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ON ALMOST PSEUDO-CONFORMALLY SYMMETRIC RICCI-RECURRENT MANIFOLDS WITH APPLICATIONS TO RELATIVITY

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Cordially dedicated to Professor W. Roter

Abstract. The object of the present paper is to study almost pseudo-conformally symmetric Ricci-recurrent manifolds. The existence of almost pseudo-conformally symmetric Ricci-recurrent manifolds has been proved by an explicit example. Some geometric properties have been studied. Among others we prove that in such a manifold the vector field ϱ corresponding to the 1-form of recurrence is irrotational and the integral curves of the vector field ϱ are geodesic. We also study some global properties of such a manifold. Finally, we study almost pseudo-conformally symmetric Ricci-recurrent spacetime. We obtain the Segre' characteristic of such a spacetime.

Keywords: pseudo-conformally symmetric manifold, almost pseudo-conformally symmetric manifold, Ricci-recurrent manifold, Einstein field equations, Segre' characteristic

MSC 2010: 53C15, 53C25, 53B20, 53B30, 53B15

1. Introduction

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric spaces was initiated in the late twenties by E. Cartan [4] who, in particular, obtained a classification of those spaces.

Let (M, g), $n = \dim M$ be a Riemannian manifold and let ∇ be the Levi-Civita connection of (M, g). A Riemannian manifold is called locally symmetric [4] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [18]. The class of Riemannian symmetric manifolds is a very natural generalization of the class of manifolds of constant curvature. The same can be extended to the class of semi-Riemannian manifolds, where g is of arbitrary signature.

During the last five decades the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as semi-symmetric manifolds by Szabo [29], Kowalski [2], conformally symmetric manifolds by Chaki and Gupta [6], recurrent manifolds introduced by Walker [31], conformally recurrent manifolds by Adati and Miyazawa [1], pseudo-symmetric manifolds introduced by Chaki [5], weakly symmetric manifolds by Tamássy and Binh [30] etc.

In 1967, Sen and Chaki [27] studied certain curvature restrictions on a certain kind of a conformally flat space of class one and obtained the following expression of the covariant derivative of the curvature tensor:

$$R^h_{ijk,l} = 2\lambda_l R^h_{ijk} + \lambda_i R^h_{ljk} + \lambda_j R^h_{ilk} + \lambda_k R^h_{ijl} + \lambda^h R_{lijk}, \tag{1.1}$$

where R^h_{ijk} are the components of the curvature tensor R, $R_{lijk} = g_{hl} R^h_{ijk}$, λ_i is a non-zero covariant vector.

Later in 1987, Chaki [5] called a manifold whose curvature tensor satisfies (1.1) a pseudo-symmetric manifold. In index-free notation this can be stated as follows:

A non-flat Riemannian or a semi-Riemannian manifold (M, g), $n \geqslant 2$ is said to be a pseudo-symmetric manifold [5] if its curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W = 2A(X)R(Y,Z)W + A(Y)R(X,Z)W \\ + A(Z)R(Y,X)W + A(W)R(Y,Z)X + g(R(Y,Z)W,X)P,\end{aligned} \tag{1.2}$$

where A is a non-zero differential 1-form, P is a vector field defined by

$$g(X,P) = A(X) \quad \text{for all vector fields } X \tag{1.3}$$

and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-form A is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of E. Cartan. An n-dimensional pseudo-symmetric manifold is denoted by $(PS)_n$.

It is to be noted that the notion of a pseudo-symmetric manifold studied in particular by Deszcz [16] is different from that of Chaki [5].

To generalize the notion of a pseudo-symmetric manifold, De and Gazi [8] introduced the notion of an almost pseudo-symmetric manifold which is defined as follows:

A non-flat Riemannian manifold (M, g) $(n \geqslant 2)$ is said to be almost pseudo-symmetric if the curvature tensor R satisfies the condition

$$(1.4)\quad (\nabla_X R)(Y,Z)W = [A(X)+B(X)]R(Y,Z)W + A(Y)R(X,Z)W + A(Z)R(Y,X)W + A(W)R(Y,Z)X + g(R(Y,Z)W,X)P,$$

where A and B are two non-zero 1-forms, defined by

$$(1.5)\qquad g(X,P) = A(X) \quad \text{and} \quad g(X,Q) = B(X) \quad \text{for all } X.$$

Here the vector fields P and Q will be called the basic vector fields of the manifold corresponding to the 1-form A and B respectively. Such a manifold is denoted by $(APS)_n$. If $B = A$, then $(APS)_n$ reduces to $(PS)_n$.

If we replace R by C in (1.2) where C is the conformal curvature tensor defined by

$$(1.6)\qquad C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[g(Y,Z)LX - g(X,Z)LY + S(Y,Z)X - S(X,Z)Y] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],$$

where R is the curvature tensor of type $(1,3)$, S is the Ricci tensor, r is the scalar curvature and L is the symmetric endomorphism corresponding to the Ricci tensor S, that is,

$$(1.7)\qquad S(X,Y) = g(LX,Y),$$

then the manifold is called a pseudo-conformally symmetric manifold introduced by De and Biswas [7]. Such an n-dimensional manifold was denoted by $(\text{PCS})_n$. If $A = 0$ on M, then the $(\text{PCS})_n$ manifold reduces to a conformally symmetric manifold [6]. For recent results on conformally symmetric manifolds we refer to [11], [12], [13], [14], [15], [28].

Conformally recurrent manifolds were introduced by Adati and Miyazawa [1] in 1967. A Riemannian manifold (M, g), $n > 3$ is called conformally recurrent if the conformal curvature tensor, defined by (1.6), satisfies the condition

$$(1.8)\qquad (\nabla_X C)(Y,Z)W = E(X)C(Y,Z)W,$$

where E is a non-zero 1-form. If in particular $E = 0$, then the manifold reduces to a conformally symmetric manifold [6].

