

Seminar Paper No. 578

ON ALTERNATIVE INTEREST RATE PROCESSES

by

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On Alternative Interest Rate Processes

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Abstract

In this paper alternative interest rate processes are estimated, using the Generalized Method of Moments (GMM), on Swedish and Danish data. In line with the study by Chan, Karolyi, Longstaff and Sanders (1992) on US data, there seems to be a positive relation between interest rate level and volatility. In contrast to their study it is found that mean-reversion is important in order to not reject different model specifications of the interest rate. In addition, there is evidence of a structural change in the Danish interest rate process in August 1985, which may be due to a change in monetary policy. The small sample properties of the GMM estimates are also studied through simulations.

Keywords: Conditional volatility, mean reversion, nominal interest rate modelling.

JEL Classification numbers: E43, G12.

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1. Introduction

A number of term structure models start by specifying a stochastic process for the instantaneous interest rate. The interest rate is the state variable in the economy and it fully describes the states of nature. Security prices, for example bond prices, are written as functions of the instantaneous interest rate and time and by using a no-arbitrage argument a partial differential equation can be derived. Assuming a certain form for the risk premium, associated with the interest rate, and by taking security specific boundary conditions into account prices might be determined. In some cases it is also possible to derive the premium in an intertemporal general equilibrium model. Common to these models is that the dynamics of the instantaneous interest rate is of great importance. Individual models have been subject to several tests but there has been a lack of a common framework for comparison.¹

In a recent paper Chan, Karolyi, Longstaff and Sanders (1992) brought together many of the proposed dynamics of the instantaneous interest rate as special cases of a general model. They estimated the general and the nested models, using the Generalized Method of Moments (GMM) framework of Hansen (1982), on US data. This paper follows that approach. The main purpose of this paper is to estimate the Swedish and Danish interest rate processes and compare the various model specifications. We will also see if the results in Chan, Karolyi, Longstaff and Sanders (1992) are robust for other market interest rates. A structural break in the Danish interest rate, perhaps due to a change in monetary policy in Denmark, will in addition be studied. In the empirical part of the paper GMM is used. The advantage of using GMM is that it provides a unified approach for the estimation of the diverse set of models and it is not necessary to make strong assumptions about the disturbance terms. Inference from GMM estimation and hypothesis testing is based on asymptotic theory. As a complement, the finite sample properties will be examined by

¹The Cox, Ingersoll and Ross (1985), CIR, model has for instance been tested by Brown and Dybvig (1986), Gibbons and Ramaswamy (1993), and Brown and Schaefer (1994) on US data. Edsparr (1992) estimates the CIR model on Swedish term structure data.

small Monte Carlo experiments. These can also be of help in examining the robustness of the results to various changes in the econometric specifications.

Chan, Karolyi, Longstaff and Sanders (1992) find a positive relation between interest rates and interest rate volatility on US data. This is also found in the present paper. In contrast to their study it seems that the mean-reversion is important in order to not reject different interest rate models. The model specification in Brennan and Schwartz (1980) captures both these features and, except for the square root process in Cox, Ingersoll and Ross (1985), appears to outperform the other models. In addition, the hypothesis of no structural change in the interest rate process in Denmark August 1985 can be rejected. Conformity of the shape of small sample distributions to the asymptotic distributions has also been studied. Simulations suggest that the small sample distributions are fairly well described by the asymptotic distributions.

The rest of the paper is organized as follows. In Section 2 different models for the instantaneous interest rate are briefly reviewed. Data and the employed estimation and testing technique are described in Section 3. The empirical results are presented in Section 4. Finally, Section 5 sets forth the conclusions and compares the results with those obtained in studies on US data.

2. Interest Rate Processes

This section gives a short review of the different interest rates models in Chan, Karolyi, Longstaff and Sanders (1992). The class of models of the instantaneous interest rate, r_t , that is considered can be represented by the following diffusion process

$$dr_t = (\alpha + \beta r_t)dt + \sigma r_t^\gamma dz_t, \quad (2.1)$$

where dz_t is an increment to a Wiener process and r_0 is assumed to be a fixed positive value.² The parameters α , σ , and γ are assumed to be non-negative, while β is negative.

²In equilibrium, nominal interest rates for small open economies like Sweden and Denmark (the countries which will be studied in the paper) can be written as the sum of a foreign nominal interest rate (with the same maturity), the expected rate of currency depreciation over the maturity and a possible foreign risk premium. If all these factors (determining the domestic nominal interest rate) would follow different

The first term describes the mean reversion feature, where β can be seen as the (negative) speed-of-adjustment and α as the product of the speed-of-adjustment and the long-run mean. The second term is a function of γ and σ . The instantaneous variance of dr_t is $\sigma^2 r_t^{2\gamma}$ and depends on the level of r_t , through the value of γ and the scale parameter for the variance, σ . In this paper, five nested models of (2.1) are considered.³ The alternative models put restrictions on α and/or γ . They are summarized in *Table 1* and ordered by the value of γ .

