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ON ALMOST PARA-COSYMPLECTIC MANIFOLDS

By

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Abstract. An almost para-cosymplectic manifold is by definition an odd-dimensional differentiable manifold endowed with an almost paracontact structure with hyperbolic metric for which the structure forms are closed. The local structure of an almost para-cosymplectic manifold is described. We also treat some special subclasses of this class of manifolds: para-cosymplectic, weakly para-cosymplectic and almost para-cosymplectic with para-Kählerian leaves. Necessary and sufficient conditions for an almost para-cosymplectic manifold to be para-cosymplectic are found. Necessary and sufficient conditions for an almost para-cosymplectic manifold with para-Kählerian leaves to be weakly para-cosymplectic are also established. We construct examples of weakly para-cosymplectic manifolds, which are not para-cosymplectic. It is proved that in dimensions ≥ 5 an almost para-cosymplectic manifold cannot be of non-zero constant sectional curvature. Main curvature identities which are fulfilled by any almost para-cosymplectic manifold are found.

1. Preliminaries

Let M be a $(2n+1)$-dimensional differentiable manifold. Suppose that (φ, ξ, η, g) is an almost paracontact hyperbolic metric structure on M. This means that (φ, ξ, η, g) is a quadruple consisting of a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a pseudo-Riemannian metric g on M satisfying the following relations

$$\varphi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y).$$

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In the above and in the sequel, $X, Y, \dots$ denote arbitrary smooth vector fields on M if it is not otherwise stated. As consequences of the above, we additionally have

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(X) = g(X, \xi), \quad g(\varphi X, Y) = -g(\varphi Y, X).$$

Thus, $\Phi(X, Y) = g(\varphi X, Y)$ is a 2-form on M, which will be said the fundamental form of the structure.

With the above terminology we follow [8]. In the papers [14], [3], [4], [1] the authors called such structures almost para-coHermitian.

The manifold M endowed with the almost paracontact hyperbolic metric structure will be called

(a) para-cosymplectic if the forms η and Φ are parallel with respect to the Levi-Civita connection ∇ of the metric g, that is, $\nabla\eta = 0$ and $\nabla\Phi = 0$;

(b) almost para-cosymplectic if the forms η and Φ are closed, that is, $d\eta = 0$ and $d\Phi = 0$.

The above notions of (almost) para-cosymplectic manifolds are paracontact—with a hyperbolic metric—analogue of (almost) cosymplectic manifolds (for almost cosymplectic manifolds see [2], [9]).

Our definition of the para-cosymplecticity differs from that used in the paper [8], in which this notion concerns even-dimensional indefinite almost Hermitian or almost para-Hermitian manifolds with coclosed fundamental forms.

For an almost para-cosymplectic manifold, define the $(1,1)$-tensor field A by

$$AX = -\nabla_X \xi.$$

PROPOSITION 1. *For an almost para-cosymplectic manifold, we have*

$$\mathscr{L}_\xi \eta = 0, \quad \mathscr{L}_\xi \Phi = 0, \quad g(AX, Y) = g(AY, X), \quad A\xi = 0,$$

$$\eta \circ A = 0, \quad (\mathscr{L}_\xi g)(X, Y) = -2g(AX, Y), \quad \nabla_\xi \varphi = 0,$$

$$A\varphi + \varphi A = 0, \quad g(\varphi AX, Y) = g(\varphi AY, X), \quad Tr(\varphi A) = Tr(A) = 0,$$

where $\mathscr{L}$ indicates the operator of the Lie differentiation.

PROOF. By $d\eta = 0$, $d\Phi = 0$, $i_\xi(\eta) = 1$ and $i_\xi\Phi(X) = g(\varphi\xi, X) = 0$, we have

$$\mathscr{L}_\xi \eta = di_\xi\eta + i_\xi\, d\eta = 0, \quad \mathscr{L}_\xi \Phi = di_\xi\Phi + i_\xi\, d\Phi = 0.$$

Moreover, by $(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) = -g(AX, Y)$,

$$0 = 2\, d\eta(X, Y) = -g(AX, Y) + g(AY, X),$$

that is, A is a symmetric operator. Consequently,

$$(\mathscr{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = -2g(AX, Y),$$

$$g(A\xi, Y) = g(AY, \xi) = -g(\nabla_Y \xi, \xi) = -\frac{1}{2}\nabla_Y g(\xi, \xi) = 0,$$

and the last line of equations implies $A\xi = -\nabla_\xi \xi = 0$ and $\eta \circ A = 0$.

For $\mathscr{L}_\xi$, we have the decomposition $\mathscr{L}_\xi = \nabla_\xi + A$, here A indicates the unique extension of the $(1,1)$-tensor field A to a derivation of the tensor algebra (see e.g. [10], p. 30).

From one hand, since $\mathscr{L}_\xi \Phi = 0$, it holds

$$\begin{aligned} 0 = (\mathscr{L}_\xi \Phi)(X, Y) &= (\nabla_\xi \Phi)(X, Y) - \Phi(AX, Y) - \Phi(X, AY) \\ &= g((\nabla_\xi \varphi)X, Y) - g((A\varphi + \varphi A)X, Y). \end{aligned}$$

Thus, we have $\nabla_\xi \varphi = A\varphi + \varphi A$.

On the other hand, since $A\xi = 0$, applying ∇_ξ to $\varphi^2 X = X - \eta(X)\xi$, we obtain

$$\varphi(\nabla_\xi \varphi)X + (\nabla_\xi \varphi)\varphi X = g(A\xi, X)\xi + \eta(X)A\xi = 0.$$

We note that $A = \varphi(\varphi A) = (A\varphi)\varphi$, by $A\xi = 0$ and $\eta \circ A = 0$. Hence

$$\begin{aligned} 0 &= \varphi(\varphi(\nabla_\xi \varphi) + (\nabla_\xi \varphi)\varphi) \\ &= \varphi(\varphi(A\varphi + \varphi A) + (A\varphi + \varphi A)\varphi) = 2\varphi(A + \varphi A\varphi) = 2(\varphi A + A\varphi). \end{aligned}$$

Consequently $\nabla_\xi \varphi = 0$. Since φ is skew-symmetric and A symmetric, then φA is traceless. Note that the trace of $A = \varphi(\varphi A)$ also vanishes, because $\varphi A + A\varphi = 0$ implies the symmetry of φA. □

2. The Local Structure

In this section, we establish a local equivalence between almost para-cosymplectic structures and certain special families of almost para-Kählerian structures.

By an almost para-Kählerian manifold it is meant a $2n$-dimensional differentiable manifold $\tilde{M}$ endowed with a pair $(\tilde{J}, \tilde{g})$, where $\tilde{J}$ is an almost para-complex structure ($\tilde{J}^2 = \tilde{I}$), $\tilde{g}$ is a pseudo-Riemannian metric such that $\tilde{g}(\tilde{J}\tilde{X}, \tilde{J}\tilde{Y}) = -\tilde{g}(\tilde{X}, \tilde{Y})$ and the fundamental form $\tilde{\Omega}(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{J}\tilde{X}, \tilde{Y})$ is closed. An almost para-Kählerian manifold with integrable almost para-complex structure $\tilde{J}$ (equivalently, $\tilde{\nabla}\tilde{J} = 0$) is said to be para-Kählerian. For almost para-Kählerian structures, we refer the survey articles [5], [6].