Prvanović called a non-flat Riemannian manifold (M, g), $n > 3$, a conformally quasi-recurrent manifold [22] if its conformal curvature tensor satisfies the same condition as pseudo-conformally symmetric manifolds. Conformally quasi-recurrent manifolds have also been studied by Buchner and Roter [3].

In a recent paper [9], De and Gazi introduced the notion of almost pseudo-conformally symmetric manifolds. If we replace R by C in (1.4), then the manifold is called almost pseudo-conformally symmetric manifold which is defined by the condition

$$\begin{aligned}(\nabla_X C)(Y,Z)W &= [A(X)+B(X)]C(Y,Z)W + A(Y)C(X,Z)W + A(Z)C(Y,X)W \\ &\quad + A(W)C(Y,Z)X + g(C(Y,Z)W,X)P, \end{aligned} \tag{1.9}$$

where A and B are two non-zero 1-forms, called the associated 1-forms, defined by

$$g(X,P) = A(X), \qquad g(X,Q) = B(X) \tag{1.10}$$

for all vector fields X, and ∇ has the meaning already mentioned. Here the vector fields P and Q will be called the basic vector fields of the manifold corresponding to the associated 1-forms A and B respectively. Such an n-dimensional manifold will be denoted by $(\mathrm{APCS})_n$. Clearly, every conformally recurrent manifold is a $(\mathrm{APCS})_n$.

If in (1.9) $A = B$, then the manifold reduces to a pseudo-conformally symmetric manifold. This justifies the name "almost pseudo-conformally symmetric manifold" and the use of the symbol $(\mathrm{APCS})_n$. In this connection it may be mentioned that in 1989 Tamássy and Binh [30] introduced weakly symmetric and weakly projectively symmetric Riemannian manifolds. A Riemannian manifold (M, g) is called weakly symmetric and denoted by $(WS)_n$ if there exist 1-forms A, B, D, E and a vector field P such that

$$\begin{aligned}(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W &+ B(Y)R(X,Z)W + D(Z)R(Y,X)W \\ &+ E(W)R(Y,Z)X + g(R(Y,Z)W,X)P,\end{aligned}$$

where R is the curvature tensor of (M, g).

In a subsequent paper [10] the first author and Bandyopadhyay introduced the weakly conformally symmetric manifold. It is to be noted that $(\mathrm{APCS})_n$ is not a particular case of $(\mathrm{WCS})_n$.

In the paper [9] De and Gazi proved the existence of an $(\mathrm{APCS})_n$ manifold by the following theorem.

Theorem A. *A conformal deformation of every conformally recurrent metric is an* $(\mathrm{APCS})_n$ *metric, provided that* $n > 3$.

Also the authors [9] cited an example of an $(\mathrm{APCS})_n$ with a metric.

In 1952, Patterson [19] introduced the notion of Ricci-recurrent manifolds.

A non-flat Riemannian manifold (M, g), $n > 2$ is called a Ricci-recurrent manifold [19] if its Ricci tensor S of type $(0, 2)$, is non-zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = T(X)S(Y, Z) \tag{1.11}$$

where T is a non-zero 1-form, called the 1-form of recurrence and defined by

$$g(X, \mu) = T(X). \tag{1.12}$$

If $T = 0$, then the manifold reduces to a Ricci symmetric manifold.

Conformally symmetric Ricci-recurrent manifolds have been studied by Roter [23] and many others. Also conformally recurrent Ricci-recurrent spaces have been studied by Roter [24].

Motivated by these works, in the present paper we have studied $(\mathrm{APCS})_n$ Ricci-recurrent manifolds, $n > 3$. Here we prove that in an $(\mathrm{APCS})_n$ Ricci-recurrent manifold, if the scalar curvature $r \neq 0$, then ϱ is an eigenvector corresponding to the eigenvalue $\frac{1}{2}r$, where $g(X, \varrho) = E(X) = 2A(X) + B(X)$, A, B are defined by (1.10). Next we prove that in an $(\mathrm{APCS})_n$ Ricci-recurrent manifold, $n > 3$, either the scalar curvature $r = 0$ or the 1-form of recurrence T satisfies $T(X) = 2A(X) + B(X)$ for all vector fields X. Also we verify the result by an example. In this section we also prove that the vector field ϱ is irrotational and the integral curves of the vector field ϱ are geodesic. Next, we study some global properties of a $(\mathrm{APCS})_n$ Ricci-recurrent manifold, $n > 3$. Finally, we study $(\mathrm{APCS})_n$ Ricci-recurrent spacetime, $n > 3$. Among others we obtain Segre' characteristic of such a spacetime.

2. $(\mathrm{APCS})_n$ Ricci-recurrent manifolds, $n > 3$

In this section we first prove the following lemma:

Lemma 2.1. *In an* $(\mathrm{APCS})_n$ *Ricci-recurrent manifold if* $r \neq 0$, *then* $\frac{1}{2}r$ *is an eigenvalue corresponding to the eigenvector* ϱ.