Model (1) is the Ornstein-Uhlenbeck process, sometimes referred to as the elastic random walk, in Vasicek (1977). It sets γ equal to zero and the instantaneous rate of variance is hence constant. The main drawback of the specification is that it can generate negative interest rates. The models below do not impose the zero restriction on γ .

Model (2) is the so called mean-reverting square root process in Cox, Ingersoll and Ross (1985), denoted CIR SR. For this process γ is equal to 0.5, so the rate of the instantaneous variance is linear in the level of the interest rate. In this model the interest rate is precluded from becoming negative.

Models (3) and (4) are the familiar Geometric Brownian Motion (GBM) model, and the model used by Brennan and Schwartz (1980), respectively, for the instantaneous interest rate. In the GBM model α and γ are restricted to zero and one, respectively. It is nested into the more general Brennan and Schwartz model, where γ is set equal to one. So, it is only the Brennan and Schwartz model that captures the whole mean reversion term in (2.1). Model (4) is also used by Courtadon (1982).

Model (5) is the Constant Elasticity of Variance model, denoted CEV, in Cox and Ross (1976). This model allows the variance to be related to the level of the instantaneous interest rate but imposes restrictions on the mean-reversion features (α equals zero).⁴

to processes, it is unclear what process the nominal interest rate follows. It is therefore preferred to model the domestic nominal interest directly.

³This is a reduced number of alternative interest rate processes compared to Chan, Karolyi, Longstaff and Sanders (1992). The models that are not considered are the arithmetic Brownian motion process in Merton (1973), the Dothan (1978) model and the variable rate model in Cox, Ingersoll and Ross (1980). They are nested in the Vasicek, GBM and CEV models, respectively, by setting β equal to zero. These models are not considered due to estimation reasons, explained in the next section. We will see that this is not so restrictive, since if the Vasicek, GBM and CEV models are false none of the nested models can be true.

⁴Marsh and Rosenfeld (1983) examine, on US data, three models where two correspond to the Vasicek

3. Data, Estimation and Testing

This section contains a brief description of the dataset, and discusses the estimation and testing procedures of a discrete-time system of (2.1).

The dataset consists of monthly Swedish and Danish interest rate series with one month to maturity.⁵ The sample covers the periods from January 1984 to August 1992 for Sweden and January 1976 to December 1991 for Denmark. They provide 103 and 192 observations, respectively. These periods are chosen because they give the longest available series of interest rates quoted in well-functioning secondary markets in these countries. The Swedish data are yields on Treasury Bills, whereas the Danish data is generated from so called spline estimation of the term structure (see Dahlquist and Jonsson

and GBM model above. US data is also used by Broze, Scaillet and Zakoian (1993), Duffee (1993) and Bliss and Smith (1994), when they study the models in Chan, Karolyi, Longstaff and Sanders (1992). Broze, Scaillet and Zakoian (1993) use maximum likelihood estimation and concentrate on the Brennan and Schwartz model. Duffee (1993) checks the sensitivity of the results, in Chan, Karolyi, Longstaff and Sanders (1992), applied to different interest rate data for the unrestricted model. Bliss and Smith (1994) focus on a regime shift in monetary policy in the US. They analyze the effects of outliers, the model used, and the assumed duration of the shift. Johanson (1994) estimates the daily Swedish overnight interest rate, with maximum likelihood, for the period January 1986 to May 1991. He uses approximately an Ornstein-Uhlenbeck process for the interest rate and hence models no conditional volatility. Hull and White (1990, 1993) consider extensions of (2.1), where some parameters are allowed to vary over time. They use the models for fitting the initial term structure of interest rates and pricing of interest rate derivatives. In the concluding section, some of the results on US data will be discussed and compared to the findings in the present paper. For more references to applications of term structure models, see Chan, Karolyi, Longstaff and Sanders (1992).