Let $(\tilde{J}_t, \tilde{g}_t)$, $t \in (a,b)$, $a < b$, be a 1-parameter family of almost para-Kählerian structures on a $2n$-dimensional manifold $\tilde{M}$ such that $\tilde{\Omega}_t = \tilde{\Omega}$ for any $t \in (a,b)$, $\tilde{\Omega}$ being a fixed closed 2-form on $\tilde{M}$. This family enables us to define an almost para-cosymplectic structure on the product $M = (a,b) \times \tilde{M}$. In fact, it is sufficient to assume that φ, ξ, η, g are given on M by

$$(1) \qquad \varphi_{(t,p)} = (\tilde{J}_t)_p, \quad g = dt \otimes dt + \tilde{g}_t, \quad \xi = \frac{\partial}{\partial t}, \quad \eta = dt.$$

The fundamental form $\Phi(X,Y) = g(\varphi X, Y)$ at any $(t,p) \in M$ is given by $\Phi_{(t,p)} = (\tilde{\Omega}_t)_p = \tilde{\Omega}_p$, and therefore it is closed. Especially, if the family $(\tilde{J}_t, \tilde{g}_t)$ collapses to a single almost para-Käler structure $(\tilde{J}, \tilde{g})$, that is, $(\tilde{J}_t, \tilde{g}_t) = (\tilde{J}, \tilde{g})$ for any $t \in (a,b)$, then we say that the almost para-cosymplectic manifold M is the product of the open interval (a,b) and the almost para-Kählerian manifold $\tilde{M}$.

We will show that any almost para-cosymplectic structure can locally be seen as that in formula (1).

In fact, let (φ, ξ, η, g) be an almost para-cosymplectic structure on M and p a fixed point of M. Since $d\eta = 0$ and $\eta(\xi) = 1$, we choose a coordinate neighbourhood U around p, which is diffeomorphic to $(-a,a) \times \tilde{U}$, $a > 0$, $\tilde{U} \subset \boldsymbol{R}^{2n}$, with coordinates $(x^0, x^1, \dots, x^{2n})$, x^0 being the coordinate on $(-a,a)$, such that

$$\xi = \frac{\partial}{\partial x^0}, \quad \eta = dx^0.$$

Since $g_{0i} = \delta_{0i}$, g can be written as

$$g = dx^0 \otimes dx^0 + \sum_{i,j=1}^{2n} g_{ij}\, dx^i \otimes dx^j.$$

By $\varphi\xi = 0$ and $\eta(\varphi X) = 0$, we find

$$\varphi = \sum_{i,j=1}^{2n} \varphi_i^j\, dx^i \otimes \frac{\partial}{\partial x^j}.$$

Moreover, $\Phi(\xi, \cdot) = 0$ and $\mathscr{L}_\xi \Phi = 0$ yield for the components of Φ

$$\Phi_{0i} = 0, \quad \frac{\partial \Phi_{ij}}{\partial x^0} = 0.$$

Thus the fundamental form Φ has the shape

$$\Phi = 2 \sum_{1 \le i < j \le 2n} \Phi_{ij}\, dx^i \wedge dx^j$$

and does not depend on x^0. For any fixed $x^0 = t$, define an almost para-Kählerian structure $(\tilde{J}_t, \tilde{g}_t)$ on $\tilde{U}$ by putting

$$\tilde{J}_t = \sum_{i,j=1}^{2n} \varphi_i^j(t,\cdot)\, dx^i \otimes \frac{\partial}{\partial x^j}, \quad \tilde{g}_t = \sum_{i,j=1}^{2n} g_{ij}(t,\cdot)\, dx^i \otimes dx^j,$$

with the fundamental form $\tilde{\Omega}_t = \Phi|\tilde{U}$.

We have just proved the following theorem.

THEOREM 1. *Let $M(\varphi, \xi, \eta, g)$ be an almost para-cosymplectic manifold. Then, for any point $p \in M$,*

(a) *there is a neighbourhood $U = (-a, a) \times \tilde{U}$ of p, where $\tilde{U}$ is a $2n$-dimensional differentiable manifold and $a > 0$;*
(b) *there exist a 1-parameter family of almost para-Kählerian structures $(\tilde{J}_t, \tilde{g}_t)$, $t \in (-a, a)$, which are defined on $\tilde{U}$ with the fundamental forms $\tilde{\Omega}_t$ not depending on the parameter t, $\tilde{\Omega}_t = \tilde{\Omega}$; and*
(c) *on $(-a, a) \times \tilde{U}$, the structure (φ, ξ, η, g) is given as in formula* (1). □

Families of almost para-Kählerian structures with the same fundamental form can be constructed in many ways. Below, we present some of them.

EXAMPLE 1. Let $(\tilde{J}, \tilde{g})$ be a fixed almost para-Kähler structure on a $2n$-dimensional differentiable manifold N and $\tilde{\Omega}$ its fundamental form. Let V be an open subset of N endowed with a frame of vector fields $(E_1, \ldots, E_{2n})$ such that $\tilde{J}E_\alpha = E_{\alpha+n}$, $\tilde{J}E_{\alpha+n} = E_\alpha$, $\tilde{g}(E_\alpha, E_\beta) = \delta_{\alpha\beta}$, $\tilde{g}(E_{\alpha+n}, E_{\beta+n}) = -\delta_{\alpha\beta}$, $\tilde{g}(E_\alpha, E_{\beta+n}) = 0$ for $\alpha, \beta = 1, \ldots, n$.

Given a family of functions $f_t : V \to \boldsymbol{R}$, $a < t < b$, define $(\tilde{J}_t, \tilde{g}_t)$ by

$$\tilde{J}_t E_\alpha = \exp(f_t) E_{\alpha+n}, \quad \tilde{J}_t E_{\alpha+n} = \exp(-f_t) E_\alpha,$$

$$\tilde{g}_t(E_\alpha, E_\beta) = \exp(f_t)\tilde{g}(E_\alpha, E_\beta), \quad \tilde{g}_t(E_{\alpha+n}, E_{\beta+n}) = \exp(-f_t)\tilde{g}(E_{\alpha+n}, E_{\beta+n})$$

for any $t \in (a, b)$. One checks that $(\tilde{J}_t, \tilde{g}_t)$ are almost para-Kählerian structures with fundamental forms $\tilde{\Omega}_t = \tilde{\Omega}$. □

EXAMPLE 2. Let $(\tilde{J}, \tilde{g})$ be an almost para-Kählerian structure on a $2n$-dimensional differentiable manifold N. Let V be an open subset of N and assume that there exist a 1-parameter family of diffeomorphisms $f_t : V \to f_t(V) \subset N$, $t \in (-a, a)$, $a > 0$, such that the fundamental form $\tilde{\Omega}$ of N is invariant with respect to all f_t's, that is, $f_t^*\tilde{\Omega} = \tilde{\Omega}$. [One should note that any point of N has a neighbourhood V with this property.] Define a family of almost para-Hermitian structures $(\tilde{J}_t, \tilde{g}_t)$ on V as follows

$$\tilde{J}_t = f_{t*}^{-1}\tilde{J}f_{t*}, \quad \tilde{g}_t = f_t^*\tilde{g}.$$

It can be checked that $(\tilde{J}_t, \tilde{g}_t)$ are almost para-Kählerian structures on V with fundamental forms $\tilde{\Omega}_t = \tilde{\Omega}$ for any $t \in (-a, a)$.

The very special case of the above construction can be obtained when X is a vector field on N satisfying $\mathscr{L}_X\tilde{\Omega} = 0$. Then, any point of N has a neighbourhood $V \subset N$ and there exists a 1-parameter group of diffeomorphisms $f_t : V \to f_t(V) \subset N$ generated by X. By $\mathscr{L}_X\tilde{\Omega} = 0$, any f_t preserves $\tilde{\Omega}$. □

REMARK 1. An almost para-cosymplectic manifold M possesses a canonical foliation $\mathscr{F}$ generated by the $2n$-dimensional, completely integrable and φ invariant distribution $\mathscr{D} = \ker\eta$. A leaf $\tilde{M}$ of $\mathscr{F}$ is a submanifold of M of codimension 1. Since $\xi|\tilde{M}$ is a vector field normal to $\tilde{M}$, we may treat $\tilde{M}$ as a pseudo-Riemannian hypersurface. Then $A = -\nabla\xi$ restricted to $\tilde{M}$ is the shape operator $\tilde{A}$ of $\tilde{M}$.