P r o o f. Contracting (1.9) with respect to X we get

$$(\operatorname{div} C)(Y, Z)W = 2A(C(Y, Z)W) + B(C(Y, Z)W) = E(C(Y, Z)W), \tag{2.1}$$

where E is a non-zero 1-form and ϱ is its corresponding vector field such that

$$E(X) = 2A(X) + B(X) \quad \text{and} \quad g(X, \varrho) = E(X) \quad \text{for all vector fields } X. \tag{2.2}$$

Now contracting (1.11) with respect to X and Z we have

$$\mathrm{d}r(Y) = T(Y)r \tag{2.3}$$

and contraction of (1.11) with respect to Z gives

$$(\nabla_Y L)(X) = T(Y)LX, \tag{2.4}$$

where L is defined by (1.7). Again contracting Y in (2.4) we get

$$(\operatorname{div} L)X = T(LX).$$

However, it is known [20] that $(\operatorname{div} L)X = \frac{1}{2}\,\mathrm{d}r(X)$. So with help of (2.3) we have

$$T(LX) = \frac{1}{2}\,\mathrm{d}r(X) = \frac{r}{2}T(X). \tag{2.5}$$

Also it is known [25] that in a Ricci-recurrent manifold the following relations hold:

$$S(R(X,Y)Z,W) + S(R(X,Y)W,Z) = 0, \tag{2.6}$$

$$S(X, LY) = \frac{r}{2}S(X,Y). \tag{2.7}$$

From the definition of the Weyl conformal curvature tensor we see that [17]

$$\begin{aligned}(\operatorname{div} C)(Y,Z)W = \frac{n-3}{n-2}\Big[&\Big\{(\nabla_Y S)(Z,W) - \frac{1}{2(n-1)}\,\mathrm{d}r(Y)g(Z,W)\Big\}\\ &- \Big\{(\nabla_Z S)(Y,W) - \frac{1}{2(n-1)}\,\mathrm{d}r(Z)g(Y,W)\Big\}\Big].\end{aligned}$$

Now using (1.11), (2.1) and (2.3) we get from the above

$$\begin{aligned}E(C(Y,Z)W) = \frac{n-3}{n-2}\Big[&\Big\{T(Y)S(Z,W) - \frac{r}{2(n-1)}T(Y)g(Z,W)\Big\}\\ &- \Big\{T(Z)S(Y,W) - \frac{r}{2(n-1)}T(Z)g(Y,W)\Big\}\Big].\end{aligned} \tag{2.8}$$

$W=\varrho$ in (2.8) implies

$$T(Y)\Big[S(Z,\varrho)-\frac{r}{2(n-1)}E(Z)\Big]=T(Z)\Big[S(Y,\varrho)-\frac{r}{2(n-1)}E(Y)\Big], \tag{2.9}$$

since

$$E(C(Y,Z)\varrho)=g(C(Y,Z)\varrho,\varrho)=0.$$

Putting $Y=LY$ in (2.9) we get

$$T(LY)\Big[S(Z,\varrho)-\frac{r}{2(n-1)}E(Z)\Big]=T(Z)\Big[S(LY,\varrho)-\frac{r}{2(n-1)}E(LY)\Big]. \tag{2.10}$$

Now using (2.5) and (2.7) we get from (2.10)

$$rT(Y)\Big[S(Z,\varrho)-\frac{r}{2(n-1)}E(Z)\Big]=rT(Z)\Big[S(Y,\varrho)-\frac{1}{(n-1)}E(LY)\Big],$$

which implies either $r=0$, or

$$T(Y)\Big[S(Z,\varrho)-\frac{r}{2(n-1)}E(Z)\Big]=T(Z)\Big[S(Y,\varrho)-\frac{1}{(n-1)}E(LY)\Big]. \tag{2.11}$$

Since $T\neq 0$, (2.9) and (2.11) together with the condition $r\neq 0$ give $E(LY)=\frac{1}{2}rE(Y)$, that is,

$$S(Y,\varrho)=\frac{r}{2}g(Y,\varrho). \tag{2.12}$$

This completes the proof. □

Now we prove the following propositions:

Proposition 2.1. *At every point p of an* $(\mathrm{APCS})_n$ *Ricci-recurrent manifold, $n>3$, either the scalar curvature at that point is zero or $T=2A+B$.*

Proof. Let $\tilde{C}$ be the conformal curvature tensor of type $(0,4)$ such that $\tilde{C}(X,Y,Z,W)=g(C(X,Y)Z,W)$, where C is the Weyl conformal curvature tensor of type $(1,3)$ defined by (1.6). Then

$$\begin{aligned}\tilde{C}&(X,Y,Z,W)\\ &=\tilde{R}(X,Y,Z,W)-\frac{1}{n-2}[g(Y,Z)S(X,W)\\ &\quad-g(X,Z)S(Y,W)+S(Y,Z)g(X,W)-S(X,Z)g(Y,W)]\\ &\quad+\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)],\end{aligned} \tag{2.13}$$

where $\tilde{R}$ is the curvature tensor of type $(0,4)$ defined by

$$\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$$

for the curvature tensor R of type $(1,3)$.

Putting $W = \varrho$ in (2.13) and using (2.12) we get

$$\begin{aligned} E(C(X,Y)Z) = E(R(X,Y)Z) - \frac{1}{n-2}[E(X)S(Y,Z) - E(Y)S(X,Z)] \\ + \frac{(n-3)r}{2(n-1)(n-2)}[E(X)g(Y,Z) - E(Y)g(X,Z)]. \end{aligned}$$

With help of (2.8) this equation gives

$$\begin{aligned} E(R(X,Y)Z) = S(Y,Z)\Big[\frac{1}{n-2}E(X) + \frac{n-3}{n-2}T(X)\Big] \\ - S(X,Z)\Big[\frac{1}{n-2}E(Y) + \frac{n-3}{n-2}T(Y)\Big] \\ + \frac{(n-3)r}{2(n-1)(n-2)}[\{E(X) - T(X)\}g(Y,Z) \\ - \{E(Y) - T(Y)\}g(X,Z)]. \end{aligned} \tag{2.14}$$