⁵As mentioned above, in the one-factor models, the term structure of interest rates is entirely determined by the specification of the instantaneous interest rate and the market price of risk. The interest rate is here under the “objective” probability measure. The market price of risk must therefore also be determined. There have been different approaches; Vasicek (1977) assumes, in his special case, it to be constant, Dothan (1978) argues in a continuous time CAPM that it is constant, while Cox, Ingersoll and Ross (1985) derive it endogenously. Alternatively, the instantaneous interest rate could be directly specified under the equivalent martingale measure. The transition between the different probability measures, however, changes the first term in the diffusion process, see for instance Duffie (1992) and Duffie and Kan (1993). Of course, the instantaneous interest rate under the martingale measure is not observable. Instead I use the one-month interest rates as approximations of the short-term interest rate. This is consistent with other empirical work in this area. For instance, Chan, Karolyi, Longstaff and Sanders (1992), and Bliss and Smith (1994) use one-month interest rates. It would, however, be possible to study interest rates with shorter maturities and more frequent observations, like the overnight interest rate in the interbank market. Unfortunately, such data tends to have idiosyncrasies (i.e. lots of missing observations, seasonalities, and higher bid-ask spreads due to reserve requirements from the central bank) and it would not be possible to cover longer time spans. In addition, overlapping observations may be problematic. Some experimenting with two and three months interest rates show that the qualitative findings in the empirical section are not affected.

(1993) and Engsted and Tanggaard (1993a) for further descriptions of the data).⁶ *Figures 1a-b* show the annualized interest rates in levels and first differences for Sweden. In *Figures 2a-b* the analogs are plotted for Denmark. From the figures we see that there appears to be a positive relation between the levels of the interest rates and the volatility in the changes of the interest rates. Engsted and Tanggaard (1993a,b) argue that there was a change in monetary policy in Denmark in August 1985. The Danish central bank changed to interest rate smoothing.⁷ The vertical dotted line in Figures 2a-b marks this possible regime shift. It is clear from the figures that the second period is characterized by lower interest rates and especially lower volatility in interest rates.

In order to estimate the interest rate process, (2.1) has to be approximated. More specifically, the maintained assumption here is that discrete changes in the interest rate can be given by the econometric specification

$$r_{t+\Delta t} - r_t = (\alpha + \beta r_t)\Delta t + \epsilon_{t+\Delta t}, \quad (3.1)$$

$$E_t[\epsilon_{t+\Delta t}] = 0 \text{ and } E_t[\epsilon_{t+\Delta t}^2] = \sigma^2 r_t^{2\gamma} \Delta t.$$

Put differently, the system embodies the assumption that it mimics the continuous time model. The specification closely parallels the continuous time dynamics of (2.1) and the potential asymptotic bias for the discrete time approximations of the models is not considered.⁸

In the empirical part of the paper the Generalized Method of Moments (GMM) of Hansen (1982) will be used.⁹ GMM estimation and testing are valid under much weaker conditions than usual. It is not necessary to make strong assumptions concerning the

⁶I am grateful to Carsten Tanggaard for making the Danish dataset available to me.

⁷There was a change in the managing of liquidity which gave a more flexible control of the short term interest rates for Danmarks Nationalbank. However, it should be kept in mind that Denmark still had the commitments as regards the exchange rate vis-à-vis the countries within the European Monetary System.

⁸An initial estimation of a simple AR(1)-model with a constant was done (for both countries). The series are characterized by mean-reversion. Tests for conditional heteroskedasticity (White's and ARCH) were also performed and the overall conclusion is that heteroskedasticity is present in both models. The results here motivate to go further and estimate system (3.1).

⁹The statistical methodology is described in Hansen and Singleton (1982) and more recently in Ogaki (1993).

disturbance terms in the system, which can be conditionally heteroskedastic of unknown form. Here this is very useful as we want to discriminate between different models, which have different distributional assumptions regarding the change in interest rates. For instance, in the Vasicek model they are normally distributed whereas in the CIR SR model they have a non-central χ^2 -distribution. The cost of using GMM compared to maximum likelihood is that statistical tests tend to have less power. This might be potentially important in small samples and motivates the Monte Carlo simulations below. Although, GMM does not rely on strong distributional assumptions, stationarity is necessary for the asymptotic theory (neither the data used in forming the moments conditions nor the instruments are allowed to be nonstationary, see for instance Ogaki (1993)). An obvious violation of the stationarity requirement is when β is restricted to zero. The models with the null hypothesis of β equal to zero (implying nonstationary interest rates) are therefore not estimated.¹⁰

Consider the unrestricted model in (3.1). A combined error term can be defined by $u_{t+\Delta t} = \{\epsilon_{t+\Delta t}, \epsilon_{t+\Delta t}^2 - \sigma^2 r_t^{2\gamma}\}$. Moments conditions are exploited for estimation and testing. The model implies that $E_t[u_{t+\Delta t}] = 0$. The moments conditions are chosen so that $u_{t+\Delta t}$ is orthogonal to variables, denoted z_t , in the information set at time t . Initially, we let z_t be equal to a two-dimensional vector of variables, $\{1, r_t\}$. The information set will be extended later.