Let $\tilde{J}$ be the $(1,1)$-tensor field defined by $\tilde{J}\tilde{X} = \varphi\tilde{X}$ and $\tilde{g}$ the induced metric on $\tilde{M}$. Then the pair $(\tilde{J}, \tilde{g})$ is an almost para-Hermitian structure on $\tilde{M}$. In fact, it is almost para-Kählerian since its fundamental form is closed, as it is the pull-back of the fundamental form of M.

Fix a point of M and choose a neighbourhood $U = (-a, a) \times \tilde{U}$, on which the structure (φ, ξ, η, g) is given as in (1), where $(\tilde{J}_t, \tilde{g}_t)$ is a suitable family of almost para-Kählerian structures on $\tilde{U}$. Then $\{t\} \times \tilde{U}$ is an open subset of a leaf. Identifying the set $\tilde{U}$ with $\{t\} \times \tilde{U}$, we note that $(\tilde{J}_t, \tilde{g}_t)$ is just the induced almost para-Kählerian structure $(\tilde{J}, \tilde{g})$ on $\{t\} \times \tilde{U}$. Moreover, by the equality $g(AX, Y) = -(1/2)(\mathscr{L}_\xi g)(X, Y)$ (see Proposition 1), for the second fundamental form h_t of $\{t\} \times \tilde{U}$, we have $h_t = -(1/2)(\partial\tilde{g}_s/\partial s)|_{s=t}$. □

3. Basic Structure Identities

LEMMA 1. *For an almost paracontact hyperbolic metric manifold $M(\varphi, \xi, \eta, g)$ with its fundamental 2-form Φ the following equations hold*

$$(\nabla_X\Phi)(Y, Z) = g((\nabla_X\varphi)Y, Z), \tag{2}$$

$$(\nabla_X\Phi)(Z, \varphi Y) + (\nabla_X\Phi)(Y, \varphi Z) = -\eta(Z)g(AX, Y) - \eta(Y)g(AX, Z), \tag{3}$$

$$(\nabla_X\Phi)(\varphi Y, \varphi Z) - (\nabla_X\Phi)(Y, Z) = -\eta(Z)g(AX, \varphi Y) + \eta(Y)g(AX, \varphi Z), \tag{4}$$

where $A = -\nabla\xi$.

Proof. Equality (2) is obvious. Differentiating the identity $\varphi^2 = I - \eta \otimes \xi$ covariantly, we obtain

$$(\nabla_X \varphi)\varphi Y + \varphi(\nabla_X \varphi) Y = g(AX, Y)\xi + \eta(Y)AX \tag{5}$$

Projecting this equality onto Z, we find (3).

To prove (4), we find at first

$$(\nabla_X \varphi)\xi = -\varphi \nabla_X \xi = \varphi AX, \tag{6}$$

whence it follows

$$(\nabla_X \Phi)(Y, \xi) = -g((\nabla_X \varphi)\xi, Y) = -g(\varphi AX, Y).$$

Replacing Z by φZ in (3), and applying the last equality, we find (4). □

Proposition 2. *For any almost para-cosymplectic manifold, we have*

$$(\nabla_{\varphi X}\varphi)\varphi Y - (\nabla_X \varphi) Y - \eta(Y)A\varphi X = 0. \tag{7}$$

Proof. Let us define $(0,3)$-tensor field B as follows

$$\begin{aligned} B(X, Y, Z) &= g((\nabla_{\varphi X}\varphi)\varphi Y - (\nabla_X \varphi) Y - \eta(Y)A\varphi X, Z) \\ &= (\nabla_{\varphi X}\Phi)(\varphi Y, Z) - (\nabla_X \Phi)(Y, Z) - \eta(Y)g(\varphi X, AZ). \end{aligned}$$

Antisymmetrizing B with respect to X, Y we have

$$\begin{aligned} B(X, Y, Z) - B(Y, X, Z) = {} & (\nabla_{\varphi X}\Phi)(\varphi Y, Z) - (\nabla_{\varphi Y}\Phi)(\varphi X, Z) \\ & - (\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(X, Z) \\ & - \eta(Y)g(\varphi X, AZ) + \eta(X)g(\varphi Y, AZ). \end{aligned}$$

Since the metric connection ∇ is torsionless and $d\Phi = 0$,

$$(\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Y) = 0.$$

Applying this in the previous formula, we obtain

$$\begin{aligned} B(X, Y, Z) - B(Y, X, Z) = {} & -(\nabla_Z \Phi)(\varphi X, \varphi Y) + (\nabla_Z \Phi)(X, Y) \\ & - \eta(Y)g(\varphi X, AZ) + \eta(X)g(\varphi Y, AZ). \end{aligned}$$

By (4), the right hand side of this equality vanishes identically, so that $B(X, Y, Z) - B(Y, X, Z) = 0$, i.e. B is symmetric with respect to X, Y.

Symmetrizing B with respect to Y, Z, we find

$$B(X, Y, Z) + B(X, Z, Y) = (\nabla_{\varphi X}\Phi)(\varphi Y, Z) + (\nabla_{\varphi X}\Phi)(\varphi Z, Y) - \eta(Y)g(\varphi X, AZ) - \eta(Z)g(\varphi X, AY).$$

This, with the help of (3), implies $B(X, Y, Z) + B(X, Z, Y) = 0$, i.e. B is anti-symmetric with respect to Y, Z. The tensor B having such symmetries must vanish identically, which implies (7). □

Lemma 2. *For an almost para-cosymplectic manifold, we also have*

$$(\nabla_{\varphi X}\varphi)Y - (\nabla_X\varphi)\varphi Y + \eta(Y)AX = 0, \tag{8}$$

$$(\nabla_{\varphi X}\varphi)Y + \varphi(\nabla_X\varphi)Y - g(AX, Y)\xi = 0. \tag{9}$$

Proof. Putting φY instead of Y in (7), we get

$$(\nabla_{\varphi X}\varphi)Y - \eta(Y)(\nabla_{\varphi X}\varphi)\xi - (\nabla_X\varphi)\varphi Y = 0.$$

By (6), $(\nabla_{\varphi X}\varphi)\xi = \varphi A\varphi X = -AX$, which applied to the above gives (8). Now, (9) follows from (8) and (5). □

Proposition 3. *For the curvature of an almost para-cosymplectic manifold, we have the following identities*

$$R_{XY}\xi = -(\nabla_X A)Y + (\nabla_Y A)X, \tag{10}$$

$$R_{\varphi X\varphi Y}\xi + R_{XY}\xi + \varphi R_{\varphi XY}\xi + \varphi R_{X\varphi Y}\xi = -\nabla_{N(X,Y)}\xi, \tag{11}$$

where $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ *and* N *is the Nijenhuis torsion tensor of* φ,

$$N(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

Proof. By $AX = -\nabla_X\xi$, we have

$$(\nabla_X A)Y = -\nabla_X\nabla_Y\xi + \nabla_{\nabla_X Y}\xi.$$

Hence, we get

$$-(\nabla_X A)Y + (\nabla_Y A)X = [\nabla_X, \nabla_Y]\xi - \nabla_{[X,Y]}\xi = R_{XY}\xi,$$

that is, formula (10).

