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On almost paracontact Riemannian manifolds of type (n, n)

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Abstract. In this paper we give a classification with eleven basic classes of almost paracontact Riemannian manifolds of type (n, n) with respect to the covariant derivative of the $(1, 1)$-tensor of the almost paracontact structure.

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1. Introduction

In 1976 I. Sato [1] introduced the concepts of almost paracontact manifolds and of almost paracontact Riemannian manifolds as analogues of almost contact manifolds and of almost contact Riemannian manifolds.

After that S. Sasaki [2] defined the notion of an almost paracontact Riemannian manifold of type (p, q) and arbitrary dimension, where p and q are the numbers of the multiplicity of the structural eigenvalues 1 and -1, respectively. In addition, there is a simple eigenvalue 0.

In this paper we consider almost paracontact Riemannian manifolds of type (n, n), i.e. $p = q = n$. We put this fixation in view of reasons of later investigations relevant to $2n$-dimensional Riemannian almost product manifolds (M^{2n}, P, g) with structural group $O(n) \times O(n)$, which are classified in [3]. In this reason the manifolds in our consideration could be construct by natural way as a direct product of (M^{2n}, P, g) and a real line or as a hypersurface of (M^{2n}, P, g).

The method used in the present paper is analogous to the methods of classification in [4] and partly to those in [5] for the almost contact metric manifolds and for the almost contact manifolds with B-metric, respectively.

2. Preliminaries

A $(2n+1)$-dimensional real differentiable manifold M is said to have an almost paracontact structure (ϕ, ξ, η) of type (n, n), if it admits a $(1, 1)$-tensor ϕ, a vector field ξ and a 1-form η satisfying the following conditions:

$$\eta(\xi) = 1, \qquad \phi^2 = I_{2n+1} - \eta \otimes \xi, \qquad tr\phi = 0. \tag{1}$$

A positive definite Riemannian metric g is said to be compatible with the almost paracontact structure if it satisfies the following conditions

$$g(x,\xi) = \eta(x), \qquad g(\phi x, \phi y) = g(x, y) - \eta(x)\eta(y) \tag{2}$$

for all vectors x, y in the tangent space T_pM.

The quadruple (ϕ, ξ, η, g) of an almost paracontact structure of type (n, n) and a compatible metric is called an almost paracontact Riemannian structure of type (n, n).

We introduce the tensor $\tilde{g}$ given by the equation

$$\tilde{g}(\cdot, \cdot) = g(\cdot, \phi\cdot) + \eta(\cdot)\eta(\cdot). \tag{3}$$

It is a compatible metric with the almost paracontact structure, too. The associated metric $\tilde{g}$ to the Riemannian metric g is a pseudo-Riemannian metric of signature $(n + 1, n)$.

Let us denote the tensor of type (0, 3) by the equation

$$F(x,y,z) = g((\nabla_x \phi)y,z) \qquad (x,\ y,\ z\ \in\ T_pM). \tag{4}$$

Because of (1) and (2) the tensor F has the following properties

$$\begin{gathered} F(x, y, z) = F(x, z, y) = -F(x, \phi y, \phi z) \\ +\eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi) \end{gathered} \tag{5}$$

for all vectors $x,\ y,\ z$ in T_pM.

If $\{e_i , \xi\}$ $(i = 1, 2, \dots, 2n)$ is a basis of the tangent space T_pM and g^{ij} are the components of the inverse matrix of g, then the following 1-forms are associated with the tensor F

$$\theta(x) = g^{ij} F(e_i, e_j, x), \quad \theta^*(x) = g^{ij} F(e_i, \phi e_j, x), \quad \omega(x) = F(\xi, \xi, x). \tag{6}$$

for an arbitrary vector $x \in T_pM$.

3. The space of covariant derivatives of the (1, 1)-tensor of the almost paracontact structure

Let (V, ϕ, ξ, η, g) be $(2n + 1)$-dimensional vector space with almost paracontact Riemannian structure of type (n, n). We consider the vector space $\mathcal{F}$ of the tensors of type (0, 3) over V having the properties (6). The metric g induces on $\mathcal{F}$ an inner product $\langle\ ,\ \rangle$, defined by

$$\langle F', F''\rangle = g^{ip}\, g^{jq} g^{kr} F'(e_i, e_j, e_k)F''(e_p, e_q, e_r)$$

for arbitrary elements F', F'' in $\mathcal{F}$ and a V's basis $\{e_i, e_{2n+1} = \xi\}$ $(i = 1, 2, \dots, 2n)$.