There are two sample moments and two instruments, giving four orthogonality restrictions. There are also four parameters to estimate $\{\alpha, \beta, \sigma, \gamma\}$, and the system is exactly identified. For the restricted cases, there are still four orthogonality conditions but only two or three parameters to be estimated. When the number of orthogonality conditions exceed the number of parameters to estimate and the system is overidentified, Hansen's (1982) so called J -test can be used. The statistic is asymptotically χ^2 -distributed with degrees of freedom equal to the number of orthogonality conditions minus the number of parameters. The test of the overidentifying restrictions provides a goodness-of-fit test for

¹⁰Chan, Karolyi, Longstaff and Sanders (1992) do not take this into account. Neither do they consider the problems with estimated models that turn out to be "explosive" (they are not non-stationary by assumption).

the models. A high value of the statistic means that the restrictions of the model are rejected.

Newey and West (1987) consider test statistics for hypothesis tests of nested models that are estimated using GMM. The Wald and D -statistic will be adapted, when we examine whether k restrictions $q(b)$ are equal to zero (possibly non-linear restrictions). To discriminate between different models the D -test is straightforward. It measures the normalized distance between the minimized unrestricted objective function in GMM and the minimized objective function where the restrictions are imposed. It is the analog to the likelihood ratio statistic in maximum likelihood estimation and testing. Both statistics are asymptotically χ^2 -distributed with k degrees of freedom.

The justification for GMM is based on large sample properties and the relatively mild assumptions needed. Using the standard errors and p-values in testing will lead to correct inference asymptotically. However, the finite-sample distributions are unknown. Therefore Monte Carlo simulation experiments will be conducted with the purpose of examining the small sample properties. The simulation procedure is as follows. The initial interest rate, r_0 , is set equal to the empirical value at the starting date. Random elements (the number corresponds to the sample size), $\eta_{t+\Delta t}$, are drawn from the standardized normal distribution and 1000 interest rate series are generated according to

$$r_{t+\Delta t} - r_t = (\alpha + \beta r_t)\Delta t + \sigma r_t^\gamma \eta_{t+\Delta t} \sqrt{\Delta t}, \quad (3.2)$$

which is the discrete-time form of (2.1). The series are simulated under the assumption that the null hypothesis is true. The GMM is then used to reestimate the parameters together with standard errors for each replication. The desired test statistics are also computed. Hence, we can use the parameters and test statistics from the simulation to obtain small sample standard errors and p-values.¹¹

¹¹We shall see that some estimated processes will be “explosive” (one model even generates negative interest rates). The standard asymptotic theory is here not valid. It is also unclear whether Monte Carlo experiments will add information about the distributions (if they exist at all) and we therefore choose to not estimate and perform the experiments for these models.

4. Empirical Results

In this section empirical results are presented for the performance of alternative interest rate models in comparison to the unrestricted model. The robustness of the results for different econometric specifications will also be examined. Inference will be conducted from both asymptotic theory and from Monte Carlo experiments. Further, a comparison is made between non-fitted moments in simulated and empirical interest rates. Finally, the small sample properties of the GMM estimates, in this setting, are summarized..

The results from the estimation of the Swedish interest rate process are presented in *Table 2*. The models are estimated on annualized interest rates. Time is measured in years so that the parameters are expressed on an annual basis. Mean-reversion seems to be of importance. The estimated α is significant and for β only negative values are covered by a 95 percent confidence band. The estimated speed-of-adjustment coefficient is 1.7098 per year and the long-run mean is about 11.5 percent per year. The conjectured relation between levels and volatility in interest rates, in Section 3, is confirmed by the positive and significant estimate of γ . The standard error of σ is, however, large compared to the point estimate and this can probably be attributed to the high correlation with γ . The estimated covariance matrix for the parameters shows that the pairs $\{\alpha, \beta\}$ and $\{\sigma, \gamma\}$ are highly negatively and positively correlated, respectively.¹² The test for overidentifying restrictions picks out the Brennan and Schwartz model and the CIR SR model. These models have both a positive γ and do not impose further restrictions on the other parameters. The other models are rejected at the 95 percent level.

In an attempt to discriminate between the Brennan and Schwartz model and the CIR SR model, a small “power” test is performed. The two models are estimated on data generated under the alternative of the other model. The experiment shows that both the mean and median of the p-values, associated with the J -statistics, are higher for the CIR SR model (0.41 and 0.35, respectively), than for the Brennan and Schwartz model

¹²This indicates an identification problem. But, the estimates are not sensitive to different starting values and J_T equals zero so the point estimates are obviously minimizing the system. In the nested models, where there are restrictions on these parameters, the high correlations are of course not present and not a problem.

(0.32 and 0.22, respectively). This indicates that the Brennan and Schwartz model in this respect has a better data fit.