Now with help of the symmetric and skew-symmetric properties of the curvature tensor, from (2.6) and (2.12) we can prove that

$$E(R(X,Y)LZ) = \frac{1}{2}rE(R(X,Y)Z). \tag{2.15}$$

Putting $Z = LZ$ in (2.14) and using (2.7), (2.15) we see that

$$\begin{aligned} rE(R(X,Y)Z) = rS(Y,Z)\Big[\frac{2}{n-1}E(X) + \frac{n-3}{n-1}T(X)\Big] \\ - rS(X,Z)\Big[\frac{2}{n-1}E(Y) + \frac{n-3}{n-1}T(Y)\Big]. \end{aligned} \tag{2.16}$$

Contracting Y and Z in (2.16) and using (2.5), (2.12) we get after simple calculation

$$r^2[E(X) - T(X)] = 0,$$

which implies that either $r = 0$ or $E(X) = T(X)$, that is, $T(X) = 2A(X) + B(X)$. This completes the proof. □

Proposition 2.2. *In an* $(\mathrm{APCS})_n$ *Ricci-recurrent manifold,* $n > 3$, *with non-zero scalar curvature the vector field* ϱ *defined by* $g(X,\varrho) = E(X)$ *is irrotational and the integral curves of the vector field* ϱ *are geodesics.*

P r o o f. From Proposition 2.1, we get in an $(\mathrm{APCS})_n$ Ricci-recurrent manifold, $n > 3$ with non-zero scalar curvature that $E = T$ where T is the 1-form of recurrence of the Ricci-Recurrent manifold. Since in a Ricci-recurrent manifold with non-zero scalar curvature the 1-form of recurrence T is closed, hence the 1-form E is also closed, that is, $\mathrm{d}E(X,Y) = 0$, which implies

$$(\nabla_X E)Y = (\nabla_Y E)X. \tag{2.17}$$

From (2.17) it follows that

$$g(\nabla_X \varrho, Y) = g(\nabla_Y \varrho, X) \quad \text{for all vector fields } X, Y, \tag{2.18}$$

which means that the vector field ϱ is irrotational.

Now putting $Y = \varrho$ in (2.18) we get

$$g(\nabla_X \varrho, \varrho) = g(\nabla_\varrho \varrho, X). \tag{2.19}$$

Suppose ϱ is a unit vector field. Then $g(\nabla_X \varrho, \varrho) = 0$. Hence from (2.19) it follows that $g(\nabla_\varrho \varrho, X) = 0$ for all vector fields X, which implies $\nabla_\varrho \varrho = 0$. This means that the integral curves of the vector field ϱ are geodesics. □

Now we are in a position to establish by an example that $r \neq 0$ is necessary condition for the conclusion of Proposition 2.1.

On the coordinate space $\mathbb{R}^n$ (with coordinates $x^1, x^2, \ldots, x^n$) we define a Riemannian space V_n. We calculate the components of the curvature tensor, the Ricci tensor, the covariant derivatives of the Ricci tensor, the conformal curvature tensor and its covariant derivatives, and then we verify the relations (1.9) and (1.11).

Let each Latin index runs over $1, 2, \ldots, n$ and each Greek index over $2, 3, \ldots, (n-1)$. We define a Riemannian metric on $\mathbb{R}^n$ $(n \geqslant 4)$ by the formula

$$\mathrm{d}s^2 = \varphi(\mathrm{d}x^1)^2 + K_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta + 2\mathrm{d}x^1\mathrm{d}x^n, \tag{2.20}$$

where $[K_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constants and φ is a function of $x^1, x^2, \ldots, x^{n-1}$ and independent of x^n. In the metric considered, the only non-vanishing components of Christoffel symbols, the curvature tensor and the Ricci tensor are, according to [23],

$$\begin{gathered} \Gamma^\beta_{11} = -\frac{1}{2}K^{\alpha\beta}\varphi_{.\alpha}, \quad \Gamma^n_{11} = \frac{1}{2}\varphi_{.1}, \quad \Gamma^n_{1\alpha} = \frac{1}{2}\varphi_{.\alpha}, \\ R_{1\alpha\beta 1} = \frac{1}{2}\varphi_{.\alpha\beta}, \quad R_{11} = \frac{1}{2}K^{\alpha\beta}\varphi_{.\alpha\beta}, \end{gathered} \tag{2.21}$$

where '.' denotes the partial differentiation with respect to the coordinates and $K^{\alpha\beta}$ are the elements of the matrix inverse to $[K_{\alpha\beta}]$.

Here we consider $K_{\alpha\beta}$ as the Kronecker symbol $\delta_{\alpha\beta}$ and

$$\varphi = (M_{\alpha\beta} + \delta_{\alpha\beta})x^\alpha x^\beta \mathrm{e}^{(x^1)^2},$$

where $M_{\alpha\beta}$ are constant and satisfy the relations

$$\begin{aligned} M_{\alpha\beta} &= 0 \quad \text{for } \alpha \neq \beta, \\ &\neq 0 \quad \text{for } \alpha = \beta, \\ \sum_{\alpha=2}^{n-1} M_{\alpha\alpha} &= 0. \end{aligned} \tag{2.22}$$

Now according to our consideration we have the following relations:

$$\varphi_{.\alpha\beta} = 2(M_{\alpha\beta} + \delta_{\alpha\beta}\mathrm{e}^{(x^1)^2},$$

$$\delta_{\alpha\beta}\delta^{\alpha\beta} = n - 2 \quad \text{and} \quad \delta^{\alpha\beta}M_{\alpha\beta} = \sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0.$$

Therefore,

$$\delta^{\alpha\beta}\varphi_{.\alpha\beta} = 2(\delta^{\alpha\beta}M_{\alpha\beta} + \delta^{\alpha\beta}\delta_{\alpha\beta})\mathrm{e}^{(x^1)^2} = 2(n-2)\mathrm{e}^{(x^1)^2}.$$

Since $\varphi_{.\alpha\beta}$ vanishes for $\alpha \neq \beta$, the only non-zero components for R_{hijk} and R_{ij} in virtue of (2.21) are

$$R_{1\alpha\alpha 1} = \frac{1}{2}\varphi_{.\alpha\alpha} = (1 + M_{\alpha\alpha})\mathrm{e}^{(x^1)^2}$$

and

$$R_{11} = \frac{1}{2}\varphi_{.\alpha\beta}\delta^{\alpha\beta} = (n-2)\mathrm{e}^{(x^1)^2}.$$

Again from (2.20) we obtain $g_{ni} = g_{in} = 0$ for $i \neq 1$, which implies $g^{11} = 0$. Hence $r = g^{ij}R_{ij} = g^{11}R_{11} = 0$. Therefore, V_n will be a space whose scalar curvature is zero. Hence the only non-zero components of the conformal curvature tensor C_{hijk} are

$$\begin{aligned} C_{1\alpha\alpha 1} &= R_{1\alpha\alpha 1} - \frac{1}{n-2}(g_{\alpha\alpha}R_{11}) \\ &= (1 + M_{\alpha\alpha})\mathrm{e}^{(x^1)^2} - \frac{1}{n-2}(n-2)\mathrm{e}^{(x^1)^2} \\ &= M_{\alpha\alpha}\mathrm{e}^{(x^1)^2} \end{aligned} \tag{2.23}$$

which never vanish. Now the only non-zero components of $C_{hijk,m}$ are

$$C_{1\alpha\alpha 1,1} = 2x^1 M_{\alpha\alpha} \mathrm{e}^{(x^1)^2} = 2x^1 C_{1\alpha\alpha 1} \neq 0. \tag{2.24}$$

Hence V_n is neither conformally flat nor conformally symmetric [6]. We shall now show that V_n is an $(\mathrm{APCS})_n$. Let us consider the associated 1-form as follows:

$$A_i(x) = \begin{cases} x^1 & \text{for } i = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.25}$$

$$B_i(x) = \begin{cases} -x^1 & \text{for } i = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.26}$$

at any point $x \in V_n$.

To verify the relation (1.9) it is sufficient to prove that

$$C_{1\alpha\alpha 1,1} = (3A_1 + B_1) C_{1\alpha\alpha 1}, \tag{2.27}$$

$$C_{11\alpha 1,\alpha} = A_1 C_{\alpha 1\alpha 1} + A_1 C_{1\alpha\alpha 1}, \tag{2.28}$$

$$C_{1\alpha 11,\alpha} = A_1 C_{1\alpha\alpha 1} + A_1 C_{1\alpha 1\alpha}, \tag{2.29}$$

as for the case other than (2.27), (2.28) and (2.29) the components of each term of (1.9) vanish identically and the relation (1.9) holds trivially. Now from (2.23), (2.24), (2.25) and (2.26) we get the following relation for the right-hand side (r.h.s.) and the left-hand side (l.h.s.) of (2.27):

$$\begin{aligned} \text{r.h.s. of (2.27)} &= (3A_1 + B_1) C_{1\alpha\alpha 1} = (3x^1 - x^1) C_{1\alpha\alpha 1} \\ &= 2x^1 C_{1\alpha\alpha 1} = C_{1\alpha\alpha 1,1} = \text{l.h.s. of (2.27)}. \end{aligned}$$

Now

$$\begin{aligned} \text{r.h.s. of (2.28)} &= x^1 (C_{\alpha 1\alpha 1} + C_{1\alpha\alpha 1}) \\ &= 0 \text{ (by skew-symmetric properties of } C_{hijk}) \\ &= \text{l.h.s. of (2.28)}. \end{aligned}$$

By an argument similar to (2.28) it can be shown that the relation (2.29) is also true.

It is to be noted that (1.9) can be satisfied by a number of 1-forms A, B, namely, by those which fulfil (2.27), (2.28), (2.29). Thus the manifold under consideration is an $(\mathrm{APCS})_n$ manifold. Now we show that this manifold is also a Ricci-recurrent manifold. For this let us consider the 1-form of recurrence

$$T_i(x) = \begin{cases} 2x^1 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.30}$$

Then we have

$$R_{11,1} = 2(n-2)x^1 \mathrm{e}^{(x^1)^2} = T_1 R_{11}.$$

Hence the manifold is a Ricci-recurrent manifold.

From this example we see that the scalar curvature is $r = 0$ and $2A_1 + B_1 = x^1 \neq T_1$. Thus we obtain an $(\mathrm{APCS})_n$ Ricci-recurrent manifold having $r = 0$ and $T_k \neq 2A_k + B_k$.

3. Some global properties of an $(\mathrm{APCS})_n$ Ricci-recurrent manifold $n > 3$

(a) Sufficient condition for a compact, orientable $(\mathrm{APCS})_n$ Ricci-recurrent manifold, $n > 3$ to be conformal to a sphere in E_{n+1}.

We begin with the definition of conformality of one Riemannian manifold to another.

Let (M, g) and $(\tilde{M}, \tilde{g})$ be two n-dimensional Riemannian manifolds. If there exists a one-one differentiable mapping $(M, g) \to (\tilde{M}, \tilde{g})$ such that the angle between any two vectors at a point p of M is always equal to that of the corresponding two vectors at the corresponding point $\tilde{p}$ of $\tilde{M}$, then (M, g) is said to be conformal to $(\tilde{M}, \tilde{g})$. Y. Watanabe [32] has given a sufficient condition of conformality of an n-dimensional Riemannian manifold to an n-dimensional sphere immersed in E_{n+1}. Its statement is as follows:

If in an n-dimensional Riemannian manifold M there exists a non parallel vector field X such that the condition

$$\int_M S(X, X)\,\mathrm{d}v = \frac{1}{2}\int_M |\mathrm{d}X|^2\,\mathrm{d}v + \frac{n-1}{n}\int_M (\partial X)^2\,\mathrm{d}v \tag{3.1}$$

holds, then M is conformal to a sphere in E_{n+1}, where $\mathrm{d}v$ is the volume element of M and $\mathrm{d}X$ and ∂X are the curl and divergence of X respectively.

Here we consider a compact and orientable $(\mathrm{APCS})_n$ Ricci-recurrent manifold, $n > 3$, without boundary and having the generator ϱ defined by $g(X, \varrho) = E(X)$, ϱ being a unit vector field. From (2.12) we see that if $r \neq 0$, then ϱ satisfies

$$S(X, \varrho) = \frac{1}{2} r E(X). \tag{3.2}$$

Hence

$$S(\varrho, \varrho) = \frac{1}{2} r, \tag{3.3}$$

since $E(\varrho) = 1$. In virtue of this and by taking ϱ for X, the condition (3.1) takes on the form

$$\frac{1}{2}\int_M r\,\mathrm{d}v = \frac{1}{2}\int_M |\mathrm{d}\varrho|^2\,\mathrm{d}v + \frac{n-1}{n}\int_M (\partial\varrho)^2\,\mathrm{d}v. \tag{3.4}$$

Suppose ϱ is a parallel vector field [26]. Then

$$\nabla_X \varrho = 0.$$

Hence by the Ricci identity we obtain

$$R(X, Y)\varrho = 0. \tag{3.5}$$

Contracting X we get from (3.5)

$$S(Y, \varrho) = 0. \tag{3.6}$$

Since $r \neq 0$ by assumption and Proposition 2.1 implies $E(X) = T(X)$, T is the 1-form recurrence of a Ricci-recurrent manifold, therefore $E \neq 0$. Therefore in an $(\mathrm{APCS})_n$ Ricci-recurrent manifold, $n > 3$, from (3.2) we get

$$S(Y, \varrho) \neq 0.$$

Hence ϱ cannot be a parallel vector field. Thus in an n-dimensional compact, orientable $(\mathrm{APCS})_n$ Ricci-recurrent manifold, $n > 3$ without boundary the vector field ϱ is a non-parallel vector field. If in such a case the condition (3.4) is satisfied, then by Watanabe's condition (3.1) M is conformal to a sphere in E_{n+1}. We can therefore state the following result.

Theorem 3.1. *If a compact, orientable* $(\mathrm{APCS})_n$ *Ricci-recurrent manifold,* $n > 3$ *without boundary, satisfies the condition* (3.4), *then the* $(\mathrm{APCS})_n$ *Ricci-recurrent manifold,* $n > 3$ *is conformal to a sphere immersed in* E_{n+1}, *provided* $r \neq 0$.

(b) Killing vector field in a compact, orientable $(\mathrm{APCS})_n$ Ricci-recurrent manifold, $n > 3$ without boundary.

Here we consider a compact, orientable $(\mathrm{APCS})_n$ Ricci-recurrent manifold, $n > 3$ without boundary. It is known ([32] or [34, p. 43]) that in such a manifold M the following relation holds:

$$(3.7)\qquad \int_M [S(X,X) - |\nabla X|^2 - (\operatorname{div} X)^2]\,\mathrm{d}v \leqslant 0 \quad \text{for all vector field } X.$$

If ϱ is a Killing vector field, then $\operatorname{div}\varrho = 0$ ([34, p. 43]) and (3.7) takes the form

$$(3.8)\qquad \int_M [S(\varrho,\varrho) - |\nabla\varrho|^2]\,\mathrm{d}v = 0.$$

Now in an $(\mathrm{APCS})_n$ Ricci-recurrent manifold, $n > 3$, if $r \neq 0$, then we see from (2.12) that

$$S(X,\varrho) = \frac{1}{2} r E(X).$$

Hence $S(\varrho,\varrho) = \frac{1}{2} r g(\varrho,\varrho)$. Therefore (3.8) becomes

$$(3.9)\qquad \int_M \Big[\frac{1}{2} r|\varrho|^2 - |\nabla\varrho|^2\Big]\,\mathrm{d}v = 0.$$

Suppose $r < 0$. Then from (3.9) it follows that $\varrho = 0$. This leads to the following result.

Theorem 3.2. *In a compact, orientable* $(\mathrm{APCS})_n$ *Ricci-recurrent manifold,* $n > 3$ *without boundary, the vector field* ϱ *cannot be a Killing vector field, provided* $r < 0$.

4. $(\mathrm{APCS})_n$ Ricci-recurrent spacetime, $n > 3$

This section is concerned with the study of $(\mathrm{APCS})_n$ Ricci-recurrent manifold, $n > 3$ in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentz metric g with signature $(-,+,+,+)$. The geometry of the Lorentz manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that the Lorentz manifold becomes a convenient choice for the study of general relativity.

Here we consider a perfect fluid $(\mathrm{APCS})_4$ Ricci-recurrent spacetime of non-zero scalar curvature and having the basic vector field ϱ as the timelike vector field of the fluid, that is, $g(\varrho,\varrho) = -1$.

For the perfect fluid spacetime, we have the Einstein equation without cosmological constant as

$$S(X,Y) - \frac{1}{2}rg(X,Y) = kT(X,Y), \tag{4.1}$$

where k is the gravitational constant, T is the energy momentum tensor of type $(0,2)$ given by [18]

$$T(X,Y) = (\sigma + p)E(X)E(Y) + pg(X,Y), \tag{4.2}$$

with σ and p the energy density and the isotropic pressure of the fluid respectively, E is a non-zero 1-form defined by $g(X,\varrho) = E(X)$ for all X, ϱ being the velocity vector field of the fluid.

The equation (4.1) implies that "matter determines the geometry of spacetime and conversely, the motion of matter is determined by the metric tensor of the space which is not flat."

It is to be noted that the basic geometric features of $(\mathrm{APCS})_n$ Ricci-recurrent manifolds, $n > 3$ are also being maintained in the Lorentzian manifold which is necessarily a semi-Riemannian manifold. Hence Lemma 2.1 is also true for an $(\mathrm{APCS})_n$ Ricci-recurrent spacetime, $n > 3$. Using (4.2) in (4.1) we get

$$S(X,Y) - \frac{1}{2}rg(X,Y) = k[(\sigma + p)E(X)E(Y) + pg(X,Y)]. \tag{4.3}$$

Putting $Y = \varrho$ in (4.3) and using Lemma 2.1 yields

$$-k\sigma E(X) = 0, \quad \text{since } g(\varrho,\varrho) = E(\varrho) = -1. \tag{4.4}$$

Since $k \neq 0$ and $E \neq 0$ due to Proposition 2.1, it follows from (4.4) that $\sigma = 0$.

But this is inadmissible when pure matter exists, because in case of existence of pure matter $\sigma > 0$. Hence the spacetime under consideration cannot contain pure matter.

Next we consider an $(\mathrm{APCS})_n$ Ricci-recurrent perfect fluid spacetime, $n > 3$ of non-zero scalar curvature with velocity vector fluid ϱ obeying Einstein's equation with cosmological constant. In this case, Einstein's equation can be written as

$$S - \frac{1}{2}rg + \lambda g = k[(\sigma + p)E \otimes E + pg],$$

where λ is the cosmological constant.

This can be expressed in the form

$$S(X,Y) - \frac{1}{2}rg(X,Y) + \lambda g(X,Y) = k[(\sigma + p)E(X)E(Y) + pg(X,Y)]. \tag{4.5}$$

Putting $Y = \varrho$ in (4.5) and using Lemma 2.1 and $g(\varrho, \varrho) = E(\varrho) = -1$, we get $\lambda E(X) = -k\sigma E(X)$ which implies

$$\sigma = -\frac{\lambda}{k}. \tag{4.6}$$

Again taking a frame field and contracting (4.5) over X and Y we get by using (4.6)

$$p = \frac{-r + 3\lambda}{3k}. \tag{4.7}$$

From (4.6) it follows that σ is constant and from (4.7) it follows that p is constant provided r is constant.

It is known [18] that the energy equation and the force equation for a perfect fluid are respectively

$$\varrho.\sigma = -(\sigma + p)\,\mathrm{div}\,\varrho \tag{4.8}$$

and

$$(\sigma + p)\nabla_\varrho \varrho = -\,\mathrm{grad}\,p - (\varrho.p)\varrho. \tag{4.9}$$

Since ϱ is constant, it follows from (4.8) that $\mathrm{div}\,\varrho = 0$, since $\sigma + p \neq 0$. Again since p is constant provided r is constant, it follows from (4.9) that $\nabla_\varrho \varrho = 0$. But $\mathrm{div}\,\varrho = 0$ represents the expansion scalar and $\nabla_\varrho \varrho$ represents the acceleration vector. Hence we conclude that the perfect fluid has vanishing expansion scalar and vanishing acceleration vector.

Therefore we can state the following result.

Theorem 4.1. *If in an* (APCS)$_4$ *Ricci-recurrent perfect fluid spacetime of non-zero constant scalar curvature the matter content is a perfect fluid whose velocity vector field is the vector field corresponding to the* 1*-form* E*, then the acceleration vector of the fluid and the expansion scalar must vanish.*

From Lemma 2.1 it follows that $\frac{1}{2}r$ is an eigenvalue of the Ricci tensor corresponding to the eigenvector ϱ.

Let $\tilde{\varrho}$ be another eigenvector of S different from ϱ. Then $\tilde{\varrho}$ must be orthogonal to ϱ. Hence $g(\varrho, \tilde{\varrho}) = 0$, that is,

$$E(\tilde{\varrho}) = 0. \tag{4.10}$$

Putting $Y = \tilde{\varrho}$ in (4.5) and using (4.10) we get

$$S(X, \tilde{\varrho}) = \Big[kp + \frac{r}{2} - \lambda\Big] g(X, \tilde{\varrho}). \tag{4.11}$$

Using (4.7) in (4.11) yields

$$S(X, \tilde{\varrho}) = \frac{1}{6} r g(X, \tilde{\varrho}). \tag{4.12}$$

From (4.12) it follows that $\frac{1}{6}r$ is another eigenvalue of S and $\tilde{\varrho}$ is an eigenvector corresponding to this eigenvalue. Since for a given eigenvector there is only one eigenvalue and $\frac{1}{2}r$ and $\frac{1}{6}r$ are different, it follows that the Ricci tensor has only two distinct eigenvalues, namely $\frac{1}{2}r$ and $\frac{1}{6}r$.

Let the multiplicity of $\frac{1}{2}r$ be m. Then the multiplicity of $\frac{1}{6}r$ must be $4 - m$, because the dimension of the space time is 4. Hence

$$m\Big(\frac{1}{2}r\Big) + (4 - m)\frac{1}{6}r = r.$$

From this we get $m = 1$. Therefore the multiplicity of $\frac{1}{2}r$ is 1 and the multiplicity of $\frac{1}{6}r$ is 3.

Hence Segre' characteristic [21] of S is $[(1, 1), 1]$. This leads to the following result.

Theorem 4.2. *An* $(\mathrm{APCS})_4$*, Ricci-recurrent perfect fluid spacetime of non-zero constant scalar curvature is of Segre' characteristic* $[(1, 1), 1]$.

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References

[1] *T. Adati, T. Miyazawa*: On a Riemannian space with recurrent conformal curvature. Tensor, N. S. *18* (1967), 348–354.

[2] *E. Boeckx, L. Vanhecke, O. Kowalski*: Riemannian Manifolds of Conullity Two. World Scientific Publishing, Singapore, 1996.

[3] *K. Buchner, W. Roter*: On conformally quasi-recurrent metrics I. Some general results and existence questions. Soochow J. Math. *19* (1993), 381–400.

[4] *E. Cartan*: Sur une classe remarquable d'espaces de Riemannian. Bull. S. M. F *54* (1926), 214–264. (In France.)

[5] *M. C. Chaki*: On pseudo symmetric manifolds. Ann. Sţiinţ. Univ. "Al. I. Cuza" Iaşi *33* (1987), 53–58.

[6] *M. C. Chaki, B. Gupta*: On conformally symmetric spaces. Indian J. Math. *5* (1963), 113–122.

[7] *U. C. De, H. A. Biswas*: On pseudo conformally symmetric manifolds. Bull. Calcutta Math. Soc. *85* (1993), 479–486.

[8] *U. C. De, A. K. Gazi*: On almost pseudo symmetric manifolds. Ann. Univ. Sci. Budap. *51* (2008), 53–68.

[9] *U. C. De, A. K. Gazi*: On almost pseudo conformally symmetric manifolds. Demonstr. Math. *42* (2009), 869–886.

[10] *U. C. De, S. Bandyopadhyay*: On weakly conformally symmetric spaces. Publ. Math. *57* (2000), 71–78.
[11] *A. Derdzinski, W. Roter*: On compact manifolds admitting indefinite metrics with parallel Weyl tensor. J. Geom. Phys. *58* (2008), 1137–1147.
[12] *A. Derdzinski, W. Roter*: Compact pseudo-Riemannian manifolds with parallel Weyl tensor. Ann. Global Anal. Geom. *37* (2010), 73–90.
[13] *A. Derdzinski, W. Roter*: Global properties of indefinite metrics with parallel Weyl tensor. Proc. Pure and Applied Differential Geometry, PADGE 2007. Shaker, Aachen, 2007, pp. 63–72.
[14] *A. Derdzinski, W. Roter*: Projectively flat surfaces, null parallel distributions, and conformally symmetric manifolds. Tohoku Math. J. *59* (2007), 566–602.
[15] *A. Derdzinski, W. Roter*: The local structure of conformally symmetric manifolds. Bull. Belg. Math. Soc. – Simon Stevin *16* (2009), 117–128.
[16] *R. Deszcz*: On pseudosymmetric spaces. Bull. Soc. Math. Belg. Soc. Sér. A *44* (1992), 1–34.
[17] *L. P. Eisenhart*: Riemannian Geometry. Princeton University Press, Princeton, 1967.
[18] *B. O'Neill*: Semi-Riemannian Geometry. With Applications to Relativity. Academic Press, New York-London, 1983.
[19] *E. M. Patterson*: Some theorems on Ricci-recurrent spaces. J. Lond. Math. Soc. *27* (1952), 287–295.
[20] *P. Petersen*: Riemannian Geometry. Springer, New York, 2006.
[21] *A. Z. Petrov*: Einstein Spaces. Pergamon Press, Oxford, 1969.
[22] *M. Prvanović*: Some theorems on conformally quasi-recurrent manifolds. Fak. Univ. u Novom Sadu, Zb. Rad. Prir.-Mat., Ser. Mat. *19* (1989), 21–31.
[23] *W. Roter*: On conformally symmetric Ricci-recurrent spaces. Colloq. Math. *31* (1974), 87–96.
[24] *W. Roter*: On the existence of conformally recurrent Ricci-recurrent spaces. Bull. Acad. Pol. Sci. Sér. Sci. Math. Astron. Phys. *24* (1976), 973–979.
[25] *W. Roter*: Some remarks on infinitesimal projective transformations in recurrent and Ricci-recurrent spaces. Colloq. Math. *15* (1966), 121–127.
[26] *J. A. Schouten*: Ricci Calculus. An Introduction to Tensor Analysis and Its Geometrical Application. 2nd ed. Springer, Berlin, 1954.
[27] *R. N. Sen, M. C. Chaki*: On curvature restrictions of a certain kind of conformally-flat Riemannian space of class one. Proc. Natl. Inst. Sci. India, Part A *33* (1967), 100–102.
[28] *Y. J. Suh, J.-H. Kwon, Y. Y. Hae*: Conformally symmetric semi-Riemannian manifolds. J. Geom. Phys. *56* (2006), 875–901.
[29] *Z. I. Szabo*: Structure theorems on Riemannian spaces satisfying $R(X,Y).R = 0$. The local version. J. Differ. Geom. *17* (1982), 531–582.
[30] *L. Tamássy, T. Q. Binh*: On weakly symmetric and weakly projectively symmetric Riemannian manifolds. Colloq. János Bolyai Math. Soc. *56* (1989), 663–670.
[31] *A. G. Walker*: On Ruse's space of recurrent curvature. Proc. London Math. Soc., II. Sér. *52* (1951), 36–64.
[32] *Y. Watanabe*: Integral inequalities in a compact orientable manifold, Riemannian or Kählerian. Kōdai Math. Semin. Rep. *20* (1968), 264–271.
[33] *K. Yano*: Integral Formulas in Riemannian Geometry. Marcel Dekker, New York, 1970.

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