The standard representation of the structural group $O(n) \times O(n) \times I$ in V induces a natural representation of the structural group in $\mathcal{F}$:

$$(\lambda(a)F)(x, y, z) = F(a^{-1}x, a^{-1}y, a^{-1}z),$$

where $a \in O(n) \times O(n) \times I$, $F \in \mathcal{F}$, $x, y, z \in V$, so that

$$\langle \lambda(a)F', \lambda(a)F'' \rangle = \langle F', F'' \rangle.$$

We consider operators h, v, w on V with the properties

$$\begin{gathered} h^2 = h, \qquad v^2 = v, \qquad w^2 = w, \\ h \circ v = v \circ h = h \circ w = w \circ h = v \circ w = w \circ v = 0. \end{gathered} \tag{7}$$

The action of these operators on the space $\mathcal{F}$ is represented by the equations

$$\begin{aligned} &hF(x, y, z) = F(\phi^2 x, \phi^2 y, \phi^2 z), \\ &vF(X, Y, Z) = \eta(x)F(\xi, y, z) + \eta(y)F(x, z, \xi) + \eta(z)F(x, y, \xi) \\ &\qquad -2\eta(x)\left[\eta(y)F(\xi, \xi, z) + \eta(z)F(\xi, \xi, y)\right], \\ &wF(x, y, z) = \eta(x)\left[\eta(y)F(\xi, \xi, z) + \eta(z)F(\xi, \xi, y)\right] \end{aligned} \tag{8}$$

The equality (8) implies

$$F = hF + vF + wF. \tag{9}$$

Now we define basic operators $F_i : \mathcal{F} \to \mathcal{F}$ $(i = 1, 2, \ldots, 10)$ by the equations:

$$\begin{aligned} &F_1(F)(x, y, z) = \eta(x)F(\xi, y, z), \\ &F_2(F)(x, y, z) = \eta(y)F(x, z, \xi) + \eta(z)F(x, y, \xi), \\ &F_3(F)(x, y, z) = \eta(x)\left[\eta(y)F(\xi, \xi, z) + \eta(z)F(\xi, \xi, y)\right], \\ &F_4(F)(x, y, z) = \eta(y)F(z, \xi, x) + \eta(z)F(y, \xi, x) \\ &\qquad -2\eta(y)\eta(z)F(\xi, \xi, x), \\ &F_5(F)(x, y, z) = \eta(y)F(\phi x, \xi, \phi z) + \eta(z)F(\phi x, \xi, \phi y), \\ &F_6(F)(x, y, z) = \eta(y)F(\phi z, \xi, \phi x) + \eta(z)F(\phi y, \xi, \phi x), \\ &F_7(F)(x, y, z) = \tfrac{\theta(F)(\xi)}{2n}\left\{\eta(y)g(\phi x, \phi z) + \eta(z)g(\phi x, \phi y)\right\}, \\ &F_8(F)(x, y, z) = \tfrac{\theta^*(F)(\xi)}{2n}\left\{\eta(y)g(x, \phi z) + \eta(z)g(x, \phi y)\right\}, \\ &F_9(F)(x, y, z) = \tfrac{1}{2n}\{g(\phi x, \phi y)\theta(\phi^2 z) + g(\phi x, \phi z)\theta(\phi^2 y) \\ &\qquad -g(x, \phi y)\theta(\phi z) - g(x, \phi z)\theta(\phi y)\}, \\ &F_{10}(F)(x, y, z) = \tfrac{1}{3}\left\{hF(x, y, z) + hF(y, z, x) + hF(z, x, y)\right\}. \end{aligned} \tag{10}$$

It is easy to check that $F_i(F) \in F$ for $F \in \mathcal{F}$ $(i = 1, 2, \ldots, 10)$.

By necessity we have to compute the compositions of the basic operators and the associated 1-forms of $F_i(F)$.

LEMMA 1. *Let* $F \in \mathcal{F}$ *and* $F_{i,j}(F) = F_i(F_j(F))$ $(i, j = 1, 2, \dots, 10)$. *Then we have:*

a) $F_{i,i}(F) = F_i(F)$ *for* $i = 1, 2, 3, 7, \dots, 10$*;*
$F_{i,i}(F) = F_2(F) - F_3(F)$ *for* $i = 4, 5, 6$*;*
$F_{1,i}(F) = F_{i,1}(F) = F_3(F)$ *for* $i = 2, 3$*;*
$F_{2,i}(F) = F_{i,2}(F) = F_i(F)$ *for* $i = 3, 4, 5, 6, 7, 8$*;*
$F_{i,j}(F) = F_k(F)$ *for* $i, j, k = 4, 5, 6$*;*
$F_{i,j}(F) = F_{j,i}(F) = F_j(F)$ *for* $i = 4, 5, 6$ *and* $j = 7, 8$
and the rest of $F_{i,j}(F)$ *are zeros;*

b) $\theta(F_i(F)) = \theta(F)(\xi)\eta$ *for* $i = 2, 4, 5, 6, 7$*;*
$\theta^*(F_i(F)) = \theta^*(F)(\xi)\eta$ *for* $i = 2, 4, 5, 6, 8$*;*
$\omega(F_i(F)) = \omega(F)$ *for* $i = 1, 2, 3$
and the rest of the associated 1*-forms are zeros.*

By virtue of the operators F_i we construct new operators L_j $(j = 1, 2, \dots, 9)$. Let

$$L_1 : \mathcal{F} \to \mathcal{F}, \qquad L_1(F) = F - 2F_3(F), \qquad F \in \mathcal{F}.$$

In a straightforward way using Lemma 1 we get that the operator L_1 is an involutive isometry on $\mathcal{F}$ and commutes with the action of $O(n) \times O(n) \times I$. Hence L_1 has two eigenvalues $+1$ and -1, and the corresponding eigenspaces $(L_1\mathcal{F})^+$ and $(L_1\mathcal{F})^-$ are orthogonal and invariant mutually complementary subspaces of $\mathcal{F}$. Besides that, the components of F in the subspaces $(L_1\mathcal{F})^+$ and $(L_1\mathcal{F})^-$ are $\frac{1}{2}\{F + L_1(F)\}$ and $\frac{1}{2}\{F - L_1(F)\}$, respectively.

It is easy to show

LEMMA 2. *If* $F \in \mathcal{F}$ *then*

a) $F_3(F) = 0$ *iff* $\omega(F) = 0$*;*
b) $F \in (L_1\mathcal{F})^+$, *i.e.* $L_1(F) = F$ *iff* $F_3(F) = 0$*;*
c) $F \in (L_1\mathcal{F})^-$, *i.e.* $L_1(F) = -F$ *iff* $F = F_3(F)$.

If we denote $\mathcal{F}_{11} = (L_1\mathcal{F})^-$ and $\mathcal{F}_{11}^\perp = (L_1\mathcal{F})^+$, then according Lemma 2 we have

$$\mathcal{F}_{11} = \{F \in \mathcal{F} \mid F = F_3(F)\}, \qquad \mathcal{F}_{11}^\perp = \{F \in \mathcal{F} \mid \omega(F) = 0\}.$$

Thereby we obtain immediately

PROPOSITION 3. *The decomposition* $\mathcal{F} = \mathcal{F}_{11} \oplus \mathcal{F}_{11}^\perp$ *is orthogonal and invariant with respect to the action of* $O(n) \times O(n) \times I$. *The component of an arbitrary element* F *in* $\mathcal{F}_{11}$ (*respectively in* $\mathcal{F}_{11}^\perp$) *is* $F_3(F)$ (*respectively* $F - F_3(F)$).

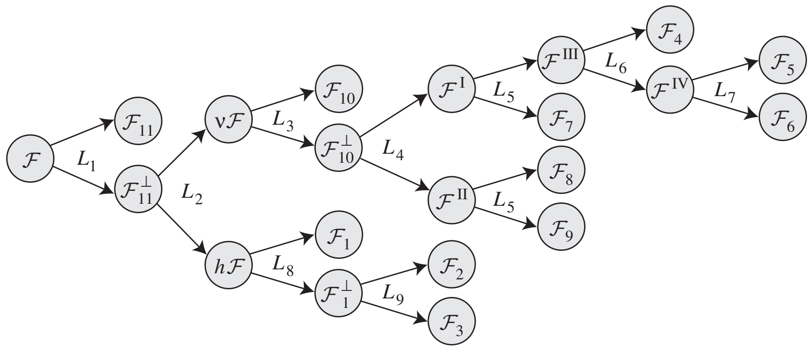

Figure 1

Using the same method we continue to decompose $\mathcal{F}$ orthogonally and invariantly with respect to the action of $O(n) \times O(n) \times I$. Thereby we obtain the complete decomposition into irreducible components by the scheme of Figure 1.

The rest of operators L_j $(j = 2, 3, \ldots, 9)$ we define by:

$$\begin{aligned} L_2(F) &= F - 2\{F_1(F) + F_2(F)\}, & L_6(F) &= F - 2F_7(F), \\ L_3(F) &= F_2(F) - F_1(F), & L_7(F) &= F - 2F_8(F), \\ L_4(F) &= -F_5(F), & L_8(F) &= F - 2F_9(F), \\ L_5(F) &= -F_4(F), & L_9(F) &= F - 2F_{10}(F). \end{aligned}$$

The operators L_j $(j = 2, 3, \ldots, 9)$ are involutive isometries on the corresponding subspaces and also commute with the action of $O(n) \times O(n) \times I$. According the applying method the corresponding eigenspaces of the eigenvalues $+1$ and -1 are orthogonal and mutually complementary subspaces of the reducible subspace of $\mathcal{F}$.

Since the endomorphism ϕ induces an almost product structure on the orthogonal complement $\{\xi\}^{\perp}$ of the subspace spanned by ξ and the restriction of g on $\{\xi\}^{\perp}$ is a Riemannian metric compatible with the almost product structure, then the decomposition of $h\mathcal{F}$ coincides with the known decomposition of the space of covariant derivatives of the traceless almost product structure [3].

Taking into account the above reasons we obtain

THEOREM 4. *The decomposition* $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_{11}$ *is orthogonal and invariant with respect to the action of* $O(n) \times O(n) \times I$. *The characterization conditions of the factors* $\mathcal{F}_i$ $(i = 1, 2, \ldots, 11)$ *for arbitrary vectors* x, y, z *in* V *are:*

$$
\begin{aligned}
\mathcal{F}_1: \quad & F(x,y,z) = \tfrac{1}{2n}\{g(\phi x,\phi y)\theta(\phi^2 z) + g(\phi x,\phi z)\theta(\phi^2 y) \\
& \qquad\qquad -g(x,\phi y)\theta(\phi z) - g(x,\phi z)\theta(\phi y)\}; \\
\mathcal{F}_2: \quad & F(x,y,\phi z) + F(y,z,\phi x) + F(z,x,\phi y) = 0, \\
& F(\xi,y,z) = F(x,y,\xi) = 0, \quad \theta = 0; \\
\mathcal{F}_3: \quad & F(x,y,z) + F(y,z,x) + F(z,x,y) = 0, \quad F(\xi,y,z) = F(x,y,\xi) = 0; \\
\mathcal{F}_4: \quad & F(x,y,z) = \tfrac{\theta(\xi)}{2n}\{g(\phi x,\phi y)\eta(z) + g(\phi x,\phi z)\eta(y)\}; \\
\mathcal{F}_5: \quad & F(x,y,z) = \tfrac{\theta^*(\xi)}{2n}\{g(x,\phi y)\eta(z) + g(x,\phi z)\eta(y)\}; \\
\mathcal{F}_6: \quad & F(x,y,z) = \eta(y)F(z,\xi,x) + \eta(z)F(y,\xi,x) - 2\eta(y)\eta(z)F(\xi,\xi,x) \\
& \qquad = \eta(y)F(\phi x,\xi,\phi z) + \eta(z)F(\phi x,\xi,\phi y), \quad \theta(\xi) = \theta^*(\xi) = 0; \\
\mathcal{F}_7: \quad & F(x,y,z) = -\eta(y)F(z,\xi,x) - \eta(z)F(y,\xi,x) + 2\eta(y)\eta(z)F(\xi,\xi,x) \\
& \qquad = \eta(y)F(\phi x,\xi,\phi z) + \eta(z)F(\phi x,\xi,\phi y); \\
\mathcal{F}_8: \quad & F(x,y,z) = \eta(y)F(z,\xi,x) + \eta(z)F(y,\xi,x) - 2\eta(y)\eta(z)F(\xi,\xi,x) \\
& \qquad = -\eta(y)F(\phi x,\xi,\phi z) - \eta(z)F(\phi x,\xi,\phi y); \\
\mathcal{F}_9: \quad & F(x,y,z) = -\eta(y)F(z,\xi,x) - \eta(z)F(y,\xi,x) + 2\eta(y)\eta(z)F(\xi,\xi,x) \\
& \qquad = -\eta(y)F(\phi x,\xi,\phi z) - \eta(z)F(\phi x,\xi,\phi y); \\
\mathcal{F}_{10}: \quad & F(x,y,z) = \eta(x)F(\xi,y,z), \quad F(x,y,\xi) = 0; \\
\mathcal{F}_{11}: \quad & F(x,y,z) = \eta(x)\{\eta(y)\omega(z) + \eta(z)\omega(y)\}.
\end{aligned}
$$

The components $p_i(F)$ of an arbitrary element F of $\mathcal{F}$ in $\mathcal{F}_i$ $(i = 1,2,\dots,11)$ are

$$
\begin{aligned}
& p_1(F) = F_9(F), \qquad p_2(F) = F_{10}(F), \\
& p_3(F) = hF - F_9(F) - F_{10}(F), \quad p_4(F) = F_7(F), \quad p_5(F) = F_8(F), \\
& p_6(F) = \tfrac{1}{4}\{F_2(F) - F_3(F) + F_4(F) + F_5(F) + F_6(F)\} - F_7(F) - F_8(F), \\
& p_7(F) = \tfrac{1}{4}\{F_2(F) - F_3(F) - F_4(F) + F_5(F) - F_6(F)\}, \\
& p_8(F) = \tfrac{1}{4}\{F_2(F) - F_3(F) + F_4(F) - F_5(F) - F_6(F)\}, \\
& p_9(F) = \tfrac{1}{4}\{F_2(F) - F_3(F) - F_4(F) - F_5(F) + F_6(F)\}, \\
& p_{10}(F) = F_1(F) - F_3(F), \qquad p_{11}(F) = F_3(F).
\end{aligned}
$$

4. Basic classes of almost paracontact Riemannian manifolds of type (*n, n*)

Let (M,ϕ,ξ,η,g) be almost paracontact Riemannian manifolds of type (n,n). The tangent space T_pM at an arbitrary point p in M is the vector space V equipped with an almost paracontact Riemannian structure of type (n,n). Then the corresponding vector space $\mathcal{F}$ considered in the previous section has eleven orthogonal and invariant subspaces $\mathcal{F}_i$. In such a way the conditions for F at every point $p \in M$ give rise to the corresponding class of manifolds under consideration. Namely, an almost paracontact Riemannian manifold of type (n,n) is said to be in the class $\mathcal{F}_i$ if the tensor F belongs to the subspace $\mathcal{F}_i$ $(i = 1,2,\dots,11)$ over T_pM at every p in M. Thus the conditions define the eleven basic classes of almost paracontact Riemannian manifolds of type (n,n). Of course, the number of all classes of manifold under conversation is 2^{11} and their defining conditions are easily obtainable by the basic ones. The special class $\mathcal{F}_0$ of almost paracontact Riemannian manifolds of type (n, n) is defined by the condition $F = 0$. This class belongs to each of the defined classes. It is an analogue to the class of cosymplectic almost contact metric manifolds [4] and to the respective class $\mathcal{F}_0$ in the classification of almost contact manifolds with B-metric [5].

The defined class $\mathcal{F}_4$ contains paracontact Riemannian manifolds of type (n, n) [1] and in particular para-Sasakian manifolds of type (n, n) [7].

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