In *Table 3* estimation of the Danish interest rate process is shown. The results are similar to the Swedish case. The speed-of-adjustment coefficient is slightly smaller for Denmark, about 0.8155 per year, but still a 95 percent confidence band includes only negative values. The estimated long-run mean is about 12.2 percent per year. The conditional standard deviation seems to be linearly related to the level of the interest rate, indicated by the value of γ close to one. With regard to the test for overidentifying restrictions, the Brennan and Schwartz model cannot be rejected at the 95 percent level. The other models seem to be incorrectly specified.¹³

The results of the estimation of the Swedish and Danish interest rate processes suggest the Brennan and Schwartz model as most data coherent. In order to test whether we can discriminate between the Brennan and Schwartz model and the GBM model, which is nested in the Brennan and Schwartz, the D -test in Newey and West (1987) is applied. The results of pair-wise testing show that the GBM model can be rejected in favor of the Brennan and Schwartz model. The corresponding p-values (for the χ^2 -distribution) to the D -statistics are 0.0011 and 0.0111 for Sweden and Denmark, respectively. Note, however, that the results must be treated with caution since the GBM model is “explosive”, in the sense that the estimated β is positive.

The unrestricted models above are exactly identified (the number of moment conditions equals the number of estimated parameters) and we cannot test for overidentifying restrictions. It is of interest to see whether the estimation results are sensitive to the choice of instruments. By extending z_t to $\{1, r_t, r_{t-1}\}$, $\{1, r_t, r_{t-1}, r_{t-2}\}$, $\{1, r_t, r_t r_{t-1}\}$ and $\{1, r_t, r_{t-1}, r_t r_{t-1}\}$, the J -statistic can be used. The results are reported in *Table 4* and *5* for Sweden and Denmark, respectively. The results do not seem to be affected by adding more instruments. The p-values associated with the J -statistics for the Swedish interest rate process are 0.52, 0.60, 0.28 and 0.14 and for the Danish process 0.17, 0.37,

¹³Once again, the mean reversion seem to be important for both Sweden and Denmark. This is also confirmed in the Monte Carlo experiments. For the unrestricted and Brennan and Schwartz models it is only for one generated series the estimated β is non-negative.

0.24 and 0.25.¹⁴ Neither the point estimates nor the standard errors for the parameters seem to be much affected. Other instruments for Denmark are considered below.

Engstedt and Tanggaard (1993a,b) argue that there was a change in monetary policy in Denmark in August 1985. The Danish central bank changed to interest rate smoothing. It appears from Figures 2a-b (recall that the two regimes are separated by the vertical dotted line) that the volatility in the interest rate is lower in the second period. A crude test of a change in the interest rate process can be performed by estimating the following system

$$r_{t+\Delta t} - r_t = ((\alpha + d_t\delta_1) + (\beta + d_t\delta_2)r_t)\Delta t + \epsilon_{t+\Delta t}, \quad (4.1)$$

$$E_t[\epsilon_{t+\Delta t}] = 0 \text{ and } E_t[\epsilon_{t+\Delta t}^2] = (\sigma + d_t\delta_3)^2 r_t^{2(\gamma + d_t\delta_4)} \Delta t,$$

where d_t is a dummy variable, one from the change in August 1985 and onwards and zero before. The dummy parameters δ_1 , δ_2 , δ_3 and δ_4 correspond to changes in α , β , σ and γ , respectively. Comparing this unrestricted system (4.1) with the estimates of (3.1) can then be seen as a test of a structural change. Since there are now four extra parameters to estimate, the instruments are extended so that z_t is equal to $\{1, r_t, d_t, d_t r_t\}$. The system is estimated for the unrestricted model and also for the model of the interest rate process that could not be rejected, that is, the Brennan and Schwartz model. The results are shown in *Table 6*. Here we have difficulties with the identification of the unrestricted model. The correlation matrix for the parameters shows that $\{\alpha, \beta\}$, $\{\sigma, \gamma\}$, and $\{\delta_3, \delta_4\}$ are all highly correlated (roughly 0.99). The high standard errors for the parameters can probably also be attributed to this. This problem is of less importance for the nested model, which fixes γ and δ_4 to one and zero, respectively. The conclusions are therefore based on the estimation of the nested model. The test for overidentifying restrictions, with associated p-value equal to 0.1836, indicates that the model cannot be rejected. The parameter estimates are in line with the ones from system (3.1). However, the dummy parameter δ_3 is significantly negative and the hypothesis of no structural change in the volatility can hence be rejected. This is also confirmed by the D -statistic,

¹⁴The p-values from the Monte Carlo experiment are even higher for both countries.

