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ON ALMOST ESSENTIALLY RUSTON ELEMENTS OF A BANACH ALGEBRA

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Abstract

We introduce the class of "almost essentially Ruston elements" with respect to a homomorphism between two Banach algebras, a class intermediate between Ruston and Fredholm elements.

1 Introduction

Let $\mathbb{C}$ denote the set of all complex numbers and let A and B denote complex Banach algebras, with identities denoted in both cases with 1, and invertible groups A^{-1} and B^{-1}, respectively. By $\sigma(x, A)$ we denote the spectrum of an element $x \in A$. The *radical* of A is the set

$$\mathrm{Rad}(A) = \{x \in A : 1 - Ax \subset A^{-1}\} = \{x \in A : 1 - xA \subset A^{-1}\}.$$

The radical is unchanged if the invertible group A^{-1} is replaced by either the semigroup A^{-1}_{left} of left invertible elements, or the semigroup A^{-1}_{right} of right invertible elements, and can also be realised as the intersection of all maximal proper left ideals, similarly right ideals.

Let S be a subset of A. The *commutant* of S is defined by $\mathrm{comm}(S) = \{x \in A : xs = sx \text{ for all } s \in S\}$. The perturbation class of S, denoted by $P(S)$, is the set

$$P(S) = \{x \in A : x + s \in S \ \text{ for every } s \in S\}.$$

The *quasinilpotents* of A form the set

$$QN(A) = \{x \in A : \sigma(x, A) = \{0\}\} = \{x \in A : 1 - \mathbb{C}x \subset A^{-1}\}. \tag{1}$$

Recall that [6, Theorem 2.11]: if $x,\ y \in A$, then

$$xy = yx \Longrightarrow \sigma(xy, A) \subset \sigma(x, A)\sigma(y, A) \tag{2}$$

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and

$$xy = yx \Longrightarrow \sigma(x+y, A) \subset \sigma(x, A) + \sigma(y, A). \tag{3}$$

From (2) it follows that

$$x \in QN(A) \Longrightarrow \mathrm{comm}(x)x \subset QN(A), \tag{4}$$

which together with (1) implies

$$QN(A) = \{x \in A : 1 - \mathrm{comm}(x)x \subset A^{-1}\}. \tag{5}$$

The radical can be recognised as the perturbation class of the invertible group, and the quasinilpotents as a sort of commutative analogue: there are the equivalencies

$$x \in \mathrm{Rad}(A) \Longleftrightarrow (\forall g)(g \in A^{-1} \Longrightarrow x + g \in A^{-1}) \tag{6}$$

and

$$x \in QN(A) \Longleftrightarrow (\forall g)(g \in A^{-1} \cap \mathrm{comm}(x) \Longrightarrow x + g \in A^{-1}). \tag{7}$$

Observe also that (6) holds separately for the left and right invertible semigroups and that (7) follows from (5) and (1).

Recall that [6, Theorem 1.43]

$$x \in A \text{ is invertible } \Longleftrightarrow x + \mathrm{Rad}(A) \text{ is invertible in } A/\mathrm{Rad}(A). \tag{8}$$

Consequently,

$$\sigma(x, A) = \sigma(x + \mathrm{Rad}(A), A/\mathrm{Rad}(A)) \tag{9}$$

and

$$x \text{ is quasinilpotent} \Longleftrightarrow x + \mathrm{Rad}(A) \text{ is quasinilpotent.} \tag{10}$$

A map $T : A \to B$ is a *homomorphism* if T is linear and satisfies $T(xy) = TxTy$, $x,\ y \in A$, and $T1 = 1$. The homomorphism T has the *Riesz property* if 0 is the only one possible point of accumulation of $\sigma(x, A)$ for every $x \in T^{-1}(0)$, that is, if $Tx = 0$, then $\sigma(x, A)$ is either finite or a sequence converging to 0 [1].

If $T : A \to B$ is a homomorphism, then $T(A^{-1}) \subset B^{-1}$ and hence

$$A^{-1} \subset A^{-1} + T^{-1}(0) \subset T^{-1}(B^{-1}).$$

Recall the following definitions from [1], [3]:

An element $a \in A$ is *T-Fredholm* if it has an invertible image,

$$a \in T^{-1}(B^{-1}),$$

and *T-Weyl* if it splits into the sum of an invertible and an essentially null element:

$$a \in A^{-1} + T^{-1}(0).$$

Thus, a is T-Weyl if

$$a = c + d \text{ with } c \in A^{-1},\ Td = 0.$$

If the previous sum is commutative, then $a \in A$ is *T-Browder*:

$$a \in A^{-1} +_c T^{-1}(0).$$

Corresponding spectra of $a \in A$ are defined as:

$\sigma_f^T(a) = \sigma(Ta, B)$-the Fredholm spectrum,

$\sigma_w^T(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not } T - \text{Weyl}\}$-the Weyl spectrum.

2 Almost essentially Ruston elements

Recall the following definitions from [4]:

An element $a \in A$ is *T-Ruston* if

$$a = c + d \text{ with } c \in A^{-1}, cd - dc \in \{0\}, Td \in QN(B),$$

an element $a \in A$ is *essentially T-Ruston* if

$$a = c + d \text{ with } c \in A^{-1}, cd - dc \in T^{-1}(0), Td \in QN(B).$$

Let us mention that essentially Ruston elements are called almost Ruston elements in [4]. We introduce the following elements which are intermediate between essentially Ruston and Fredholm elements:

An element $a \in A$ is *almost essentially T-Ruston* if

$$a = c + d \text{ with } c \in A^{-1}, cd - dc \in T^{-1}(\text{Rad}(B)), Td \in QN(B).$$

2.1 Theorem. Let $T : A \to B$ be a homomorphism. If $a \in A$ is an almost essentially T-Ruston element, that is $a = c + d$ where $c \in A^{-1}$, $Td \in QN(B)$ and $cd - dc \in T^{-1}(\text{Rad}(B))$, then $\sigma(Ta, B) = \sigma(Tc, B)$ and a is T-Fredholm.

Proof. Let $\pi : B \to B/\text{Rad}(B)$ denote the quotient map and let $a = c + d$ where $c \in A^{-1}$, $Td \in QN(B)$ and $cd - dc \in T^{-1}(\text{Rad}(B))$. Hence $\pi(Tc)$ and $\pi(Td)$ commute. By (9) we have $\sigma(Ta, B) = \sigma(\pi(Ta), B/\text{Rad}(B))$, $\sigma(\pi(Td), B/\text{Rad}(B)) = \sigma(Td, B) = \{0\}$ and $\sigma(\pi(Tc), B/\text{Rad}(B)) = \sigma(Tc, B)$. Then, according to (3), we have

$$\begin{aligned}
\sigma(Ta, B) &= \sigma(\pi(Ta), B/\text{Rad}(B)) \\
&= \sigma(\pi(Tc) + \pi(Td), B/\text{Rad}(B)) \\
&\subset \sigma(\pi(Tc), B/\text{Rad}(B)) + \sigma(\pi(Td), B/\text{Rad}(B)) \\
&= \sigma(\pi(Tc), B/\text{Rad}(B)) \\
&= \sigma(Tc, B).
\end{aligned}$$

As $c = a - d$ and $ad - da \in T^{-1}(\mathrm{Rad}(B))$, we get $\sigma(Tc, B) \subset \sigma(Ta, B)$. Hence $\sigma(Ta, B) = \sigma(Tc, B)$. Since $c \in A^{-1}$, it follows that $0 \notin \sigma(Tc, B)$, which implies that $0 \notin \sigma(Ta, B)$, i.e. $Ta \in B^{-1}$. Thus a is T-Fredholm. □

Let us remark that the fact that every almost essentially T-Ruston element is T-Fredholm can be proved also by using (7):

Suppose that $a \in A$ is an almost essentially T-Ruston element, i.e. $a = c + d$ where $c \in A^{-1}$, $Td \in QN(B)$ and $T(cd - dc) \in \mathrm{Rad}(B)$. Then $\pi(Ta) = \pi(Tc) + \pi(Td)$, $\pi(Tc)$ is invertible, $\pi(Td)$ is quasinilpotent and $\pi(Tc)$ and $\pi(Td)$ commute. According to (7) we conclude that $\pi(Ta)$ is invertible which by (8) implies that Ta is invertible. Hence a is T-Fredholm.

Therefore we have:

$$\text{Browder} \Rightarrow \begin{matrix}\text{Ruston} \\ \text{Weyl}\end{matrix} \Rightarrow \text{essentially Ruston} \Rightarrow \text{almost essentially Ruston} \Rightarrow \text{Fredholm} \tag{11}$$

The essentially Ruston spectrum of $a \in A$ is defined as:

$$\sigma^T_{er}(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not essentially } T - \text{Ruston}\},$$

and the almost essentially Ruston spectrum of $a \in A$ is defined as:

$$\sigma^T_{aer}(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not almost essentially } T - \text{Ruston}\}.$$

Let us remark that

$$\sigma^T_{aer}(a) = \cap\{\sigma(a - d, A) : Td \in QN(B), ad - da \in T^{-1}(\mathrm{Rad}(B))\},$$

and consequently, this spectrum is compact. Clearly,

$$\sigma^T_f(a) \subset \sigma^T_{aer}(a) \subset \sigma^T_{er}(a) \subset \sigma^T_w(a), \tag{12}$$

and we conclude that $\sigma^T_{aer}(a)$ is nonempty.

In [4, Theorem 6.6] it is proved that if $T : A \to B$ is a homomorphism with closed range which satisfies the Riesz property, then Ruston elements are Browder and essentially Ruston elements are Weyl. We can improve the second assertion:

2.2 Theorem. If $T : A \to B$ is a homomorphism with closed range which satisfies the Riesz property, then every almost essentially T-Ruston element is T-Weyl.
Proof. Let $a = c+d$, where $c \in A^{-1}$, $d \in T^{-1}(QN(B))$ and $cd-dc \in T^{-1}(\mathrm{Rad}(B))$. As $TcTd - TdTc \in \mathrm{Rad}(B)$, it follows that $Tc^{-1}Td - TdTc^{-1} \in \mathrm{Rad}(B)$, since $\mathrm{Rad}(B)$ is a two-sided ideal. If $\pi : B \to B/\mathrm{Rad}(B)$ denotes the quotient map, then $\pi(Tc^{-1})$ and $\pi(Td)$ commute, and by (10), $\pi Td \in QN(B/\mathrm{Rad}(B))$. From

(4) it follows that $\pi(T(c^{-1}d)) \in QN(B/\mathrm{Rad}(B))$, and so $T(c^{-1}d) \in QN(B)$ by (10). Therefore, $c^{-1}a = 1 + c^{-1}d$ is T-Ruston and by [4, Theorem 6.6.1], $c^{-1}a$ is T-Browder. Thus $a = c(c^{-1}a)$ is T-Weyl. □

2.3 Corollary. If $T : A \to B$ is a homomorphism with closed range and satisfies the Riesz property, then the set of T-Weyl elements, the set of essentially T-Ruston elements and the set of almost essentially T-Ruston elements are equal.

Proof. Follows from Theorem 2.2 and the diagram (11). □

2.4 Corollary. If $T : A \to B$ is a homomorphism with closed range and satisfies the Riesz property, then for $a \in A$

$$\sigma^T_{aer}(a) = \sigma^T_{er}(a) = \sigma^T_w(a).$$

If $K \subset \mathbb{C}$ is compact, then ηK denotes the connected hull of K, i.e. ηK is the union of K and the bounded components of the complement of K.

2.5 Corollary. If $T : A \to B$ is an onto homomorphism and $a \in A$, then $\sigma^T_{aer}(a) \subset \eta\sigma^T_f(a)$ and hence $\eta\sigma^T_{aer}(a) = \eta\sigma^T_f(a)$.

Proof. Follows from the first and the second inclusion in (12), and [4, Corollary 7.4]. □

2.6 Corollary. If $T : A \to B$ is a homomorphism with closed range and $a \in A$, then $\sigma^T_{aer}(a) \subset \eta\sigma^T_f(a)$ and hence $\eta\sigma^T_{aer}(a) = \eta\sigma^T_f(a)$.

Proof. Follows from the first and the second inclusion in (12), and [4, Corollary 7.5]. □

V. Müller [5, Definition 1] introduced the following concepts:

A subset R of a Banach algebra A is an *upper semiregularity* if

(i) $a \in R,\ n \in \mathbb{N} \Rightarrow a^n \in R$,
(ii) if a, b, c, d are mutually commuting elements of A satisfying $ac + bd = 1$ and $a, b \in R$, then $ab \in R$,
(iii) R contains a neighbourhood of 1,

and an *lower semiregularity* if

(i) $a \in R,\ n \in \mathbb{N}, a^n \in R \Rightarrow a \in R$,
(ii) if a, b, c, d are mutually commuting elements of A satisfying $ac + bd = 1$ and $ab \in R$, then $a, b \in R$.

2.7 Corollary. Let $T : A \to B$ be a homomorphism. If either B is commutative or T has closed range and satisfies the Riesz property, then the set of almost essentially T-Ruston elements is an upper semiregularity.

Proof. If B is commutative, then from the definitions it is obvious that the set of almost essentially T-Ruston elements is equal to the set of essentially T-Ruston elements. If T has closed range and satisfies the Riesz property, then, by Corollary 2.3, these sets are equal again. Now the assertion follows from [4, Corollary 8.4]. □

2.8 Corollary. Let $T : A \to B$ be a homomorphism. If either B is commutative or T has closed range and satisfies the Riesz property, then

$$\sigma^T_{aer}(f(a)) \subset f(\sigma^T_{aer}(a))$$

for all $a \in A$ and all functions f analytic on a neighbourhood of $\sigma(a)$ and non-constant on each component of its domain of definition.

Proof. Follows from Corollary 2.7 and [4, Corollary 8.5]. □

In [4] it is remarked that from [2, Example 4.4] it follows that the set of T-Weyl elements is not a lower semiregularity even if the homomorphism T has the Riesz property and closed range. From Corollary 2.3 and the same example it follows that, in general, the set of almost essentially Ruston elements is not a lower semiregularity.

In [4] it is proved that if $a \in A$ is a Ruston (essentially Ruston) element, then a^n is a Ruston (essentially Ruston) element for every $n \in \mathbb{N}$. We prove:

2.9 Theorem. Let $T : A \to B$ be a homomorphism and $a \in A$. If a is almost essentially T-Ruston, then a^n is almost essentially T-Ruston for every $n \in \mathbb{N}$.

Proof. Suppose that $a = c + d$, where $c \in A^{-1}$, $Td \in QN(B)$ and $cd - dc \in T^{-1}(\mathrm{Rad}(B))$. Then $\pi(Tc)$ and $\pi(Td)$ commute, and by (9), $\sigma(\pi(Td), B/\mathrm{Rad}(B)) = \sigma(Td, B) = \{0\}$, where $\pi : B \to B/\mathrm{Rad}(B)$ denotes the quotient map. For $n \in \mathbb{N}$, $a^n = c_1 + d_1$, with $c_1 = c^n \in A^{-1}$ and $d_1 = \sum_{k=1}^n \binom{n}{k} c^{n-k} d^k$. Hence, according to (2) and (3), we get

$$\begin{aligned}\sigma(\pi(Td_1), B/\mathrm{Rad}(B)) &= \sigma(\sum_{k=1}^n \binom{n}{k} (\pi(Tc))^{n-k} (\pi(Td))^k, B/\mathrm{Rad}(B))\\ &\subset \sum_{k=1}^n \binom{n}{k} (\sigma(\pi(Tc), B/\mathrm{Rad}(B)))^{n-k} (\sigma(\pi(Td), B/\mathrm{Rad}(B)))^k\\ &= \{0\}.\end{aligned}$$

By (9), $\sigma(Td_1, B) = \sigma(\pi(Td_1), B/\mathrm{Rad}(B))$, and so $Td_1 \in QN(B)$. Since $\pi(Tc)$ and $\pi(Td)$ commute, it follows that $\pi(Tc_1)$ and $\pi(Td_1)$ commute, i.e. $c_1 d_1 - d_1 c_1 \in T^{-1}(\mathrm{Rad}(B))$. Therefore, a^n is almost essentially T-Ruston. □

In [4] it is noticed that product of two Ruston elements does not have to be Ruston. We remark that also product of two essentially Ruston element is not always essentially Ruston:

Suppose that $a = c + d$ where $c \in A^{-1}$, $Td \in QN(B)$ and $TcTd - TdTc \in \mathrm{Rad}(B) \backslash \{0\}$, in other words a is almost essentially T-Ruston but not essentially

T-Ruston. As in the proof of Theorem 2.2 we conclude that $T(c^{-1}d) \in QN(B)$. Thus $1 + c^{-1}d$ is T-Ruston and so it is an essentially T-Ruston element. Since $a = c(1 + c^{-1}d)$, it follows that a is a product of two essentially T-Ruston elements which is not essentially T-Ruston.

Moreover, we see that every almost essentially T-Ruston element is the product of two T-Ruston elements.

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