Proposition 1 leads to

$$\nabla_{\varphi X}\xi = -A\varphi X = \varphi AX = -\varphi\nabla_X\xi. \tag{12}$$

Moreover, using (8), we obtain

$$-(\nabla_{\varphi X}\varphi)\nabla_Y\xi + (\nabla_X\varphi)\varphi\nabla_Y\xi = 0.$$

Hence this, together with (12), gives

$$\nabla_{\varphi X}\nabla_{\varphi Y}\xi + \varphi\nabla_{\varphi X}\nabla_Y\xi + \nabla_X\nabla_Y\xi + \varphi\nabla_X\nabla_{\varphi Y}\xi = 0. \tag{13}$$

Now, using (12) and (13), we find

$$\begin{aligned}-\nabla_{N(X,Y)}\xi &= \nabla_{\varphi X}\nabla_{\varphi Y}\xi + \varphi\nabla_{\varphi X}\nabla_Y\xi + \nabla_X\nabla_Y\xi + \varphi\nabla_X\nabla_{\varphi Y}\xi \\ &\quad - \nabla_{\varphi Y}\nabla_{\varphi X}\xi - \varphi\nabla_{\varphi Y}\nabla_X\xi - \nabla_Y\nabla_X\xi - \varphi\nabla_Y\nabla_{\varphi X}\xi \\ &\quad - \nabla_{[X,Y]}\xi - \nabla_{[\varphi X,\varphi Y]}\xi - \varphi\nabla_{[\varphi X,Y]}\xi - \varphi\nabla_{[X,\varphi Y]}\xi \\ &= R_{\varphi X\varphi Y}\xi + R_{XY}\xi + \varphi R_{\varphi XY}\xi + \varphi R_{X\varphi Y}\xi,\end{aligned}$$

that is (11). □.

4. Para-cosymplectic Manifolds

In this section, we prove various necessary and sufficient conditions for an almost para-cosymplectic manifold to be para-cosymplectic.

At first, we prove the following proposition

PROPOSITION 4. *For the Nijenhuis torsion tensor N of an almost para-cosymplectic manifold, we have the following*

$$N(X,Y) = 2((\nabla_{\varphi X}\varphi)Y - (\nabla_{\varphi Y}\varphi)X) = -2\varphi((\nabla_X\varphi)Y - (\nabla_Y\varphi)X), \tag{14}$$

$$N(\varphi X,\varphi Y) = N(X,Y) - 2\eta(X)AY + 2\eta(Y)AX, \tag{15}$$

$$(\nabla_{\varphi Z}\Phi)(X,Y) = -\frac{1}{2}g(Z,N(X,Y)), \tag{16}$$

$$\eta(N(X,Y)) = 0, \quad N(\xi,X) = 2AX. \tag{17}$$

PROOF. Writing the Nijenhuis torsion tensor of φ with the help of the Levi-Civita connection, we get

$$N(X,Y) = -\varphi(\nabla_X\varphi)Y + \varphi(\nabla_Y\varphi)X + (\nabla_{\varphi X}\varphi)Y - (\nabla_{\varphi Y}\varphi)X.$$

Formula (14) follows from the above in view of (9). Using (14) and (7), we find (15). Moreover, using (14), we compute

$$(\nabla_{\varphi Z}\Phi)(X,Y) = -g((\nabla_X\varphi)Y - (\nabla_Y\varphi)X, \varphi Z)$$
$$= g(\varphi((\nabla_X\varphi)Y - (\nabla_Y\varphi)X), Z) = -\frac{1}{2}g(Z, N(X,Y)),$$

which gives (16). Formulas (17) are immediate consequences of (14) and (15), respectively. □

Theorem 2. *For an almost para-cosymplectic manifold M, the following conditions are equivalent*

(a) *M is para-cosymplectic,*
(b) $N = 0$,
(c) *φ is parallel,*
(d) *M is locally a product of an open interval and a para-Kählerian manifold,*
(e) *the leaves $\tilde{M}$ of the canonical foliation $\mathscr{F}$ are totally geodesic and the induced structures $(\tilde{J}, \tilde{g})$ are para-Kählerian.*

Proof. (a) ⇒ (e): Note that $A = 0$ since $AZ = \varphi(\nabla_Z\varphi)\xi$ and $\nabla\varphi = 0$. Therefore, for the shape operator of a leaf $\tilde{M}$ of $\mathscr{F}$, it holds $\tilde{A} = A|\tilde{M} = 0$. Thus, $\tilde{M}$ is totally geodesic and $\tilde{\nabla} = \nabla|\tilde{M}$ by the Gauss equation. Consequently, $\tilde{\nabla}\tilde{J} = 0$, that is, $(\tilde{J}, \tilde{g})$ is para-Kählerian.

(e) ⇒ (d): By Theorem 1, choose a neighbourhood $U = (-a, a) \times \tilde{U}$ on which the structure (φ, ξ, η, g) is given as in (1), where $(\tilde{J}_t, \tilde{g}_t)$ is a family of almost para-Kählerian structures on $\tilde{U}$ with $\tilde{\Omega}_t$ not depending on t. Restrict further considerations to the set U. As we have already known, $(\tilde{J}_t, \tilde{g}_t)$'s are the induced structures on leaves. By our assumption, they are para-Kählerian. Since the leaves are also totally geodesic, their second fundamental forms h_t vanish identically and consequently $(\partial/\partial t)\tilde{g}_t = -2h_t = 0$. Hence $(\tilde{J}_t, \tilde{g}_t)$ are independent of t. Thus, M is locally a product of an open interval and a para-Kählerian manifold.

(d) ⇒ (c): It is obvious.

(c) ⇒ (b): It follows from (2) and (16).

(b) ⇒ (a): Since $N = 0$, (2) and (16) give $\nabla_{\varphi Z}\Phi = 0$ and $(\nabla_{\varphi Z}\varphi) = 0$. But then, by the virtue of Proposition 1 that $\nabla_\xi\varphi = 0$, we get $\nabla\varphi = 0$ which in turn implies $\nabla\xi = 0$. Also $\nabla_\xi\Phi = 0$ by (2), so we have $\nabla\Phi = 0$. On the other hand, since $(\nabla_X\eta)(Y) = g(\nabla_X\xi, Y)$ and $\nabla\xi = 0$ one gets $\nabla\eta = 0$. Thus (a) follows. □

Let us call an almost para-cosymplectic manifold, satisfying the condition

$$[R_{XY}, \varphi] = R_{XY} \circ \varphi - \varphi \circ R_{XY} = 0, \tag{18}$$

weakly para-cosymplectic.

It is obvious that a para-cosymplectic manifold is weakly para-cosymplectic. The converse implication does not hold in general. Indeed, the almost para-cosymplectic manifolds given in the below example fulfill (18) and are not para-cosymplectic.

EXAMPLE 3. Consider the flat pseudo-Riemannian metric g on $\boldsymbol{R}^3$ of signature $(++-)$,

$$g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 - dx^3 \otimes dx^3.$$

Let $h = 1 - x^1 - x^3$ and define a frame of vector fields (E_0, E_1, E_2) on $\boldsymbol{R}^3$ by

$$E_0 = h\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - h\frac{\partial}{\partial x^3},$$

$$E_1 = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^3},$$

$$E_2 = \frac{1}{2}(1 - h^2)\frac{\partial}{\partial x^1} - h\frac{\partial}{\partial x^2} + \frac{1}{2}(1 + h^2)\frac{\partial}{\partial x^3}.$$

For these vector fields, we have $g(E_0, E_0) = g(E_1, E_2) = g(E_2, E_1) = 1$, otherwise $g(E_i, E_j) = 0$. Let $(\omega^0, \omega^1, \omega^2)$ be the dual frame of 1-forms. Then g can be written in the form

$$g = \omega^0 \otimes \omega^0 + \omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1.$$

Let us define φ, ξ, η by

$$\xi = E_0, \quad \eta = \omega^0, \quad \varphi = \omega^1 \otimes E_1 - \omega^2 \otimes E_2.$$

Then $M_0 = \boldsymbol{R}^3(\varphi, \xi, \eta, g)$ is a 3-dimensional flat almost para-cosymplectic manifold. By the flatness, M_0 realizes (18). Since $AE_2 = -\nabla_{E_2}\xi = \partial/\partial x^1 - \partial/\partial x^3 \neq 0$, then $\nabla\varphi \neq 0$. Thus, M_0 is weakly para-cosymplectic but not para-cosymplectic.