which rejects system (3.1) in favor of system (4.1) with dummy-variables.¹⁵ The figures also indicate that there seems to be a lower long-run mean in the second period. For the nested model, the estimated long-run means are 12.5 and 10.0 percent per year in the first and second regime respectively. However, the null hypothesis of equal long-run means cannot be rejected.¹⁶

In order to provide further diagnostics for the estimated models, we study how well the models fit other moments. This is done by comparing moments that GMM estimation does not impose restrictions on, both for the simulated and empirical interest rates and interest rate changes. That is to say, how well the estimated processes describe moments other than the first and second. Based on the results above the simulation is performed for the unrestricted model for Sweden. For Denmark, the specification has been extended so as to handle the structural break. Due to the problem with highly correlated parameters in the unrestricted model, the Brennan and Schwartz model with dummies is simulated. In each replication the mean, standard deviation, minimum, maximum, skewness and kurtosis (excess of the normal distribution) were calculated for the interest series and their changes. The fit can first be illustrated by comparing the empirical series together with randomly chosen generated series, shown in *Figures 3a-b* and *Figures 4a-b*. The series are visually indistinguishable in their time series properties. In *Table 7* the empirical estimates and the averages from the simulations with the standard deviations within parenthesis are presented. For Sweden, the unconditional means and standard deviations for the empirical and simulated interest rates (both for levels and changes) are similar.

¹⁵Following Andrews (1993), a test for parameter change with unknown break point is also undertaken. The timing of the shift was allowed to be between June 1978 and July 1989. The Wald statistics indicate that there could have been a shift before August 1985. It appears to be two extreme periods. The first occurred about January 1981 (before a rise in the interest rate) and the second occurred in May 1983 (before a fall in the interest rate). The supremum Wald statistic shows that there was a significant shift in June 1981. Notable is that the Wald statistic cannot reject the hypothesis of stable parameters if the point-wise break is in August 1985 (even though the point estimates of σ differ substantial).

¹⁶A test of equal long-run mean boils down to test whether $q(b) = \alpha\delta_2 - \beta\delta_1$ is equal to zero. The hypothesis can be tested by the Wald statistic, which here is asymptotically χ^2 -distributed with one degree of freedom. It is well known that the Wald statistic is sensitive to the parameterization of non-linear hypothesis. The p-values for the test statistics vary between 0.12 and 0.24 for both models, depending on how the hypothesis is formulated. The estimated small sample p-values are slightly higher for the Brennan and Schwartz model. For the unrestricted model they are as high as 0.90 (maybe due to the numerical problems).

The measure of skewness indicates that the symmetry in the empirical series is captured by the simulated series. With regards to the kurtosis, we have both wrong signs and magnitudes. The empirical series show a lower (higher) kurtosis for the levels (changes) of the interest rates. Recall that a higher measure of kurtosis means that the peakness is higher for the distribution and that it has thicker tails relative to the normal distribution. The results for Sweden and Denmark are alike. In the first regime for Denmark, the empirical means and standard deviations are well described, even though the volatility seems to be lower according to the minima and maxima. In the second regime, this is not apparent. The symmetry in the empirical series appears to be present also for the simulated series, except for the interest level in the second regime. For both regimes, the estimated models fail to match the kurtosis. On the other hand, the variability is high in this measure for the simulated series. The general impression is, however, that the most important moments are those used in the optimization.

With regard to the small sample properties of the GMM estimation, the overall impression is that the asymptotic standard errors for the parameters are smaller than the estimated small sample standard errors but that the different test statistics mimic the asymptotic distribution well. By comparing the standard errors in Table 4 and 5, we see that the standard errors from the Monte Carlo are on average 40 and 30 percent larger than the asymptotic for Sweden and Denmark, respectively. Hence, there seems to be a difference in using 102 (Sweden) or 191 (Denmark) observations. However, this is not a general result. In models with restrictions on γ the estimated standard errors are on average about 15 percent larger for both countries. This is due to the smaller standard errors for σ . With regard to the distributions for the test statistics, the estimated small sample and asymptotic distribution are surprisingly close. In fact, in almost half of the cases, the hypothesis of equal distributions cannot be rejected by the Kolmogorov-Smirnov test at the one percent significance level. *Figures 5a-p* show smoothed density estimates for some of the test statistics.¹⁷ As a comparison, the χ^2 -distributions with corresponding degrees of freedoms are plotted. The vertical lines mark the asymptotic (solid) and the

¹⁷The estimation is performed with a Gaussian kernel and the bandwidth parameter is automatically chosen, see Silverman (1986).

estimated small sample (dotted) five percent right-tail critical values. We see that there is conformity of the shape for the J -statistics and the critical values are close to the asymptotic values. This result seems to be independent of the degree of freedom for these statistics. The Wald-statistic fits the asymptotic distribution fairly well. Note that the second Wald-statistic used is not plotted due to the numerical problems discussed above. With regard to the D -statistics they appear to be less skewed than the asymptotic distributions. The critical values are much higher, 12.93 and 11.94 compared to 7.81 and 9.49 for the χ^2 -distribution with 3 and 4 degrees of freedom, respectively. These statistics seem to have worse small sample properties in this setting. It would, however, be too hasty to make any general conclusions regarding the performance of the D -statistic compared to the J -statistic, based on only two cases for the D -statistic.

