogeneous holonomy group $H(O)$ which is a real representation of $CL(n, Q) \otimes T^1$ or one of its subgroups transforms the three matrices $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ satisfying (2. 1) and $\overset{(1)}{F}{}^2 = \overset{(2)}{F}{}^2 = \overset{(3)}{F}{}^2 = - E_{4n}$ according to (1. 14), taking account of Lemma 1. 2. And we attach to each point of P of A_{4n} a frame obtained from R_0 by parallel translation along an arbitrary but fixed curve joining O to P. Then at each P there are uniquely determined three tensors $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ which are transformed according to (1. 14) by the restricted homogeneous holonomy group $H(P)$ at P. The connection of the tangent spaces at infinitely near two points P and P' is given by an infinitesimal transformation of $H(P)$, which is easily verified by considering the above fixed curve $\widehat{OP}$ and $\widehat{OP'}$ and a closed curve $POP'P$. Hence (2. 2) holds true. When A_{4n} is not simply connected, we consider the universal covering manifold $\widetilde{A}_{4n}$ of A_{4n} admitting the affine connection introduced naturally from that of A_{4n}, then the conclusion for the $\widetilde{A}_{4n}$ induces naturally the same conclusion for A_{4n}.

The sufficiency follows from Lemma 1. 3 and 1. 4. q. e. d.

COROLLARY 2.1. *The affinely connected manifold A_{4n} in the Theorem admits three complex tensor fields $\overset{(1)}{\mathfrak{F}}{}^i{}_j, \overset{(2)}{\mathfrak{F}}{}^i{}_j$ and $\overset{(3)}{\mathfrak{F}}{}^i{}_j$ satisfying*

$$\overset{(1)}{\mathfrak{F}}{}^i{}_{j|k} = i\varphi_k \overset{(1)}{\mathfrak{F}}{}^i{}_j,\ \overset{(2)}{\mathfrak{F}}{}^i{}_{j|k} = -i\varphi_k \overset{(2)}{\mathfrak{F}}{}^i{}_j,\ \overset{(3)}{\mathfrak{F}}{}^i{}_{j|k} = 0,$$

where

$$\overset{(1)}{\mathfrak{F}}{}^i{}_j = \frac{1}{2}(\overset{(2)}{F}{}^i{}_j - i\overset{(1)}{F}{}^i{}_j),\ \overset{(2)}{\mathfrak{F}}{}^i{}_j = \frac{1}{2}(i\overset{(2)}{F}{}^i{}_j - \overset{(1)}{F}{}^i{}_j),\ \overset{(3)}{\mathfrak{F}}{}^i{}_j = \overset{(3)}{F}{}^i{}_j.$$

The proof is almost evident from (1.13), (1.16) and (1.17).

COROLLARY 2.2. *If a 4n-dimensional affinely connected manifold A_{4n} admits two almost complex structures $\overset{(1)}{F}{}^i{}_j$ and $\overset{(2)}{F}{}^i{}_j$ satisfying*

$$\overset{(1)}{F}{}^i{}_k \overset{(2)}{F}{}^k{}_j = -\overset{(2)}{F}{}^i{}_k \overset{(1)}{F}{}^k{}_j,\ \overset{(1)}{F}{}^i{}_{j|k} = -\varphi_k \overset{(2)}{F}{}^i{}_j,\ \overset{(2)}{F}{}^i{}_{j|k} = \varphi_k \overset{(1)}{F}{}^i{}_j,$$

then the restricted homogeneous holonomy group is the real representation of $CL(n, Q) \otimes T^1$ or one of its subgroups.

PROOF. Put

$$\overset{(1)}{F}{}^i{}_k \overset{(2)}{F}{}^k{}_j = -\overset{(2)}{F}{}^i{}_k \overset{(1)}{F}{}^k{}_j \equiv \overset{(3)}{F}{}^i{}_j,$$

then $\overset{(3)}{F}{}^i{}_j$ is also an almost complex structure and there are relations (2.1) among $\overset{(1)}{F}{}^i{}_j, \overset{(2)}{F}{}^i{}_j$ and $\overset{(3)}{F}{}^i{}_j$. Further, we see that

$$\overset{(3)}{F}{}^i{}_{j|k} = (\overset{(1)}{F}{}^i{}_h \overset{(2)}{F}{}^h{}_j)_{|k} = \overset{(1)}{F}{}^i{}_{h|k} \overset{(2)}{F}{}^h{}_j + \overset{(1)}{F}{}^i{}_h \overset{(2)}{F}{}^h{}_{j|k} = \varphi_k \delta^i_j - \varphi_k \delta^i_j = 0,$$

by virtue of the assumptions for $\overset{(1)}{F}{}^i{}_j$ and $\overset{(2)}{F}{}^i{}_j$ and $\overset{(1)}{F}{}^i{}_k \overset{(1)}{F}{}^k{}_j = \overset{(2)}{F}{}^i{}_k \overset{(2)}{F}{}^k{}_j = -\delta^i_j$. The Theorem completes the proof.

THEOREM 2.2. *In Theorem 2.1, the necessary and sufficient condition that the restricted homogeneous holonomy group H be contained in the real representation of $CL(n, Q)$ is that the vector φ_k be a gradient vector.*

PROOF. If H does not contain the real representation of T^1, then from Lemma 1.1 we see that $\rho^{\frac{1}{2}} = 1$, which shows that H leaves invariant all $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$. In this case, $\varphi_k = 0$ and of course gradient. We will prove the sufficiency. Assume that A_{4n} admit three almost complex structures $\overset{(1)}{F}, \overset{(2)}{F}$ and $\overset{(3)}{F}$ satisfying (2.1) and (2.2) and assume furthermore that the φ_k in (2.2) is gradient, i.e. there exist a scalar function φ such that

$$\varphi_k = \partial\varphi/\partial x^k. \tag{2.3}$$

Then we can find three almost complex structures satisfying the same relations as (2.1) and all invariant by H, that is, covariant constant. In fact, put

$$\overset{(1)}{G}{}^i{}_j = \cos\varphi \cdot \overset{(1)}{F}{}^i{}_j + \sin\varphi \cdot \overset{(2)}{F}{}^i{}_j,$$
$$\overset{(2)}{G}{}^i{}_j = -\sin\varphi \cdot \overset{(1)}{F}{}^i{}_j + \cos\varphi \cdot \overset{(2)}{F}{}_j, \qquad \overset{(3)}{G}{}^i{}_j = \overset{(3)}{F}{}^i{}_j,$$

by using the scalar function φ of (2. 3). Then $\overset{(1)}{G}{}^i{}_j$, $\overset{(2)}{G}{}^i{}_j$ and $\overset{(3)}{G}{}^i{}_j$ are all almost complex structures i. e.

$$\overset{(1)}{G}{}^i{}_k \overset{(1)}{G}{}^k{}_j = \overset{(2)}{G}{}^i{}_k \overset{(2)}{G}{}^k{}_j = \overset{(3)}{G}{}^i{}_k \overset{(3)}{G}{}^k{}_j = -\delta^i_j,$$

and satisfy the quaternic relations

$$\overset{(1)}{G}{}^i{}_k \overset{(2)}{G}{}^k{}_j = -\overset{(2)}{G}{}^i{}_k \overset{(1)}{G}{}^k{}_j = \overset{(3)}{G}{}^i{}_j,\ \overset{(2)}{G}{}^i{}_k \overset{(3)}{G}{}^k{}_j = -\overset{(3)}{G}{}^i{}_k \overset{(2)}{G}{}^k{}_j = \overset{(1)}{G}{}^i{}_j,$$
$$\overset{(3)}{G}{}^i{}_k \overset{(1)}{G}{}^k{}_j = -\overset{(1)}{G}{}^i{}_k \overset{(3)}{G}{}^k{}_j = \overset{(2)}{G}{}^i{}_j,$$

which are easily verified by (2. 1) and by the fact that $\overset{(1)}{F}{}^i{}_j$, $\overset{(2)}{F}{}^i{}_j$ and $\overset{(3)}{F}{}^i{}_j$ are almost complex structures. And lastly, we see that

$$\overset{(1)}{G}{}^i{}_{j|k} = 0,\ \overset{(2)}{G}{}^i{}_{j|k} = 0,\ \overset{(3)}{G}{}^i{}_{j|k} = 0,$$

which completes the proof.

3. Some identities. In this section we introduce some identities obtained from the Ricci's identities for $\overset{(1)}{F}{}^i{}_j$, $\overset{(2)}{F}{}^i{}_j$ and $\overset{(3)}{F}{}^i{}_j$

We have already known that

$$(3.1)\quad \begin{cases} \overset{(1)}{F}{}^i{}_k \overset{(1)}{F}{}^k{}_j = \overset{(2)}{F}{}^i{}_k \overset{(2)}{F}{}^k{}_j = \overset{(3)}{F}{}^i{}_k \overset{(3)}{F}{}^k{}_j = -\delta^i_j, \\ \overset{(1)}{F}{}^i{}_k \overset{(2)}{F}{}^k{}_j = -\overset{(2)}{F}{}^i{}_k \overset{(1)}{F}{}^k{}_j = \overset{(3)}{F}{}^i{}_j,\ \overset{(2)}{F}{}^i{}_k \overset{(3)}{F}{}^k{}_j = -\overset{(3)}{F}{}^i{}_k \overset{(2)}{F}{}^k{}_j = \overset{(1)}{F}{}^i{}_j, \\ \overset{(3)}{F}{}^i{}_k \overset{(1)}{F}{}^k{}_j = -\overset{(1)}{F}{}^i{}_k \overset{(3)}{F}{}^k{}_j = \overset{(2)}{F}{}^i{}_j, \end{cases}$$

and

$$(3.2)\qquad \overset{(1)}{F}{}^i{}_{i|k} = -\varphi_k \overset{(2)}{F}{}^i{}_j,\ \overset{(2)}{F}{}^i{}_{j|k} = \varphi_k \overset{(1)}{F}{}^i{}_j,\ \overset{(3)}{F}{}^i{}_{i|k} = 0.$$

Differentiating covariantly the first relation of (3. 2) and making use of the second, we have

$$\overset{(1)}{F}{}_{j|k|h} = -\varphi_{k|h} \overset{(2)}{F}{}^i{}_j - \varphi_k \varphi_h \overset{(1)}{F}{}^i{}_j.$$

Consequently we see that

$$(3.3)\qquad -(\varphi_{k|h} - \varphi_{h|k}) \overset{(2)}{F}{}^i{}_j = \overset{(1)}{F}{}^l{}_j R^i{}_{lkh} - \overset{(1)}{F}{}^i{}_l R^l{}_{jkh} + 2\overset{(2)}{F}{}^i{}_j \varphi_l S^l_{kh},$$

by virtue of the Ricci's identity

$$\overset{(1)}{F}{}^i{}_{j|k|h} - \overset{(1)}{F}{}^i{}_{j|h|k} = \overset{(1)}{F}{}^l{}_j R^i{}_{lkh} - \overset{(1)}{F}{}^i{}_l R^l{}_{jkh} - 2\overset{(1)}{F}{}^i{}_{j|l} S^l_{kh},$$

where $R^i{}_{jkh}$ is the curvature tensor obtained from the connection parameter Γ^i_{jk} of A_{4n} and S^l_{kh} is the torsion tensor, i. e.

$$R^i{}_{jkh} = \frac{\partial \Gamma^i_{jk}}{\partial x^h} - \frac{\partial \Gamma^i_{jh}}{\partial x^k} + \Gamma^l_{jk}\Gamma^i_{lh} - \Gamma^l_{jh}\Gamma^i_{lk},\ S^l_{kh} = \frac{1}{2}(\Gamma^i_{kh} - \Gamma^l_{hk}).$$

Multiplying $\overset{(1)}{F}{}^j{}_m$ to the both sides of (3. 3) and contracting with respect to j and making use of (3. 1), we have

$$(\varphi_{k|h} - \varphi_{h|k})\overset{(3)}{F}{}^i{}_m = - R^i{}_{mkh} - \overset{(1)}{F}{}^i{}_l\overset{(1)}{F}{}^j{}_m R^l{}_{jkh} - 2\overset{(3)}{F}{}^i{}_m\varphi_l S^l_{kh},$$

or renewing some indices

$$[(\varphi_{k|h} - \varphi_{h|k}) + 2\varphi_l S^l_{kh}]\overset{(3)}{F}{}^i{}_j + R^i{}_{jkh} + \overset{(1)}{F}{}^i{}_l\overset{(1)}{F}{}^m{}_j R^l{}_{mkh} = 0. \tag{3. 4}$$

And similarly from

$$(\varphi_{k|h} - \varphi_{h|k})\overset{(1)}{F}{}^i{}_j = \overset{(2)}{F}{}^l{}_j R^i{}_{lkh} - \overset{(2)}{F}{}^i{}_l R^l{}_{jkh} - 2\overset{(1)}{F}{}^i{}_j\varphi_l S^l_{kh}. \tag{3. 5}$$

we have

$$[(\varphi_{k|h} - \varphi_{h|k}) + 2\varphi_l S^l_{kh}]\overset{(3)}{F}{}^i{}_j + R^i{}_{jkh} + \overset{(2)}{F}{}^i{}_l\overset{(2)}{F}{}^m{}_j R^l{}_{mkh} = 0. \tag{3. 6}$$

For the covariant constant $\overset{(3)}{F}{}^i{}_j$, it is known that

$$\overset{(3)}{F}{}^l{}_j R^i{}_{lkh} - \overset{(3)}{F}{}^i{}_l R^l{}_{jkh} = 0, \tag{3. 7}$$

or

$$R^i{}_{jkh} + \overset{(3)}{F}{}^i{}_l\overset{(3)}{F}{}^m{}_j R^l{}_{mkh} = 0, \tag{3. 8}$$

which is proved by $\overset{(3)}{F}{}^i{}_{j|k} = 0$ and the Ricci's identity.

Multiplying $\overset{(3)}{F}{}^i{}_j$ to (3. 4) and contracting in i and j, we get

$$- 4n(\varphi_{k|h} - \varphi_{h|k} + 2\varphi_l S^l_{kh}) + \overset{(3)}{F}{}^m{}_l R^l{}_{mkh} + \overset{(3)}{F}{}^m{}_l R^l{}_{mkh} = 0$$

or

$$\varphi_{k|h} - \varphi_{h|k} + 2\varphi_l S^l_{kh} = \frac{1}{2n}\overset{(3)}{F}{}^m{}_l R^l{}_{mkh}, \tag{3. 9}$$

by virtue of (3. 1).

If especially A_{4n} is *without torsion*, then (3. 3), (3. 4), (3. 5), (3. 6) become the following forms:

$$(\varphi_{k|h} - \varphi_{h|k})\overset{(2)}{F}{}^i{}_j + \overset{(1)}{F}{}^l{}_j R^i{}_{lkh} - \overset{(1)}{F}{}^i{}_l R^l{}_{jkh} = 0, \tag{3. 10}$$

$$(\varphi_{k|h} - \varphi_{h|k})\overset{(3)}{F}{}^i{}_j + R^i{}_{jkh} + \overset{(1)}{F}{}^i{}_l\overset{(1)}{F}{}^m{}_j R^l{}_{mkh} = 0, \tag{3. 11}$$

$$(\varphi_{k|h} - \varphi_{h|k})\overset{(\ )}{F}{}^i{}_j - \overset{(2)}{F}{}^l{}_j R^i{}_{lkh} + \overset{(2)}{F}{}^i{}_l R^l{}_{jkh} = 0, \tag{3. 12}$$

$$(\varphi_{k|h} - \varphi_{h|k})\overset{(3)}{F}{}^i{}_j + R^i{}_{jkh} + \overset{(2)}{F}{}^i{}_l\overset{(2)}{F}{}^m{}_j R^l{}_{mkh} = 0, \tag{3. 13}$$

$$\varphi_{k|h} - \varphi_{h|k} = \frac{1}{2n}\overset{(3)}{F}{}^m{}_l R^l{}_{mkh} \tag{3. 14}$$

A covariant vector φ_k is gradient if and only if

$$\varphi_{k|h} - \varphi_{h|k} + 2\varphi_l S_{kh}^{\ l} = 0.$$

Therefore, from (3. 9) and the results in §2, we see that, *the necessary and sufficient condition that the restricted homogeneons holonomy group* H *of* A_{4n} *(with torsion or without torsion) in consideration be contained in the real representation of* $CL(n, Q)$ *is that*

$$\overset{(3)}{F}{}^m{}_l R^l{}_{mkh} = 0.$$

REMARK. It is known that, if A_{4n} is a metric connection without torsion, then H can not be the real representation of $Sp(n) \otimes T^1$, for $n > 1$, which is the unitary restriction of $CL(n, Q) \otimes T^1$(M. Berger, [9]). This is also proved from metric conditions and the above identities (the proof is omitted). For $n = 1$, however, there exist actually 4-dimensional Riemannian manifolds whose restricted homogeneous holonomy groups are real representations of $Sp(n) \otimes T^1$ or one of its subgroups, whose examples have been already shown by T. Ôtsuki, [10]. The fundamental form ds^2 of such a Riemannian manifold is given by

$$ds^2 = a^2\{du_1^2 + du_2^2 + (b_1^2 + b_2^2 + b_3^2)(du_3^2 + du_4^2) + 2b_1(du_1du_3 + du_2du_4) - 2b_2(du_1du_4 - du_2du_3)\},$$

where $a = a(u)$, $b_1 = b_1(u)$, $b_2 = b_2(u)$ and $b_3 = b_3(u)$ are arbitrary functions of u's.

We shall study in the following paper, the converse problem, that is, in a manifold with quaternion structure to introduce affine connections whose restricted homogeneous holonomy groups are real representations of $CL(n, Q) \otimes T^1$.

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