It is interesting to point out that the vector fields E_0, E_1, E_2 form a basis of a 3-dimensional Lie algebra isomorphic to the Lie algebra of the Heisenberg group $\boldsymbol{H}^3$. Explicitly, the Poisson brackets are the following

$$[E_0, E_1] = 0, \quad [E_0, E_2] = E_1, \quad [E_1, E_2] = 0.$$

Moreover, E_0, E_1, E_2 are complete. Thus, there is a unique Lie group structure G on $\boldsymbol{R}^3$ with $(0,0,0) \in \boldsymbol{R}^3$ as the identity element, for which E_0, E_1, E_2 are left-invariant [15]. Because the group G is connected and simply connected, G is isomorphic to the Heisenberg group $\boldsymbol{H}^3$. By the above construction, the structure (φ, ξ, η, g) is left-invariant.

Let G_0 be a discrete, cocompact subgroup of G and $M_1 = G_0 \backslash G$ be a compact cosets manifold. Via the canonical projection, we obtain a flat non para-cosymplectic, almost para-cosymplectic structure on M_1, which will be denoted also by (φ, ξ, η, g).

Examples of strictly weakly para-cosymplectic manifolds in higher dimensions can be obtained in the following way. Let $M = M_0$ or $M = M_1$ with the suitable structure (φ, ξ, η, g) defined in the above and $\tilde{M}(\tilde{J}, \tilde{g})$ be an arbitrary para-Kählerian manifold. On the product manifold $M' = M \times \tilde{M}$, define an almost para-cosymplectic structure $(\varphi', \xi', \eta', g')$ as the product structure

$$\varphi' = (\varphi, \tilde{J}), \quad \xi' = (\xi, 0), \quad \eta' = (\eta, 0), \quad g' = (g, \tilde{g}).$$

Then, clearly, $[R'_{XY}, \varphi'] = 0$ and $\nabla' \varphi' \neq 0$. Thus, M' is weakly para-cosymplectic non para-cosymplectic. If $M = M_1$ and $\tilde{M}$ is compact, then M' is compact too. □

5. Manifolds with Para-Kählerian Leaves

In this section, we study almost para-cosymplectic manifolds, whose leaves of the canonical foliation are para-Kählerian submanifolds. We will call such manifolds almost para-cosymplectic with para-Kählerian leaves.

THEOREM 3. *An almost para-cosymplectic manifold M has para-Kählerian leaves if and only if any of the following equivalent conditions holds*

$$N(X, Y) = 2\eta(X)AY - 2\eta(Y)AX, \tag{19}$$

$$(\nabla_X \varphi)Y = g(A\varphi X, Y)\xi - \eta(Y)A\varphi X, \tag{20}$$

PROOF. Note that the Nijenhuis tensors N and $\tilde{N}$ of φ and the induced para-complex structure $\tilde{J}$ of a leaf $\tilde{M} \in \mathscr{F}$ are related by $N|_{\tilde{M}} = \tilde{N}$. If the induced structures $(\tilde{J}, \tilde{g})$ are para-Kählerian, then $\tilde{N} = 0$, and consequently $N(X, Y) = 0$ for any vector fields X, Y tangent to $\tilde{M}$. Therefore, $N(\varphi X, \varphi Y) = 0$ for any vector fields on M, whence (19) follows by (15).

Let us assume (19) for an almost para-cosymplectic manifold. Then, by (16), we have

$$(\nabla_{\varphi Z}\Phi)(X, Y) = -\eta(X)g(AZ, Y) + \eta(Y)g(AZ, X),$$

and hence

$$(\nabla_{\varphi Z}\varphi)X = g(AZ, X)\xi - \eta(X)AZ.$$

If we replace Z by φZ in the last equation and use $\nabla_\xi\varphi = 0$, we get (20).

Now we prove that (20) implies the leaves of the manifold M are para-Kählerian. Let $(\tilde{J}, \tilde{g})$ be the induced almost para-Kählerian structure on a leaf $\tilde{M}$ and $\tilde{\nabla}$ be the Levi-Civita connection of $\tilde{g}$. By the Gauss equation

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \nabla_{\tilde{X}}\tilde{Y} - g(A\tilde{X}, \tilde{Y})\xi$$

and (20), we find

$$(\tilde{\nabla}_{\tilde{X}}\tilde{J})\tilde{Y} = (\nabla_{\tilde{X}}\varphi)\tilde{Y} + g(\varphi A\tilde{X}, \tilde{Y})\xi = g(A\varphi\tilde{X}, \tilde{Y})\xi + g(\varphi A\tilde{X}, \tilde{Y})\xi = 0,$$

hence $(\tilde{J}, \tilde{g})$ is para-Kählerian. □

PROPOSITION 5. *For an almost para-cosymplectic manifold with para-Kählerian leaves, we have the following curvature identity*

$$\begin{aligned}(21)\quad R_{ZX}\varphi Y - \varphi R_{ZX}Y = g(A\varphi Z, Y)AX - g(A\varphi X, Y)AZ + g(AZ, Y)A\varphi X\\ - g(AX, Y)A\varphi Z - g(R_{ZX}\xi, \varphi Y)\xi - \eta(Y)\varphi R_{ZX}\xi.\end{aligned}$$

PROOF. By $\varphi A = -A\varphi$, (20) and $\eta(AX) = 0$ we find

$$(\nabla_Z(A\varphi))X = -(\nabla_Z\varphi)AX - \varphi(\nabla_Z A)X = -g(A\varphi Z, AX)\xi - \varphi(\nabla_Z A)X.$$

Differentiating covariantly (20) and using the relation above, we obtain

$$\begin{aligned}(\nabla^2_{ZX}\varphi)Y = -g(A\varphi X, Y)AZ + g(AZ, Y)A\varphi X\\ + g((\nabla_Z A)X, \varphi Y)\xi + \eta(Y)\varphi(\nabla_Z A)X.\end{aligned}$$

Now, the result follows if we antisymmetrize the last relation with respect to Z, X and use (10). □

THEOREM 4. *Almost para-cosymplectic manifolds of constant non-zero sectional curvature do not exist in dimensions* ≥ 5.

PROOF. Let M be an almost para-cosymplectic manifold of non-zero constant sectional curvature $\lambda \neq 0$ of dimension $2n + 1 \geq 5$. Then

$$R_{\xi X}\xi = \varphi R_{\xi\varphi X}\xi = \lambda\eta(X)\xi - \lambda X.$$

On the other hand, by (10) and Proposition 1, $R_{\xi X}\xi = A^2X - (\nabla_\xi A)X$ and $\varphi R_{\xi\varphi X}\xi = A^2X + (\nabla_\xi A)X$. Hence, $(\nabla_\xi A)X = 0$ and

$$A^2X = -\lambda X + \lambda\eta(X)\xi. \tag{22}$$

Formula (11) implies

$$2\lambda\eta(Y)X - 2\lambda\eta(X)Y = -\nabla_{N(X,Y)}\xi = AN(X,Y).$$

Applying A to the both sides of the above equation and using (22), (17) and $\lambda \neq 0$, we find

$$2\eta(Y)AX - 2\eta(X)AY = -N(X,Y).$$

Then by the virtue of Theorem 3, M has para-Kählerian leaves. By $Tr(A) = Tr(A\varphi) = Tr(Z \mapsto \varphi R_{ZX}\xi) = 0$, the trace of (21) with respect to Z gives

$$(2n-1)\lambda g(X,\varphi Y) = -g(R_{\xi X}\xi, \varphi Y) = \lambda g(X, \varphi Y).$$

This is a contradiction since $n \geq 2$ and $\lambda \neq 0$. □

LEMMA 3. *Let u, v be bilinear symmetric forms on a real s-dimensional vector space W, $s \geq 2$. If* $\operatorname{rank}(u) = \operatorname{rank}(v) = p$, *$u$ and v have a common diagonalizing basis and*

$$u(z,y)u(x,w) - u(x,y)u(z,w) + v(z,y)v(x,w) - v(x,y)v(z,w) = 0$$

for any $x, y, z, w \in W$, then $p \leq 2$.

PROOF. Choose a basis $(e_i, i = 1, 2, \ldots, s)$, so that $u(e_i, e_j) = a_i\delta_{ij}$, $v(e_i, e_j) = b_i\delta_{ij}$ for certain a_i, b_i. We may assume that $a_i \neq 0$, $b_i \neq 0$ for $i = 1, \ldots, p$, otherwise $a_i = b_i = 0$. Let us suppose that $p \geq 3$. From (3), for $z = y = e_i$, $x = w = e_j$, $1 \leq i \neq j \leq p$, we obtain $a_ia_j + b_ib_j = 0$. Hence $b_i = -a_ia_1/b_1$ for $2 \leq i \leq p$. This applied in the previous equation gives $a_ia_j = 0$, $2 \leq i \neq j \leq p$, which is a contradiction. □

THEOREM 5. *Let M be an almost para-cosymplectic manifold with para-Kählerian leaves. Then M is weakly para-cosymplectic if and only if the following two conditions* (I) *and* (II) *are fulfilled*

(I) *the tensor field A is a Codazzi tensor, that is, $(\nabla_X A)Y = (\nabla_Y A)X$;*
(II) *at any point $p \in M$, the operator A has one of the following shape*

(a) $A = 0$,

(b) $AX = \varepsilon g(X, V)V$, *where* $|\varepsilon| = 1$ *and* V *is a non-zero null vector such that* $\varphi V = V$ *or* $\varphi V = -V$,

(c) $AX = \varepsilon_1 g(X, V_1)V_1 + \varepsilon_2 g(X, V_2)V_2$, *where* V_1, V_2 *are non-zero orthogonal null vectors such that* $\varphi V_1 = -V_1$, $\varphi V_2 = V_2$ *and* $|\varepsilon_i| = 1$.

Proof. Let M be the weakly para-cosymplectic. Then $\varphi R_{XY}\xi = R_{XY}\varphi\xi - [R_{XY}, \varphi]\xi = 0$, and hence $R_{XY}\xi = 0$. Now (I) follows by (10). Observe, $0 = \varphi R_{\xi\varphi X}\xi = A^2X + (\nabla_\xi A)X = A^2X + (\nabla_X A)\xi = 2A^2X$. Thus $A^2X = 0$.

By $R_{XY}\xi = 0$ the identity (21) simplifies to

$$g(A\varphi Z, Y)AX - g(A\varphi X, Y)AZ + g(AZ, Y)A\varphi X - g(AX, Y)A\varphi Z = 0.$$

Projecting the last relation onto φW we find

$$\begin{aligned} g(A\varphi Z, Y)g(A\varphi X, W) - g(A\varphi X, Y)g(A\varphi Z, W)& \\ + g(AZ, Y)g(AX, W) - g(AX, Y)g(AZ, W) &= 0. \end{aligned} \tag{23}$$

Now, let $(E_0, E_\alpha, E_{\alpha+n})$, $\alpha = 1, \dots, n$, be a basis of the tangent space at a point $p \in M$, such that

$$\begin{gathered} E_0 = \xi_p, \quad \varphi E_\alpha = E_\alpha, \quad \varphi E_{\alpha+n} = -E_{\alpha+n}, \\ g(E_0, E_0) = 1, \quad g(E_\alpha, E_{\alpha+n}) = 1. \end{gathered} \tag{24}$$

For $X = \sum_{i=0}^{2n} X^i E_i$, $Y = \sum_{i=0}^{2n} Y^i E_i$, put $g(AX, Y) = \sum_{i,j=0}^{2n} c_{ij} X^i Y^j$. By $A\xi = 0$, $c_{0i} = g(A\xi_p, E_i) = 0$ and by $\varphi A = -A\varphi$, (24)

$$\begin{aligned} c_{\alpha(\beta+n)} &= g(AE_\alpha, E_{\beta+n}) = -g(\varphi AE_\alpha, \varphi E_{\beta+n}) \\ &= g(A\varphi E_\alpha, \varphi E_{\beta+n}) = -g(AE_\alpha, E_{\beta+n}) = -c_{\alpha(\beta+n)} = 0. \end{aligned}$$

Hence

$$g(AX, Y) = \sum_{\alpha,\beta=1}^{n} (c_{\alpha\beta} X^\alpha Y^\beta + c_{(\alpha+n)(\beta+n)} X^{\alpha+n} Y^{\beta+n}). \tag{25}$$

Observe the following

$$\begin{aligned} g(A\varphi E_\alpha, E_\beta) &= g(AE_\alpha, E_\beta) = c_{\alpha\beta}, \\ g(A\varphi E_\alpha, E_{\beta+n}) &= g(AE_\alpha, E_{\beta+n}) = c_{\alpha(\beta+n)} = 0, \\ g(A\varphi E_{\alpha+n}, E_{\beta+n}) &= -g(AE_{\alpha+n}, E_{\beta+n}) = -c_{(\alpha+n)(\beta+n)}. \end{aligned}$$

Thus we have

$$(26)\qquad g(A\varphi X, Y) = \sum_{\alpha,\beta=1}^{n} (c_{\alpha\beta} X^{\alpha} Y^{\beta} - c_{(\alpha+n)(\beta+n)} X^{\alpha+n} Y^{\beta+n}).$$

The forms $g(A\cdot,\cdot), g(A\varphi\cdot,\cdot)$ have the same rank r, common diagonalizing basis (by (25) and (26)) and fulfill (23). That means, they realize the assumptions of Lemma 3, therefore it must hold $r \leq 2$. Note that $r = \operatorname{rank}(c_{\alpha\beta}) + \operatorname{rank}(c_{(\alpha+n)(\beta+n)})$. If $r = 0$, then $A = 0$. Let $r = 1$. We will show the assertion (II)(b). At first, consider the case $\operatorname{rank}(c_{\alpha\beta}) = 1$. Then $c_{(\alpha+n)(\beta+n)} = 0$, and $c_{\alpha\beta} = \varepsilon d_{\alpha} d_{\beta}$ for $|\varepsilon| = 1$ and certain d_{α}. Define a 1-form ω and a vector V by assuming $\omega(X) = \sum_{\alpha=1}^{n} d_{\alpha} X^{\alpha}$ and $V = \sum_{\alpha=1}^{n} d_{\alpha} E_{\alpha+n}$. One checks that $A = \varepsilon\omega \otimes V$, $\varphi V = -V$, $\omega(X) = g(X, V)$ and $\omega(V) = g(V, V) = 0$. Similarly, in the case $\operatorname{rank}(c_{\alpha\beta}) = 0$ and $\operatorname{rank}(c_{(\alpha+n)(\beta+n)}) = 1$, we find a 1-form ω and a vector V for which $A = \varepsilon\omega \otimes V$, $\varphi V = V$, $\omega(X) = g(X, V)$ and $g(V, V) = 0$. Let now $r = 2$. Suppose $\operatorname{rank}(c_{\alpha\beta}) = 2$. Then $c_{(\alpha+n)(\beta+n)} = 0$, which together with (25) and (26) implies $g(A\varphi X, Y) = g(AX, Y)$. Applying the last relation into (23), we find

