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ON ALGEBRAIC EXPRESSIONS OF SIGMA FUNCTIONS FOR (n, s) CURVES*

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Dedicated to Masaki Kashiwara on his sixtieth birthday

Abstract. An expression of the multivariate sigma function associated with an (n,s)-curve is given in terms of algebraic integrals. As a corollary the first term of the series expansion around the origin of the sigma function is directly proved to be Schur function determined from the gap sequence at infinity.

Key words. Sigma Function, Schur function, gap sequence.

AMS subject classifications. 14H42, 14H55, 14H70

1. Introduction. One of the prominent features of Weierstrass' elliptic sigma functions is their algebraic nature directly related to the defining equation of the elliptic curve. Klein [19, 20] extended the elliptic sigma functions to the case of hyperelliptic curves from this point of view. Since they are defined, it had been one of the central problems to determine the coefficients of the series expansion of the sigma functions. This problem was studied mainly by making linear differential equations satisfied by sigma functions [28, 29, 4, 5, 6, 7], by making non-linear equations [3] for genus two and by using algebraic expressions [19, 20, 13]. To determine a solution of linear differential equations it is necessary to specify an initial condition which requires separate consideration. Therefore the expansion was mainly studied for the sigma functions with non-singular even half periods as characteristics.

Recently Klein's sigma function is further generalized to the case of more general plane algebraic curves called (n, s)-curves by Buchstaber, Enolski and Leykin [9, 10, 11, 12, 8]. They made an important observation that the first term, with respect to certain degree introduced in the theory of soliton equations, of the series expansion of the sigma function, which corresponds to the most singular characteristic, is described by Schur function. Although such connection is expected from the theory of the KP-hierarchy [27, 15], a concrete description of the degeneration of the quasi-periodic solutions to singular curves has not been done before. In order to establish the connection to Schur functions Buchstaber et al. [10] have developed the rational theory of abelian integrals and characterized Schur functions by Riemann's vanishing theorem. Moreover Buchstaber and Leykin [8] have proposed a system of linear differential equations satisfied by sigma functions, which is independent of characteristics. Combining those results is expected to be effective for the further study of sigma functions with singular characteristics. Unfortunately a basic formula (the formula (4) in [11]), on which some of results of [8, 10] including the assertion related to Schur functions mentioned above depend, is not correct.

The purpose of this paper is to generalize Klein's algebraic formulas for the hyperelliptic sigma functions to the case of (n, s)-curves and is to establish the relation with Schur functions directly, that is, without using the results of [10, 11]. The sigma

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function, in this paper, signifies the sigma function with Riemann's constant as its characteristic [9].

The heart of the algebraic formula for the extended sigma function is in the formula for the elliptic sigma function given by Klein [19]. Therefore let us briefly explain it. Let $\sigma(u)$ and $\wp(u)$ be Weierstrass' sigma and elliptic functions associated with the periods $2\omega_1$, $2\omega_2$. Consider two variables u_1, u_2 and set $p_i = (x_i, y_i) = (\wp(u_i), \wp'(u_i))$. They are points on the elliptic curve $y^2 = 4x^3 - g_2 x - g_3$. By making use of the addition theorem for $\wp(u)$ the bilinear form

$$\widehat{\omega} = \wp(u_2 - u_1) du_1 du_2$$

can be written in an algebraic form as

$$\widehat{\omega} = \frac{2y_1y_2 + 4x_1x_2(x_1 + x_2) - g_2(x_1 + x_2) - 2g_3}{4y_1y_2(x_1 - x_2)^2} dx_1 dx_2.$$

With this $\widehat{\omega}$ the elliptic sigma function is expressed as

$$\sigma(u_2 - u_1) = \frac{x_1 - x_2}{\sqrt{y_1y_2}} \exp\left(\frac{1}{2}\int_{\bar{p}_1}^{\bar{p}_2}\int_{p_1}^{p_2} \widehat{\omega}\right), \tag{1}$$

where $\bar{p}_i = (x_i, -y_i)$. What is remarkable for this formula is that the sigma function is expressed by the algebraic functions x_i, y_i and the integral of the algebraic differential form $\widehat{\omega}$. This formula very clearly manifests the algebraic structure of the sigma function. For example, prescribing degree $2i$ to g_i, one can deduce that the coefficients of the series expansion of $\sigma(u)$ at the origin become homogeneous polynomials of g_2 and g_3 directly from this formula without using differential equations.

For higher genus curves one needs to introduce g variables in the sigma function. Already in the case of genus one it is possible to introduce arbitrary number of variables. In fact the generalized addition formula due to Frobenius and Stickelberger makes it possible to express the "n-point function" in terms of the "2-point function":

$$\sigma\left(\sum_{i=1}^{N}(u_i - v_i)\right) = \frac{\prod_{i,j=1}^{N} \sigma(u_i - v_j) \quad \det\left(\wp^{(i-1)}(u_j)\right)_{1\le i,j\le 2N}}{\prod_{i<j}\sigma(u_i - u_j)\sigma(v_j - v_i)\prod_{i,j=1}^{N}\left(\wp(v_j) - \wp(u_i)\right)},$$

where in the determinant we set $u_{N+j} = -v_j$, $1 \le j \le N$. This formula suggests how one should increase the number of variables in the sigma function in general.

Consider the algebraic curve X defined by

$$y^n - x^s - \sum_{ni+sj<ns} \lambda_{ij} x^i y^j = 0,$$

with n and s being relatively coprime integers satisfying $2 \le n < s$. We call it a (n, s) curve. If it is non-singular its genus is $g = 1/2(n-1)(s-1)$. The sigma function for X is defined as the holomorphic function on $\mathbb{C}^g$ which satisfies certain quasi-periodicity and normalization conditions (see (43), (44)). It can be considered as a holomorphic section of some line bundle on the Jacobian $J(X)$ of X or as a multi-valued holomorphic function on $J(X)$ whose multivalued property is specified by the quasi-periodicity. In turn it can also be considered as a symmetric multi-valued holomorphic function on X^g through Abel-Jacobi map. More generally we construct a symmetric multi-valued holomorphic function on X^N with the required quasi-periodicity properties for any $N \geq 1$.

The building block of the formula is the prime function which is a certain modification of the prime form [17]. It takes a similar form to the right hand side of (1):

$$\tilde{E}(p_1, p_2) = \frac{x(p_2) - x(p_1)}{\sqrt{f_y(p_1) f_y(p_2)}} \exp\left(\frac{1}{2} \sum_{i=1}^{n-1} \int_{p_1^{(i)}}^{p_2^{(i)}} \int_{p_1}^{p_2} \widehat{\omega} \right),$$

for certain algebraic bilinear form $\widehat{\omega}$, where $\{p^{(0)}, ..., p^{(n-1)}\}$ is the inverse image of $p = p^{(0)}$ by the map $x : X \longrightarrow \mathbb{P}^1$, $x : (x, y) \mapsto x$. It is skew symmetric and has the same transformation rule as that of the sigma function when one of the argument goes round cycles of X. Therefore this function can be considered as something like the sigma function restricted to the Abel-Jacobi image of $X \times X$ although the restriction of the sigma function itself vanishes identically. Then the function on X^N $(N \geq 3)$ is constructed in a form suggested by Frobenius-Stickelberger's formula using the "2-point function" $\tilde{E}(p_1, p_2)$. In this way the problem of constructing the sigma function reduces to finding certain meromorphic function on X^N. This problem is solved for (n, s)-curves.

To write explicitly the formula we need to describe a basis of meromorphic functions on X which are singular only at ∞. Prescribe degrees n and s to x and y, order the functions $x^i y^j$, $i \geq 0$, $0 \leq j \leq n-1$ from lower degrees and name them as $f_1 = 1$, f_2, f_3,... Then the formula for the sigma function takes the form (Theorem 2):

$$\sigma\left(\sum_{i=1}^{N} \int_{q_i}^{p_i} du \right) = C_N \; M_N \; F_N,$$

where du is the vector of a basis of holomorphic one forms (12), C_N is the explicit constant (62) and

$$\begin{aligned}
M_N &= \frac{\prod_{i,j=1}^{N} \tilde{E}(p_i, q_j)}{\prod_{i<j} \left(\tilde{E}(p_i, p_j) \tilde{E}(q_i, q_j) \right) \prod_{i,j=1}^{N} \left(x(p_i) - x(q_j) \right)}, \\
F_N &= \frac{D_N}{\prod_{i<j} \left(x(q_i) - x(q_j) \right)^{n-2} \prod_{k=1}^{N} \prod_{1 \leq i < j \leq n-1} \left(y(q_k^{(i)}) - y(q_k^{(j)}) \right)}, \\
D_N &= \det \left(f_i(p_j) \right)_{1 \leq i, j \leq nN},
\end{aligned}$$

where we set

$$p_{N+(n-1)(k-1)+j} = q_k^{(j)}, \qquad 1 \leq k \leq N, \quad 1 \leq j \leq n-1.$$

In the case of hyperelliptic curves of genus g, that is, the case of $(n, s) = (2, 2g+1)$, $F_N = D_N$, $q_k^{(1)} = (x, -y)$ for $q_k = (x, y)$ and the formula coincides with that given by Klein [20].

It follows from this formula that, prescribing degree $ns - ni - sj$ to λ_{ij}, Taylor coefficients of the sigma function become homogeneous polynomials of λ_{ij} and the first term, with respect to certain degrees, of the expansion of the sigma function is

Schur function corresponding to the partition determined from the gap sequence at ∞ (Theorem 3). The results have important applications in the study of differential structure of abelian functions [14, 24, 25].

The plan of the present paper is as follows. In section 2 necessary facts on Riemann surfaces and related objects on them such as flat line bundles, prime form and normalized bilinear form are reviewed. The meromorphic functions and differentials on (n,s)-curves are studied in section 3. The Important object here is the algebraic bilinear form $\widehat{\omega}$. The existence of it is proved in section 3.3 and the relation with the symplectic basis of the first cohomology group of an (n,s)-curve is given in section 3.4. In section 4 the properties of Schur functions are reviewed. The sigma function of an (n,s)-curve is defined and studied in section 5. After giving the definition and an analytic expression of the sigma function in section 5.1, an algebraic expression of the prime form is given in section 5.2. In section 5.3 the prime function is introduced and its properties are established using those of the prime form. The algebraic expressions of the sigma function are given in section 5.4. Theorems 1 and 2 are main results of this paper. In section 5.5 the series expansion of the sigma function is studied and the proportionality constants in the proofs of main theorems are determined. Examples of $(2,3)$ curve and more generally $(2,2g+1)$ curves are given in section 5.6 and 5.7. In section 6 some comments are given.

2. Preliminaries.

2.1. Riemann's theta function. Let τ be a $g\times g$ symmetric matrix whose imaginary part is positive definite and $a,b\in\mathbb{R}^g$. Riemann's theta function with characteristics ${}^t(a,b)$ is defined by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z)=\sum_{n\in\mathbb{Z}^g}\exp\left(\pi i\,{}^t(n+a)\tau(n+a)+2\pi i(n+a)(z+b)\right).$$

The theta function with zero characteristic is simply denoted by $\theta(z)$. We list here some of the fundamental properties of Riemann's theta functions.

(i)

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z+m_1+\tau m_2)=$$

$$\exp\left(2\pi i({}^t a m_1-{}^t b m_2)-\pi i\,{}^t m_2\tau m_2-2\pi i\,{}^t m_2 z\right)\theta\begin{bmatrix} a \\ b \end{bmatrix}(z),\quad m_1,m_2\in\mathbb{Z}. \tag{2}$$

(ii)

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(-z)=(-1)^{4\,{}^t ab}\theta\begin{bmatrix} a \\ b \end{bmatrix}(z),\qquad a,b\in\frac{1}{2}\mathbb{Z}^g \tag{3}$$

(iii)

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z)=\exp\left(\pi i\,{}^t a\tau a+2\pi i\,{}^t a(z+b)\right)\theta(z+\tau a+b). \tag{4}$$

2.2. Flat line bundle. We briefly review some fundamental facts about Riemann surfaces and the description of flat line bundles on them. We refer to [17, 18] for more details.

Let X be a compact Riemann surface of genus g, $\tilde{X}$ the universal cover of X and π: $\tilde{X} \longrightarrow X$ be the projection. We fix a marking of X. It means that we fix a base point p_0 on X, a base point $\tilde{p}_0$ on $\tilde{X}$ which lies over p_0 and a canonical basis $\{\alpha_i, \beta_j\}$ of $\pi_1(X, p_0)$, . Then the covering transformation group can be canonically identified with $\pi_1(X, p_0)$. For $k = 0, 1$, a holomorphic k-form on X can be identified with that on $\tilde{X}$ which is $\pi_1(X, p_0)$-invariant.

Let dv_j, $1 \leq j \leq g$ be the basis of holomorphic one forms normalized as $\int_{\alpha_j} dv_i = \delta_{ij}$ and τ the period matrix, $\tau = (\int_{\beta_j} dv_i)$. Set $dv = {}^t(dv_1, ..., dv_g)$. The Jacobian variety $J(X)$ is defined by $J(X) = \mathbb{C}^g/\tau\mathbb{Z}^g + \mathbb{Z}^g$.

Let $S^kX = X^k/S_k$ be the k-th symmetric product of X. An element of it can be considered as a positive divisor on X of degree k. We denote by I_k the Abel-Jacobi map with the base point p_0:

$$I_k : S^kX \longrightarrow J(X), \qquad I_k(p_1 + \cdots + p_k) = \sum_{i=1}^{k} \int_{p_0}^{p_i} dv.$$

Then $J(X)$ can be identified with $\mathrm{Pic}^0(X)$ of linear equivalence classes of divisors of degree zero by Abel-Jacobi map: for $A = \sum_{i=1}^{d} p_i$,$B = \sum_{i=1}^{d} q_i$,

$$\begin{aligned} I : \mathrm{Pic}^0(X) &\longrightarrow J(X), \\ B - A &\mapsto I(B - A) = I_d(B) - I_d(A). \end{aligned}$$

We sometimes use I_k for the map $X^k \longrightarrow J(X)$.

A flat line bundle on X is described by a representation $\chi : \pi_1(X, p_0) \longrightarrow \mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of non-zero complex numbers. Namely a meromorphic section of the line bundle defined by χ is described by a meromorphic function F on $\tilde{X}$ which satisfies

$$F(\gamma\tilde{p}) = \chi(\gamma)F(\tilde{p}).$$

Since $\mathbb{C}^*$ is abelian, the image $\chi(\gamma)$ of $\gamma \in \pi_1(X, p_0)$ depends only on the image of γ in the homology group $H_1(X, \mathbb{Z})$, which we call the abelian image of γ.

Two representations χ_1 and χ_2 defines a holomorphically equivalent line bundle if and only if

$$\chi_1(\gamma)\chi_2(\gamma)^{-1} = \exp\left(\int_{\gamma} \omega\right)$$

for some holomorphic one form ω and any $\gamma \in \pi_1(X, p_0)$.

The Jacobian variety can also be identified with the set of holomorphic equivalence classes of flat line bundles on X. The flat line bundle corresponding to the degree zero divisor $A - B$ with A, B positive divisors as before, is described by

$$\chi(\alpha_i) = 1, \qquad \chi(\beta_i) = \exp\left(-2\pi i \int_A^B dv_i\right), \tag{5}$$

where

$$\int_A^B dv = \sum_{i=1}^{g} \int_{p_i}^{q_i} dv,$$

with a path from p_i to q_i being specified. Another choice of paths gives an equivalent line bundle. We denote the equivalence class of this bundle by $\mathcal{L}_\alpha$, where $\alpha = \int_A^B dv \in J(X)$.

For $\alpha \in \mathbb{C}^g$ there exists a unique set of vectors $\alpha', \alpha'' \in \mathbb{R}^g$ such that

$$\alpha = \tau\alpha' + \alpha''.$$

The vector ${}^t(\alpha', \alpha'')$ is called the characteristic of α. We sometimes identify α with its characteristic. Let ${}^t(\alpha', \alpha'')$ be the characteristic of $\int_A^B dv \in \mathbb{C}^g$, where the integration paths are specified. Then the function on $\tilde{X}$

$$\frac{\theta(\int_{\tilde{p}_0}^{\tilde{p}} dv + \tau\alpha' + \alpha'' + e)}{\theta(\int_{\tilde{p}_0}^{\tilde{p}} dv + e)},$$

is a meromorphic section of $\mathcal{L}_\alpha$ corresponding to χ, where $e \in \mathbb{C}^g$ is taken such that numerators and denominators are not identically zero as a function of $\tilde{p}$.

There exists a unique unitary representation for each equivalence class of line bundles. The unitary representation for $\mathcal{L}_\alpha$ is given by

$$\chi'(\alpha_j) = \exp(2\pi i \alpha'_j), \qquad \chi'(\beta_j) = \exp(-2\pi i \alpha''_j). \tag{6}$$

A Meromorphic sections of $\mathcal{L}_\alpha$ corresponding to χ' is given by

$$\frac{\theta[\alpha](\int_{\tilde{p}_0}^{\tilde{p}} dv + e)}{\theta(\int_{\tilde{p}_0}^{\tilde{p}} dv + e)},$$

where e satisfies the same conditions as before.

2.3. Prime form. Let δ_0 be Riemann divisor for the choice $\{\alpha_i, \beta_j\}$ and L_0 the corresponding holomorphic line bundle of degree $g-1$. For $\alpha \in J(X)$ set $L_\alpha = \mathcal{L}_\alpha \otimes L_0$.

There exists a non-singular odd half period α [23, 17]. By Riemann's theorem there is a unique divisor $p_1 + \cdots + p_{g-1}$ such that

$$\alpha = p_1 + \cdots + p_{g-1} - \delta_0,$$

in $J(X)$. Considering the function $\theta[\alpha](\int_x^y dv)$ we see that the divisor of the holomorphic one form

$$\sum_{i=1}^{g} \frac{\partial\theta[\alpha]}{\partial z_i}(0) dv_i(p)$$

is $2\sum_{i=1}^{g-1} p_i$. Since α is non-singular, there is a unique, up to constant, holomorphic section of L_α which vanishes on $p_1 + \cdots + p_{g-1}$. Thus there exists a holomorphic section h_α of L_α such that

$$h_\alpha^2(p) = \sum_{i=1}^{g} \frac{\partial\theta[\alpha]}{\partial z_i}(0) dv_i(p).$$

We use the same symbol h_α for the pull back of h_α to $\tilde{X}$. Then the prime form [17, 23, 1] is defined as

$$E(\tilde{p}_1, \tilde{p}_2) = \frac{\theta[\alpha](\int_{\tilde{p}_1}^{\tilde{p}_2} dv)}{h_\alpha(\tilde{p}_1) h_\alpha(\tilde{p}_2)}, \quad \tilde{p}_1, \tilde{p}_2 \in \tilde{X}. \tag{7}$$

By construction it vanishes to the first order at $\pi(\tilde{p}_1) = \pi(\tilde{p}_2)$ and at no other divisors. Let $\pi_j : X \times X \longrightarrow X$ be the projection to the j-th component and $I_{2-1} : X \times X \longrightarrow J(X)$ the map defined by $I_{2-1}(p_1, p_2) = I_1(p_2) - I_1(p_1)$. Then $E(\tilde{p}_1, \tilde{p}_2)$ can be considered as a holomorphic section of the line bundle $\pi_1^* L_0^{-1} \otimes \pi_2^* L_0^{-1} \otimes I_{2-1}^* \Theta$ on $X \times X$, where Θ is the line bundle on $J(X)$ defined by the theta divisor $\Theta = \{\theta(z) = 0\}$. Notice that the prime form does not depend on the choice of α.

We list some fundamental properties of the prime form.

(i) $E(\tilde{p}_2, \tilde{p}_1) = -E(\tilde{p}_1, \tilde{p}_2)$.

(ii) $E(\tilde{p}_1, \tilde{p}_2) = 0 \Longleftrightarrow \pi(\tilde{p}_1) = \pi(\tilde{p}_2)$.

(iii) For $\tilde{p} \in \tilde{X}$ take a local coordinate t around $\tilde{p}$. Then the expansion in $t(\tilde{p}_2)$ at $t(\tilde{p}_1)$ is of the form

$$E(\tilde{p}_1, \tilde{p}_2)\sqrt{dt(\tilde{p}_1)dt(\tilde{p}_2)} = t(\tilde{p}_2) - t(\tilde{p}_1) + O\left((t(\tilde{p}_2) - t(\tilde{p}_1))^3\right).$$

(iv) Consider the function

$$F(\tilde{p}) = \frac{E(\tilde{p}, \tilde{p}_2)}{E(\tilde{p}, \tilde{p}_1)},$$

for $\tilde{p}_1, \tilde{p}_2 \in \tilde{X}$. If the abelian image of $\gamma \in \pi_1(X, p_0)$ is $\sum_{i=1}^{g} m_{1,i}\alpha_i + \sum_{i=1}^{g} m_{2,i}\beta_i$,

$$F(\gamma\tilde{p}) = \exp\left(2\pi i {}^t m_2 \int_{\tilde{p}_1}^{\tilde{p}_2} dv\right) F(\tilde{p}),$$

where $m_i = {}^t(m_{i,1}, ..., m_{i,g})$.

2.4. Normalized fundamental form. Let K_X be the canonical bundle of X. A section of $\pi_1^* K_X \otimes \pi_2^* K_X$ is called a bilinear form on $X \times X$ and a bilinear form $w(p_1, p_2)$ is called symmetric if $w(p_2, p_1) = w(p_1, p_2)$. Since

$$H^0(X \times X, \pi_1^* K_X \otimes \pi_2^* K_X) \simeq \pi_1^* H^0(X, K_X) \otimes \pi_2^* H^0(X, K_X),$$

any holomorphic symmetric bilinear form can be written as

$$\sum c_{ij} dv_i(p_1) dv_j(p_2), \qquad c_{ij} = c_{ji}, \tag{8}$$

where c_{ij}'s are constants.

We denote by Δ the diagonal set of $X \times X$:

$$\Delta = \{(p, p) \,|\, p \in X\}.$$

Definition 1. *A meromorphic symmetric bilinear form $\omega(p_1, p_2)$ on $X \times X$ is called a normalized fundamental form if the following conditions are satisfied.*

(i) $\omega(p_1, p_2)$ is holomorphic except Δ where it has a double pole. For $p \in X$ take a local coordinate t around p. Then the expansion in $t(p_1)$ at $t(p_2)$ is of the form

$$\omega(p_1, p_2) = \left(\frac{1}{(t(p_1) - t(p_2))^2} + \textit{regular}\right) dt(p_1) dt(p_2). \tag{9}$$

(ii) $\int_{\alpha_j} \omega = 0$, *where the integration is with respect to any one of the variables.*

Normalized fundamental form exists and unique. It can be expressed explicitly using the prime form as [17]

$$\omega(p_1, p_2) = d_{\tilde{p}_1} d_{\tilde{p}_2} \log E(\tilde{p}_1, \tilde{p}_2), \tag{10}$$

where $p_i = \pi(\tilde{p}_i)$. Integrating this formula we get

Proposition 1. *[17, 18] For* $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \tilde{X}$,

$$\exp\left(\int_{\tilde{a}}^{\tilde{b}} \int_{\tilde{c}}^{\tilde{d}} \omega\right) = \frac{E(\tilde{b}, \tilde{d}) E(\tilde{a}, \tilde{c})}{E(\tilde{a}, \tilde{d}) E(\tilde{b}, \tilde{c})}.$$

3. (n, s) curve.

3.1. Definition. For relatively coprime integers n and s satisfying $s > n \geq 2$ consider the polynomial [11]

$$f(x, y) := y^n - x^s - \sum_{in+js<ns} \lambda_{ij} x^i y^j. \tag{11}$$

Let X^{aff} be the plane algebraic curve defined by $f(x, y) = 0$. We assume that X^{aff} is non-singular. Denote X the corresponding compact Riemann surface which can be considered as X^{aff} completed by one point ∞. The point ∞ becomes a ramification point with the ramification index n. The genus of X becomes $g = 1/2(n-1)(s-1)$. Hereafter we take ∞ as a base point and fix a marking of X, $(\infty, \tilde{\infty}, \{\alpha_i, \beta_i\})$.

A basis of holomorphic one form on X is given by

$$du_i = -\frac{x^{a_i - 1} y^{n-1-b_i} dx}{f_y}, \tag{12}$$

where $\{(a_i, b_i)\}$ is the set of non-negative integers satisfying

$$1 \leq b \leq n-1, 1 \leq a \leq [\frac{sb-1}{n}],$$

and ordered as $-na_1 + sb_1 < \cdots < -na_g + sb_g$ [10]. This order is specified in such a way that the order of zeros at ∞ is increasing.

Example. $du_g = -\dfrac{dx}{f_y}$, $du_{g-1} = -\dfrac{xdx}{f_y}$.

3.2. Meromorphic functions on X. The space of meromorphic functions on X which are holomorphic on $X \backslash \{\infty\}$ coincides with the space of polynomials of x and y. We describe a basis of this space. Let $w_1 < \cdots < w_g$ be the gap sequence at ∞. It means that there is no meromorphic function on X which has poles only at ∞ of order w_i. Then

Lemma 1. [10] *(i)* $w_1 = 1$ *and* $w_g = 2g - 1$.

(ii) Let $0 = w_1^* < \cdots < w_g^*$ *be integers such that* $\{w_i^*, w_i | i = 1, ...g\} = \{0, 1, ..., 2g-1\}$. *Then* $(2g - 1 - w_1^*, ..., 2g - 1 - w_g^*) = (w_g, ..., w_1)$.

Notice that $\{w_i^*\}$ are non-gaps between 0 and $2g - 2$.

A local parameter t around ∞ can be taken in such a way that

$$x = \frac{1}{t^n}, \qquad y = \frac{1}{t^s}\left(1 + O(t)\right). \tag{13}$$

In particular x and y have poles at ∞ of order n and s respectively. For a meromorphic function h on X we denote by $\mathrm{ord}_\infty h$ the order of poles at ∞. Then

$$\mathrm{ord}_\infty x^i y^j = ni + sj.$$

Let $L(k\infty)$ be the vector space of meromorhic functions on X which are holomorphic on $X \backslash \{\infty\}$ and have poles at ∞ of order at most k. Set $L(*\infty) = \cup_{k=0}^\infty L(k\infty)$, which is the space of meromorphic functions on X holomorphic outside ∞. A basis of $L(*\infty)$ is given by

$$x^i y^j, \qquad i \geq 0, \qquad 0 \leq j \leq n - 1. \tag{14}$$

There are exactly g-gaps in the set $\{\mathrm{ord}_\infty x^i y^j\}$:

$$\mathbb{Z}_{\geq 0} \backslash \{\, ni + sj \,|i \geq 0,\, 0 \leq j \leq n-1\,\} = \{w_1 < \cdots < w_g\}.$$

Let f_i be the monomial basis (14) such that

$$0 = \mathrm{ord}_\infty f_1 < \mathrm{ord}_\infty f_2 < \mathrm{ord}_\infty f_3 < \cdots.$$

In particular $f_1 = 1$, $f_2 = x$. By Riemann-Roch theorem

$$\dim L((N + g - 1)\infty) = N \qquad \text{for} \quad N \geq g.$$

Explicitly, using the local coordinate t, we have

$$f_i = \begin{cases} \dfrac{1}{t^{w_i^*}}(1 + O(t)) & 1 \leq i \leq g \\ \dfrac{1}{t^{g-1+i}}(1 + O(t)) & g + 1 \leq i \end{cases} \tag{15}$$

Notice that $\mathrm{ord}_\infty f_g = 2g - 2$.

Example. $(n, s) = (2, 2g + 1)$:

$(w_1, ..., w_g) = (1, 3, ..., 2g - 1), \quad (w_1^*, ..., w_g^*) = (0, 2, 4, ..., 2g - 2),$

$(f_1, f_2, ...) = (1, x, ..., x^g, y, x^{g+1}, xy, x^{g+2}, x^2 y, ...).$

$(n, s) = (3, 4)$, $g = 3$:

$(w_1, w_2, w_3) = (1, 2, 5), \quad (w_1^*, w_2^*, w_3^*) = (0, 3, 4),$

$(f_1, f_2, ...) = (1, x, y, x^2, xy, y^2, x^3, x^2 y, xy^2, ...).$

$(n, s) = (3, 5)$, $g = 4$:

$(w_1, w_2, w_3, w_4) = (1, 2, 4, 7), \quad (w_1^*, w_2^*, w_3^*, w_4^*) = (0, 3, 5, 6),$

$(f_1, f_2, ...) = (1, x, y, x^2, xy, x^3, y^2, x^2y, x^4, xy^2, x^3y, ...)$.

$(n, s) = (3, 7)$, $g = 6$:

$(w_1, ...w_6) = (1, 2, 4, 5, 8, 11)$, $\quad (w_1^*, ..., w_6^*) = (0, 3, 6, 7, 9, 10)$,

$(f_1, f_2, ...) = (1, x, x^2, y, x^3, xy, x^4, x^2y, y^2, ...)$.

$(n, s) = (4, 5)$, $g = 6$:

$(w_1, ...w_6) = (1, 2, 3, 6, 7, 11)$, $\quad (w_1^*, ..., w_6^*) = (0, 4, 5, 8, 9, 10)$,

$(f_1, f_2, ...) = (1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, ...)$.

In terms of f_i the holomorphic one form du_i is simply described as

$$du_i = -\frac{f_{g+1-i}}{f_y}dx, \qquad 1 \le i \le g.$$

By Lemma 1 (ii) we have, around ∞,

$$du_i = \left(t^{w_i-1} + O(t^{w_i})\right) dt. \tag{16}$$

3.3. Algebraic fundamental form. A meromorphic symmetric bilinear form which satisfies the condition (i) of Definition 1 can explicitly be constructed in terms of algebraic functions. Such algebraic form plays a central role in the construction of the sigma function.

Let $p_i = (x_i, y_i)$, $i = 1, 2$ and [11]

$$\Omega(p_1, p_2) = \frac{\sum_{i=0}^{n-1} y_1^i [\frac{f(z,w)}{w^{i+1}}]_+|_{(z,w)=(x_2,y_2)}}{(x_1 - x_2)f_y(p_1)}dx_1,$$

where

$$[\sum_{n\in\mathbb{Z}} a_n w^n]_+ = \sum_{n\ge 0} a_n w^n.$$

Consider

$$\widehat{\omega}(p_1, p_2) = d_{p_2}\Omega(p_1, p_2) + \sum c_{i_1j_1;i_2j_2} \frac{x_1^{i_1} y_1^{j_1}}{f_y(p_1)} \frac{x_2^{i_2} y_2^{j_2}}{f_y(p_2)} dx_1 dx_2, \tag{17}$$

where (i_1, j_1) runs over $(a_i - 1, n - 1 - b_i)$, $1 \le i \le g$ and $i_2 \ge 0$, $0 \le j_2 \le n - 1$, $c_{i_1j_1;i_2j_2}$'s are constants. Assign degrees as

$$\deg \lambda_{ij} = ns - ni - sj, \quad \deg x = \deg dx = n, \quad \deg y = s.$$

Proposition 2. *(i) If $c_{ij;kl}$ is taken such that $\widehat{\omega}(p_1, p_2) = \widehat{\omega}(p_2, p_1)$ then $\widehat{\omega}$ defined by (17) satisfies the conditions (i) of Definition 1.*

(ii) There exists a set of $c_{i_1j_1;i_2j_2}$ such that $\widehat{\omega}(p_1, p_2) = \widehat{\omega}(p_2, p_1)$, non-zero $c_{i_1j_1;i_2j_2}$ is a homogeneous polynomial of $\{\lambda_{kl}\}$ of degree $2sn - n(i_1 + i_2 + 2) - s(j_1 + j_2 + 2)$ and $c_{i_1j_1;i_2j_2} = 0$ if $2ns - n(i_1 + i_2 + 2) - s(j_1 + j_2 + 2) < 0$.

Notice that, if we take $c_{i_1j_1;i_2j_2}$ as in (ii) in the proposition, then $\widehat{\omega}$ becomes homogeneous of degree 0. For the proof of the proposition we need several lemmas.

Let B be the set of branch points for the map $x : X \longrightarrow \mathbb{P}^1$, $(x,y) \mapsto x$. For $p \in X$ set $x^{-1}(x(p)) = \{p^{(0)},...,p^{(n-1)}\}$ with $p = p^{(0)}$, where the same $p^{(i)}$ is listed according to its multiplicity.

LEMMA 2. *The one form $\Omega(p_1,p_2)$ is holomorphic except $\Delta \cup \{(p^{(i)},p) | p \in B, i \neq 0\} \cup X \times \{\infty\} \cup \{\infty\} \times X$.*

Proof. It is sufficient to prove that Ω does not have a pole at $p_1 = p_2^{(i)}$,$i \neq 0$, for $p_2 \notin B$. Let $p_2^{(i)} = (x_2, y_2^{(i)})$ and

$$f(x,y) = \sum_{j=0}^{n} f_j(x) y^j, \quad f_n = 1. \tag{18}$$

Then, for $i \neq 0$,

$$\begin{aligned}\sum_{k=0}^{n-1} (y_2^{(i)})^k [\frac{f(z,w)}{w^{k+1}}]_+|_{(z,w)=(x_2,y_2)} &= \sum_{k=0}^{n-1} (y_2^{(i)})^k \sum_{j \geq k+1} f_j(x_2) y_2^{j-k-1} \\ &= \sum_{j=1}^{n} f_j(x_2) \frac{(y_2^{(i)})^j - y_2^j}{y_2^{(i)} - y_2} \\ &= \frac{f(x_2,y_2) - f(x_2,y_2^{(i)})}{y_2^{(i)} - y_2} \\ &= 0, \end{aligned} \tag{19}$$

where we use $y_2 \neq y_2^{(i)}$, $i \neq 0$ which follows from the assumption $p_2 \notin B$. Thus Ω is holomorphic at $p_1 = p_2^{(i)}$, $i \neq 0$ as desired. □

LEMMA 3. *Let $p \notin B$, t a local coordinate around p and $t_i = t(p_i)$. Then the expansion of Ω in t_2 at t_1 is of the form*

$$\Omega(p_1,p_2) = \left(\frac{-1}{t_2 - t_1} + O\left((t_2 - t_1)^0\right) \right) dt_1. \tag{20}$$

Proof. Since $p \notin B$ one can take x as a local coordinate around p. Therefore it is sufficient to prove

$$\sum_{k=0}^{n-1} y_1^k [\frac{f(z,w)}{w^{k+1}}]_+|_{(z,w)=(x_1,y_1)} = f_y(x_1,y_1). \tag{21}$$

Let us write $f(x,y)$ as in (18). Then the left hand side of (21) is equal to

$$\sum_{k=0}^{n-1} y_1^k \sum_{j=k+1}^{n} f_j(x_1) y_1^{j-k-1} = \sum_{j=1}^{n} j f_j(x_1) y_1^{j-1} = f_y(x_1,y_1).$$

□

LEMMA 4. *The meromorphic bilinear form $d_{p_2}\Omega(p_1,p_2)$ is holomorphic except $\Delta \cup \{(p^{(i)},p) | p \in B, i \neq 0\} \cup X \times \{\infty\}$.*

Proof. Due to Lemma 2 it is sufficient to prove that $d_{p_2}\Omega(p_1,p_2)$ is holomorphic at (∞,p_2), $p_2\neq\infty$.

Let t be the local coordinate around ∞ such that $x=1/t^n$, $y=(1/t^s)(1+O(t))$ and $t_i=t(p_i)$. Then at (∞,p_2), $p_2\neq\infty$, the expansion of Ω in t_1 takes the form

$$\Omega=-\frac{dt_1}{t_1}\left(1+O(t_1)\right).$$

Thus $d_{p_2}\Omega$ is holomorphic at (∞,p_2), $p_2\neq\infty$. □

LEMMA 5. *There exist second kind differentials $d\widehat{r}_i$, $1\leq i\leq g$ which are holomorphic outside $\{\infty\}$ and satisfy the equation*

$$\omega(p_1,p_2)-d_{p_2}\Omega(p_1,p_2)=\sum_{i=1}^{g}du_i(p_1)d\widehat{r}_i(p_2). \tag{22}$$

Proof. Let us set

$$\omega_1(p_1,p_2)=\omega(p_1,p_2)-d_{p_2}\Omega(p_1,p_2).$$

By Lemma 2, 3, 4 and (9), the singularities of ω_1 are contained in $B_2\cup X\times\{\infty\}$, where $B_2=\{(b^{(i)},b)|b\in B\backslash\{\infty\},0\leq i\leq n-1\}$. Since B_2 is a finite set and $B_2\cap(X\times\{\infty\})=\phi$, ω_1 is holomorphic except $X\times\{\infty\}$. Thus one can write, for $p_2\neq\infty$,

$$\omega_1(p_1,p_2)=\sum_{i=1}^{g}du_i(p_1)d\tilde{r}_i(p_2),$$

for some one forms $d\tilde{r}_i$. Let us describe $d\tilde{r}_i$ more neatly in terms of $\omega(p_1,p_2)$. To this end let us take $q_1,...,q_g\in X\backslash B$ such that $\sum_{j=1}^{g}q_i$ is a general divisor and q_j's are in some small neighborhood of ∞. Take the local coordinate t around ∞ such that $x=1/t^n$, $y=(1/t^s)(1+O(t))$ and write

$$\begin{aligned}du_i(p)&=h_i(t)dt,\\ \omega_1(p_1,p_2)&=K_1(t(p_1),p_2)dt(p_1).\end{aligned}$$

Then we have a set of linear equations

$$\sum_{i=1}^{g}h_i(t(q_j))d\tilde{r}_i(p_2)=K_1(t(q_j),p_2).$$

Since $\sum_{j=1}^{g}q_j$ is a general divisor, $\det(h_i(t(q_j)))\neq0$. Thus $d\tilde{r}_i$ can be expressed as

$$d\tilde{r}_i(p_2)=\sum c_{ij}K_1(t(q_j),p_2)$$

for some constants, with respect to p_2, c_{ij}.

Notice that $K_1(t(q_j),p_2)$ is a second kind differential whose only singularity is ∞. In fact the coefficient of $\omega(q_j,p_2)$ of $dt(p_1)$ is a second kind differential due to the property (i) of ω and $d_{p_2}\Omega(q_j,p_2)$ is obviously of second kind. As already proved the only singularity of $K_1(t(q_j),p_2)$ is $p_2=\infty$.

Let us set

$$d\widehat{r}_i(p) = \sum_{j=1}^{g} c_{ij} K_1(t(q_j), p),$$

which is a second kind differential singular only at ∞, and set

$$\omega_2(p_1, p_2) = \omega_1(p_1, p_2) - \sum_{i=1}^{g} du_i(p_1) d\widehat{r}_i(p_2).$$

Then $\omega_2 = 0$ on $X \times (X \backslash \{\infty\})$. Thus $\omega_2 = 0$ on $X \times X$. Consequently

$$\omega(p_1, p_2) - d_{p_2}\Omega(p_1, p_2) = \sum_{i=1}^{g} du_i(p_1) d\widehat{r}_i(p_2),$$

which proves the lemma. □

Proof of Proposition 2. (ii) Let us write

$$d_{p_2}\Omega(p_1, p_2) = \frac{\sum_{j_1, j_2 \le n-1} a_{i_1 j_1; i_2 j_2} x_1^{i_1} y_1^{j_1} x_2^{i_2} y_2^{j_2}}{(x_1 - x_2)^2 f_y(p_1) f_y(p_2)} dx_1 dx_2.$$

It can be easily verified that $a_{i_1 j_1; i_2 j_2} \in \mathbb{Z}[\{\lambda_{kl}\}]$ and $a_{i_1 j_1; i_2 j_2}$ is homogeneous of degree $2(n-1)s - n(i_1 + i_2) - s(j_1 + j_2)$.

On the other hand

$$\sum c_{i_1 j_1; i_2 j_2} \frac{x_1^{i_1} y_1^{j_1} x_2^{i_2} y_2^{j_2}}{f_y(p_1) f_y(p_2)} = \frac{\sum (c_{i_1-2, j_1; i_2 j_2} - 2c_{i_1-1, j_1; i_2-1, j_2} + c_{i_1, j_1; i_2-2, j_2}) x_1^{i_1} y_1^{j_1} x_2^{i_2} y_2^{j_2}}{(x_1 - x_2)^2 f_y(p_1) f_y(p_2)}.$$

Thus $\widehat{\omega}(p_1, p_2) = \widehat{\omega}(p_2, p_1)$ is equivalent to

$$c_{i_1-2, j_1; i_2 j_2} - 2c_{i_1-1, j_1; i_2-1, j_2} + c_{i_1, j_1; i_2-2, j_2} - c_{i_2-2, j_2; i_1 j_1} + 2c_{i_2-1, j_2; i_1-1, j_1} - c_{i_2, j_2; i_1-2, j_1}$$
$$= a_{i_2, j_2; i_1 j_1} - a_{i_1 j_1; i_2 j_2}. \tag{23}$$

This is a system of linear equations for $c_{i_1 j_1; i_2 j_2}$ whose coefficient matrix has integers as components.

Lemma 6. *Any meromorphic differential on X which is singular only at ∞ is a linear combination of $(x^i y^j / f_y) dx$, $i \ge 0$, $0 \le j \le n-1$.*

Proof. Let η be a meromorphic differential which has a pole at ∞ of order k and is holomorphic on $X \backslash \{\infty\}$. Consider the meromorphic function $\eta f_y / dx$. It has a pole only at ∞ of order $k - (2g-2)$ since the zero divisor of dx/f_y is $(2g-2)\infty$. Any meromorphic function holomorphic except ∞ is a polynomial of x and y. Thus $\eta \in \mathbb{C}[x, y] dx / f_y$. □

By Lemma 5 and 6, the system of linear equations (23) has a solution, that is, it consists of compatible equations. Moreover it has a solution such that each $c_{i_1 j_1; i_2 j_2}$

is a linear combination of $a_{i'_1 j'_1; i'_2 j'_2}$ satisfying $i_1+i_2+2=i'_1+i'_2$, $j_1+j_2=j'_1+j'_2$. In particular one can set $c_{i_1 j_1; i_2 j_2}=0$ if $2ns-n(i_1+i_2+2)-s(j_1+j_2+2)<0$ and has

$$\deg c_{i_1 j_1; i_2 j_2} = 2ns - n(i_1+i_2+2) - s(j_1+j_2+2),$$

if $c_{i_1 j_1; i_2 j_2}$ is non-zero. □

(i) It is sufficient to prove the property (i) for ω. By Lemma 5 $d_{p_2}\Omega(p_1,p_2)$ is holomorphic except $\{(p,p)|p\in X\}\cup X\times\{\infty\}$ and so is $\widehat{\omega}$. Since $\widehat{\omega}(p_1,p_2)=\widehat{\omega}(p_2,p_1)$, $\widehat{\omega}$ does not have a pole at $p_2=\infty$ and therefore is holomorphic except $\{(p,p)|p\in X\}$ where it has a double pole.

Let us prove that the expansion of $\widehat{\omega}$ at the diagonal has the required form. Set

$$dr_i = -\sum c_{a_i-1,n-1-b_i;kl}\frac{x^k y^l}{f_y}dx. \tag{24}$$

Then

$$\widehat{\omega}-\omega = \sum_{i=1}^{g} du_i(p_1)(dr_i(p_2)-d\widehat{r}_i(p_2)). \tag{25}$$

Both hand sides of (25) are meromorphic on $X\times X$. The singularities of the left hand side are contained in $\{(p,p)|p\in X\}$ and those of the right hand side are contained in $X\times\{\infty\}$. Thus the possible singularity of $\widehat{\omega}-\omega$ is $\{\infty\}\times\{\infty\}$. Therefore $\widehat{\omega}-\omega$ and $dr_i-d\widehat{r}_i$ are holomorphic on $X\times X$ and X respectively. Then the required expansion of $\widehat{\omega}$ at (p,p), $p\in X$ follows from (9). □

Take one set of $c_{i_1 j_1; i_2 j_2}$ satisfying (ii) of the proposition and define dr_i by (24). Notice that dr_i is a second kind differential. In fact $dr_i=d\widehat{r}_i$ modulo holomorphic one form as is just proved and $d\widehat{r}_i$ is a second kind differential by Lemma 5. We have

$$\widehat{\omega}(p_1,p_2) = d_{p_2}\Omega(p_1,p_2) + \sum_{i=1}^{g} du_i(p_1)dr_i(p_2).$$

Define period matrices $\omega_1,\omega_2,\eta_1,\eta_2$ by

$$2\omega_1 = \left(\int_{\alpha_j} du_i\right), \quad 2\omega_2 = \left(\int_{\beta_j} du_i\right), \quad -2\eta_1 = \left(\int_{\alpha_j} dr_i\right), \quad -2\eta_2 = \left(\int_{\beta_j} dr_i\right).$$

Notice that ω_1 is invertible due to Riemann's inequality. We set $\tau=\omega_1^{-1}\omega_2$. It is symmetric and satisfies $\operatorname{Im}\tau>0$.

3.4. Relation between ω and $\widehat{\omega}$. We give the relation between ω and $\widehat{\omega}$ using the period matrices.

LEMMA 7. *Let ω_1 and ω_2 be meromorphic symmetric bilinear form satisfying the condition (i) of Definition 1. Then*

$$\omega_1-\omega_2 = \sum_{i,j=1}^{g} c_{ij}\,du_i(p_1)du_j(p_2), \tag{26}$$

for some constants c_{ij} such that $c_{ij}=c_{ji}$.

Proof. The left hand side of (26) is holomorphic symmetric bilinear form. Thus it can be written as desired by (8). □

Set

$$du = {}^t(du_1, ..., du_g).$$

LEMMA 8. *We have*

$$\omega(p_1, p_2) = \widehat{\omega}(p_1, p_2) + {}^t du(p_1)\eta_1\omega_1^{-1}du(p_2).$$

In particular $\eta_1\omega_1^{-1}$ *is symmetric.*

Proof. By Lemma 7

$$\omega - \widehat{\omega} = \sum_{i,j=1}^{g} {}^t du(p_1)\, C\, du(p_2), \tag{27}$$

where $C = (c_{ij})$ is a constant symmetric $g \times g$ matrix. Since $\int_{\alpha_k} \omega(p_1, p_2) = 0$ we have

$$\sum_{i=1}^{g} du_i(p_1)(\eta_1)_{ik} = \sum_{i=1}^{g} c_{ij} du_i(p_1)(\omega_1)_{jk}.$$

Thus

$$(\eta_1)_{ik} = \sum_{j=1}^{g} c_{ij}(\omega_1)_{jk},$$

and

$$C = \eta_1\omega_1^{-1}.$$

□

3.5. Symplectic basis of cohomology. For the sake of simplicity we call, hereafter, a meromorphic differential on X second kind if it is locally exact. In this terminology a first kind differential is of second kind, the space of differentials of the second kind becomes a vector space and the first cohomology group $H^1(X, \mathbb{C})$ is described as the space of second kind differentials modulo exact forms.

The intersection form on $H^1(X, \mathbb{C})$ is given by

$$\eta \circ \eta' = \sum \mathrm{Res}\left(\int^p \eta\right)\eta'(p),$$

where η, η' are second kind differentials, summation is over all singular points of η and η' and Res means taking a residue at a point.

Riemann's bilinear relation can be written as

$$2\pi i \eta \circ \eta' = \sum_{i=1}^{g}\left(\int_{\alpha_i}\eta\int_{\beta_i}\eta' - \int_{\alpha_i}\eta'\int_{\beta_i}\eta\right).$$

Proposition 3. *We have*

$$du_i \circ du_j = 0, , \quad du_i \circ dr_j = \delta_{ij}, \quad dr_i \circ dr_j = 0, \tag{28}$$

which means that $\{du_i, dr_j\}$ *is a symplectic basis of* $H^1(X, \mathbb{C})$.

Proof. It is sufficient to prove (28), since the linear independence follows from it.

The relation $du_i \circ du_j = 0$ is obvious. Let us prove $du_i \circ dr_j = \delta_{ij}$. We calculate $\widehat{\omega}(p_1, p_2) \circ du_j(p_2)$ in two ways. By Proposition 2 (i)

$$\begin{aligned}\widehat{\omega}(p_1, p_2) \circ du_j(p_2) &= \operatorname*{Res}_{p_2=p_1}(\int^{p_2} \widehat{\omega}) du_j(p_2) \\ &= -du_j(p_1).\end{aligned} \tag{29}$$

On the other hand

$$\begin{aligned}\widehat{\omega}(p_1, p_2) \circ du_j(p_2) &= \left(d_{p_2}\Omega(p_1, p_2) + \sum_{i=1}^{g} du_i(p_1) dr_i(p_2) \right) \circ du_j(p_2) \\ &= \sum_{i=1}^{g} du_i(p_1)(dr_i \circ du_j).\end{aligned} \tag{30}$$

Comparing (29) and (30) we have

$$dr_i \circ du_j = -\delta_{ij},$$

since $\{du_i\}$ are linearly independent.

Next let us prove $dr_i \circ dr_j = 0$. Similarly to (30) we have

$$\widehat{\omega}(p_1, p_2) \circ dr_j(p_2) = \sum_{i=1}^{g} du_i(p_1)(dr_i \circ dr_j). \tag{31}$$

Since $du_k \circ dr_j = \delta_{kj}$ as already proved, we have, using Lemma 8,

$$\widehat{\omega}(p_1, p_2) \circ dr_j(p_2) = \omega(p_1, p_2) \circ dr_j(p_2) - \sum_{i=1}^{g} du_i(p_1)(\eta_1 \omega_1^{-1})_{ij}. \tag{32}$$

Let us calculate $\omega(p_1, p_2) \circ dr_j(p_2)$. By Riemann's bilinear relation

$$\begin{aligned}2\pi i \omega(p_1, p_2) \circ dr_j(p_2) &= \sum_{k=1}^{g} \left(\int_{\alpha_k} \omega \int_{\beta_k} dr_j - \int_{\alpha_k} dr_j \int_{\beta_k} \omega \right) \\ &= 2 \sum_{k=1}^{g} (\eta_1)_{jk} \int_{\beta_k} \omega,\end{aligned} \tag{33}$$

since $\int_{\alpha_k} \omega = 0$.

Lemma 9. *We have*

$$\int_{\beta_k} \omega = 2\pi i \sum_{i=1}^{g} (2\omega_1)^{-1}_{ki} du_i(p_1), \tag{34}$$

where the integral of the left hand side is with respect to p_2 *and* $(2\omega_1)^{-1}_{ki}$ *denotes the* (k,i)*-component of* $(2\omega_1)^{-1}$.

Proof. Similarly to (29) we have

$$\omega(p_1,p_2)\circ du_i(p_2) = -du_i(p_1),$$

and similarly to (33)

$$2\pi i\omega(p_1,p_2)\circ du_i(p_2) = -\sum_{k=1}^{g}(2\omega_1)_{ik}\int_{\beta_k}\omega.$$

The assertion of the lemma follows from these. □

Substitute (34) into (33) and get

$$\omega(p_1,p_2)\circ dr_j(p_2) = \sum_{i=1}^{g}(\eta_1\omega_1^{-1})_{ji}du_i(p_1). \tag{35}$$

Then we have, by (32),

$$\widehat{\omega}(p_1,p_2)\circ dr_j(p_2) = \sum_{i=1}^{g}du_i(p_1)\left((\eta_1\omega_1^{-1})_{ji}-(\eta_1\omega_1^{-1})_{ij}\right),$$

which becomes zero since $\eta_1\omega_1^{-1}$ is symmetric by Lemma 8. It follows from (31) that $dr_i\circ dr_j=0$. □

Due to the relation (28) Riemann's bilinear equations take the form

$$-{}^t\eta_1\omega_1 + {}^t\omega_1\eta_1 = 0, \tag{36}$$

$$-{}^t\eta_2\omega_2 + {}^t\omega_2\eta_2 = 0, \tag{37}$$

$$-{}^t\eta_1\omega_2 + {}^t\omega_1\eta_2 = -\frac{\pi i}{2}I_g, \tag{38}$$

where I_g denotes the unit matrix of degree g. If we introduce the matrix

$$M = \begin{pmatrix}\omega_1 & \omega_2\\ \eta_1 & \eta_2\end{pmatrix},$$

those relations can be written compactly as

$${}^tM\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}M = -\frac{\pi i}{2}\begin{pmatrix}0 & I_g\\ -I_g & 0\end{pmatrix}.$$

4. Schur function. Let $p_n(T)$ be the polynomial of T_1, T_2,... defined by

$$\exp\left(\sum_{n=1}^{\infty}T_nk^n\right) = \sum_{n=0}^{\infty}p_n(T)k^n,$$

where k is a variable making a generating function [15].

EXAMPLE. $p_0 = 1, \quad p_1 = T_1, \quad p_2 = T_2 + \frac{T_1^2}{2}, \quad p_3 = T_3 + T_1T_2 + \frac{T_1^3}{6}.$

A sequence of non-negative integers $\lambda = (\lambda_1, ..., \lambda_l)$ is called a partition if $\lambda_1 \geq \cdots \geq \lambda_l$. We set $|\lambda| = \lambda_1 + \cdots + \lambda_l$. Denote by $\lambda' = (\lambda'_1, ..., \lambda'_{l'})$, $l' = \lambda_1$, the conjugate of λ [21]:

$$\lambda'_i = \sharp\{j \,|\, \lambda_j \geq i\,\}.$$

For a partition $\lambda = (\lambda_1, ..., \lambda_l)$ define the polynomial $S_\lambda(T)$ of T_1, T_2, T_3,... by

$$S_\lambda(T) = \det(p_{\lambda_i - i + j}(T))_{1 \leq i,j \leq l}, \tag{39}$$

which we call Schur function. Notice that, for any $r \geq 0$, we have

$$S_{(\lambda, 0^r)}(T) = S_\lambda(T), \tag{40}$$

where $(\lambda, 0^r) = (\lambda_1, ..., \lambda_l, 0, ..., 0)$.

EXAMPLE. $S_{(1)}(T) = T_1$, $S_{(2,1)}(T) = -T_3 + \frac{T_1^3}{3}$,
$S_{(3,2,1)}(T) = T_1T_5 - T_3^2 - \frac{1}{3}T_1^3T_3 + \frac{1}{45}T_1^6, \quad S_{(3,1,1)}(T) = T_5 - T_1T_2^2 + \frac{1}{20}T_1^5.$

We prescribe the degree $-i$ to T_i:

$$\deg T_i = -i.$$

The following properties are well known (see for example [15]).

LEMMA 10. *(i) $S_\lambda(T)$ is a homogeneous polynomial of degree $-|\lambda|$.*
(ii) $S_\lambda(-T) = (-1)^{|\lambda|} S_{\lambda'}(T)$.

For a partition $\lambda = (\lambda_1, ..., \lambda_l)$ we define a symmetric polynomial of t_1, t_2, ... t_l by

$$s_\lambda(t) = \frac{\det(t_j^{\lambda_i + l - i})_{1 \leq i,j \leq l}}{\prod_{1 \leq i < j \leq l}(t_i - t_j)}, \tag{41}$$

which we also call Schur function.

Two Schur functions are related by

$$S_\lambda(T) = s_\lambda(t) \quad \text{if } T_i = \frac{\sum_{j=1}^l t_j^i}{i}. \tag{42}$$

If one takes $l' \geq |\lambda|$ for $\lambda = (\lambda_1, ..., \lambda_l)$, then the symmetric function $s_{(\lambda, 0^{l'-l})}(t)$ can be expressed uniquely as a polynomial of power sum symmetric functions $T_i = \frac{\sum_{j=1}^{l'} t_j^i}{i}$, $1 \leq i \leq l'$. This polynomial coincides with $S_\lambda(T)$.

We define a partition associated with an (n, s)-curve by

$$\lambda(n, s) = (w_g, ..., w_1) - (g - 1, ..., 1, 0).$$

Then

Proposition 4. [10] *(i)* $S_{\lambda(n,s)}(T)$ *does not depend on the variables other than* T_{w_1}, ..., T_{w_g}, *that is, it is a polynomial of the variables* T_{w_1}, ..., T_{w_g}.

(ii) $\lambda(n, s)' = \lambda(n, s)$.

(iii) $|\lambda(n, s)| = \dfrac{1}{24}(n^2 - 1)(s^2 - 1)$.

(iv) $S_{\lambda(n,s)}(-T) = (-1)^{\frac{1}{24}(n^2-1)(s^2-1)} S_{\lambda(n,s)}(T)$.

Notice that properties (iii) and (iv) follow from (ii) and Lemma 10.

5. Sigma function.

5.1. Definition. Riemann's constant of an (n, s)-curve with the base point ∞ becomes a half period, since the divisor of the holomorphic one form du_g is $(2g-2)\infty$. Let

$$\left[\begin{array}{c} \delta' \\ \delta'' \end{array} \right], \quad \delta',\, \delta'' \in \{0, \frac{1}{2}\},$$

be the characteristic of Riemann's constant $\delta = \delta_0 - (g - 1)\infty \in J(X)$ with respect to our choice $(\infty, \{\alpha_i, \beta_j\})$. We define the degree of u_i to be $-w_i$:

$$\deg u_i = -w_i.$$

Definition 2. *The fundamental sigma function or simply the sigma function* $\sigma(u)$ *is the holomorphic function on* $\mathbb{C}^g$ *of the variables* $u = {}^t(u_1, ..., u_g)$ *which satisfies the following conditions.*

(i)
$$\sigma(u + 2\omega_1 m_1 + 2\omega_2 m_2)/\sigma(u) = (-1)^{{}^t m_1 m_2 + 2({}^t\delta' m_1 - {}^t\delta'' m_2)} \times \exp\left({}^t(2\eta_1 m_1 + 2\eta_2 m_2)(u + \omega_1 m_1 + \omega_2 m_2)\right). \tag{43}$$

(ii) The expansion at the origin takes the form

$$\sigma(u) = S_{\lambda(n,s)}(T)|_{T_{w_i}=u_i} + \sum_d f_d(u), \tag{44}$$

where $f_d(u)$ *is a homogeneous polynomial of degree* d *and the range of the summation is* $d < -|\lambda(n, s)|$.

It is possible to give an analytic expression of a function satisfying the condition (i) in terms of Riemann's theta function.

Proposition 5. [9] *Let* $\tau = \omega_1^{-1}\omega_2$. *Then a holomorphic function satisfying (43) is a constant multiple of the function*

$$\exp\left(\frac{1}{2}{}^t u \eta_1 \omega_1^{-1} u\right) \theta \left[\begin{array}{c} \delta' \\ \delta'' \end{array} \right] ((2\omega_1)^{-1}u, \tau). \tag{45}$$

The proposition can easily be proved using (36), (37), (38) and the uniqueness of Riemann's theta function $\theta(z)$ [22].

The existence of the sigma function is not obvious because of the condition (ii). In the succeeding subsections we shall construct the sigma function explicitly using the algebraic integrals.

5.2. Algebraic expression of prime form. In this section an algebraic expression of the prime form is given using the map $x : X \longrightarrow \mathbf{P}^1$, $(x,y) \mapsto x$.

Let $B \subset X$ be the set of branch points of the map x, $B' = B\backslash\{\infty\}$, $\tilde{B} = \pi^{-1}(B) \subset \tilde{X}$, $\tilde{B}' = \pi^{-1}(B')$ $\bar{B} = x(B) \subset \mathbb{P}^1$ and $\bar{B}' = x(B')$. Let $\tilde{p} \in \tilde{X}$ and $\tilde{\gamma}$ be a path from $\tilde{\infty}$ to $\tilde{p}$ in $\tilde{X}\backslash\tilde{B}'$. Then $\bar{\gamma} = (x \circ \pi)(\tilde{\gamma})$ is a path from ∞ to $(x \circ \pi)(\tilde{p})$. Let $\gamma^{(i)}$ be the lift of $\bar{\gamma}$ to $X\backslash B'$ connecting $p^{(i)}$ to ∞ and $\tilde{\gamma}^{(i)}$ the lift of $\gamma^{(i)}$ to $\tilde{X}$ beginning at $\tilde{\infty}$. Denote $\tilde{p}^{(i)}$ the end point of $\tilde{\gamma}^{(i)}$. Then $\tilde{p}^{(i)}$ lies over $p^{(i)}$.

In this way for each path $\tilde{\gamma} = \tilde{\gamma}^{(0)}$ from $\tilde{\infty}$ to $\tilde{p}$ in $\tilde{X}\backslash\tilde{B}'$ we have a uniquely determined path from $\tilde{\infty}$ to $\tilde{p}^{(i)}$ in $\tilde{X}\backslash\tilde{B}'$. Let $\tilde{\gamma}_i = \tilde{\gamma}_i^{(0)}$, $i = 1, 2$ be paths from $\tilde{\infty}$ to $\tilde{p}_i$ and $\tilde{\gamma}_i^{(j)}$ be the corresponding path from $\tilde{\infty}$ to $\tilde{p}_i^{(j)}$. The path from $\tilde{p}_1^{(i)}$ to $\tilde{p}_2^{(i)}$ is defined by $\tilde{\gamma}_2^{(i)} \circ (\tilde{\gamma}_1^{(i)})^{-1}$. Hereafter, for $\tilde{p} \in X$, we denote $\pi(\tilde{p})$ by p if there is no fear of confusion.

Proposition 6. *[17] We have*

$$E(\tilde{p}_1, \tilde{p}_2)^2 = \frac{(x(p_2) - x(p_1))^2}{dx(p_1)dx(p_2)} \exp\left(\sum_{i=1}^{n-1} \int_{\tilde{p}_1^{(i)}}^{\tilde{p}_2^{(i)}} \int_{\tilde{p}_1}^{\tilde{p}_2} \omega\right).$$

Remark. In Fay's book [17] (p17, formula (v)), the prime form is expressed by the right hand side of Proposition 6 multiplied by the term $\exp(\sum_{i=1}^{g} \int_{p_1}^{p_2} m_i dv_i)$. This difference stems from the fact that $E(\tilde{p}_1, \tilde{p}_2)$ is described on the universal covering $\tilde{X}$ in this paper while it is described in the fundamental polygon cut out along the homology basis $\{\alpha_i, \beta_i\}$ in [17].

For the sake to be complete and self-contained we give a proof of this proposition.

Lemma 11. *We have*

$$\frac{(x(w) - x(p_2))(x(z) - x(p_1))}{(x(w) - x(p_1))(x(z) - x(p_2))} = \exp\left(\sum_{i=0}^{n-1} \int_{\tilde{z}}^{\tilde{w}} \int_{\tilde{p}_1^{(i)}}^{\tilde{p}_2^{(i)}} \omega\right). \tag{46}$$

Proof. By Proposition 1

$$\exp\left(\sum_{i=0}^{n-1} \int_{\tilde{z}}^{\tilde{w}} \int_{\tilde{p}_1^{(i)}}^{\tilde{p}_2^{(i)}} \omega\right) = \prod_{i=0}^{n-1} \frac{E(\tilde{w}, \tilde{p}_2^{(i)})E(\tilde{z}, \tilde{p}_1^{(i)})}{E(\tilde{w}, \tilde{p}_1^{(i)})E(\tilde{z}, \tilde{p}_2^{(i)})}.$$

Let us consider the right hand side of this equation as a function of $\tilde{w}$ and denote it by $F(\tilde{w})$. By the property (iv) of the prime form, if the abelian image of $\gamma \in \pi_1(X, \infty)$ is $\sum_{i=1}^{g} m_{1,i}\alpha_i + \sum_{i=1}^{g} m_{2,i}\beta_i$ then

$$F(\gamma\tilde{w}) = F(\tilde{w}) \exp\left(2\pi i \sum_{j=0}^{n-1} {}^t m_2 \int_{\tilde{p}_1^{(j)}}^{\tilde{p}_2^{(j)}} dv\right). \tag{47}$$

Lemma 12.

$$\sum_{j=0}^{n-1} \int_{\tilde{p}_1^{(j)}}^{\tilde{p}_2^{(j)}} dv_i = 0, \quad 1 \leq i \leq g.$$

Proof. It is sufficient to prove

$$\sum_{j=0}^{n-1} \int_{\tilde{p}_1^{(j)}}^{\tilde{p}_2^{(j)}} du_i = 0. \tag{48}$$

Let us fix $\tilde{p_1}$ and consider the left hand side of (48) as a function of $\tilde{p_2}$. We denote it by $G(\tilde{p_2})$. Then $G(\tilde{p_2})$ is a holomorphic function on $Y = \tilde{X} - \tilde{B}$. At each $\tilde{p}^{(i)} \in Y$ one can take x-coordinate as a local coordinate around it. By differentiating $G(\tilde{p_2})$ with respect to the local coordinate x we get

$$dG(\tilde{p_2}) = \sum_{j=0}^{n-1} \frac{x^{a_i-1}(y^{(j)})^{n-1-b_i}}{f_y(x, y^{(j)})} dx, \tag{49}$$

where $p_2^{(j)} = (x, y^{(j)})$.

Lemma 13. *Let $q \in X$ and $q^{(i)} = (x, y^{(i)})$. Suppose that $q \notin B$. Then*

$$\sum_{j=0}^{n-1} \frac{(y^{(j)})^b}{f_y(x, y^{(j)})} = 0, \qquad 0 \le b \le n-2.$$

Proof. Let

$$f(x,y) = \prod_{j=0}^{n-1} (y - y^{(j)}).$$

Since q is not a branch point, $y^{(i)} \neq y^{(j)}$, $i \neq j$. For $0 \le b \le n-2$ we have

$$\operatorname*{Res}_{z=\infty} \frac{z^b}{\prod_{j=0}^{n-1}(z - y^{(j)})} dz = 0,$$

where z is a variable on $\mathbb{P}^1$. Thus, by the residue theorem on $\mathbb{P}^1$,

$$\sum_{j=0}^{n-1} \operatorname*{Res}_{z=y^{(j)}} \frac{z^b}{\prod_{i=0}^{n-1}(z - y^{(i)})} dz = \sum_{j=0}^{n-1} \frac{(y^{(j)})^b}{f_y(x, y^{(j)})} = 0.$$

□

By Lemma 13 and (49) we have

$$dG(\tilde{p_2}) = 0,$$

on Y. Since $G(\tilde{p_2})$ is continuous at each point of $\tilde{B}$, it is a constant on $\tilde{X}$. By the definition of the integration path $G(\tilde{p_1}) = 0$. Therefore $G(\tilde{p_2})$ is identically zero as desired. □

Let us continue the proof of Lemma 11. By Lemma 12 and (47) $F(\tilde{w})$ is $\pi_1(X, \infty)$-invariant and can be considered as a meromorphic function on X. By comparing zeros and poles,

$$F(\tilde{w}) = C \frac{x(w) - x(p_2)}{x(w) - x(p_1)},$$

for some constant C. Since $F(\tilde{z})=1$

$$C=\frac{x(z)-x(p_1)}{x(z)-x(p_2)}$$

which proves the lemma. ▯

Proof of Proposition 6. In Lemma 11 take the limit $\tilde{z}\to\tilde{p}_1$, $\tilde{w}\to\tilde{p}_2$ and use

$$\lim_{\tilde{w}\to\tilde{q}}\frac{x(w)-x(q)}{E(\tilde{w},\tilde{q})}=-dx(q),$$

$$\exp\left(\int_{\tilde{z}}^{\tilde{w}}\int_{\tilde{p}_1}^{\tilde{p}_2}\omega\right)=\frac{E(\tilde{w},\tilde{p}_2)E(\tilde{z},\tilde{p}_1)}{E(\tilde{w},\tilde{p}_1)E(\tilde{z},\tilde{p}_2)}. \tag{50}$$

Then we easily get the desired result. ▯

5.3. Prime function. Let $\sqrt{du_g}$ be the holomorphic section of the line bundle on X defined by the divisor $(g-1)\infty$ satisfying

$$(\sqrt{du_g})^2=du_g, \tag{51}$$

$$\sqrt{du_g}=t^{g-1}\left(1+O(t)\right)\sqrt{dt}, \tag{52}$$

where t is the local parameter (13) around ∞.

Definition 3. *We define the prime function $\tilde{E}(\tilde{p}_1,\tilde{p}_2)$ on $\tilde{X}\times\tilde{X}$ by*

$$\tilde{E}(\tilde{p}_1,\tilde{p}_2)=-E(\tilde{p}_1,\tilde{p}_2)\sqrt{du_g(p_1)}\sqrt{du_g(p_2)}\exp\left(\frac{1}{2}\int_{\tilde{p}_1}^{\tilde{p}_2}{}^t du\cdot\eta_1\omega_1^{-1}\cdot\int_{\tilde{p}_1}^{\tilde{p}_2}du\right). \tag{53}$$

Since

$$\delta=\delta'\tau+\delta''=\delta_0-(g-1)\infty=(g-1)\infty-\delta_0\quad\text{in}\quad J(X),$$

$\tilde{E}(\tilde{p}_1,\tilde{p}_2)$ can be considered as a holomorphic section of the line bundle $\pi_1^*\mathcal{L}_\delta\otimes\pi_2^*\mathcal{L}_\delta\otimes I_{2-1}^*\Theta$ on $X\times X$.

By Proposition 6 and Lemma 8 we have

$$\tilde{E}(\tilde{p}_1,\tilde{p}_2)^2=\frac{\left(x(p_2)-x(p_1)\right)^2}{f_y(p_1)f_y(p_2)}\exp\left(\sum_{i=1}^{n-1}\int_{\tilde{p}_1^{(i)}}^{\tilde{p}_2^{(i)}}\int_{\tilde{p}_1}^{\tilde{p}_2}\widehat{\omega}\right). \tag{54}$$

We need to put one of the variables in $\tilde{E}(\tilde{p}_1,\tilde{p}_2)$ to be $\tilde{\infty}$ in order to describe the sigma function. Since $\tilde{E}(\tilde{p}_1,\tilde{p}_2)$ becomes zero at $\tilde{p}_i=\tilde{\infty}$, it is defined in the following manner. Take the local coordinate t (13) and the local frame $\sqrt{dt}$ as above and define

$$E(\tilde{\infty},\tilde{p}_2)=E(\tilde{p}_1,\tilde{p}_2)\sqrt{dt(p_1)}|_{t(p_1)=0}, \tag{55}$$

$$\tilde{E}(\tilde{\infty},\tilde{p})=E(\tilde{\infty},\tilde{p})\sqrt{du_g(p)}\exp\left(\frac{1}{2}\int_{\tilde{\infty}}^{\tilde{p}}{}^t du\cdot\eta_1\omega_1^{-1}\cdot\int_{\tilde{\infty}}^{\tilde{p}}du\right). \tag{56}$$

Notice that $E(\tilde{\infty}, \tilde{p})$ and $\tilde{E}(\tilde{\infty}, \tilde{p}_2)$ can be considered as holomorphic sections of $L_0^{-1} \otimes I_1^* \Theta$ and $\mathcal{L}_\delta \otimes I_1^* \Theta$ respectively. By (52) and the property (iii) of $E(\tilde{p}_1, \tilde{p}_2)$ we have

$$-\tilde{E}(\tilde{p}_1, \tilde{p}_2) = \tilde{E}(\tilde{\infty}, \tilde{p}_2) t(p_1)^{g-1} + O\left(t(p_1)^g\right).$$

The properties of the prime form imply those of $\tilde{E}(\tilde{p}_1, \tilde{p}_2)$ and $\tilde{E}(\tilde{\infty}, \tilde{p})$. The next proposition follows from the properties (i) and (ii) of the prime form.

Proposition 7. *(i)* $\tilde{E}(\tilde{p}_2, \tilde{p}_1) = -\tilde{E}(\tilde{p}_1, \tilde{p}_2)$.

(ii) As a section of a line bundle on $X \times X$, the zero divisor of $\tilde{E}(\tilde{p}_1, \tilde{p}_2)$ is

$$\Delta + (g-1)(\{\infty\} \times X + X \times \{\infty\}).$$

(iii) As a section of a line bundle on X the zero divisor of $\tilde{E}(\tilde{\infty}, \tilde{p})$ is $g\infty$.

Later we shall study the series expansion of those functions (see Lemma 16).

Proposition 8. *Let the abelian image of $\gamma \in \pi_1(X, \infty)$ be $\sum m_{1,i}\alpha_i + \sum m_{2,i}\beta_i$. Then*

(i) $$\tilde{E}(\tilde{p}_1, \gamma\tilde{p}_2)/\tilde{E}(\tilde{p}_1, \tilde{p}_2) = (-1)^{{}^t m_1 m_2 + 2({}^t\delta' m_1 - {}^t\delta'' m_2)}$$
$$\times \exp\left({}^t(2\eta_1 m_1 + 2\eta_2 m_2)(\int_{\tilde{p}_1}^{\tilde{p}_2} du + \omega_1 m_1 + \omega_2 m_2)\right).$$

(ii) The equation (i) substituted by $\tilde{p}_1 = \tilde{\infty}$ holds for $\tilde{E}(\tilde{\infty}, \tilde{p}_2)$.

Proof. (i) Consider

$$F_1(\tilde{p}_1, \tilde{p}_2) = E(\tilde{p}_1, \tilde{p}_2)\sqrt{du_g(p_1) du_g(p_2)}.$$

It is a section of the bundle $\pi_1^* \mathcal{L}_\delta \otimes \pi_2^* \mathcal{L}_\delta \otimes I_{2-1}^* \Theta$. For a non-singular odd half period $\alpha = \tau\alpha' + \alpha''$ set

$$F_2 = F_1 / \theta[\alpha](\int_{\tilde{p}_1}^{\tilde{p}_2} dv),$$

which is a section of the line bundle $\pi_1^* \mathcal{L}_{\delta-\alpha} \otimes \pi_2^* \mathcal{L}_{\delta-\alpha}$. Let

$$F_2(\tilde{p}_1, \gamma\tilde{p}_2) = \chi(\gamma) F_2(\tilde{p}_1, \tilde{p}_2), \quad \gamma \in \pi_1(X, \infty),$$

where $\chi : \pi_1(X, \infty) \longrightarrow \mathbb{C}^*$ is a representation of $\pi_1(\infty, X)$. Since

$$\frac{du_g}{h_\alpha^2} = \frac{1}{\sum_{i,j=1}^g \frac{\partial\theta[\alpha]}{\partial z_i}(0)(2\omega_1)_{ij}^{-1} x^{a_j - 1} y^{n-1-b_j}}$$

is $\pi_1(X, \infty)$-invariant, so is F_2^2. Thus $\chi(\gamma)^2 = 1$ and χ is a unitary representation. Therefore, if the abelian image of γ is $\sum m_{1,i}\alpha_i + \sum m_{2,i}\beta_i$, we have

$$\chi(\gamma) = \exp\left(2\pi i({}^t(\delta' - \alpha')m_1 - {}^t(\delta'' - \alpha'')m_2)\right),$$

by (6) and consequently

$$\frac{F_1(\tilde{p}_1,\gamma\tilde{p}_2)}{F_1(\tilde{p}_1,\tilde{p}_2)}=(-1)^{2({}^t\delta' m_1-{}^t\delta'' m_2)}\exp\left(-\pi i{}^t m_2\tau m_2-2\pi i{}^t m_2\int_{\tilde{p}_1}^{\tilde{p}_2}dv\right), \quad (57)$$

by (2).

Next let

$$F_3(\tilde{p}_1,\tilde{p}_2)=\exp\left(\frac{1}{2}\int_{\tilde{p}_1}^{\tilde{p}_2}du\ \ \eta_1\omega_1^{-1}\int_{\tilde{p}_1}^{\tilde{p}_2}du\right). \quad (58)$$

By calculations we have

$$\frac{F_3(\tilde{p}_1,\gamma\tilde{p}_2)}{F_3(\tilde{p}_1,\tilde{p}_2)}=\exp\left({}^t(2\eta_1 m_1+2\eta_2 m_2)(\int_{\tilde{p}_1}^{\tilde{p}_2}du+\omega_1 m_1+\omega_2 m_2)+\pi i{}^t m_1 m_2\right)$$
$$\times\exp\left(\pi i{}^t m_2\tau m_2+2\pi i{}^t m_2\int_{\tilde{p}_1}^{\tilde{p}_2}dv\right). \quad (59)$$

Here we use (36), (38) and the relation

$$dv=(2\omega_1)^{-1}du.$$

Multiplying (57) and (59) we get the desired result.

(ii) The statement is obvious from (55), (56) and (i). □

5.4. Algebraic expression of sigma function. We now state our main theorems of this paper.

THEOREM 1. *For $N\geq g$*

$$\sigma(\sum_{i=1}^{N}\int_{\tilde{\infty}}^{\tilde{p}_i}du\ )\ =\ \frac{\prod_{i=1}^{N}\tilde{E}(\tilde{\infty},\tilde{p}_i)^N}{\prod_{i<j}\tilde{E}(\tilde{p}_i,\tilde{p}_j)}\ \det(f_i(p_j))_{1\leq i,j\leq N}. \quad (60)$$

It is possible to derive a more general formula which contains Theorem 1 as a limit.

Let $N\geq g$, p_i, q_i, $i=1,...,N$ be points on X and f_i, $i=1,...,nN$ the basis of $L((nN+g-1)\infty)$ defined before. Consider the function

$$F_N=\frac{D_N}{\prod_{i<j}(x(q_i)-x(q_j))^{n-2}\prod_{k=1}^{N}\prod_{1\leq i<j\leq n-1}\left(y(q_k^{(i)})-y(q_k^{(j)})\right)},$$

$$D_N=\begin{vmatrix} f_1(p_1) & \cdots & f_1(p_N) & f_1(q_1^{(1)}) & \cdots & f_1(q_1^{(n-1)}) & \cdots & f_1(q_N^{(1)}) & \cdots & f_1(q_N^{(n-1)}) \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ f_{nN}(p_1) & \cdots & f_{nN}(p_N) & f_{nN}(q_1^{(1)}) & \cdots & f_{nN}(q_1^{(n-1)}) & \cdots & f_{nN}(q_N^{(1)}) & \cdots & f_{nN}(q_N^{(n-1)}) \end{vmatrix}.$$

Notice that, for each j, F_N is symmetric in $q_j^{(1)},...,q_j^{(n-1)}$ and does not depend on the way of labeling the points $x^{-1}(x(q_j))\backslash\{q_j\}$.

THEOREM 2. *Suppose that $N \geq g$. Then*

$$\sigma(\sum_{i=1}^{N} \int_{\tilde{q}_i}^{\tilde{p}_i} du \ ) \ = \ C_N \ M_N F_N, \tag{61}$$

where

$$M_N = \frac{\prod_{i,j=1}^{N} \tilde{E}(\tilde{p}_i,\tilde{q}_j)}{\prod_{i<j}\left(\tilde{E}(\tilde{p}_i,\tilde{p}_j)\tilde{E}(\tilde{q}_i,\tilde{q}_j)\right)\prod_{i,j=1}^{N}(x(p_i)-x(q_j))},$$

$$C_N = (-1)^{\frac{1}{2}nN(N-1)}\left(\frac{\epsilon(s)}{\epsilon(1)}\right)^N \epsilon_n^{\frac{1}{2}N(N-1)-\frac{1}{4}N(N-1)(n-1)(n-2)+\frac{1}{2}Nn(n-1)-\frac{1}{2}gNn(n-3)}, \tag{62}$$

$$\epsilon_n = \exp(2\pi i/n), \qquad \epsilon(r) = \prod_{1\leq i<j\leq n-1}(\epsilon_n^{ri}-\epsilon_n^{rj}).$$

Proof of Theorem 1. Let G be the right hand side of (60) divided by the left hand side. Obviously G is a symmetric function of $\tilde{p}_1,...,\tilde{p}_N$. Using Proposition 8 and Proposition 7 one can easily verify the following properties.

(i) $G(\gamma\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_N) = G(\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_N)$ for any $\gamma \in \pi_1(X,\infty)$.

(ii) The right hand side of (60) is holomorphic as a function of $\tilde{p}_1$.

Let us consider G as a function of $\tilde{p}_1,...,\tilde{p}_g$. By (i) it can be considered as a meromorphic function on the g-th symmetric product $S^gX = X^g/S_g$ and therefore on the Jacobian $J(X)$. As a meromorphic function on $J(X)$ G has poles only on $\Sigma = \{\sigma(u)=0\}$ of order at most one by (ii). Thus it is a constant which means that it is independent of $\tilde{p}_i$, $1 \leq i \leq g$. Since G is symmetric, it does not depend on all of the variables. The constant is calculated in the proof of Theorem 3 (i) in the next section. □

In order to prove Theorem 2 we have to study the properties of the function F_N.

LEMMA 14. *(i) F_N is skew symmetric with respect to $\{p_i\}$ and $\{q_i\}$ respectively.*

(ii) In each of the variables $\{p_i,q_j\}$ F_N is a meromorphic function .

(iii) As a function of p_1 F_N has poles only at ∞ of order at most $nN+g-1$ and zeros at p_j $(j\geq 2)$, $q_k^{(i)}$ $(1\leq k\leq N, 1\leq i\leq n-1)$.

(iv) As a function of q_1 F_N has poles only at ∞ of order at most $nN+g-1$ and zeros at q_j $(j\geq 2)$, $p_k^{(i)}$ $(1\leq k\leq N, 1\leq i\leq n-1)$.

Proof. (i) The skew symmetry in $\{p_i\}$ is obvious and that in $\{q_i\}$ can be easily verified.

(ii) It is obvious that F_N is a meromorphic function with respect to p_i. Let us prove that F_N is a meromorphic function of q_1. To this end we first prove that F_N is a symmetric polynomial of $y(q_1^{(1)})$,...,$y(q_1^{(n-1)})$. Notice that D_N is a skew symmetric polynomial of $y(q_1^{(1)})$,...,$y(q_1^{(n-1)})$, since $f_k(q_1^{(i)}) = x(q_1)^a y(q_1^{(i)})^b$ for some a, b and permuting $q_1^{(1)}, ..., q_1^{(n-1)}$ is the same as permuting columns. Thus

$$D_N / \prod_{i<j} \left(y(q_1^{(i)}) - y(q_1^{(j)}) \right), \tag{63}$$

is a symmetric polynomial of $y(q_1^{(1)})$,...,$y(q_1^{(n-1)})$. Obviously its coefficients are polynomials of $x(q_1)$.

Now it is sufficient to prove that any symmetric polynomial of $y(q^{(1)})$,...,$y(q^{(n-1)})$ is a polynomial of $x(q)$ and $y(q)$. Let us write

$$f(x(q), y) = \sum_{i=0}^{n} A_i(x(q)) y^{n-i} = \prod_{i=0}^{n-1} (y - y_i),$$

where $A_0 = 1$, $A_i(x)$ is a polynomial of x and $y_i = y(q^{(i)})$. Then

$$e_i(y_0, ..., y_{n-1}) = (-1)^i A_i(x(q)),$$

where $e_i(t_1, ..., t_n)$ is the i-th elementary symmetric polynomial,

$$\prod_{i=1}^{n} (t + t_i) = \sum_{i=0}^{n} e_i(t_1, ..., t_n) t^{n-i}.$$

Using

$$e_i(y_0, ..., y_{n-1}) = y_0 e_{i-1}(y_1, ..., y_{n-1}) + e_i(y_1, ..., y_{n-1}),$$

one can easily prove that every $e_i(y_1, ..., y_{n-1})$ is a polynomial of $y_0 = y(q)$ and $x(q)$. Thus any symmetric polynomial of $y_1, ..., y_{n-1}$ is a polynomial of $x(q)$ and $y(q)$.

(iii) This is obvious.

(iv) As proved in (ii) (63) is a polynomial of $x(q_1)$, $y(q_1)$ and therefore its only singularity is ∞. Let us examine the zeros of D_N.

Notice that D_N has zeros at $q_1 = q_j$, $j \neq 1$ of order at least $n-1$. In fact $q_1 = q_j$ implies that $\{q_1^{(1)}, ..., q_1^{(n-1)}\} = \{q_j^{(1)}, ..., q_j^{(n-1)}\}$. Therefore D_N has zeros of order at least $n-1$ at $q_1 = q_j$.

In a similar manner D_N has zeros at $q_1 = q_j^{(i)}$, $i \neq 0$, of order at least $n-2$, because, in this case, the number of elements in $\{q_1^{(1)}, ..., q_1^{(n-1)}\} \cap \{q_j^{(1)}, ..., q_j^{(n-1)}\}$ is $n-2$.

Consequently the only singularity of F_N is at most $q_1 = \infty$. Moreover F_N has zeros at $q_1 = q_j$, $j \neq 1$. The fact that F_N is zero at $q_1 = p_j^{(i)}$, $i \neq 0$ can be easily verified.

The order of poles at ∞, which we denote by $\mathrm{ord}_\infty F_N$, is estimated as

$$\begin{aligned}
\mathrm{ord}_\infty F_N &\leq O_1 - O_2 - O_3, \\
O_1 &= (nN + g - 1) + \cdots + (nN + g - (n-2)), \\
O_2 &= (n-2)n(N-1), \\
O_3 &= s \begin{pmatrix} n-1 \\ 2 \end{pmatrix},
\end{aligned}$$

where O_1 is the maximal possible order of poles of D_N, O_2 is the order of poles of $\prod_{j=2}^{N}(x(q_1)-x(q_j))^{n-2}$ and O_3 is that of $\prod_{i<j}(y(q_1^{(i)})-y(q_1^{(j)}))$. By calculation we get

$$O_1-O_2-O_3=nN+g-1.$$

□

Proof of Theorem 2. The proof is similar to that of Theorem 1. Let $G(p_1,...,p_N|q_1,...,q_N)$ be the right hand side divided by the left hand side of (61).

The function G is $\pi_1(X,\infty)$-invariant in each of the variables $\{\tilde{p}_i,\tilde{q}_j\}$ by Proposition 8. Consider G as a function of $p_1,...,p_g$. By Lemma 14 G can be considered as a meromorphic function on $J(X)$ which has poles only on Σ of order at most one. Thus it does not depend on $p_1,...,p_g$. Since G is symmetric in $\{p_i\}$ it does not depend on any p_i. Similarly G does not depend on $\{q_i\}$. Thus G is a constant. The constant is calculated in the next section. □

5.5. The series expansion of sigma function. In this section we study the series expansion of the sigma function using the expression obtained in the previous section.

Let us use the multi-index notations like

$$u^\alpha=u_1^{\alpha_1}\cdots u_g^{\alpha_g},\qquad \alpha=(\alpha_1,...,\alpha_g).$$

We set

$$|\alpha|=\sum_{i=1}^{g}w_i\alpha_i,$$

so that $\deg u^\alpha=-|\alpha|$.

THEOREM 3. *(i) The expansion of $\sigma(u)$ at the origin takes the form*

$$\sigma(u)=S_{\lambda(n,s)}(T)|_{T_{w_i}=u_i}+\sum a_\alpha u^\alpha,$$

where $a_\alpha\in\mathbb{Q}[\{\lambda_{ij}\}]$ and the sum is taken for α such that $|\alpha|>|\lambda(n,s)|$.

(ii) In (i) a_α is homogeneous of degree $-|\lambda(n,s)|+|\alpha|$. In particular $\sigma(u)$ is homogeneous of degree $-|\lambda(n,s)|$ with respect to the variables $\{u_i,\lambda_{jk}\}$.

(iii) $\sigma(-u)=(-1)^{|\lambda(n,s)|}\sigma(u)$.

In the remaining of this section t denotes the local parameter around ∞ specified by (13). Set

$$\deg t=-1.$$

LEMMA 15. *(i) If we write $y=\frac{1}{t^s}\sum_{i=0}^{\infty}c_it^i$, $c_0=1$, then c_i belongs to $\mathbb{Q}[\{\lambda_{kl}\}]$ and is homogeneous of degree i.*

(ii) If we write $dx/f_y=-t^{2g-2}(1+\sum_{i=1}^{\infty}c_i't^i)dt$, then c_i' belongs to $\mathbb{Q}[\{\lambda_{kl}\}]$ and is homogeneous of degree i.

(iii) Let $t_i = t(p_i)$ *and*

$$\widehat{\omega}(p_1, p_2) = \left(\frac{1}{(t_1 - t_2)^2} + \sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j \right) dt_1 dt_2.$$

Then $a_{ij} = a_{ji}$*,* a_{ij} *belongs to* $\mathbb{Q}[\{\lambda_{kl}\}]$ *and is homogeneous of degree* $i+j+2$*.*

Proof. (i) Substitute the expressions of x, y in t to the defining equation $f(x,y) = 0$ with f being given by (11). Then

$$y = \frac{1}{t^s} \left(1 + \sum_{ni+sj<ns} \lambda_{ij} t^{ns-ni-sj} (\sum_{r=0}^{\infty} c_r t^r)^j \right)^{\frac{1}{n}}.$$

If we write

$$1 + \sum_{ni+sj<ns} \lambda_{ij} t^{ns-ni-sj} (\sum_{r=0}^{\infty} c_r t^r)^j = \sum_{i=0}^{\infty} b_i t^i, \quad b_0 = 1, \tag{64}$$

then each b_i is a polynomial of $c_1, ..., c_{i-1}$ and $\{\lambda_{kl}\}$ with the coefficients in $\mathbb{Q}$, since $ns - ni - sj > 0$.

In the expansion

$$(\sum_{i=0}^{\infty} b_i t^i)^{1/n} = 1 + \sum_{j=1}^{\infty} \begin{pmatrix} 1/n \\ j \end{pmatrix} (\sum_{i=1}^{\infty} b_i t^i)^j, \tag{65}$$

the coefficient of t^r is a polynomial of $b_1, ..., b_r$ with the coefficient in $\mathbb{Q}$ and consequently is a polynomial of $c_1, ..., c_{r-1}$, $\{\lambda_{kl}\}$ with the coefficient in $\mathbb{Q}$. Thus the equation

$$\sum_{i=0}^{\infty} c_i t^i = (\sum_{i=0}^{\infty} b_i t^i)^{1/n} \tag{66}$$

implies that $c_r \in \mathbb{Q}[c_1, ..., c_{r-1}, \{\lambda_{kl}\}]$ for $r \geq 1$. This proves $c_r \in \mathbb{Q}[\{\lambda_{kl}\}]$.

Next let us prove $\deg c_l = l$ by the induction on l. The case $l = 0$ is obvious. Suppose that $l \geq 1$ and $\deg c_i = i$ for $i \leq l-1$. Then the equation (64) to determine b_i, $i \leq l$ can be written as

$$1 + \sum_{ni+sj<ns} \lambda_{ij} t^{ns-ni-sj} (\sum_{r=0}^{l-1} c_r t^r)^j = \sum_{i=0}^{l} b_i t^i \mod. \ (t^{l+1}). \tag{67}$$

Since the left hand side of (67) is of degree 0 by the induction hypothesis, $\deg b_i = i$ for any $i \leq l$. Similarly the equation (66) determining c_l from $b_1, ..., b_l$ can be written as

$$\sum_{i=0}^{l} c_i t^i = (\sum_{i=0}^{l} b_i t^i)^{1/n} \mod. \ (t^{l+1}) \tag{68}$$

The right hand side of this equation is of degree 0. Thus $\deg c_l = l$. $\square$

(ii) This easily follows from (i).

(iii) By the property (9) for $\widehat{\omega}$,

$$\widehat{\omega} - \frac{dt_1 dt_2}{(t_1 - t_2)^2}$$

is holomorphic near $\{\infty\} \times \{\infty\}$. Thus it is possible to expand it as

$$\widehat{\omega}(p_1, p_2) - \frac{dt_1 dt_2}{(t_1 - t_2)^2} = \left(\sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j \right) dt_1 dt_2. \tag{69}$$

Let us prove that a_{ij} is a homogeneous polynomial in $\{\lambda_{kl}\}$ of degree $2+i+j$.

Since $\widehat{\omega}(p_1,p_2)(t_1-t_2)^2$ is holomorphic near $\{\infty\} \times \{\infty\}$ one can write

$$\widehat{\omega}(p_1, p_2)(t_1^n - t_2^n)^2 = (\sum_{i,j=0}^{\infty} b_{ij} t_1^i t_2^j) dt_1 dt_2.$$

We first prove $b_{ij} \in \mathbb{Q}[\{\lambda_{kl}\}]$.

Using

$$f_x dx + f_y dy = 0, \quad (x_1 - x_2)^2 = (t_1 t_2)^{-2n} (t_1^n - t_2^n)^2,$$

we see that

$$\widehat{\omega}(p_1, p_2)(t_1^n - t_2^n)^2 = (t_1 t_2)^{2n} \, P \, \frac{dx(p_1) dx(p_2)}{f_y(p_1) f_y(p_2)}$$

for some homogeneous polynomial P in $x_1, y_1, x_2, y_2, \{\lambda_{kl}\}$ with the coefficient in $\mathbb{Q}$ of degree $2s(n-1)$. By (i), (ii) one has $b_{ij} \in \mathbb{Q}[\{\lambda_{kl}\}]$ and $\deg b_{ij} = -2n+2+i+j$. Therefore one can write

$$\left(\widehat{\omega} - \frac{dt_1 dt_2}{(t_1 - t_2)^2} \right) (t_1^n - t_2^n)^2 = \left(\sum_{i,j=0}^{\infty} c_{ij} t_1^i t_2^j \right) dt_1 dt_2,$$

where $c_{ij} \in \mathbb{Q}[\{\lambda_{kl}\}]$ and $\deg c_{ij} = -2n+2+i+j$. By (69) we have

$$(t_1^n - t_2^n)^2 \sum_{i,j=0}^{\infty} a_{ij} t_1^i t_2^j = \sum_{i,j=0}^{\infty} c_{ij} t_1^i t_2^j,$$

which is equivalent to

$$a_{ij} - 2a_{i+n,j-n} + a_{i+2n,j-2n} = c_{i+2n,j}. \tag{70}$$

Here we set $a_{ij} = 0$ if $i < 0$ or $j < 0$.

Define $(i,j) < (k,l)$ if and only if $i+j < k+l$ or $i+j = k+l$ and $j < l$. It defines a total order on the set $\{(i,j) | \, i,j \geq 0\}$. The equation (70) expresses a_{ij} as a linear combination of a_{kl} with $(k,l) < (i,j)$, $k+l = i+j$ and $\{c_{rs}\}$ with $r+s = i+j+2n$. Since $a_{k0} = c_{k+2n,0}$, any a_{ij} is a linear combination of $\{c_{rs}\}$ with $r+s = 2n+i+j$. Thus a_{ij} is in $\mathbb{Q}[\{\lambda_{kl}\}]$ and is homogeneous of degree $2+i+j$. $\square$

LEMMA 16. *(i) The expansion of $\tilde{E}(\tilde{p}_1,\tilde{p}_2)$ near $(\tilde{\infty},\tilde{\infty})$ is of the form*

$$\tilde{E}(\tilde{p}_1,\tilde{p}_2)=(t_1-t_2)(t_1t_2)^{g-1}(1+\sum_{i+j\geq 1}c_{ij}t_1^i t_2^j),$$

where c_{ij} is a homogeneous element in $\mathbb{Q}[\{\lambda_{kl}\}]$ of degree $i+j$.

(ii) The expansion of $\tilde{E}(\tilde{\infty},\tilde{p})$ near $\tilde{\infty}$ is of the form

$$\tilde{E}(\tilde{\infty},\tilde{p})=t^g(1+\sum_{j=1}^{\infty}c_{0j}t^j),$$

where c_{0j} is the same as that in (i).

Proof. (i) Using the definition, property (iii) of the prime form and (52) we have the expansion of the form

$$\tilde{E}(\tilde{p}_1,\tilde{p}_2)=(t_1-t_2)(t_1t_2)^{g-1}(1+\sum_{i+j\geq 1}c_{ij}t_1^i t_2^j).$$

In order prove that c_{ij} has the required properties we use (54). The right hand side of (54) is calculated in the following way.

Let $\varepsilon_n=\exp(2\pi i/n)$. Since $x(p^{(i)})=x(p)=1/t^n$, we take $t^{(i)}=\varepsilon_n^{-i}t$ as a local parameter of $p^{(i)}$ by rearranging i of $p^{(i)}$ if necessary. Then

$$x(p_k^{(i)})=1/t_k^n,\quad y(p_k^{(i)})=\frac{\varepsilon_n^{is}}{t_k^s}(1+\cdots)=y(p_k)|_{t_k\longrightarrow\varepsilon_n^{-i}t_k}.$$

Using these local parameters we get

$$\exp\left(\sum_{i=1}^{n-1}\int_{\tilde{p}_1^{(i)}}^{\tilde{p}_2^{(i)}}\int_{\tilde{p}_1}^{\tilde{p}_2}\widehat{\omega}\right)$$

$$=n^2\frac{(t_1t_2)^{n-1}}{\prod_{i=1}^{n-1}(t_1-t_2^{(i)})^2}\exp\left(\sum_{i=1}^{n-1}\sum_{k,l=0}^{\infty}\frac{a_{kl}}{(k+1)(l+1)}((t_2^{(i)})^{k+1}-(t_1^{(i)})^{k+1})(t_2^{l+1}-t_1^{l+1})\right),$$

and

$$\frac{(x(p_2)-x(p_1))^2}{f_y(p_1)f_y(p_2)}=\frac{1}{n^2}(t_1t_2)^{2g-n-1}(t_1^n-t_2^n)^2\prod_{j=1}^{2}\left(1+\sum_{i=1}^{\infty}c_i't_j^i\right),$$

where c_i' is that in Lemma 15, (ii). The assertions for c_{ij} follow from these expressions and Lemma 15. ∎

Proof of Theorem 3. (i), (ii): Let $t_i=t(p_i)$. By Lemma 16 we have

$$\frac{\prod_{i=1}^{N}\tilde{E}(\infty,p_i)^N}{\prod_{i<j}\tilde{E}(p_i,p_j)}=\frac{(\prod_{i=1}^{N}t_i)^{N+g-1}}{\prod_{i<j}(t_i-t_j)}(1+\sum_{k_1+\cdots+k_N\geq 1}\tilde{c}_{k_1\ldots k_N}t_1^{k_1}\cdots t_N^{k_N}),$$

where $\tilde{c}_{k_1 \ldots k_N} \in \mathbb{Q}[\{\lambda_{kl}\}]$ and $\deg \tilde{c}_{k_1 \ldots k_N} = \sum k_i$. By (15) we have

$$
\begin{aligned}
&(f_1(t), ..., f_N(t)) \\
&\quad = (1, \frac{1}{t^{w_2^*}}(1+O(t)), ..., \frac{1}{t^{w_g^*}}(1+O(t)), \frac{1}{t^{2g}}(1+O(t)), ..., \frac{1}{t^{N+g-1}}(1+O(t))),
\end{aligned}
$$

where all $O(t)$ parts are series in t with the coefficients in $\mathbb{Q}[\{\lambda_{kl}\}]$ and are homogeneous of degree 0 with respect to $\{t, \lambda_{kl}\}$. Using Lemma 1 we get

$$
\begin{aligned}
&(N+g-1, ..., N+g-1) + (0, -w_2^*, ..., -w_g^*, -2g, ..., -(N+g-1)) \\
&\qquad = (\lambda(n,s)_1, ..., \lambda(n,s)_g, 0, ..., 0) + (N-1, N-2, ..., 1, 0).
\end{aligned}
$$

Let us denote the partition $(\lambda(n,s), 0^{N-g})$ by $\lambda^{(N)}(n,s)$. Then

$$
\frac{(\prod_{i=1}^N t_i)^{N+g-1}}{\prod_{i<j}(t_i - t_j)} \det(f_i(t_j))_{1 \le i,j \le N} = s_{\lambda^{(N)}(n,s)}(t_1, ..., t_N) + \sum \widehat{c}_{k_1 \ldots k_N} t_1^{k_1} \cdots t_N^{k_N}, \tag{71}
$$

where $\widehat{c}_{k_1 \ldots k_N} \in \mathbb{Q}[\{\lambda_{kl}\}]$, $\deg \widehat{c}_{k_1 \ldots k_N} = -|\lambda(n,s)| + \sum k_i$ and the summation is taken for k_i's satisfying $\sum k_i > |\lambda(n,s)|$.

By (16) we have

$$
\int_\infty^{\tilde{p}} du_i = \frac{t^{w_i}}{w_i} + \sum_{j=1}^\infty c_{i,j} t^{j+w_i}, \quad c_{i,j} \in \mathbb{Q}[\{\lambda_{kl}\}], \quad \deg c_{ij} = j.
$$

Let

$$
T_k = T_k(t_1, ..., t_N) = \frac{\sum_{j=1}^N t_j^k}{k}.
$$

Then $T_1, ..., T_N$ are algebraically independent and become a generator of the ring of symmetric polynomials of $t_1, ..., t_N$ with the coefficients in $\mathbb{Q}$,

$$
\mathbb{Q}[t_1, ..., t_N]^{S_N} = \mathbb{Q}[T_1, ..., T_N].
$$

Moreover if we prescribe degrees for t_i and T_i by

$$
\deg t_i = -1, \qquad \deg T_i = -i,
$$

a symmetric homogeneous polynomial of $t_1, ..., t_N$ of degree k can be uniquely written as a homogeneous polynomial of $T_1, ..., T_N$ of degree k.

We have

$$
\begin{aligned}
u_i = \sum_{k=1}^N \int_\infty^{\tilde{p}_k} du_i &= T_{w_i} + \sum_{j=1}^\infty (j+w_i) c_{ij} T_{j+w_i} \\
&= T_{w_i} + \sum_{\sum j k_j > w_i} \tilde{c}_{k_1 \ldots k_N} T_1^{k_1} \cdots T_N^{k_N},
\end{aligned} \tag{72}
$$

where $\tilde{c}_{k_1 \ldots k_N} \in \mathbb{Q}[\{\lambda_{kl}\}]$, deg, $\tilde{c}_{k_1 \ldots k_N} = -w_i + \sum j k_j$ and the second expression is unique.

Let us take $N \geq 2g-1 = w_g$. Then $T_{w_1},...,T_{w_g}$ are algebraically independent. We denote the right hand side of (60) by $F(\tilde{p}_1,...,\tilde{p}_N)$. By Theorem 1 the series expansion of F in $t_1,...,t_N$ can be rewritten as a series in $u_1,...,u_g$.

Let

$$F(p_1,...,p_N) = s_{\lambda^{(N)}(n,s)}(t_1,...,t_N) + \sum F_{-k} \tag{73}$$

be the homogeneous decomposition of the series in t_i and

$$F(p_1,...,p_N) = \sum_{d\geq 0} \widehat{F}_{-d} \tag{74}$$

the homogeneous decomposition of the series in u_i, where in (73) the sum in k is taken for $k > |\lambda(n,s)|$. Let d_0 be the smallest integer such that $\widehat{F}_{-d} \neq 0$. Write

$$\widehat{F}_{-d}(u) = \sum_{\sum l_i w_i = d} \widehat{c}_{d;l_1\ldots l_g} u_1^{l_1} \cdots u_g^{l_g}.$$

Then

$$\widehat{F}_{-d_0}(u) = \sum_{\sum l_i w_i = d_0} \widehat{c}_{d_0;l_1\ldots l_g} T_{w_1}^{l_1} \cdots T_{w_g}^{l_g} + \sum_{\sum j k_j > d_0} c'_{k_1\ldots k_N} T_1^{k_1} \cdots T_N^{k_N}, \tag{75}$$

for some constants, with respect to T_i's, $c'_{k_1\ldots k_N}$. By comparing (73) with (75) we have $d_0 = |\lambda(n,s)|$ and

$$\sum_{\sum l_i w_i = d_0} \widehat{c}_{d_0;l_1\ldots l_g} T_{w_1}^{l_1} \cdots T_{w_g}^{l_g} = s_{\lambda^{(N)}(n,s)}(t_1,...,t_N).$$

Since

$$s_{\lambda^{(N)}(n,s)}(t_1,...,t_N) = S_{\lambda(n,s)}(T),$$

by (40) and (42) and $T_{w_1},...,T_{w_g}$ are algebraically independent, we have

$$\widehat{F}_{-d_0}(u) = \sum_{\sum l_i w_i = d_0} \widehat{c}_{d_0;l_1\ldots l_g} u_1^{l_1} \cdots u_g^{l_g} = S_{\lambda(n,s)}(T)|_{T_{w_i}=u_i}.$$

For $d > d_0$ one can write

$$F_{-d}(t_1,...,t_N) = \sum b_{d;l_1\ldots l_N} T_1^{l_1} \cdots T_N^{l_N},$$

with $b_{d;l_1\ldots l_N} \in \mathbb{Q}[\{\lambda_{kl}\}]$ and $\deg b_{d;l_1\ldots l_N} = -d_0 + d$. Comparing this with the expression of $\widehat{F}_{-d}(u)$ and using the algebraic independence of $T_1,...,T_N$, we see inductively that $\widehat{F}_{-d}$ is a polynomial in $\{u_i\}$ with the coefficients in $\mathbb{Q}[\{\lambda_{kl}\}]$ which is homogeneous of degree $-d_0$ with respect to $\{u_i, \lambda_{kl}\}$. Consequently the equation (60) holds and (i),(ii) of Theorem 3 is proved.

(iii) Since Riemann's constant $\tau\delta' + \delta''$ is a half period, $\sigma(u)$ is even or odd by Proposition 5 and (4). Thus the relation in (iii) follows from Proposition 4 (iv). ∎

Calculation of C_N in Theorem 2. We set $t_{1i} = t_i = t(\tilde{p}_i)$ and $t_{2i} = t(\tilde{q}_i)$. We take the limit $\tilde{q}_i \to \tilde{\infty}$, $i = 1,...,N$. Since we know that F_N is holomorphic in $\tilde{q}_i$ by the already proved part of Theorem 2, we can calculate the limit by taking the limits $t_{21} \to 0$, $t_{22} \to 0$,... $t_{2N} \to 0$ in this order. Therefore we assume $|t_{21}| < \cdots < |t_{2N}|$ and expand everything first into Laurent series in t_{21} and take the top term of the expansion. Next we expand this top term in t_{22} and so on. Below, if we simply write $A + \cdots$, then it means such an expansion. Let us carry out such calculations.

We have

$$D_N = (-1)^{\frac{1}{2}(n-1)N(N-1)} \epsilon_n^{-\frac{1}{2}N(N-1)-\frac{1}{4}N(N-1)(n-1)^2(n-2)+\frac{1}{2}N(n-1)(2g-n)+\frac{1}{2}gN(n-1)(n-2)}$$
$$\times \epsilon(1)^N \det(f_i(p_j))_{1\le i,j\le N} \prod_{j=1}^{N} t_{2j}^{(j-1)(n-1)^2-\sum_{i=1}^{n-1}(nN+g-i)} + \cdots . \quad (76)$$

By Lemma 16 (i) we have

$$M_N = (-1)^N \frac{\prod_{i=1}^{N}(t_{1i}t_{2i})^{nN+g-1}\prod_{i,j=1}^{N}(t_{1i}-t_{2j})}{\prod_{i<j}(t_{1i}-t_{1j})(t_{2i}-t_{2j})\cdot\prod_{i,j=1}^{N}(t_{1i}^n-t_{2j}^n)}\left(1+(\text{regular term})\right). \quad (77)$$

By calculation

$$F_N/D_N = (-1)^{\frac{1}{2}N(N-1)(n-2)}\epsilon(s)^{-N}$$
$$\times \frac{(\prod_{i=1}^{N} t_{2i})^{(N-1)n(n-2)+\frac{1}{2}(n-1)(n-2)s}}{\prod_{i<j}(t_{2i}^n - t_{2j}^n)^{n-2}}\left(1+(\text{regular term})\right). \quad (78)$$

Let H_N be the product of F_N/D_N and M_N. Then

$$H_N = (-1)^{\frac{1}{2}N(N+1)}\epsilon(s)^{-N}\frac{(\prod_{i=1}^{N} t_{1i})^{N+g-1}}{\prod_{i<j}(t_{1i}-t_{1j})}$$
$$\times \prod_{i=1}^{N} t_{2i}^{-(i-1)(n-1)^2+nN+g-1+(N-1)n(n-2)+\frac{1}{2}(n-1)(n-2)s} + \cdots . \quad (79)$$

Multiplying (76) and (79) we have

$$M_N F_N = C_N^{-1}\ \frac{(\prod_{i=1}^{N} t_{1i})^{N+g-1}}{\prod_{i<j}(t_{1i}-t_{1j})}\ \det(f_i(p_j))_{1\le i,j\le N} + \cdots .$$

Thus, by (71), in the limit $t_{21} \to 0$,..., $t_{2N} \to 0$ we get

$$\begin{aligned} M_N F_N &= C_N^{-1}\ s_{\lambda^{(N)}(n,s)}(t_1,...,t_N) + \cdots \\ &= C_N^{-1}\ S_{\lambda(n,s)}(T)|_{T_{w_i}=u_i} + \cdots , \end{aligned}$$

where in the last $+\cdots$ part is a series in t_i, $i=1,...,N$ of degree less than $-|\lambda(n,s)|$. $\square$

5.6. Example$-g=1-$**.** We take $f(x,y)=y^2-x^3-\lambda_{10}x-\lambda_{00}$, that is, we set $\lambda_{20}=\lambda_{01}=\lambda_{11}=0$. In this case $c_{ij;kl}$ satisfying (ii) of Proposition 2 is unique, $c_{00;10}=1$ and other $c_{ij;kl}$'s are all zero. Then

$$\begin{aligned}\widehat{\omega}(p_1,p_2)&=d_{p_2}\left(\frac{y_1+y_2}{2y_1(x_1-x_2)}dx_1\right)+\frac{dx_1}{2y_1}\frac{x_2dx_2}{2y_2}\\&=\frac{2y_1y_2+x_1x_2(x_1+x_2)+\lambda_{10}(x_1+x_2)+2\lambda_{00}}{4y_1y_2(x_1-x_2)^2}dx_1dx_2,\end{aligned}$$

and

$$du_1=-\frac{dx}{2y},\quad dr_1=-\frac{xdx}{2y}.$$

Theorem 2 gives

$$\sigma\left(\int_{\tilde{q}_1}^{\tilde{p}_1}du_1\right)^2=\tilde{E}(\tilde{p}_1,\tilde{q}_1)^2=\frac{(x(p_1)-x(q_1))^2}{4y(p_1)y(q_1)}\exp\left(\int_{\tilde{q}_1^{(1)}}^{\tilde{p}_1^{(1)}}\int_{\tilde{q}_1}^{\tilde{p}_1}\widehat{\omega}\right).$$

Notice that $p^{(1)}=(x,-y)$ for $p=(x,y)$. One can transform the defining equation of the elliptic curve to Weierstrass form

$$Y^2=4X^3-g_2X-g_3,$$

by

$$X=x,\quad Y=-2y,\quad g_2=-4\lambda_{10},\quad g_3=-4\lambda_{00}.$$

The sigma function in this case coincides with that of Weierstrass. Let $\wp(u)$ be the Weierstrass elliptic function. The symplectic basis du_1, dr_1 are transformed to

$$du=\frac{dX}{Y},\quad dr=\frac{XdX}{Y}.$$

and $\widehat{\omega}$ becomes

$$\begin{aligned}\widehat{\omega}&=\frac{2Y_1Y_2+4X_1X_2(X_1+X_2)-g_2(X_1+X_2)-2g_3}{4Y_1Y_2(X_1-X_2)^2}dX_1dX_2\\&=\wp(v_2-v_1)dv_1dv_2,\end{aligned}$$

where

$$v_i=\int_{\tilde{\infty}}^{\tilde{p}_i}du.$$

The formula for the sigma function gives

$$\sigma\left(v_2-v_1\right)^2=\frac{(X_1-X_2)^2}{Y_1Y_2}\exp\left(\int_{\tilde{p}_1^{(1)}}^{\tilde{p}_2^{(1)}}\int_{\tilde{p}_1}^{\tilde{p}_2}\widehat{\omega}\right).$$

This formula coincides with that given by Klein [19].

5.7. Example: Hyperelliptic case. Consider the case

$$f(x,y) = y^2 - x^{2g+1} - \sum_{i=0}^{2g} \lambda_{i,0} x^i.$$

We set $\lambda_{2g+1,0} = 1$. One can take $c_{i_1,0;i_2,0} = (i_2 - i_1)\lambda_{i_1+i_2+2,0}$ for $0 \le i_1 \le g-1$, $i_1 + 1 \le i_2 \le 2g - i_1$ and all other $c_{i_1 j_1; i_2 j_2}$ to be zero. Then [9, 2]

$$\begin{aligned}\widehat{\omega} &= d_{p_2}\left(\frac{y_1 + y_2}{2y_1(x_1 - x_2)}\right) + \sum_{i=1}^{g} du_i(p_1) dr_i(p_2)\\ &= \frac{2y_1y_2 + \sum_{i=0}^{g} x_1^i x_2^i \left(2\lambda_{2i,0} + \lambda_{2i+1,0}(x_1 + x_2)\right)}{4y_1y_2(x_1 - x_2)^2} dx_1 dx_2,\end{aligned}$$

and

$$\begin{aligned}du_i &= -\frac{x^{g-i}dx}{2y},\\ dr_i &= -\sum_{k=g+1-i}^{g+i} (k - g + i)\lambda_{k+g+2-i,0} \frac{x^k dx}{2y}.\end{aligned}$$

Set $p_i = (x_i, y_i)$ and $q_i = (X_i, Y_i)$. The formula of the sigma function is given by (61) with

$$F_N = \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_1 & \cdots & x_N & X_1 & \cdots & X_N \\ \vdots & & \vdots & \vdots & & \vdots \\ x_1^g & \cdots & x_N^g & X_1^g & \cdots & X_N^g \\ y_1 & \cdots & y_N & -Y_1 & \cdots & -Y_N \\ x_1^{g+1} & \cdots & x_N^{g+1} & X_1^{g+1} & \cdots & X_N^{g+1} \\ x_1 y_1 & \cdots & x_N y_N & -X_1 Y_1 & \cdots & -X_N Y_N \\ \vdots & & \vdots & \vdots & & \vdots \end{vmatrix},$$

and

$$C_N = (-1)^{\frac{1}{2}N(N+1)+gN}.$$

This is Klein's formula [19, 20].

6. Concluding remarks. In this paper we have established the formula for the sigma function associated to an (n,s)-curve X in terms of algebraic integrals. Some properties of the series expansion of the sigma function are deduced from it. Namely it is shown that the first term of the expansion becomes Schur function corresponding to the partition determined from the gap sequence at infinity and the expansion coefficients are homogeneous polynomials of the coefficients of the defining equation of the curve.

The building block of the formula is the prime function. It is a multi-valued function on $X \times X$ with some vanishing property and has the same transformation rule as that of the sigma function if one of the variables goes round a cycle of X. Remarkably, in the case of hyperelliptic curves, Ônishi [26] has constructed a function with the same properties as a certain derivative of the sigma function. By the uniqueness of such a function they coincide. In general it is expected that the prime function can be expressed as a derivative of the sigma function.

Fay's determinant formula ((43) in [17]) expresses Riemann's theta function in terms of prime form and some determinant. To get a formula of the sigma function one needs to take a limit sending some parameter to a singular point of the theta divisor. To this end one has to know the structure of sigularities of the theta divisor. So it is difficult to get the formula of the sigma function by taking a limit of Fay's formula in general. We have avoided this task and directly constructed a formula of the sigma function.

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