

ON ALMOST CONTACT 3-STRUCTURE

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Introduction. An almost contact structure (ϕ, ξ, η) ¹⁾ on a differentiable manifold is an aggregate consisting of a tensor field ϕ of type (1,1), a contravariant vector field ξ and a covariant vector field η which satisfy

$$\eta(\xi) = 1, \quad \phi\phi = -I + \xi \otimes \eta,$$

where \otimes means the tensor product, I is the identity tensor.

It is known that almost contact structure has many similarities to almost complex one. The main purpose of this paper is to discuss a structure of contact type similar to almost quaternion structure²⁾.

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1. Almost contact 3-structure. Suppose a differentiable manifold admits 3 almost contact structures (ϕ_i, ξ_i, η_i) , $i = 1, 2, 3$, satisfying

$$\begin{aligned} \eta_i(\xi_j) &= \eta_j(\xi_i) = 0, \\ \phi_i \xi_j &= -\phi_j \xi_i = \xi_k, \\ \eta_i \circ \phi_j &= -\eta_j \circ \phi_i = \eta_k, \\ \phi_i \phi_j - \xi_i \otimes \eta_j &= -\phi_j \phi_i + \xi_j \otimes \eta_i = \phi_k, \end{aligned}$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$. We shall call such a structure an almost contact 3-structure. First we have the following

THEOREM 1. *If a differentiable manifold admits 2 almost contact structures (ϕ_i, ξ_i, η_i) , $i = 1, 2$, satisfying*

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- 1) S. Sasaki, [2], S. Sasaki and Y. Hatakeyama, [3].
 - 2) H. Wakakuwa, [5], M. Obata, [6].

$$(1.1) \quad \eta_1(\xi_2) = \eta_2(\xi_1) = 0,$$

$$(1.2) \quad \phi_1\xi_2 = -\phi_2\xi_1,$$

$$(1.3) \quad \eta_1 \circ \phi_2 = -\eta_2 \circ \phi_1,$$

$$(1.4) \quad \phi_1\phi_2 - \xi_1 \otimes \eta_2 = -\phi_2\phi_1 + \xi_2 \otimes \eta_1,$$

it admits an almost contact 3-structure.

In fact, we can get the third structure by putting

$$\xi_3 = \phi_1\xi_2, \quad \eta_3 = \eta_1 \circ \phi_2,$$

$$\phi_3 = \phi_1\phi_2 - \xi_1 \otimes \eta_2.$$

2. Associated metric. Let (ϕ, ξ, η) be an almost contact structure on a differentiable manifold M^n . A Riemannian metric (positive definite) g is called to be associated to the structure if it satisfies

$$g(\xi, X) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vectors X and Y .

The following theorem is easily obtained.

THEOREM 2. *Suppose a differentiable manifold M^n admits 2 almost contact structure (ϕ_i, ξ_i, η_i) , $i = 1, 2$. If there is a Riemannian metric associated to both of the structures, then (1.1), (1.2) and (1.3) follows from (1.4).*

REMARK. If a Riemannian metric is associated to 2 structures of an almost contact 3-structure, it is associated to the third one.

When a Riemannian metric is associated to each structure of an almost contact 3-structure, it will be called an associated metric of the 3-structure. We shall discuss the existence of such metrics.

LEMMA. *For an almost contact 3-structure (ϕ_i, ξ_i, η_i) , $i = 1, 2, 3$, there exists a Riemannian metric h such that*

$$h(\xi_i, X) = \eta_i(X), \quad i = 1, 2, 3,$$

hold good for any vector X .

PROOF. Let j be an associated metric to (ϕ_1, ξ_1, η_1) and define a new metric f by

$$f(X, Y) = j(X - \eta_2(X)\xi_2, Y - \eta_2(Y)\xi_2) + \eta_2(X)\eta_2(Y).$$

Next define h by

$$h(X, Y) = f(X - \eta_3(X)\xi_3, Y - \eta_3(Y)\xi_3) + \eta_3(X)\eta_3(Y),$$

and we can see that it has the required property.

Q. E. D.

If we define a new metric g from h by

$$4g(X, Y) = h(X, Y) + \sum_{i=1}^3 \{h(\phi_i(X), \phi_i(Y)) + \eta_i(X)\eta_i(Y)\},$$

we can check that g is associated to (ϕ_i, ξ_i, η_i) , $i = 1, 2, 3$. Thus we have

THEOREM 3. *In a differentiable manifold of almost contact 3-structure, there is an associated metric of the structure.*

3. Dimension. It is well known³⁾ that the dimension of an almost contact manifold is odd. We shall study the corresponding problem in this section.

Consider M^n with an almost contact 3-structure (ϕ_i, ξ_i, η_i) , $i = 1, 2, 3$. Let $M^{n+1} = M^n \times R$ be the product manifold of M^n with the real line R . If we define a tensor field Φ_i of type $(1, 1)$ on M^{n+1} by

$$\Phi_i = \begin{pmatrix} \phi_i & \xi_i \\ -\eta_i & 0 \end{pmatrix}, \quad i = 1, 2, 3,$$

they define an almost quaternion structure, i. e., they satisfy

$$\Phi_1\Phi_1 = \Phi_2\Phi_2 = \Phi_3\Phi_3 = -I,$$

$$\Phi_i\Phi_j = -\Phi_j\Phi_i = \Phi_k,$$

where I denotes the unit tensor and (i, j, k) is any cyclic permutation of $(1, 2, 3)$.

On the other hand, it is known⁴⁾ that the dimension of a manifold with an almost quaternion structure is of the form $4(m+1)$. Thus we get

3) As to the proof, see S. Sasaki, [2].

4) M. Obata, [6].

THEOREM 4. *The dimension of a manifold with almost contact 3-structure is $4m+3$ for a non-negative integer m .*

Let g be an associated metric of an almost contact 3-structure and define a metric \tilde{g} on M^{n+1} by

$$\tilde{g} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Then \tilde{g} satisfies

$$\tilde{g}(\tilde{X}, \tilde{Y}) = \tilde{g}(\Phi_i \tilde{X}, \Phi_i \tilde{Y}), \quad i = 1, 2, 3,$$

for any vectors \tilde{X} and \tilde{Y} on M^{n+1} .

4. Structure group of tangent bundle. Let g be an associated metric to an almost contact 3-structure (ϕ_i, ξ_i, η_i) , $i = 1, 2, 3$. Put $e_{4m+3} = \xi_1$, $e_{4m+2} = \xi_2$, $e_{4m+1} = \xi_3$, then they are an orthonormal field over M^{4m+3} globally. Let e_{4m} be a unit vector field defined locally in a neighborhood U such that e_{4m}, \dots, e_{4m+3} are orthonormal. If we put

$$e_{4m-1} = \phi_1 e_{4m}, \quad e_{4m-2} = \phi_2 e_{4m}, \quad e_{4m-3} = \phi_3 e_{4m},$$

these 7 vectors $e_{4m-3}, \dots, e_{4m+3}$ are orthonormal. In this way we finally have an orthonormal frame field in U consisting of

$$\begin{aligned} e_{4\lambda}, \quad \lambda &= 1, \dots, m, \\ e_{4\lambda-i} &= \phi_i e_{4\lambda}, \quad i = 1, 2, 3, \\ e_{4m+1}, \quad e_{4m+2}, \quad e_{4m+3}. \end{aligned}$$

The product manifold $M^{4m+4} = M^{4m+3} \times R$ has the almost quaternion structure Φ_i and the metric \tilde{g} stated in §3. Let e_{4m+4} be the unit vector on R . Moreover let E_1, \dots, E_{4m+4} be the orthonormal frame in $U \times R$ which are obtained by natural extension from e_1, \dots, e_{4m+4} . We have easily

$$(4.1) \quad \Phi_i E_{4\lambda} = E_{4\lambda-i}, \quad \lambda = 1, \dots, m+1, \quad i = 1, 2, 3.$$

Let U' be a neighborhood on M^{4m+3} such that $U \cap U' \neq \emptyset$. Let e'_1, \dots, e'_{4m+4} , E'_1, \dots, E'_{4m+4} be the corresponding vectors. There exists an orthogonal matrix $A = (a_{\alpha\beta})$ such that

$$(4.2) \quad \sum_{\beta=1}^{4m+4} a_{\alpha\beta} E_{\beta} = E'_{\alpha}, \quad \alpha = 1, \dots, 4m+4$$

and it is known⁵⁾ that $A \in Sp(m+1)$ = the real representation of the symplectic group, by taking account of the choice (4.1) of frames.

As $E_{4m+i} = E'_{4m+i}$ for $i = 1, 2, 3, 4$, we know that

$$A = \begin{pmatrix} B & 0 \\ 0 & I_4 \end{pmatrix}, \quad B \in Sp(m).$$

If we restrict our attention to M^{4m+3} , we get from (4.2)

$$\sum_{\beta=1}^{4m+3} c_{\alpha\beta} e_{\beta} = e'_{\alpha}, \quad \alpha = 1, \dots, 4m+3,$$

$$C = (c_{\alpha\beta}) = \begin{pmatrix} B & 0 \\ 0 & I_3 \end{pmatrix}, \quad B \in Sp(m).$$

Thus we have

THEOREM 5. *Let M^{4m+3} be a differentiable manifold of almost contact 3-structure. Then the structure group of the tangent bundle is reducible to $Sp(m) \times I_3$.*

5. Sasakian structure and Betti numbers. Consider a Riemannian manifold with metric g . For a vector field ξ we shall define a covariant vector η by

$$(5.1) \quad g(\xi, X) = \eta(X),$$

where X is any vector. The condition for ξ to be a Killing vector field is that

$$(\nabla_X \eta)Y + (\nabla_Y \eta)X = 0$$

holds good for any vectors X and Y .

A Sasakian manifold is a Riemannian manifold M^n with metric g such that it admits a unit Killing vector field ξ satisfying

$$(\nabla_Y \phi)X = \eta(X)Y - g(Y, X)\xi,$$

where ϕ is defined by

5) M. Obata, [6], [7]. H. Wakakuwa, [5].

$$(5.2) \quad \phi X = \nabla_X \xi$$

and X and Y are any vectors. In this case ξ is called a Sasakian structure. For a Sasakian structure ξ , (ϕ, ξ, η) defined by (5.1) and (5.2) is an almost contact structure, and g is an associated metric.

The following theorem is easily obtained.

THEOREM 6. *Let ξ_1 and ξ_2 be Sasakian structures orthogonal to each other. Then $\xi_3 = \phi_1 \xi_2$ is a Sasakian structure orthogonal to ξ_1 and ξ_2 . In this case, (ϕ_i, ξ_i, η_i) , $i = 1, 2, 3$, is an almost contact 3-structure.*

We shall call Sasakian structure stated in Theorem 6 a Sasakian 3-structure.

In a Sasakian manifold M^n with Sasakian structure ξ , let Ψ be an operator defined by

$$(5.3) \quad (\Psi u)(X_1, \dots, X_p) = u(\phi X_1, \dots, \phi X_p)$$

for differential p -form u , where X_1, \dots, X_p are vectors.

The following theorems are known⁶⁾.

THEOREM A. *In a compact Sasakian manifold M^n , Ψu is harmonic for a harmonic p -form u , provided that $p < (n+1)/2$.*

THEOREM B. *In a compact Sasakian manifold M^n , a harmonic p -form u , ($p < (n+1)/2$), is orthogonal to ξ , i.e., $u(\xi, X_2, \dots, X_p) = 0$ holds good for any vectors X_2, \dots, X_p .*

By making use of these theorems, we can get

THEOREM 7⁷⁾. *For a compact manifold M^{4m+3} of Sasakian 3-structure ξ_1, ξ_2, ξ_3 , the p -th Betti number is of the form $4q$ for a non-negative integer q , if p is odd and satisfies $p < (n+1)/2 = 2m+2$.*

PROOF. Let $\mathfrak{X}_p(M)$ be the vector space of all harmonic p -form over M^{4m+3} . For ξ_i we define Ψ_i similarly to (5.3). These operators Ψ_i define an almost quaternion structure in $\mathfrak{X}_p(M)$ and hence $\dim \mathfrak{X}_p(M) = 4q$. Q. E. D.

6. Examples. (i) R^{4m+3} . Consider the 3 dimensional Euclidean space R^3 .

6) S. Tachibana, [4]. S. Tachibana and Y. Ogawa, [9]. T. Fujitani, [10].

7) As to Kählerian manifold with quaternion structure, see H. Wakakuwa, [5].

Let e_1, e_2, e_3 be the fundamental vectors and define

$$\begin{aligned}
 \phi_1 &= (-e_1, 0, e_3), & \phi_2 &= (0, e_3, -e_2), & \phi_3 &= (-e_2, e_1, 0), \\
 (6.1) \quad \xi_1 &= e_2, & \xi_2 &= e_1, & \xi_3 &= -e_3, \\
 \eta_1 &= e_2^*, & \eta_2 &= e_1^*, & \eta_3 &= -e_3^*,
 \end{aligned}$$

where e_i^* means the dual of e_i . Then these structures give an almost contact 3-structure and $g = I_3$ is an associated metric.

Next consider the $4m$ dimensional Euclidean space with standard quaternion structure $\Phi_i (i = 1, 2, 3)$ and define

$$\hat{\phi}_i = \begin{pmatrix} \phi_i & 0 \\ 0 & \Phi_i \end{pmatrix}, \quad \hat{\xi}_i = \begin{pmatrix} \xi_i \\ 0 \end{pmatrix}, \quad \hat{\eta}_i = (\eta_i, 0),$$

where (ϕ_i, ξ_i, η_i) are given by (6.1). They are an almost contact 3-structure in R^{4m+3} with associated metric I_{4m+3} .

(ii) S^{4m+3} . Consider the Euclidean space R^{2n+2} with the standard complex structure Φ . Let N be the unit normal vector field on the unit sphere S^{2n+1} with the origin as its center. It is known⁸⁾ that $\xi = \Phi N$ gives S^{2n+1} a Sasakian structure and the induced metric on S^{2n+1} is just an associated one. If we regard S^{2n+1} as a projective space P^{2n+1} by identifying anti-podal points, ξ defined above induces a Sasakian structure on P^{2n+1} .

Now consider the Euclidean space R^{4m+4} with standard quaternion structure $\Phi_i, i = 1, 2, 3$. They satisfy

$$\Phi_i \Phi_i = -I, \quad {}^t \Phi_i = -\Phi_i, \quad i = 1, 2, 3, \quad \Phi_1 \Phi_2 = \Phi_3.$$

Define ξ_i by $\xi_i = \Phi_i N, i = 1, 2, 3$. Then the inner product of ξ_1 and ξ_2 satisfies

$$\begin{aligned}
 I(\xi_1, \xi_2) &= I(\Phi_1 N, \Phi_2 N) = ({}^t \Phi_1 \Phi_2)(N, N) \\
 &= -(\Phi_1 \Phi_2)(N, N) = -\Phi_3(N, N) = 0,
 \end{aligned}$$

where $I = I_{4m+4}$ is the Euclidean metric in R^{4m+4} . Thus ξ_1 and ξ_2 are orthogonal to each other, and hence S^{4m+3} and P^{4m+3} have a Sasakian 3-structure by virtue of Theorem 6.

8) Y. Tashiro and S. Tachibana, [11].

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