$$g(AZ, Y)g(AX, W) - g(AX, Y)g(AZ, W) = 0,$$

which clearly yields $\operatorname{rank}(g(A\cdot,\cdot)) = \operatorname{rank}(c_{\alpha\beta}) \leq 1$, a contradiction. Hence $\operatorname{rank}(c_{\alpha\beta}) < 2$. Similar arguments show that $\operatorname{rank}(c_{(\alpha+n)(\beta+n)}) < 2$. Thus

$$\operatorname{rank}(c_{\alpha\beta}) = \operatorname{rank}(c_{(\alpha+n)(\beta+n)}) = 1,$$

that is, $c_{\alpha\beta} = \varepsilon_1 d_{\alpha} d_{\beta}$, $c_{(\alpha+n)(\beta+n)} = \varepsilon_2 h_{\alpha} h_{\beta}$, $|\varepsilon_i| = 1$. Define 1-forms ω_1, ω_2 and vectors V_1, V_2 by assuming

$$\omega_1(X) = \sum_{\alpha=1}^{n} d_{\alpha} X^{\alpha}, \quad \omega_2(X) = \sum_{\alpha=1}^{n} h_{\alpha} X^{\alpha+n}, \quad V_1 = \sum_{\alpha=1}^{n} d_{\alpha} E_{\alpha+n}, \quad V_2 = \sum_{\alpha=1}^{n} h_{\alpha} E_{\alpha}.$$

We verify that

$$A = \varepsilon_1 \omega_1 \otimes V_1 + \varepsilon_2 \omega_2 \otimes V_2, \quad \omega_1(X) = g(X, V_1), \quad \omega_2(X) = g(X, V_2),$$

$$\varphi V_1 = -V_1, \quad \varphi V_2 = V_2, \quad g(V_1, V_1) = g(V_2, V_2) = 0.$$

Finally, $A^2 = 0$ implies $A^2 V_1 = \varepsilon_1 \varepsilon_2 g(V_1, V_2)^2 V_1 = 0$, hence $g(V_1, V_2) = 0$. Thus, the assertion (II)(c) holds.

Conversely, by (10) and (I), $R_{XY}\xi = 0$. Moreover, (II) implies (23). Consequently, $R_{ZX}\varphi Y - \varphi R_{ZX} Y = 0$ follows from (21). □

EXAMPLE 4. Let $M = \boldsymbol{R}^5$ with coordinates (z, u_1, u_2, v_1, v_2). Define a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a metric g as follows

$$\varphi\frac{\partial}{\partial z} = 2u_1\frac{\partial}{\partial v_1} - 2u_2\frac{\partial}{\partial v_2},$$

$$\varphi\frac{\partial}{\partial u_1} = -2u_1\frac{\partial}{\partial z} - \frac{\partial}{\partial u_1} + 4u_1u_2\frac{\partial}{\partial v_2},$$

$$\varphi\frac{\partial}{\partial u_2} = 2u_2\frac{\partial}{\partial z} + \frac{\partial}{\partial u_2} - 4u_1u_2\frac{\partial}{\partial v_1},$$

$$\varphi\frac{\partial}{\partial v_1} = \frac{\partial}{\partial v_1}, \quad \varphi\frac{\partial}{\partial v_2} = -\frac{\partial}{\partial v_2},$$

$$\xi = \frac{\partial}{\partial z} - 2u_1\frac{\partial}{\partial v_1} - 2u_2\frac{\partial}{\partial v_2},$$

$$\eta = dz - 2u_1\,du_1 - 2u_2\,du_2,$$

$$g = dz^2 + 2\,du_1dv_1 + 2\,du_2dv_2.$$

By straightforward computations we verify that (φ, ξ, η, g) is the almost para-cosymplectic structure on M with the fundamental form Φ given by

$$\Phi = 4u_1\,dz \wedge du_1 - 4u_2\,dz \wedge du_2 + 8u_1u_2\,du_1 \wedge du_2 - 2\,du_1 \wedge dv_1 + 2\,du_2 \wedge dv_2.$$

For the tensor field $A = -\nabla\xi$, we have

$$A = 2\,du_1 \otimes \frac{\partial}{\partial v_1} + 2\,du_2 \otimes \frac{\partial}{\partial v_2},$$

so that A is of rank 2 everywhere on M. The covariant derivative $\nabla\varphi$, which is nonzero, satisfies the relation (20) and therefore by Theorem 3, M has para-Kählerian leaves. M is weakly para-cosymplectic since the metric g is flat. □

6. Curvature Identities

PROPOSITION 6. *For any almost para-cosymplectic manifold, we have*

$$(27)\qquad [R_{Z\varphi X}, \varphi] + [R_{\varphi ZX}, \varphi] - [R_{\varphi Z\varphi X}, \varphi]\varphi - [R_{ZX}, \varphi]\varphi = \nabla_{\varphi N(Z,X)}\varphi + \eta \otimes (R_{\varphi Z\varphi X}\xi + R_{ZX}\xi).$$

PROOF. At first, by (8), we have

$$(\nabla_X\varphi)\varphi Y - (\nabla_{\varphi X}\varphi)Y = \eta(Y)AX.$$

Next, differentiating this covariantly, we obtain

$$(28)\quad (\nabla^2_{ZX}\varphi)\varphi Y-(\nabla^2_{Z\varphi X}\varphi)Y = (\nabla_{(\nabla_Z\varphi)X}\varphi)Y-(\nabla_X\varphi)(\nabla_Z\varphi)Y-g(AZ,Y)AX+\eta(Y)(\nabla_Z A)X.$$

Replacing in (28) Z, X, Y by $\varphi V, U, \varphi W$, respectively, we obtain

$$(29)\quad (\nabla^2_{\varphi VU}\varphi)W-\eta(W)(\nabla^2_{\varphi VU}\varphi)\xi-(\nabla^2_{\varphi V\varphi U}\varphi)\varphi W = (\nabla_{(\nabla_{\varphi V}\varphi)U}\varphi)\varphi W-(\nabla_U\varphi)(\nabla_{\varphi V}\varphi)\varphi W-g(AV,W)AU.$$

On the other hand, using (7) and (9), we find

$$(\nabla_U\varphi)(\nabla_{\varphi V}\varphi)\varphi W=(\nabla_U\varphi)(\nabla_V\varphi)W+\eta(W)(\nabla_U\varphi)A\varphi V,$$

$$(\nabla_{(\nabla_{\varphi V}\varphi)U}\varphi)\varphi W=(\nabla_{-\varphi(\nabla_V\varphi)U+g(AV,U)\xi}\varphi)\varphi W=-(\nabla_{\varphi(\nabla_V\varphi)U}\varphi)\varphi W$$
$$=-(\nabla_{(\nabla_V\varphi)U}\varphi)W-\eta(W)A\varphi(\nabla_V\varphi)U.$$

By these relations, (29) turns into

$$(30)\quad (\nabla^2_{\varphi VU}\varphi)W-(\nabla^2_{\varphi V\varphi U}\varphi)\varphi W = -(\nabla_{(\nabla_V\varphi)U}\varphi)W-(\nabla_U\varphi)(\nabla_V\varphi)W-g(AV,W)AU + \eta(W)((\nabla^2_{\varphi VU}\varphi)\xi-(\nabla_U\varphi)A\varphi V-A\varphi(\nabla_V\varphi)U).$$

Putting $V=X,\ U=Z,\ W=Y$ in (30) and adding the obtained relation to (28), we have

$$-[R_{Z\varphi X},\varphi]Y-(\nabla^2_{\varphi X\varphi Z}\varphi)\varphi Y+(\nabla^2_{ZX}\varphi)\varphi Y$$
$$=-(\nabla_{(\nabla_X\varphi)Z}\varphi)Y+(\nabla_{(\nabla_Z\varphi)X}\varphi)Y-(\nabla_X\varphi)(\nabla_Z\varphi)Y-(\nabla_Z\varphi)(\nabla_X\varphi)Y$$
$$-g(AX,Y)AZ-g(AZ,Y)AX+\eta(Y)((\nabla^2_{\varphi XZ}\varphi)\xi-(\nabla_Z\varphi)A\varphi X$$
$$-A\varphi(\nabla_X\varphi)Z+(\nabla_Z A)X).$$

Antisymmetrization of the last equality with respect to Z, X and application of (14), gives

$$(31)\quad [R_{Z\varphi X},\varphi]Y+[R_{\varphi ZX},\varphi]Y-[R_{\varphi Z\varphi X},\varphi]\varphi Y-[R_{ZX},\varphi]\varphi Y$$
$$=-2(\nabla_{(\nabla_Z\varphi)X}\varphi)Y+2(\nabla_{(\nabla_X\varphi)Z}\varphi)Y-\eta(Y)S(Z,X)$$
$$=(\nabla_{\varphi N(Z,X)}\varphi)Y-\eta(Y)S(Z,X),$$

where S is a $(1,2)$ skew-symmetric tensor field. Put $Y=\xi$ in (31) and find

$$[R_{Z\varphi X},\varphi]\xi+[R_{\varphi ZX},\varphi]\xi-(\nabla_{\varphi N(Z,X)}\varphi)\xi=-S(Z,X).$$

This implies $g(S(Z,X),\xi)=0$. Having this in mind and projecting (31) onto ξ, we find

$$[R_{Z\varphi X},\varphi]\xi+[R_{\varphi ZX},\varphi]\xi+\varphi[R_{\varphi Z\varphi X},\varphi]\xi+\varphi[R_{ZX},\varphi]\xi=(\nabla_{\varphi N(Z,X)}\varphi)\xi,$$

which having substituted to the previous equation, gives

$$S(Z,X)=\varphi[R_{\varphi Z\varphi X},\varphi]\xi+\varphi[R_{ZX},\varphi]\xi=-R_{\varphi Z\varphi X}\xi-R_{ZX}\xi.$$

But then this reduces (31) to (27). □

Let Ric and Ric^* be the Ricci and $*$-Ricci tensors defined by

$$Ric(X,Y)=Tr\{Z\mapsto R_{ZX}Y\},\quad Ric^*(X,Y)=Tr\{Z\mapsto \varphi R_{ZX}\varphi Y\}.$$

Let $\widetilde{Ric},\widetilde{Ric^*}$ be the Ricci and $*$-Ricci operators and r,r^* be the scalar and $*$-scalar curvatures given by

$$Ric(X,Y)=g(\widetilde{Ric}X,Y),\quad Ric^*(X,Y)=g(\widetilde{Ric^*}X,Y),$$
$$r=Tr(\widetilde{Ric}),\quad r^*=Tr(\widetilde{Ric^*}).$$

THEOREM 6. *For an almost para-cosymplectic manifold, we have*

(32) $$R_{Z\varphi X}\varphi Y-\varphi R_{Z\varphi X}Y+R_{\varphi ZX}\varphi Y-\varphi R_{\varphi ZX}Y - R_{\varphi Z\varphi X}Y+\varphi R_{\varphi Z\varphi X}\varphi Y-R_{ZX}Y+\varphi R_{ZX}\varphi Y=(\nabla_{\varphi N(Z,X)}\varphi)Y,$$

(33) $$Ric^*(X,Y)+Ric^*(Y,X)-Ric(X,Y)+Ric(\varphi X,\varphi Y) +\frac{1}{2}(R_{\xi XY\xi}-R_{\xi\varphi X\varphi Y\xi})+\sum_{j=0}^{2n}\varepsilon_j g((\nabla_{E_j}\varphi)X,(\nabla_{E_j}\varphi)Y)=0,$$

(34) $$r^*-r+Ric(\xi,\xi)+\frac{1}{2}g(\nabla\varphi,\nabla\varphi)=0.$$

where $(E_j, 0\le j\le 2n)$ is an orthonormal frame, ε_j's are the indicators of E_j's and

$$g(\nabla\varphi,\nabla\varphi)=\sum_{i,j=0}^{2n}\varepsilon_i\varepsilon_j g((\nabla_{E_j}\varphi)E_i,(\nabla_{E_j}\varphi)E_i).$$

PROOF. Formula (32) is in fact a direct consequence of the identity (27).

Taking the trace of (32) with respect to Z, we find

$$
(35)\qquad 2Ric^*(X,Y) - 2Ric(X,Y) + 2Ric(\varphi X,\varphi Y) - 2Tr\{Z \mapsto \varphi R_{Z\varphi X}Y\} + R_{\xi XY\xi} - R_{\xi\varphi X\varphi Y\xi} = Tr\{Z \mapsto (\nabla_{\varphi N(Z,X)}\varphi)Y\}.
$$

Let (E_i) be a local orthonormal frame and compute

$$
Tr\{Z \mapsto \varphi R_{Z\varphi X}Y\} = -\sum_{i=0}^{2n}\varepsilon_i R_{\varphi E_i Y\varphi XE_i} = -Ric^*(Y,X).
$$

By (16), we have

$$
\begin{aligned}
Tr\{Z \mapsto (\nabla_{\varphi N(Z,X)}\varphi)Y\} &= \sum_{i=0}^{2n}\varepsilon_i g((\nabla_{\varphi N(E_i,X)}\varphi)Y, E_i) \\
&= \frac{1}{2}\sum_{i=0}^{2n}\varepsilon_i g(N(E_i,X), N(E_i,Y)) = -\frac{1}{2}\sum_{i=0}^{2n}\varepsilon_i g(\varphi N(E_i,X), \varphi N(E_i,Y)) \\
&= -\frac{1}{2}\sum_{i,j=0}^{2n}\varepsilon_i\varepsilon_j g(E_j, \varphi N(E_i,X))g(E_j,\varphi N(E_i,Y)) \\
&= -2\sum_{i,j=0}^{2n}\varepsilon_i\varepsilon_j g((\nabla_{E_j}\varphi)X, E_i)g((\nabla_{E_j}\varphi)Y, E_i) = -2\sum_{j=0}^{2n}\varepsilon_j g((\nabla_{E_j}\varphi)X, (\nabla_{E_j}\varphi)Y).
\end{aligned}
$$

Applying these relations into (35), we find (33).

Taking the trace of (33) with respect to g, we obtain

$$
(36)\qquad 2r^* - r + Tr_g\{(X,Y) \mapsto Ric(\varphi X,\varphi Y)\} + \frac{1}{2}Ric(\xi,\xi) - \frac{1}{2}Tr_g\{(X,Y) \mapsto R_{\xi\varphi X\varphi Y\xi}\} + g(\nabla\varphi,\nabla\varphi) = 0.
$$

On the other hand, we find

$$
\begin{aligned}
Tr_g\{(X,Y) \mapsto Ric(\varphi X,\varphi Y)\} &= -Tr\{X \mapsto \varphi\widetilde{Ric}\varphi X\} \\
&= -Tr\{X \mapsto \widetilde{Ric}\varphi^2 X\} = -r + Ric(\xi,\xi).
\end{aligned}
$$

Moreover

$$
\begin{aligned}
Tr_g\{(X,Y) \mapsto R_{\xi\varphi X\varphi Y\xi}\} &= -Tr\{X \mapsto \varphi R_{\varphi X\xi}\xi\} \\
&= -Tr\{X \mapsto R_{X\xi}\xi\} = -Ric(\xi,\xi).
\end{aligned}
$$

The last two relations reduce (36) to (34). □

FINAL REMARKS. Certain of our results are para-cosymplectic analogies of theorems concerning almost cosymplectic manifolds proved in [12], [13], [7].

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