5. Concluding Comments

In this paper interest rate processes for Sweden and Denmark have been estimated for the periods January 1984 to August 1992 and January 1976 to December 1991, respectively. Alternative interest rate models have been compared using the available tests within the Generalized Method of Moments (GMM). The results of the comparison show that the specification in Brennan and Schwartz (1980) seems to outperform the other models. This is because it takes into account the mean-reverting feature of interest rates and the fact that the variance in interest rate changes appears to be dependent on the interest rate level. The other models, with the square root process in Cox, Ingersoll and Ross (1985) as an exception, impose severe restrictions on the parameters and can thus be rejected. It has been argued that there was a structural change in the Danish interest rate process in August 1985, due to interest rate smoothing. The hypothesis of no structural change can be rejected on the basis of the data. The small sample properties of the GMM framework in this setting is also studied by simulations. It is found that the estimated small sample distributions, for the test statistics used, are fairly well described by the asymptotic distributions.¹⁸

¹⁸As mentioned above, we do not consider the temporal aggregation and assume that the discrete-time system correctly describes the instantaneous interest rate. How this maintained assumption affects

Chan, Karolyi, Longstaff and Sanders (1992) use US data for the period June 1964 to December 1989. They also find that the volatility for the interest rate is sensitive to the level of interest rates, and stress the importance for financial applications. They conclude, however, that there is only weak evidence of mean-reversion. The models that capture the conditional volatility cannot be rejected on the basis of the test for overidentifying restrictions. However, Chan, Karolyi, Longstaff and Sanders (1992) neglect the problems with “explosive” processes and care should be taken using the conventional asymptotic distributions. Their examination is extended by both Duffee (1993), and Bliss and Smith (1994). Duffee (1993) shows that the results in Chan, Karolyi, Longstaff and Sanders (1992) are sensitive to the chosen interest rate. Further, he finds that the inclusion, or not, of the potential regime shift in the US between October 1979 and September 1982, affects especially the relationship between the level and the volatility of interest rates. The estimates of γ then vary between 0.1 and 1.7. Bliss and Smith (1994) study the strength of the results in relation to outliers, the model used, and the duration of the regime shift. For instance, they find that exclusion of one observation (a potential outlier) lowers the estimated γ from 1.5 to 1.2. Hence, the emphasized relationship between the volatility and the level of the interest rate can actually be questioned. Evans, Keef and Okunev (1994) model the real interest rate, rather than the nominal interest rate. They estimate a version of the Ornstein-Uhlenbeck process and tests for conditional volatility on annual US and UK data for the period 1870-1975. They find a significant mean-reversion in real interest rates (realized) for both countries, but there is no evidence of conditional volatility. However, their examination of the inflation and nominal interest rates reveals that the conditional volatility is present in both these series. So, the conditional volatility seems to cancel out when real interest rates are formed. This point may be worthy of further inquiry (together with a check of the robustness of the modelling of inflation expectations).

the estimation is also an interesting question. I performed a small experiment for the model of the Swedish interest rate. Hourly and daily data were generated 1000 times under the hypothesis that the estimated model is the “true” one. Then monthly data was sampled and estimated. A comparison of the distributions for the parameters shows that they are similar except for the σ parameter, which has a flatter distribution for the monthly data. Overall, the temporal aggregation problem does not seem to be too severe for this model. This issue could, however, be further studied.

In most models referred to above there is only one state variable that drives the term structure, namely the instantaneous interest rate. The results presented are not inconsistent with multi-state variable models. Longstaff and Schwartz (1992), for example, extend the one state variable model of Cox, Ingersoll and Ross (1985) to a two-state variable model.¹⁹ The state variables are the instantaneous interest rate and the interest rates instantaneous rate of variance, specified as interdependent mean-reverting square root processes. Based on the results in this paper, their specification seems to be a promising direction of development for modelling the term structure of interest rates.

¹⁹The Longstaff and Schwartz (1992) model can actually be seen as a special case of the multi-factor model in Duffie and Kan (1993).

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Table 1. Alternative Interest Rate Processes

Interest rate models, in their continuous-time formulation, nested within the unrestricted model,

$$dr_t = (\alpha + \beta r_t)dt + \sigma r_t^\gamma dz_t.$$

Model		α	β	σ	γ
(1)	Vasicek				0
(2)	CIR SR				0.5
(3)	GBM	0			1
(4)	Brennan-Schwartz				1
(5)	CEV	0			

Model (1), (2) and (4) refer to the model specifications in Vasicek (1977), Cox, Ingersoll and Ross (1985), and Brennan and Schwartz (1980), respectively. Model (3) and (5) are the Geometric Brownian Motion (GBM) and the Constant Elasticity of Variance (CEV), respectively. A blank means that no restriction is imposed.

Table 2. Estimation of the Swedish Interest Rate Process

Results are based on monthly data from January 1984 to August 1992 (providing 102 observations in total) for the annualized one month Swedish Treasury Bill. The following system is estimated by the generalized method of moments:

$$r_{t+\Delta t} - r_t = (\alpha + \beta r_t)\Delta t + \epsilon_{t+\Delta t},$$

$$E_t[\epsilon_{t+\Delta t}] = 0 \text{ and } E_t[\epsilon_{t+\Delta t}^2] = \sigma^2 r_t^{2\gamma} \Delta t.$$

Model	α	β	σ	γ	J -statistic ^a	d.f. ^b
Unrestricted	0.1964 (0.0620) [0.0924]	-1.7098 (0.5749) [0.8373]	0.4000 (0.4133) [0.8277]	1.1544 (0.4794) [0.4983]		
Vasicek	0.1905 (0.0619) [0.0901]	-1.6286 (0.5738) [0.7654]	0.0295 (0.0295) [0.0022]	0	4.2694 (0.0388) [0.0220]	1
CIR SR	0.1964 (0.0620) [0.0932]	-1.6842 (0.5746) [0.8071]	0.0938 (0.0105) [0.0066]	0.5	1.6919 (0.1934) [0.1950]	1
GBM	0	0.1018 (0.0986)	0.2602 (0.0329)	1	8.0431 (0.0179)	2
Brennan-Schwartz	0.1973 (0.0620) [0.0883]	-1.7101 (0.5749) [0.7789]	0.2864 (0.0305) [0.0207]	1	0.1017 (0.7498) [0.7830]	1
CEV	0	0.0993 (0.0990)	0.4464 (0.5819)	1.2470 (0.6010)	7.9634 (0.0048)	1

Asymptotic and Monte Carlo standard errors for the coefficients are given within parentheses and square brackets, respectively.

^aThe J -statistics are the tests for overidentifying restrictions. Asymptotic and Monte Carlo p-values are given within parentheses and square brackets, respectively.

^bDegrees of freedom associated with the J -statistics.

Table 3. Estimation of the Danish Interest Rate Process

Results are based on monthly data from January 1976 to December 1991 (providing 191 observations in total) for the annualized one month Danish interest rate. The following system is estimated by the generalized method of moments:

$$r_{t+\Delta t} - r_t = (\alpha + \beta r_t)\Delta t + \epsilon_{t+\Delta t},$$

$$E_t[\epsilon_{t+\Delta t}] = 0 \text{ and } E_t[\epsilon_{t+\Delta t}^2] = \sigma^2 r_t^{2\gamma} \Delta t.$$

Model	α	β	σ	γ	J -statistic ^a	d.f. ^b
Unrestricted	0.0990 (0.0390) [0.0474]	-0.8155 (0.3458) [0.4238]	0.2600 (0.1120) [0.2417]	0.9703 (0.2226) [0.2910]		
Vasicek	0.0649 (0.0384) [0.0454]	-0.4923 (0.3385) [0.3524]	0.0315 (0.0029) [0.0016]	0	12.2885 (0.0005) [0.0000]	1
CIR SR	0.0902 (0.0390) [0.0467]	-0.7037 (0.3429) [0.3928]	0.0992 (0.0076) [0.0052]	0.5	4.3631 (0.0367) [0.0250]	1
GBM	0	0.0341 (0.0688)	0.2571 (0.0216)	1	6.2240 (0.0445)	2
Brennan-Schwartz	0.0989 (0.0390) [0.0629]	-0.8177 (0.3454) [0.5570]	0.2751 (0.0198) [0.0204]	1	0.0175 (0.8947) [0.9060]	1
CEV	0	0.0449 (0.0703)	0.1587 (0.0879)	0.7554 (0.2737)	5.6471 (0.0175)	1

Asymptotic and Monte Carlo standard errors for the coefficients are given within parentheses and square brackets, respectively.

^aThe J -statistics are the tests for overidentifying restrictions. