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ON ALMOST COSYMPLECTIC (κ, μ, ν)-SPACES

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Abstract. An almost cosymplectic (κ, μ, ν)-space is by definition an almost cosymplectic manifold whose structure tensor fields φ, ξ, η, g satisfy a certain special curvature condition (see formula (16)). This condition is invariant with respect to the so-called $\mathcal{D}$-homothetic transformations of almost cosymplectic structures. For such manifolds, the tensor fields φ, h $(= (1/2)\mathcal{L}_\xi\varphi)$, A $(= -\nabla\xi)$ fulfill a certain system of differential equations. It is proved that the leaves of the canonical foliation of an almost cosymplectic (κ, μ, ν)-space with $\kappa < 0$ are locally flat Kählerian manifolds. A local characterization of such manifolds is established up to a $\mathcal{D}$-homothetic transformation of the almost cosymplectic structures.

1. Preliminaries. Let M be a connected, differentiable manifold of dimension $2n+1$, $n \geqslant 1$. A quadruple (φ, ξ, η, g) is called an almost contact metric structure [1] on M if φ, ξ, η, g are, respectively, a $(1,1)$-tensor field, a vector field, a 1-form, a Riemannian metric on M and

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta(X) = g(X, \xi), \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

In the above and in the sequel, $X, Y, \ldots$ denote arbitrary vector fields on M if not otherwise stated. Let Φ denote the fundamental 2-form associated to an almost contact metric structure by $\Phi(X, Y) = g(\varphi X, Y)$.

Given an almost contact metric structure (φ, ξ, η, g) on M, we say that the manifold M and the structure (φ, ξ, η, g) are:

(a) almost cosymplectic if the forms η and Φ are closed [10];

(b) cosymplectic if they are almost cosymplectic and the almost contact structure (φ, ξ, η) is normal (equivalently, $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection determined by g [1]).

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Let M be an almost cosymplectic manifold. Let $\mathcal{F}$ be the codimension 1 foliation of M, which is generated by the integrable distribution $\mathcal{D} = \operatorname{Ker}\eta$. Since $\mathcal{D} = \operatorname{Im}\varphi$, $\mathcal{D}$ is φ-invariant and a leaf (a maximal integral submanifold) N of $\mathcal{F}$ is φ-invariant. Hence, φ induces an almost complex structure J ($J^2 = -I$) on N by $J\widetilde{X} = \varphi\widetilde{X}$ for any vector field $\widetilde{X}$ tangent to N. Let G be the Riemannian metric induced on N, $G(\widetilde{X},\widetilde{Y}) = g(\widetilde{X},\widetilde{Y})$. Then the pair (J,G) becomes an almost Hermitian structure on N ($G(J\widetilde{X},J\widetilde{Y}) = G(\widetilde{X},\widetilde{Y})$). The fundamental form Ω ($\Omega(\widetilde{X},\widetilde{Y}) = g(J\widetilde{X},\widetilde{Y})$) of (J,G) is closed since it is the pull-back of the closed form Φ. Therefore, (J,G) is an almost Kählerian structure on N. In the case when J is a complex structure, (J,G) becomes a Kählerian structure on N. If (J,G) is Kählerian on every leaf of $\mathcal{F}$, we will say that M is an almost cosymplectic manifold with Kählerian leaves [5, 13].

2. Auxiliary tensor fields. Let M be an almost cosymplectic manifold. Consider the $(1,1)$-tensor field A defined on M by

$$AX = -\nabla_X\xi. \tag{1}$$

This is a geometric interpretation of A: for an arbitrary leaf N of $\mathcal{F}$, the vector field ξ restricted to N is its normal vector field and $A\widetilde{X} = -\nabla_{\widetilde{X}}\xi$ is the shape operator with respect to ξ.

Another application of A follows from the following fact [11]: an almost cosymplectic manifold M has Kählerian leaves if and only if

$$(\nabla_X\varphi)Y = -g(\varphi AX,Y)\xi + \eta(Y)\varphi AX. \tag{2}$$

The main algebraic properties of A can be found in [13],

$$g(AX,Y) = g(AY,X), \quad A\varphi + \varphi A = 0, \quad A\xi = 0, \quad \eta\circ A = 0. \tag{3}$$

For further use, we also define the (1,1)-tensor field h by

$$h = \frac{1}{2}\,\mathcal{L}_\xi\varphi. \tag{4}$$

Observe that the tensor fields A and h are related by

$$h = A\varphi, \qquad A = \varphi h. \tag{5}$$

In fact, using (3) and $\nabla_\xi\varphi = 0$ (cf. eq. (2.10) in [11]), we get

$$2hX = (\mathcal{L}_\xi\varphi)X = [\xi,\varphi X] - \varphi[\xi,X] = 2A\varphi X.$$

As a consequence of (3) and (5), one finds the following algebraic properties of h (cf. also [7, 8], however the tensor field h defined in those papers differs in sign from ours)

$$g(hX,Y) = g(hY,X), \quad h\varphi + \varphi h = 0, \quad hA + Ah = 0, \quad h\xi = 0, \quad \eta\circ h = 0. \tag{6}$$

In the sequel, we also need the following lemma:

LEMMA 1. *For the tensor field A, we have*

$$(\nabla_\xi A)\varphi + \varphi(\nabla_\xi A) = 0, \tag{7}$$

$$\mathcal{L}_\xi A = \nabla_\xi A \tag{8}$$

$$R(X,Y)\xi = -(\nabla_X A)Y + (\nabla_Y A)X, \tag{9}$$

$$R(\xi,Y)\xi = -(\nabla_\xi A)Y + A^2 Y, \tag{10}$$

where $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ are the curvature operators.

Proof. The covariant differentiation ∇_ξ of the second equality of (3) and an application of $\nabla_\xi\varphi = 0$ give (7). (8) can be found by a straightforward computation using (1). (9) is just the integrability condition of (1). Finally, (10) follows from (9) by applying $X = \xi$ and the formulas (3), (1). ∎

3. $\mathcal{D}$-homothetic transformations. Let M be an almost cosymplectic manifold and (φ,ξ,η,g) its almost cosymplectic structure. Let $\mathcal{R}_\eta(M)$ be the subring of the ring of smooth functions f on M for which $df \wedge \eta = 0$, or equivalently $df = df(\xi)\eta$.

Consider a $\mathcal{D}$-homothetic transformation of (φ,ξ,η,g) into an almost contact metric structure (φ',ξ',η',g') defined by

$$\varphi' = \varphi, \quad \xi' = \frac{1}{\beta}\,\xi, \quad \eta' = \beta\eta, \quad g' = \alpha g + (\beta^2 - \alpha)\eta\otimes\eta, \tag{11}$$

where α is a positive constant and $\beta \in \mathcal{R}_\eta(M)$, $\beta \neq 0$ at any point of M. Since $d\beta\wedge\eta = 0$, it follows that $d\eta' = 0$. Moreover $d\Phi' = 0$, since the fundamental forms Φ, Φ' of the structures are related by $\Phi' = \alpha\,\Phi$.

Thus, a $\mathcal{D}$-homothetic transformation of an almost cosymplectic structure (φ,ξ,η,g) always gives a new almost cosymplectic structure (φ',ξ',η',g') on the same manifold. For two almost cosymplectic structures (φ,ξ,η,g) and (φ',ξ',η',g') related by (11), we will say that they are $\mathcal{D}$-homothetic. In the sequel, geometric invariants corresponding to the structure (φ',ξ',η',g') will be marked by primes.

$\mathcal{D}$-homothetic transformations of almost contact metric structures with $\alpha,\beta = \text{const.}$ were studied in many papers (see [1, 12, 14], etc.).

PROPOSITION 1. *For $\mathcal{D}$-homothetic almost cosymplectic structures, the Levi-Civita connections ∇' and ∇ are related by*

$$\nabla'_X Y = \nabla_X Y - \frac{\beta^2 - \alpha}{\beta^2}\, g(AX,Y)\xi + \frac{d\beta(\xi)}{\beta}\,\eta(X)\eta(Y)\xi. \tag{12}$$

Proof. Clearly, the operation ∇' defined by the formula (12) is an affine connection on M. ∇' is symmetric by the symmetries of ∇ and A (cf. (3)). Next, using (11), we find

$$\begin{aligned}(\nabla'_X g')(Y,Z) &= \alpha(\nabla'_X g)(Y,Z) + 2\beta d\beta(\xi)\eta(X)\eta(Y)\eta(Z)\\ &\quad + (\beta^2-\alpha)((\nabla'_X\eta)(Y)\eta(Z) + \eta(Y)(\nabla'_X\eta)(Z)),\end{aligned}$$

whence, by applying (12), (1), (3), we obtain $\nabla' g' = 0$, that is, ∇' is metric. Thus, ∇' is the Levi-Civita connection with respect to g', which completes the proof. ∎

PROPOSITION 2. *For $\mathcal{D}$-homothetic almost cosymplectic structures, we have*

$$A' = \frac{1}{\beta}\,A, \qquad h' = \frac{1}{\beta}\,h \tag{13}$$

$$R'(X,Y)\xi' = \frac{1}{\beta}\;R(X,Y)\xi + \frac{d\beta(\xi)}{\beta^2}\;(\eta(X)AY - \eta(Y)AX). \tag{14}$$

Proof. Using (1), (3), (11) and (12), we find

$$A'X = -\nabla'_X\xi' = -\frac{1}{\beta}\nabla_X\xi = \frac{1}{\beta}AX.$$

By the above and (5), (11), we get also the second equality of (13). To prove (14), we need the formula

$$(\nabla'_X A')Y = \frac{1}{\beta}\;(\nabla_X A)Y - \frac{\beta^2-\alpha}{\beta^3}\;g(AX,AY)\xi - \frac{d\beta(\xi)}{\beta^2}\;\eta(X)AY, \tag{15}$$

which is a consequence of (12), (13) and (3). Now, using (9) and (15), we find

$$\begin{aligned} R'(X,Y)\xi' &= -(\nabla'_X A')Y + (\nabla'_Y A')X \\ &= \frac{1}{\beta}\;(-(\nabla_X A)Y + (\nabla_Y A)X) + \frac{d\beta(\xi)}{\beta^2}\;(\eta(X)AY - \eta(Y)AX) \\ &= \frac{1}{\beta}\;R(X,Y)\xi + \frac{d\beta(\xi)}{\beta^2}\;(\eta(X)AY - \eta(Y)AX), \end{aligned}$$

completing the proof. ∎

4. Auxiliary results. We are specially interested in almost cosymplectic manifolds whose almost cosymplectic structure (φ,ξ,η,g) satisfies the condition

$$R(X,Y)\xi = \eta(Y)(\kappa I + \mu h + \nu A)X - \eta(X)(\kappa I + \mu h + \nu A)Y \tag{16}$$

with $\kappa,\mu,\nu \in \mathcal{R}_\eta(M)$. In the sequel, such a manifold will be called an almost cosymplectic (κ,μ,ν)-space and (φ,ξ,η,g) will be called an almost cosymplectic (κ,μ,ν)-structure.

Almost cosymplectic manifolds satisfying the condition (16) with $\kappa = \text{const.}$, $\mu = \nu = 0$ were studied in [4]; and with $\kappa,\mu = \text{const.}$, $\nu = 0$ in [7, 8, 9].

Contact metric manifolds fulfilling the condition (16) with $\kappa,\mu = \text{const.}$ and $\nu = 0$ were extensively studied in [2, 3] and many other papers; see also the monograph [1] for conditions of this type.

PROPOSITION 3. *For $\mathcal{D}$-homothetic almost cosymplectic structures, if (φ,ξ,η,g) is an almost cosymplectic (κ,μ,ν)-structure, then (φ',ξ',η',g') is an almost cosymplectic (κ',μ',ν')-structure with $\kappa',\mu',\nu' \in \mathcal{R}_{\eta'}(M) = \mathcal{R}_\eta(M)$ being related to κ,μ,ν by*

$$\kappa' = \frac{\kappa}{\beta^2}, \quad \mu' = \frac{\mu}{\beta}\;, \quad \nu' = \frac{\nu\beta - d\beta(\xi)}{\beta^2}, \tag{17}$$

that is,

$$R'(X,Y)\xi' = \eta'(Y)(\kappa' I + \mu' h' + \nu' A')X - \eta'(X)(\kappa' I + \mu' h' + \nu' A')Y. \tag{18}$$

Proof. By applying (16) and next (11), (13) in (14) and making some computations, we get both (18) and (17). ∎

The following algebraic lemma will be useful.

LEMMA 2. *Let B be a symmetric (1,1)-tensor field on an almost contact metric manifold such that $B\xi = 0$. Then B has a unique decomposition into a sum $B = B^- + B^+$, where B^-, B^+ are symmetric (1,1)-tensor fields such that*

$$B^-\xi = B^+\xi = 0, \quad \varphi B^- - B^-\varphi = 0, \quad \varphi B^+ + B^+\varphi = 0.$$

Proof. Given B define

$$B^- = \frac{1}{2}(B - \varphi B\varphi), \quad B^+ = \frac{1}{2}(B + \varphi B\varphi).$$

It is a straightforward verification that we obtained the desired decomposition. The uniqueness of the decomposition can also be easily seen. ∎

PROPOSITION 4. *For an almost cosymplectic (κ,μ,ν)-space, the tensor field A and the function κ satisfy the relations*

$$A^2Y = -\kappa(Y - \eta(Y)\xi)\ , \tag{19}$$

$$(\nabla_\xi A)Y = \mu hY + \nu AY, \tag{20}$$

$$d\kappa(\xi) = 2\nu\kappa. \tag{21}$$

Proof. Let us suppose $B = R(\xi,\cdot)\xi$. First, note that for the tensor field B, the formula (10) gives the decomposition mentioned in Lemma 2 with $B^- = A^2$ and $B^+ = -\nabla_\xi A$. This can be easily verified with the help of (3) and (7).

On the other hand, putting $X = \xi$ in (16) and using (3) and (6), we have

$$B = R(\xi,\cdot)\xi = -\kappa(I - \eta\otimes\xi) - \mu h - \nu A.$$

Considering the right hand side of the above formula and the formulas (3), (6), we find $B^- = -\kappa(I - \eta\otimes\xi)$ and $B^+ = -\mu h - \nu A$. Hence, by the uniqueness, we obtain (19) and (20).

The covariant differentiation ∇_ξ of (19) and an application of the relations $\nabla_\xi\xi = 0$, $\nabla_\xi\eta = 0$ (which can be found in [11]), (6) give $d\kappa(\xi)(I - \eta\otimes\xi) = -2\nu A^2$, which again by (19) leads to (21). ∎

From (19) and (3) it follows that at every point of an almost cosymplectic (κ,μ,ν)-space: (1) $\kappa \leqslant 0$; (2) $\kappa = 0$ if and only if $A = 0$; (3) if $\kappa < 0$, then the eigenvalues of A are 0 of multiplicity 1 and $\pm\sqrt{-\kappa}$ both of multiplicity n.

LEMMA 3. *For an almost cosymplectic (κ,μ,ν)-space, if $\kappa = 0$ at a certain point of M, then κ vanishes identically on M.*

Proof. Let Z be the closed subset of M containing the points q at which $\kappa(q) = 0$. Suppose that $p \in Z$. Choose a coordinate neighborhood $U = (-a,a)\times\widetilde{U}$ around p such that t is the coordinate on the open interval $(-a,a)$, $(x^1,\ldots,x^{2n})$ are the coordinates on $\widetilde{U}$ and $\xi = \partial/\partial t$, $\eta = dt$. Since $d\kappa\wedge\eta = 0$, the function κ restricted to Z depends on $t \in (-a,a)$ only; and by (21) it satisfies the linear differential equation $d\kappa/dt = 2\nu\kappa$. Since κ vanishes at a certain t, $\kappa = 0$ identically on $(-a,a)$. Hence $\kappa = 0$ on the whole of U. Therefore, the set Z is open. Finally, $Z = M$ since M is connected and Z is nonempty. ∎

PROPOSITION 5. *For an almost cosymplectic (κ,μ,ν)-space, the tensor fields φ, h, A fulfill the following system of differential equations:*

$$\mathcal{L}_\xi\varphi = 2h, \quad \mathcal{L}_\xi h = -2\kappa\varphi + \nu h - \mu A, \quad \mathcal{L}_\xi A = \mu h + \nu A. \tag{22}$$

Proof. The first equation follows from (4). The third equation follows from (8) and (20). Now, taking the Lie derivative of the first relation of (5), next using the just obtained third equation and (4), we find

$$\mathcal{L}_\xi h = (\mathcal{L}_\xi A)\varphi + A(\mathcal{L}_\xi\varphi) = (\mu h + \nu A)\varphi + 2Ah,$$

which with the help of (3), (5) and (19) leads to the second equation. ∎

5. Main results. By virtue of Lemma 3, the following two typical situations should be treated for almost cosymplectic (κ,μ,ν)-spaces M: $\kappa = 0$ identically on M or $\kappa < 0$ at every point of M.

PROPOSITION 6. *An almost cosymplectic $(0,\mu,\nu)$-space is locally a product of an open interval and an almost Kählerian manifold.*

Proof. When $\kappa = 0$, then $A = 0$ by (19), and next $\nabla\xi = 0$ by (1). Hence the assertion follows. ∎

In the sequel, we restrict our investigations to the case when $\kappa < 0$ because of the above proposition.

THEOREM 1. *Let M be an almost cosymplectic (κ,μ,ν)-space with $\kappa < 0$. Then the leaves of the canonical foliation $\mathcal{F}$ of M are locally flat Kählerian manifolds.*

Proof. For an arbitrary almost cosymplectic manifold, the following curvature identity is well known [11]:

$$\begin{aligned} R(X,Y,\varphi Z,\xi) - R(\varphi X,\varphi Y,\varphi Z,\xi) - R(\varphi X,Y,Z,\xi)& \\ - R(X,\varphi Y,Z,\xi) = -2(\nabla_{AZ}\Phi)(X,Y)&, \end{aligned}$$

where $R(\cdot,\cdot,\cdot,\cdot)$ denotes the Riemann curvature $(0,4)$-tensor,

$$R(X_1,X_2,X_3,X_4) = g(R(X_1,X_2)X_3,X_4).$$

On the other hand, for our almost cosymplectic (κ,μ,ν)-space, using (16), we find

$$\begin{aligned} R(X,Y,\varphi Z,\xi) - R(\varphi X,\varphi Y,\varphi Z,\xi) - R(\varphi X,Y,Z,\xi)& \\ - R(X,\varphi Y,Z,\xi) = -2\kappa(\eta(Y)g(X,\varphi Z) - \eta(X)g(Y,\varphi Z))&, \end{aligned}$$

which applied to the previous relation gives

$$(\nabla_{AZ}\Phi)(X,Y) = \kappa(\eta(Y)g(X,\varphi Z) - \eta(X)g(Y,\varphi Z)).$$

Putting AZ instead of Z into the last equation and taking into account (19), $\kappa < 0$ and $\nabla_\xi\Phi$, we get

$$(\nabla_Z\Phi)(X,Y) = \eta(X)g(Y,\varphi AZ) - \eta(Y)g(X,\varphi AZ),$$

which is equivalent to (2). Thus, by a result of [13], M is almost cosymplectic with Kählerian leaves.

The rest of the proof will be divided into two parts.

In the first part, we will prove that the leaves of $\mathcal{F}$ are flat in the case when $\kappa = -1$. Let N be an arbitrary leaf of the canonical foliation $\mathcal{F}$ and (J, G) be the induced Kählerian structure on N. Let $\widetilde{A}$ be the Weingarten operator of N so that we have $\widetilde{A}\widetilde{X} = A\widetilde{X}$ for any vector field tangent to N.

By $A\varphi + \varphi A = 0$ and (19), $\widetilde{A}$ fulfills the following relations:

$$\widetilde{A}J + J\widetilde{A} = 0, \quad \widetilde{A}^2 = I. \tag{23}$$

They imply that ± 1 are the eigenvalues of $\widetilde{A}$ both of the same multiplicity. The corresponding eigendistributions will be denoted by $\mathcal{D}_1$ and $\mathcal{D}_2$, $\dim \mathcal{D}_1 = \dim \mathcal{D}_2 = n$.

The tensor $\widetilde{A}$, being the Weingarten operator of N, satisfies the Codazzi equation

$$R(\widetilde{X}, \widetilde{Y})\xi = -(\widetilde{\nabla}_{\widetilde{X}}\widetilde{A})\widetilde{Y} + (\widetilde{\nabla}_{\widetilde{Y}}\widetilde{A})\widetilde{X},$$

where $\widetilde{\nabla}$ is the Levi-Civita connection with respect to G. However, by (16), $R(\widetilde{X}, \widetilde{Y})\xi = 0$ and the last identity turns into

$$(\widetilde{\nabla}_{\widetilde{X}}\widetilde{A})\widetilde{Y} - (\widetilde{\nabla}_{\widetilde{Y}}\widetilde{A})\widetilde{X} = 0.$$

Now, the tensor field $\tilde{A}$ must be parallel since it is a Codazzi tensor field and has two different constant eigenvalues. Equivalently, the distributions $\mathcal{D}_1$, $\mathcal{D}_2$ are parallel.

In what follows we denote by $\widetilde{X}_1, \widetilde{Y}_1, \widetilde{Z}_1, \ldots$ and $\widetilde{X}_2, \widetilde{Y}_2, \widetilde{Z}_2, \ldots$ vector fields belonging to $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

For the curvature tensor $\widetilde{R}$ of $\widetilde{\nabla}$, the parallelity of $\mathcal{D}_1$ and $\mathcal{D}_2$ implies

$$\widetilde{R}(\widetilde{X}_i, \widetilde{Y}_j)\widetilde{Z}_k = 0, \tag{24}$$

if $\widetilde{X}_i, \widetilde{Y}_j, \widetilde{Z}_k$ do not belong to the same distribution. Thus, $\widetilde{R}$ is completely determined by its behavior on $\mathcal{D}_i$, $i = 1, 2$. However, we will show that $\widetilde{R}|_{\mathcal{D}_i} = 0$ for $i = 1, 2$. Indeed, by (23), we have $J\widetilde{Z}_1 \in \mathcal{D}_2$ and $J\widetilde{Z}_2 \in \mathcal{D}_1$. Consequently, by virtue of the identity $\widetilde{R}(\widetilde{X}, \widetilde{Y}) = \widetilde{R}(J\widetilde{X}, J\widetilde{Y})$ and (24), we have

$$\widetilde{R}(\widetilde{X}_1, \widetilde{Y}_1)\widetilde{Z}_1 = \widetilde{R}(J\widetilde{X}_1, J\widetilde{Y}_1)\widetilde{Z}_1 = 0, \qquad \widetilde{R}(\widetilde{X}_2, \widetilde{Y}_2)\widetilde{Z}_2 = \widetilde{R}(J\widetilde{X}_2, J\widetilde{Y}_2)\widetilde{Z}_2 = 0.$$

Thus, $\widetilde{R} = 0$ identically on N, that is, G is locally flat.

In the second part, we consider the case $\kappa \neq \text{const}$. We make a $\mathcal{D}$-homothetic transformation (11) of the structure (φ, ξ, η, g) with $\alpha = 1$ and $\beta = \sqrt{-\kappa}$. We obtain an almost cosymplectic (κ', μ', ν')-structure $(\varphi', \xi', \eta', g')$ with $\kappa' = -1$. By virtue of the first part, the metric G' induced from g' on N is locally flat. But the metrics G and G' induced from g and g' on the same leaf N are exactly the same. Thus, G is locally flat. ∎

PROPOSITION 7. *An almost cosymplectic (κ, μ, ν)-structure, $\kappa < 0$, can be $\mathcal{D}$-homothetically transformed to an almost cosymplectic $(-1, \mu', 0)$-structure with $\mu' = \mu/\sqrt{-\kappa}$.*

Proof. Let (φ, ξ, η, g) be an almost cosymplectic (κ, μ, ν)-structure. Make the $\mathcal{D}$-homothetic transformation of the structure (φ, ξ, η, g) with $\alpha = 1$ and $\beta = \sqrt{-\kappa}$. Then by Proposition 3, we obtain an almost cosymplectic (κ', μ', ν')-structure $(\varphi', \xi', \eta', g')$ with $\kappa' = -1$, $\mu' = -\mu/\sqrt{-\kappa}$ and a certain ν'; cf. formula (17). But by Proposition 4, formula (21), for the structure $(\varphi', \xi', \eta', g')$, we must have $d\kappa'(\xi') = 2\nu'\kappa'$. This clearly implies $\nu' = 0$. ∎

For almost cosymplectic $(-1, \mu, 0)$-spaces, we have the following local characterization.

Theorem 2. *Let M be an almost cosymplectic manifold of dimension $2n+1$. Given $\mu \in \mathcal{R}_\eta(M)$, the following two conditions* (I) *and* (II) *are equivalent:*

(I) *M is an almost cosymplectic $(-1,\mu,0)$-space, that is,*

$$R(X,Y)\xi = \eta(Y)(-I+\mu h)X - \eta(X)(-I+\mu h)Y. \tag{25}$$

(II) *At any point $p \in M$, there is a neighborhood $U = (-a,a)\times\widetilde{U}$ of p with coordinates $(t,x^1,\dots,x^{2n})$, t being a coordinate on $(-a,a)$ and $(x^1,\dots,x^{2n})$ coordinates on $\widetilde{U}$, and on U the structure tensor fields φ, ξ, η, g can be expressed as*

$$\varphi = \sum \varphi_i^j\, dx^i \otimes \frac{\partial}{\partial x^j}, \quad \xi = \frac{\partial}{\partial t}, \quad \eta = dt, \quad g = dt\otimes dt + \sum g_{ij}\, dx^i \otimes dx^j, \tag{26}$$

where the Latin indices take on values from the range $\{1,2,\dots,2n\}$, the sum is over the repeated indices and φ_i^j, g_{ij} are functions depending on t only and such that

$$\sum \varphi_i^k g_{kj} = +1 \text{ if } j=i+n,\ -1 \text{ if } i=j+n,\ 0 \text{ otherwise}. \tag{27}$$

Moreover, on U the tensor fields A and h can be written as

$$A = \sum A_i^j\, dx^i \otimes \frac{\partial}{\partial x^j}, \qquad h = \sum h_i^j\, dx^i\otimes\frac{\partial}{\partial x^j}, \tag{28}$$

where A_j^i, h_j^i are functions of t only, which satisfy the condition $\sum A_i^s A_s^j = \delta_i^j$ and the following system of differential equations:

$$\frac{d\varphi_i^j}{dt} = 2h_i^j, \qquad \frac{dh_i^j}{dt} = 2\varphi_i^j - \mu A_i^j, \qquad \frac{dA_i^j}{dt} = \mu h_i^j. \tag{29}$$

Proof. (I) $\Rightarrow$ (II). Let M be an almost cosymplectic $(-1,\mu,0)$-space and p be an arbitrary point on M. According to Theorem 1 of [13], choose a coordinate neighborhood U' around p with Darboux coordinates $(t,x^1,\dots,x^{2n})$ such that $U' = (-a,a)\times\widetilde{U}$, $a>0$, where t is a coordinate on $(-a,a)$ and $(x^1,\dots,x^{2n})$ are coordinates on $\widetilde{U}$. With respect to these coordinates, the structure tensor fields φ, ξ, η, g are expressed as in the formulas (26) and (27), but φ_i^j, g_{ij} are functions depending on all coordinates $t,x^1,\dots,x^{2n}$ in general. Note additionally that, by $A\xi=0$ and $h\xi=0$, we also have (28) but with A_i^j, h_i^j depending on the all coordinates $t,x^1,\dots,x^{2n}$ in general. With respect to this coordinate system, (22) takes the form

$$\frac{\partial\varphi_i^j}{\partial t} = 2h_i^j, \qquad \frac{\partial h_i^j}{\partial t} = 2\varphi_i^j - \mu A_i^j, \qquad \frac{\partial A_i^j}{\partial t} = \mu h_i^j. \tag{30}$$

Observe that on U', μ is a function depending on t only.

For any fixed $t\in(-a,a)$, the subset $\{t\}\times\widetilde{U}\subset U_p$ is an open part of a leaf of $\mathcal{F}$. The induced complex structure J and the shape operator $\widetilde{A}$ can be written on $\{t\}\times\widetilde{U}$ in the following way:

$$J = \sum \varphi_i^j(t,\cdot)\, dx^i\otimes\frac{\partial}{\partial x^j}, \qquad \widetilde{A} = \sum A_i^j(t,\cdot)\, dx^i \otimes \frac{\partial}{\partial x^j}.$$

Now, by Theorem 1 (the formula (23) should be considered too), we may assume that $(x^1,\dots,x^{2n})$ are chosen such that on $\{t\}\times\widetilde{U}$

$$J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^{i+n}}, \quad J\frac{\partial}{\partial x^{i+n}} = -\frac{\partial}{\partial x^i}, \quad \widetilde{A}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}, \quad \widetilde{A}\frac{\partial}{\partial x^{i+n}} = -\frac{\partial}{\partial x^{i+n}}.$$

This shows that φ_i^j and A_i^j depend on t only. Consequently, h_i^j are functions of t only in view of (5). And since the components Φ_{ij} of the fundamental form Φ are constants and $g_{ij} = -\sum \varphi_i^k \Phi_{kj}$, then g_{ij} depend on t only. Finally, (30) gives (29), and (19) gives $\sum A_i^s A_s^j = \delta_i^j$.

(II) $\Rightarrow$ (I). We have only to prove that (25) holds under the additional assumptions (28) and (29).

Let $X_i = \partial/\partial x^i$. Then $\nabla_{X_i} X_j = \nabla_{X_j} X_i$, $\nabla_\xi X_i = \nabla_{X_i}\xi$. And since X_i's are Killing vector fields, $g(\nabla_{X_j} X_i, X_k) = 0$ for any triple (X_i, X_j, X_k). Consequently, we have for the Levi-Civita connection

$$\nabla_{X_i} X_j = \nabla_{X_j} X_i = -g(X_i, AX_j)\xi, \quad \nabla_\xi X_i = \nabla_{X_i}\xi = -AX_i, \quad \nabla_\xi \xi = 0.$$

By the above formula and (28), (29), we compute

$$\begin{aligned} R(X_i, X_j)\xi &= [\nabla_{X_i}, \nabla_{X_j}]\xi = -\nabla_{X_i} AX_j + \nabla_{X_j} AX_i \\ &= \sum(-A_j^k \nabla_{X_i} X_k + A_i^k \nabla_{X_j} X_k) = 0, \\ R(\xi, X_i)\xi &= \nabla_\xi \nabla_{X_i}\xi = -\nabla_\xi AX_i = -\sum \Big(\frac{dA_i^k}{dt} X_k + A_i^k \nabla_\xi X_k\Big) \\ &= -\mu \sum h_i^k X_k + A^2 X_i = -\mu h X_i + X_i. \end{aligned}$$

Using the two last formulas, we find

$$\begin{aligned} R(Y,Z)\xi &= Y^i Z^j R(X_i, X_j)\xi + Z^i \eta(Y) R(\xi, X_i)\xi - Y^i \eta(Z) R(\xi, X_i)\xi \\ &= Z^i \eta(Y)(-\mu h X_i + X_i) - Y^i(-\mu h X_i + X_i) \\ &= -\eta(Z)Y + \eta(Y)Z + \mu(\eta(Z)hY - \eta(Y)hZ), \end{aligned}$$

which is just the same as the formula (25). ∎

Investigations of the class of almost cosymplectic (κ,μ,ν)-spaces will be continued in our forthcoming paper [6].

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