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Research Article

On Almost Hyper-Para-Kähler Manifolds

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In this paper it is shown that a $2n$-dimensional almost symplectic manifold (M,ω) can be endowed with an almost paracomplex structure K, $K^2 = \mathrm{Id}_{TM}$, and an almost complex structure J, $J^2 = -\mathrm{Id}_{TM}$, satisfying $\omega(JX, JY) = \omega(X,Y) = -\omega(KX, KY)$ for $X, Y \in TM$, $\omega(X, JX) > 0$ for $X \neq 0$ and $KJ = -JK$, if and only if the structure group of TM can be reduced from $Sp(2n)$ (or $U(n)$) to $O(n)$. In the symplectic case such a manifold (M,ω,J,K) is called an almost hyper-para-Kähler manifold. Topological and metric properties of almost hyper-para-Kähler manifolds as well as integrability of (J,K) are discussed. It is especially shown that the Pontrjagin classes of the eigenbundles $P_\pm$ of K to the eigenvalues ± 1 depend only on the symplectic structure and not on the choice of K.

1. Introduction

While it is well known (see [1–4]) that every symplectic manifold (M,ω) can be made into an almost Kähler manifold by choosing an almost complex structure $J : TM \to TM$ that satisfies $J \circ J = -\mathrm{Id}_{TM}$ and the compatibility condition $\omega(JX, JY) = \omega(X,Y)$ for every $X, Y \in TM$ (Moreover, for an almost Kähler manifold $g(X,Y) := \omega(X, JY)$ is required to be a positive definite Riemannian metric on M, that is, J is required to be tame. If g is merely pseudo-Riemannian, then (M,ω,J) is called an almost pseudo-Kähler manifold.), it is more difficult to find in the literature a concise answer to the corresponding question for almost paracomplex structures.

Definition 1.1. Let (M,ω) be a (almost) symplectic manifold. A bundle automorphism $K : TM \to TM$ satisfying $K \circ K = \mathrm{Id}_{TM}$ and $\omega(KX, KY) = -\omega(X,Y)$ for every $X, Y \in TM$ is called a compatible almost paracomplex structure on (M,ω).

An introduction to paracomplex geometry can be found in [5–7]. As illustrated by [6, Theorem 6, Proposition 7], compatible almost paracomplex structures K on symplectic manifolds (M, ω) correspond on the one hand to almost bi-Lagrangian structures (the eigenbundles $P_{\pm} \subset TM$ of K to the eigenvalues ± 1 are transversal Lagrangian distributions, i.e., $P_+ \oplus P_- = TM$ and $\omega|_{P_\pm \times P_\pm} = 0$ hold) and on the other hand to almost para-Kähler structures (by $h(X, Y) := \omega(KX, Y)$ a neutral metric is defined, which satisfies $h(KX, KY) = -h(X, Y)$).

Definition 1.2. A symplectic manifold (M, ω) endowed with a compatible almost paracomplex structure K is called an almost para-Kähler manifold (an almost bi-Lagrangian manifold).

Existence of compatible almost paracomplex structures is characterized by the following theorem.

Theorem 1.3. *On a (almost) symplectic manifold (M, ω) of dimension $2n$ there exists a compatible almost paracomplex structure K if and only if the structure group of TM can be reduced from $Sp(2n)$ to the paraunitary group $U(n, \mathbb{A})$.*

The validity of this theorem is mentioned in [6, Section 2.5]. For the convenience of the reader a proof of Theorem 1.3 is given in Section 2. An aim of this paper is to characterize (almost) symplectic manifolds (M, ω) that admit a compatible almost paracomplex structure K and a tame compatible almost complex structure J such that $K \circ J = -J \circ K$ is valid.

Definition 1.4. A pair (J, K) of an almost complex structure $J : TM \to TM$ and an almost paracomplex structure $K : TM \to TM$ on a manifold M is called an almost hyperparacomplex structure if and only if $K \circ J = -J \circ K$ is valid.

Note that on an almost hyperparacomplex manifold (M, J, K) the bundle automorphism $J \circ K$ is another almost paracomplex structure. In analogy to the case of almost hyper-Kähler manifolds where a symplectic manifold (M, ω) is endowed with a pair (I, J) of two tame compatible almost complex structures satisfying $J \circ I = -I \circ J$, symplectic manifolds are called almost hyper-para-Kähler manifolds, if the almost (para)complex structures J, K are compatible and J is tame.

Definition 1.5. A symplectic manifold (M, ω) endowed with a pair (J, K) of a compatible almost paracomplex structure K and a tame compatible almost complex structure J satisfying $K \circ J = -J \circ K$ is called an almost hyper-para-Kähler manifold.

Existence of tame compatible almost hyper-paracomplex structures (J, K) is characterized by the following theorem.

Theorem 1.6. *On a (almost) symplectic manifold (M, ω) of dimension $2n$ there exists a compatible almost paracomplex structure K and a tame compatible almost complex structure J such that $K \circ J = -J \circ K$ if and only if the structure group of TM can be reduced from $Sp(2n)$(or $U(n)$) to $O(n)$.*

In the symplectic case, the following corollary is an immediate consequence of Definition 1.5.

Corollary 1.7. *A symplectic manifold (M, ω) of dimension $2n$ can be made into an almost hyper-para-Kähler manifold if and only if the structure group of TM can be reduced from $Sp(2n)$(or $U(n)$) to $O(n)$.*

Note that a reduction of the structure group of TM from $Sp(2n)$ to $U(n)$ is always possible and corresponds to the choice of a tame compatible almost complex structure J on (M,ω). Theorem 1.6 is proved in Section 3 and can be viewed as a combination of [6, Theorem 1], where it is shown that the existence of a Lagrangian distribution on (M,ω) implies the existence of infinitely many different Lagrangian distributions, and [8, Corollary 2.1], where a one-to-one correspondence between Lagrangian distributions on (M,ω,J) and reductions of the structure group of TM from $U(n)$ to $O(n)$ is established. Especially, due to $U(n)\cap U(n,\mathbb{A}) = O(n)$ existence of compatible almost paracomplex structures on a (almost) symplectic manifold (M,ω) can alternatively be characterized as follows.

Corollary 1.8. *On a (almost) symplectic manifold (M,ω) of dimension $2n$ there exists a compatible almost paracomplex structure K if and only if the structure group of TM can be reduced from $Sp(2n)$(or $U(n)$) to $O(n)$.*

In the final section topological and metric properties of almost hyper-para-Kähler manifolds as well as some facts about integrability are discussed and applications are mentioned. Especially, it is shown in Proposition 4.3 and Corollary 4.4 that the Pontrjagin classes of the vector bundles $P_\pm$ over M do not depend on the chosen compatible almost paracomplex structure K but only on the symplectic structure. This result may initiate a deeper study of the question of which manifolds admit a symplectic structure with structure group reducible to $O(n)$.

In the appendix a paracomplex analogue of polarization is formulated.

2. Existence of Compatible Almost Paracomplex Structures

In this section the existence of a compatible almost paracomplex structure K on a symplectic manifold (M,ω) is characterized. Recall that a bundle automorphism $K : TM \to TM$ on a manifold M is called an almost product structure if $K\circ K = \mathrm{Id}_{TM}$ (often the trivial case $K = \pm\mathrm{Id}_{TM}$ is excluded). Obviously, K merely has the eigenvalues ± 1, and if the corresponding eigenbundles $P_\pm$ satisfy $\dim(P_+) = \dim(P_-)$, then K is called an almost paracomplex structure. In this case, necessarily M has even dimension. On an almost symplectic manifold (M,ω) every almost product structure K that satisfies the compatibility condition $\omega(KX,KY) = -\omega(X,Y)$ is automatically an almost paracomplex structure.

To prove Theorem 1.3, some information about the frame bundle $Gl(TM)$ of TM is needed. If M has dimension $2n$, then the fiber of the frame bundle $Gl(TM)$ at a point $m\in M$ consists of the ordered bases (frames) $(X_1,\dots,X_{2n})$ of T_mM, and $Gl(TM)$ is a principal $Gl(2n)$-bundle. The choice of an almost symplectic form ω on M, that is, a nondegenerate (but not necessarily closed) 2-form ω, corresponds to a reduction of the structure group of TM from $Gl(2n)$ to $Sp(2n)$ by selecting only those frames $(X_1,\dots,X_n,Y_1,\dots,Y_n)$ with $\omega(X_i,X_j) = 0 = \omega(Y_i,Y_j)$ and $\omega(X_i,Y_j) = \delta_{ij}$ for $i,j = 1,\dots,n$, that is, ω has the matrix representation $\left(\begin{smallmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{smallmatrix}\right)$ in these so-called symplectic frames.

The following proof of Theorem 1.3 shows that the choice of a compatible almost paracomplex structure K on (M,ω) corresponds to a reduction of the structure group of TM from $\mathrm{Sp}(2n)$ to the paraunitary group

$$U(n,\mathbb{A}) := \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix} \mid A\in Gl(n,\mathbb{R}) \right\}, \tag{2.1}$$

where $\mathbb{A} = \mathbb{R}[k]$ is used as symbol for the paracomplex numbers $a + kb$, $k^2 = 1$, $a, b \in \mathbb{R}$, and $\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$ is considered as (almost) paracomplex structure on $\mathbb{R}^{2n}$.

Proof of Theorem 1.3. As already mentioned in the introduction, compatible almost paracomplex structures K correspond to almost bi-Lagrangian structures $P_\pm$ by assigning to K the eigenbundles $P_\pm$ to the eigenvalues ± 1, and conversely to an almost bi-Lagrangian structure $P_\pm$ the unique almost product structure K which has $P_\pm$ as eigenbundles to the eigenvalue ± 1.

For a given almost bi-Lagrangian structure $P_\pm$ on (M, ω), select only those symplectic frames $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ at $m \in M$ for which $(X_1, \dots, X_n)$ is a base of P_+ and $Y_1, \dots, Y_n$ is a base of P_-. If $(\widetilde{X}_1, \dots, \widetilde{X}_n)$, respectively, $(\widetilde{Y}_1, \dots, \widetilde{Y}_n)$, is another base of P_+, respectively, P_-, then there exist matrices $A, B \in Gl(n)$ with $\widetilde{X}_i = \sum_j a_{ij} X_j$, respectively, $\widetilde{Y}_i = \sum_j b_{ij} Y_j$, and from $\omega(X_i, Y_j) = \delta_{ij} = \omega(\widetilde{X}_i, \widetilde{Y}_j)$ we conclude $B^* A = \mathrm{Id}$, that is, $B = (A^*)^{-1}$. Therefore, the frames $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ and $(\widetilde{X}_1, \dots, \widetilde{X}_n, \widetilde{Y}_1, \dots, \widetilde{Y}_n)$ are related by the matrix $\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$. Thus, the selected frames define a reduction of the structure group of TM from $Sp(2n)$ to $U(n, \mathbb{A})$.

Conversely, if the structure group of TM is reduced from $Sp(n)$ to $U(n, \mathbb{A})$, then two transversal distributions $P_\pm$ can be defined by assigning to a frame $(X_1, \dots, X_n, Y_1, \dots, Y_n)$ at $m \in M$ the subspace $P_+(m) := \mathrm{span}(X_1, \dots, X_n)$ and $P_-(m) := \mathrm{span}(Y_1, \dots, Y_n)$. Note that $P_\pm$ does not depend on the chosen frame because if $(\widetilde{X}_1, \dots, \widetilde{X}_n, \widetilde{Y}_1, \dots, \widetilde{Y}_n)$ is a different frame, then $(\widetilde{X}_1, \dots, \widetilde{X}_n)$ is related to $(X_1, \dots, X_n)$ by a matrix $A \in Gl(n)$ and $(\widetilde{Y}_1, \dots, \widetilde{Y}_n)$ is related to $(Y_1, \dots, Y_n)$ by a $(A^*)^{-1}$. Especially, $\mathrm{span}(\widetilde{X}_1, \dots, \widetilde{X}_n) = \mathrm{span}(X_1, \dots, X_n)$ and $\mathrm{span}(\widetilde{Y}_1, \dots, \widetilde{Y}_n) = \mathrm{span}(Y_1, \dots, Y_n)$ are valid. Further, $P_\pm$ is Lagrangian as $\omega(X_i, X_j) = 0 = \omega(Y_i, Y_j)$ for every $i, j = 1, \dots, n$, and therefore $P_\pm$ are transversal Lagrangian distributions.

Thus, almost bi-Lagrangian structures (and hence compatible almost paracomplex structures) are in one-to-one correspondence with reductions of the structure group of TM from $Sp(2n)$ to $U(n, \mathbb{A})$. □

Although it seems that Theorem 1.3 completely characterizes the existence of compatible almost paracomplex structures on symplectic manifolds, there is a small gap in this characterization. In fact, the analytic conditions required from a symplectic manifold (M, ω), that is, closedness of ω, may already imply that the structure group of TM can be reduced from $Sp(2n)$ to $U(n, \mathbb{A})$. However, this is not the case as there are many symplectic manifolds that do not admit a compatible almost paracomplex structure, see also [6, Section 2.5].

Example 2.1. The 2-sphere S^2 is an example of a symplectic manifold that does not admit any compatible almost paracomplex structure, see also [9, Corollary 2.5]. In fact, the 2-form ω on S^2 given in polar coordinates $(\phi, \theta) \in (-\pi, \pi) \times (-\pi/2, \pi/2)$ by the surface area

$$\omega = \cos(\theta) d\phi \wedge d\theta \tag{2.2}$$

is nondegenerate and closed, that is, a symplectic form on S^2, but there does not exist a Lagrangian distribution on S^2 because else TS^2 would split into two one-dimensional bundles, contradicting nontriviality of the bundle TS^2 over S^2.

3. Existence of Almost Hyper-Para-Kähler Structures

Given a (almost) symplectic manifold (M,ω) the question arises whether a compatible almost paracomplex structure K and a tame compatible almost complex structure J exist such that $K \circ J = -J \circ K$ holds. Hereby, J is called tame if $g(X,Y) := \omega(X,JY)$ is positive definite.

Recall that the choice of a tame almost complex structure J on M is always possible and corresponds to a reduction of the structure group of TM from $Sp(2n)$ to $U(n)$. In fact, if the structure group of TM has already been reduced from $Gl(2n)$ to $Sp(2n)$, that is, if M has been endowed with an almost symplectic form ω, then it can further be reduced to $U(n)$, and this reduction corresponds to the choice of a tame compatible almost complex structure J on (M,ω) by selecting only those symplectic frames $(X_1,\ldots,X_n,Y_1,\ldots,Y_n)$ that additionally satisfy $Y_i = JX_i$ for $i = 1,\ldots,n$, that is, J has the matrix representation $\left(\begin{smallmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{smallmatrix}\right)$ in these so-called unitary frames. Consequently, the positive definite Riemannian metric g defined by $g(X,Y) := \omega(X,JY)$ has in unitary frames the matrix representation $\left(\begin{smallmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{smallmatrix}\right)$. For the convenience of the reader and later reference let us give a short proof of the existence of a compatible almost complex structure on an almost symplectic manifold (see also [1–4]).

Lemma 3.1. *On every almost symplectic manifold (M,ω) there exists a tame compatible almost complex structure J.*

Proof. Choose an arbitrary positive definite Riemannian metric $\langle\cdot,\cdot\rangle$ on M and define a bundle automorphism $A : TM \to TM$ by $\omega(X,Y) = \langle AX,Y\rangle$, which represents ω with respect to $\langle\cdot,\cdot\rangle$. Let $A = G \circ J$ be the unique polar decomposition of A into a positive definite symmetric G and an orthogonal J with respect to $\langle\cdot,\cdot\rangle$. Then the $(0,2)$-tensor g defined by $g(X,Y) := \langle GX,Y\rangle$ is positive definite symmetric and satisfies $g(JX,Y) = \omega(X,Y)$, Further, as A is skew symmetric w.r.t $\langle\cdot,\cdot\rangle$ due to

$$\langle AX,Y\rangle = \omega(X,Y) = -\omega(Y,X) = -\langle AY,X\rangle = -\langle X,AY\rangle, \tag{3.1}$$

the bundle automorphisms G and J obtained by polar decomposition commute, that is, also A and G (or G^{-1}) commute. Thus, not only $J^* = J^{-1}$ holds by orthogonality of J, but symmetry of G^{-1} also implies

$$\langle X,JY\rangle = \left\langle X,G^{-1}AY\right\rangle = -\left\langle AG^{-1}X,Y\right\rangle = -\left\langle G^{-1}AX,Y\right\rangle = -\langle JX,Y\rangle, \tag{3.2}$$

that is, $J^* = -J$. Hence, $J^2 = -\mathrm{Id}_{TM}$ is valid and compatibility of J follows from

$$\omega(JX,JY) = g\left(J^2X,JY\right) = -g(X,JY) = -g(JY,X) = -\omega(Y,X) = \omega(X,Y). \tag{3.3}$$

□

As already stated in the introduction, Theorem 1.6 can be considered as a combination of [8, Corollary 2.1] and [6, Theorem 1]. The following two lemmata are reformulations of these results.

Lemma 3.2. *On an almost symplectic manifold (M,ω) of dimension $2n$ there exists a Lagrangian distribution $P \subset TM$ if and only if the structure group of TM can be reduced from $Sp(2n)$ (or $U(n)$) to $O(n)$.*

Proof. Due to Lemma 3.1 without restriction it can be assumed that the structure group of TM has already been reduced from $Sp(2n)$ to $U(n)$ by choosing a tame compatible almost complex structure J and the corresponding positive definite Riemannian metric g on (M,ω).

For a given Lagrangian distribution P select only those unitary frames $(X_1,\dots,X_n,JX_1,\dots,JX_n)$ at $m\in M$ for which $(X_1,\dots,X_n)$ is an orthonormal base of $P_m\subset T_mM$ with respect to g. If $(\widetilde{X}_1,\dots,\widetilde{X}_n)$ is another base of P_m that is orthonormal w.r.t. g, then there exists a real orthogonal matrix $A\in O(n)$ such that $\widetilde{X}_i=\sum_j a_{ij}X_j$, and due to $J\widetilde{X}_i=\sum_j a_{ij}JX_j$ the corresponding frames are related by the matrix $\left(\begin{smallmatrix}A&0\\0&A\end{smallmatrix}\right)$. Thus, the selected frames define a reduction of the structure group of TM from

$$U(n):=\left\{\begin{pmatrix}A&-B\\B&A\end{pmatrix}\mid A+iB\in U(n,\mathbb{C})\right\}. \tag{3.4}$$

to the subgroup $\{\left(\begin{smallmatrix}A&0\\0&A\end{smallmatrix}\right)\mid A\in O(n)\}$.

Conversely, if the structure group of TM is reduced from $U(n)$ to $O(n)$, then by assigning to a frame $(X_1,\dots,X_n,JX_1,\dots,JX_n)$ at $m\in M$ the subspace $P_m:=\operatorname{span}(X_1,\dots,X_n)$ a Lagrangian distribution P can be defined. Note that P_m does not depend on the chosen frame because if $(\widetilde{X}_1,\dots,\widetilde{X}_n,JX_1,\dots,JX_n)$ is a different frame, then the equation $\widetilde{X}_i=\sum_j a_{ij}X_j$ is valid with an orthogonal matrix $A\in O(n)$, and especially $\operatorname{span}(\widetilde{X}_1,\dots,\widetilde{X}_n)=\operatorname{span}(X_1,\dots,X_n)$. Further, P_m is Lagrangian as $\omega(X_i,X_j)=0$ for every $i,j=1,\dots,n$, and therefore P is a Lagrangian distribution. □

Remark 3.3. The proof of Lemma 3.2 even shows that there is a one-to-one correspondence of Lagrangian distributions and different reductions of the bundle $U(TM)$ of unitary frames on (M,ω,J) to a principal $O(n)$-bundle.

Lemma 3.4. *Let (M,ω) be a (almost) symplectic manifold. If there exists a Lagrangian distribution P on (M,ω), then there exists a tame compatible almost complex structure J and a compatible almost paracomplex structure K having P as eigenbundle to the eigenvalue 1 and satisfying $K\circ J=-J\circ K$.*

Proof. By Lemma 3.1 there exists a tame compatible almost complex structure J on (M,ω). Denote by g the corresponding positive definite Riemannian metric. Let $P_+:=P$, let $P_-:=P^\perp$ be the orthogonal complement of P w.r.t. g, and let K be the the almost product structure with $P_\pm$ as eigenbundles to the eigenvalues ±1. Then $JP_+=P_-$ due to $g(JX,Y)=\omega(X,Y)=0$ for every $X,Y\in P=P_+$ and $\dim(P_+)=\dim(P_-)$. Thus, not only $P_+=P$ is Lagrangian but also P_-, as $\omega(JX,JY)=\omega(X,Y)=0$ holds for $X,Y\in P=P_+$. Hence, K is a compatible almost paracomplex structure with P as eigenbundle to the eigenvalue 1, and $K\circ J=-J\circ K$ holds due to

$$\begin{gathered}JKP_+=JP_+=P_-=-KP_-=-KJP_+,\\ JKP_-=-JP_-=-JJP_+=P_+=KP_+=-KJJP_+=-KJP_-.\end{gathered} \tag{3.5}$$

□

Remark 3.5. The proof of Lemma 3.4 even shows that to a tame compatible almost complex structure J and a Lagrangian distribution P on (M,ω) there exists a unique compatible almost paracomplex structure K such that P is the eigenbundle of K to the eigenvalue 1 and JP is the eigenbundle to −1. Further, this unique K satisfies $K\circ J=-J\circ K$. Especially, every compatible almost paracomplex structure $\widetilde{K}$ on an almost Kähler manifold (M,ω,J) can be changed to a unique compatible almost paracomplex structure K having the same eigenbundle P_+ but satisfying additionally $K \circ J = -J \circ K$.

The two former lemmata directly imply Theorem 1.6 and Corollary 1.8.

Proof of Theorem 1.6 respectively Corollary 1.8. If there exists an almost hyper-para-Kähler structure (J, K) (resp., a compatible almost paracomplex structure K) on (M, ω), then the eigenbundle P_+ of K to the eigenvalue 1 is Lagrangian and by Lemma 3.2 the structure group of TM can be reduced from $Sp(2n)$(or $U(n)$) to $O(n)$.

Conversely, if the structure group of TM can be reduced from $Sp(2n)$ (or $U(n)$) to $O(n)$, then by Lemma 3.2 there exists a Lagrangian distribution P on M, and by Lemma 3.4 there exists a hyper-para-Kähler structure (J, K) (resp., a compatible almost paracomplex structure K) on (M, ω). □

Theorem 1.6 shows that tame compatible almost hyperparacomplex structures (J, K) on an almost symplectic manifold (M, ω) correspond to a reduction of the structure group of TM from $Sp(2n)$ to $O(n)$. In the corresponding frames K is represented by the matrix $\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$ as the condition $K \circ J = -J \circ K$ implies $JP_+ = P_-$ due to $K(JP_+) = -JKP_+ = -JP_+$ with the eigenbundles $P_\pm$ of K to the eigenvalues ± 1. Especially, the neutral metric h defined by $h(X, Y) := \omega(KX, Y)$ has the representation $\begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$ in these frames.

4. Properties of Almost Hyper-Para-Kähler Manifolds

4.1. Topological Properties

In Lemma 3.1 polarization w.r.t. an arbitrary positive definite Riemannian metric $\langle \cdot, \cdot \rangle$ was used to associate with an almost symplectic form ω on M a tame compatible almost complex structure J. Especially, the space of all tame compatible almost complex structures J is contractible. In fact, the space of all positive definite Riemannian metrics is contractible, and composition of the mappings $J \mapsto g$ (where the positive definite Riemannian metric g is defined by $g(X, Y) := \omega(X, JY)$) and $\langle \cdot, \cdot \rangle \mapsto J$ (where J is obtained from polarization w.r.t. $\langle \cdot, \cdot \rangle$) is the identity $J \mapsto J$. As a consequence, the Chern classes associated with the complex vector bundle (TM, J) over M do not depend on the choice of J but only on (M, ω). Therefore, the Chern classes can be used to formulate topological obstructions to the existence of a (almost) symplectic form on a manifold M, but also to the existence of compatible almost paracomplex structures.

Proposition 4.1. *A necessary condition for the existence of a compatible almost paracomplex structure K on a symplectic manifold (M, ω) is that the odd Chern classes of (M, ω) vanish.*

Proof. By Corollary 1.8 a compatible almost paracomplex structure K exists on (M, ω) if and only if the structure group of TM can be reduced from $U(n)$ to $O(n)$. In this case the Chern classes are not only real but vanish for odd k because the Chern polynomial is odd for $A \in \mathfrak{o}(n)$, as $A^T = -A$ implies

$$\det\left(\lambda \mathrm{Id} - \frac{1}{2\pi i} A\right) = \det\left(\lambda \mathrm{Id} + \frac{1}{2\pi i} A^T\right) = \det\left(\lambda \mathrm{Id} + \frac{1}{2\pi i} A\right). \tag{4.1}$$

□

Example 4.2. The symplectic sphere S^2 of Example 2.1 can be identified with $\mathbb{C} \cup \{\infty\}$. Thus, it admits a (integrable) compatible almost complex structure J. Further, the Chern class $c_1(TS^2, J) = -2$ does not vanish. This again shows that the symplectic sphere S^2 does not admit any compatible almost paracomplex structure K.

While on a symplectic manifold (M, ω) the Chern classes of the complex vector bundle (TM, J) do not depend on the choice of the tame compatible almost complex structure J, it is a priori not clear whether the Pontrjagin classes of the eigenbundles $P_\pm$ of K to the eigenvalues ± 1 depend on the choice of the compatible almost paracomplex structure K. This is not the case as the following proposition and its corollary show that the Pontrjagin classes of $P_\pm$ do not depend on the choice of K but only on the symplectic structure.

Proposition 4.3. *On an almost hyper-para-Kähler manifold (M, ω, J, K) the odd Chern classes vanish and the even Chern classes $c_{2k}(TM, J)$ are related to the Pontrjagin classes $p_k(P_\pm)$ of the eigenbundles $P_\pm$ of K to the eigenvalues ± 1 by*

$$(-1)^k c_{2k}(TM) = p_k(P_+) = p_k(P_-). \tag{4.2}$$

Proof. Because K satisfies $K \circ J = -J \circ K$, the eigenbundles $P_\pm$ of K satisfy $JP_+ = P_-$ and $JP_- = P_+$. Thus $J : P_+ \to P_-$ is a bundle isomorphism and therefore $p_k(P_+) = p_k(P_-)$ holds. Moreover, the tangential bundle TM of M can be identified via the bundle isomorphism

$$(P_+)_\mathbb{C} \ni X + iY \longmapsto X + JY \in TM \tag{4.3}$$

with the complexification $(P_+)_\mathbb{C}$, and hence $(-1)^k c_{2k}(TM) = p_k(P_+)$ holds. □

Corollary 4.4. *On an almost para-Kähler manifold (M, ω, K) the Pontrjagin classes of the eigenbundles $P_\pm$ of K are identical and do not depend on the choice of K but only on the symplectic structure.*

Proof. By Remark 3.5 for a chosen tame compatible almost complex structure J on (M, ω) the compatible almost paracomplex structure K can be changed to a compatible almost paracomplex structure $\widetilde{K}$ with the same eigenbundle P_+ to 1 such that $(M, \omega, J, \widetilde{K})$ is an almost hyper-para-Kähler manifold. Thus, by Proposition 4.3 the Pontrjagin classes of P_+ are related to the Chern classes of (TM, J) by $(-1)^k c_{2k}(TM) = p_k(P_+)$. Especially, $p_k(P_+)$ depends only on the symplectic structure of (M, ω). The same argument applied to $-K$ shows $(-1)^k c_{2k}(TM) = p_k(P_-)$. □

Because polarization implies the independence of the Chern classes of (TM, J) of the chosen tame compatible complex structure J, the question arises whether there is a paracomplex analogue of polarization. This question is discussed in the appendix.

4.2. Metric Properties

As already mentioned in the introduction, on a (almost) symplectic manifold (M, ω) endowed with a compatible almost paracomplex structure K a neutral metric h can be defined by $h(X, Y) := \omega(KX, Y)$, and h satisfies $h(KX, KY) = -h(X, Y)$. Recall that a nondegenerate symmetric $(0, 2)$-tensor h on a manifold M is called a pseudo-Riemannian metric and if h has signature (n, n), then h is said to be a neutral metric. If additionally J is a compatible almost complex structure on (M, ω) and $g(X, Y) := \omega(X, JY)$ is the associated metric, then by definition of g and h the equation

$$g(KX, Y) = \omega(KX, JY) = h(X, JY) \tag{4.4}$$

is valid. On an almost hyper-para-Kähler manifold (M, ω, J, K) moreover J is symmetric w.r.t. h and K is symmetric w.r.t. g.

Lemma 4.5. *On an almost hyper-para-Kähler manifold (M, ω, J, K) with associated metrics g to J, respectively, h to K the compatible almost complex structure J is symmetric with respect to h and the compatible almost paracomplex structure K is symmetric with respect to g.*

Proof. Symmetry of J with respect to h follows from

$$\begin{aligned} h(X, JY) &= \omega(KX, JY) = -\omega(JKX, Y) = \omega(KJX, Y) = -\omega(JX, KY) \\ &= \omega(KY, JX) = h(Y, JX) = h(JX, Y), \end{aligned} \tag{4.5}$$

and symmetry of K with respect to h holds due to

$$\begin{aligned} g(KX, Y) &= \omega(KX, JY) = -\omega(JKX, Y) = \omega(KJX, Y) = -\omega(JX, KY) \\ &= \omega(KY, JX) = g(KY, X) = g(X, KY). \end{aligned} \tag{4.6}$$

□

In applications it may be worthwhile to calculate the signature of the restriction of the neutral metric h to a Lagrangian submanifold L of (M, ω) as parts of L with different signature of h may be interpreted as different "phases" of a mechanical systems with state space modeled by (M, ω) and configuration space given by $L \subset M$, and a change of signature of h may indicate a kind of "phase transition."

Example 4.6. If the almost bi-Lagrangian structure $P_\pm$ is integrable (see Section 4.3) and given by $P_+ = \operatorname{span}(\partial/\partial q_k)$, $P_- = \operatorname{span}(\partial/\partial p_k)$, in local canonical coordinates (q, p) with $\omega = \sum_k dq_k \wedge dp_k$, then $h = \sum_k dq_k \otimes_{\text{sym}} dp_k$. Thus, if L is a Lagrangian submanifold locally given by $p = b(q)$ with the derivative b of a function $Q \ni q \mapsto \phi(q) \in \mathbb{R}$, then the pullback of h to Q by $d\phi : q \mapsto (q, b(q))$ is

$$(d\phi)^* h = \sum_{kj} \frac{\partial^2 \phi}{\partial q_k \partial q_j} dq_k \otimes_{\text{sym}} dq_j. \tag{4.7}$$

Therefore, h is positive (resp., negative) definite if and only if ϕ is convex (resp., concave), and the signature of h changes along those hypersurfaces where the second-order derivative of ϕ does not have full rank.

Associated with h and g are the corresponding Levi-Cita connections ∇^h and ∇^g, but there are also other useful connections ∇ (possibly with torsion) like the almost Kähler connection uniquely determined by $\nabla\omega = 0$, $\nabla J = 0$ and $\mathrm{Tor}_\nabla(X,Y) = (1/4)\,[J,J]$ or the almost para-Kähler connection uniquely determined by $\nabla\omega = 0$, $\nabla K = 0$ and $\mathrm{Tor}_\nabla(X,Y) = 0$ for $X, Y \in P_+$, respectively, $X, Y \in P_-$. For a study of connections on almost para-Kähler manifolds and their curvature see [5–7] and the references therein.

4.3. Integrability

A compatible almost paracomplex structure K on a symplectic manifold (M,ω) is said to be integrable if the eigenbundles $P_\pm$ of K to the eigenvalues ±1 are involutive. Symplectic manifolds endowed with such a structure were first studied by [10], see also [11, Chapter 10]. Recall that each $P_\pm$ is a Lagrangian distribution by compatibility of K. An involutive Lagrangian distribution is also called a real polarization and induces by Frobenius' theorem a foliation of (M,ω) into Lagrangian submanifolds. Therefore, if a compatible almost paracomplex structure K on (M,ω) is integrable, then the eigenbundles $P_\pm$ induce two transversal Lagrangian foliations and (M,ω,K) is called a bi-Lagrangian manifold.

Note that with equal right (M,ω,K) could be called a para-Kähler manifold. In fact, K is integrable on (M,ω) if and only if the Levi-Cita connection ∇^h associated with the unique neutral metric h satisfying $h(KX,Y) = \omega(X,Y)$ does not only parallelize h but also K (and thus ω), that is, $\nabla^h h = 0$, $\nabla^h K = 0$, and $\nabla^h\omega = 0$ are valid, see [6, Theorem 6] or [11, Definition 10.2]. Another possibility to test the integrability of a compatible almost paracomplex structure K on a symplectic manifold (M,ω) is to use the $(1,2)$-tensor defined by

$$[K,K](X,Y) = [KX,KY] + K^2[X,Y] - K[KX,Y] - K[X,KY] \tag{4.8}$$

for vector fields X, Y on M, which is called the Nijenhuis tensor of K. In fact, K is integrable if and only if the Nijenhuis tensor of K vanishes, that is, if and only if $[K,K](X,Y) = [KX,KY] + [X,Y] - K[KX,Y] - K[X,KY] = 0$ holds.

In the case that the structure group of the tangential bundle TM of a symplectic manifold (M,ω) (endowed with a tame compatible almost complex structure J) can be reduced from $Sp(2n)$ to $U(n,\mathbb{A})$ (resp., from $U(n)$ to $O(n)$), the existence of a compatible almost paracomplex structure K is guaranteed by Theorem 1.3, but by no means K has to be integrable. For example, [12] shows that there exist symplectic manifolds that do not admit any polarization, regardless whether they are real, complex, or of mixed type. Further, there also are manifolds that admit an integrable complex polarization but not any real Lagrangian distribution, see Example 4.2.

For an almost hyper-para-Kähler manifold (M,ω,J,K) it may happen that neither the almost complex structure J nor the almost paracomplex structure K is integrable. Similarly, integrability of J does not imply integrability of K, and conversely from integrability of K it does not follow that J is integrable. However, if J and K are integrable, then also the almost paracomplex structure $J \circ K$ is integrable, and in this case (M,ω,J,K) is called a hyper-para-Kähler manifold. Such manifolds are, for example, studied in the context of supersymmetry, see [13].

Proposition 4.3 shows that in the chain of proper inclusions

$$\text{hyper-para-Kähler} \subsetneq O(n) - \text{symplectic} \subsetneq \text{almost hyper-paracomplex}, \tag{4.9}$$

(where a manifold is called $O(n)$-symplectic if it is symplectic and its structure group can be reduced to $O(n)$) topologically the second inclusion does not depend on the choice of (J, K). In the complex case the analogous chain of inclusions

$$\text{Kähler} \subsetneq \text{symplectic} \subsetneq \text{almost complex} \tag{4.10}$$

is widely used to study topological obstructions to the existence of symplectic forms on manifolds. The corresponding chain of inclusions for symplectic manifolds, whose structure group is reducible to $O(n)$, does not seem to be intensively studied in the literature. However, see [14], where topological obstructions to the existence of compatible almost paracomplex structures are given by means of the Euler class.

Another possible application of compatible almost paracomplex structures is geometric quantization, where symplectic manifolds (M, ω) with integral cohomology class $[\omega] \in H^2(M, \mathbb{Z})$ are considered, because only in this case there exists a complex line bundle of M. However, in geometric quantization not every section of such a line bundle is considered as a wave function of the quantized system, but only those sections that vanish along a polarization. Now an integrable compatible almost paracomplex structure K just defines two transversal real polarizations, that is, intrinsically a dual real polarization is given, while there is only one real polarization in the ordinary setting. There are some efforts to generalize geometric quantization with complex polarizations, that is, Kähler quantization, to almost Kähler quantization, see [15, 16], and it may be worthwhile to study in analogy almost para-Kähler quantization.

5. Conclusion

In this paper the existence of compatible almost paracomplex structures K (almost bi-Lagrangian structures) and almost hyper-para-Kähler structures (J, K) on a symplectic manifold (M, ω) was characterized. Further, topological and metric properties of such manifolds were discussed. Especially, the result that the second inclusion in

$$\text{hyper-para-Kähler} \subsetneq O(n) - \text{symplectic} \subsetneq \text{almost hyper-paracomplex}, \tag{5.1}$$

(where a manifold is called $O(n)$-symplectic if it is symplectic and its structure group can be reduced to $O(n)$) is topologically independent of the choice of (J, K) may initiate a deeper study of the topological obstructions to the existence of compatible almost paracomplex structures on symplectic manifolds.

Appendix

A Paracomplex Analogue of Polarization

In this appendix it is discussed whether there is a paracomplex analogue of polarization. Note that the polarization $A = G \circ J$ of a skew symmetric A representing the (almost) symplectic form ω via $\omega(X, Y) = \langle AX, Y \rangle$ w.r.t. a chosen positive definite Riemannian metric $\langle \cdot, \cdot \rangle$ on M is obtained from the (complex) eigenvalue decomposition $A = \sum_k (i\lambda_k \mathrm{Id}_{V_k}) \oplus (-i\lambda_k \mathrm{Id}_{\overline{V}_k})$ with the eigenbundles $V_k \subset TM_{\mathbb{C}}$ of A to the eigenvalues $i\lambda_k$, $\lambda_k > 0$, by $G := \sum_k \lambda_k \mathrm{Id}_{V_k + \overline{V}_k}$

and $J := \sum_k (i\,\mathrm{Id}_{V_k}) \oplus (-i\,\mathrm{Id}_{\overline{V_k}})$. It is simple to see that the complex linear automorphisms G and J of $TM_{\mathbb{C}}$ are in fact real, that is, they are induced by real linear automorphisms on TM denoted again by G, J and allow a decomposition $A = G \circ J$ on TM.

A paracomplex analogue is the decomposition $A = \widetilde{H} \circ \widetilde{K}$ with $\widetilde{H} := \sum_k (-i\lambda_k \bar{\cdot}|_{V_k}) \oplus (i\lambda_k \bar{\cdot}|_{\overline{V_k}})$ and $\widetilde{K} := \sum_k \bar{\cdot}|_{V_k} \oplus \bar{\cdot}|_{\overline{V_k}}$ of $TM_{\mathbb{C}}$, where $\bar{\cdot}$ denotes conjugation on $TM_{\mathbb{C}}$ and maps V_k onto $\overline{V_k}$, respectively, $\overline{V_k}$ onto V_k. Note that $\widetilde{H}$ has the real eigenvalues $\pm\lambda_k$, that is, $\widetilde{H}$ is neutral, while $\widetilde{K}$ satisfies $\widetilde{K} \circ \widetilde{K} = \mathrm{Id}_{TM_{\mathbb{C}}}$. However, $\widetilde{H}$ and $\widetilde{K}$ are merely real linear automorphisms on $TM_{\mathbb{C}}$ and not complex linear, that is, they are not induced by real linear automorphisms H and K on TM.

Nevertheless, with a Lagrangian distribution P on (M, ω, J) a real neutral H, respectively, a real K on TM can be associated such that the complexification of H, respectively, K coincides with $\widetilde{H}$, respectively, $\widetilde{K}$ on $P + iJP$. In fact, let $P_+ := P$ and $P_- := JP_+$, then the real dimension of $(P_+ + iP_-) \cap V_k$ is the same as the complex dimension of V_k because if v is an eigenvector of J to i and $v = v_1 + v_2 \in (P_+ + iP_-) \oplus (P_- + iP_+) = TM_{\mathbb{C}}$, then due to $JP_+ = P_-$, $JP_- = P_+$ the decomposition

$$Jv_2 + Jv_1 = Jv = iv = iv_2 + iv_1 \in (P_+ + iP_-) \oplus (P_- + iP_+) \tag{A.1}$$

implies $Jv_1 = iv_1$, $Jv_2 = iv_2$. Especially, $v_1 \in (P_+ + iP_-)$ is an eigenvector of J to i, and as the eigenspace of J to i is the sum of the V_k, the real subspace $(P_+ + iP_-) \cap V_k$ of $TM_{\mathbb{C}}$ is nonempty and $\dim_{\mathbb{R}}((P_+ + iP_-) \cap V_k) = \dim_{\mathbb{C}}(V_k)$. Thus, associated with P there are unique real linear automorphisms H and K on TM such that the complexification of H coincides with $\widetilde{H}$ on $(P_+ + iP_-) \cap V_k$, and the complexification of K coincides on $(P_+ + iP_-) \cap V_k$ with $\widetilde{K}$. As a consequence, the decomposition $A = H \circ K$ holds, K is orthogonal w.r.t. $\langle \cdot, \cdot \rangle$ and satisfies $K^2 = \mathrm{Id}_{TM}$, and a neutral metric h satisfying $h(KX, Y) = \omega(X, Y)$ can be defined by $h(X, Y) := \langle HX, Y \rangle$. However, note that the decomposition $A = H \circ K$ into a nondegenerate neutral symmetric H and an orthogonal K w.r.t. $\langle \cdot, \cdot \rangle$ was merely made unique by the choice of P, in general there are many such decompositions.

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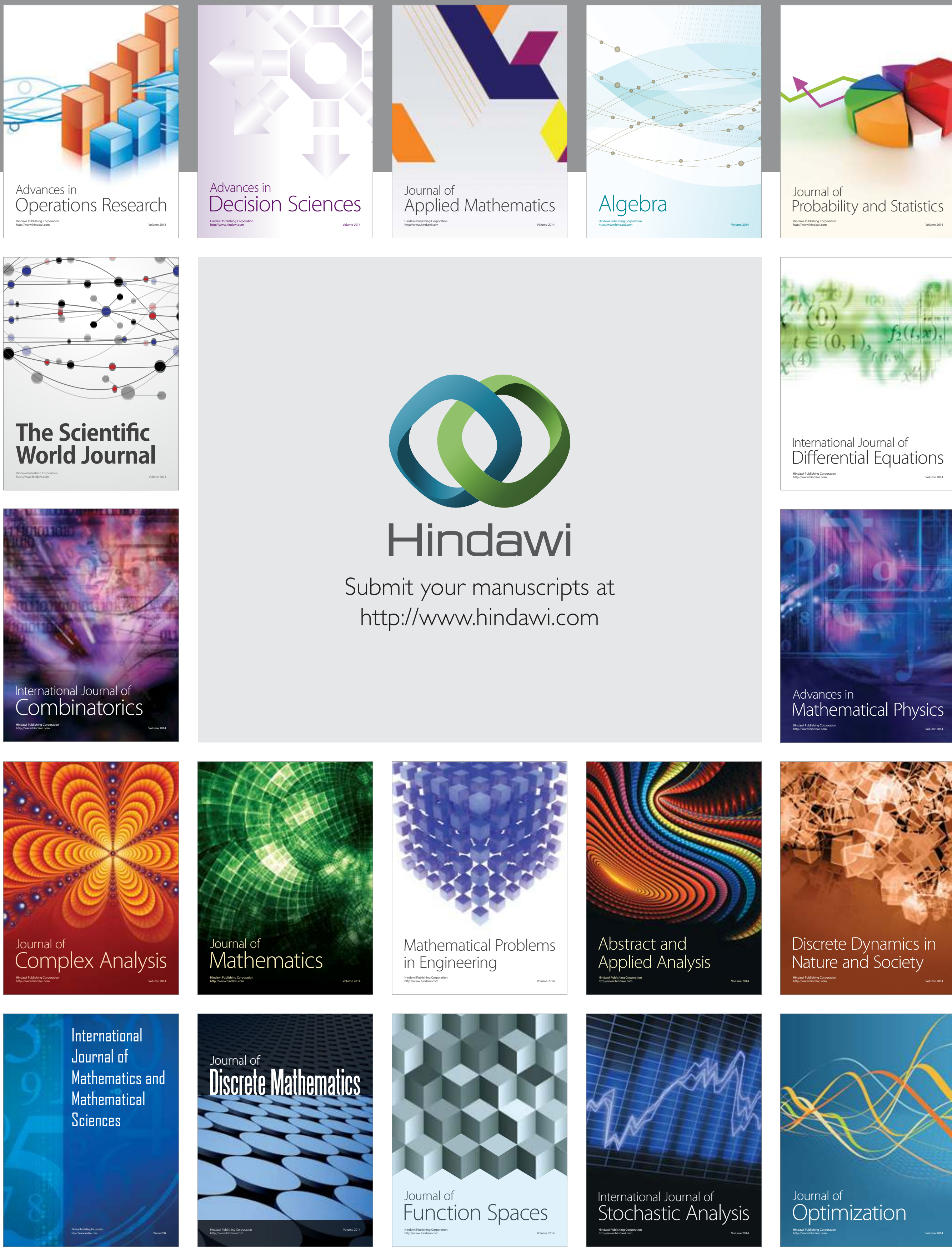
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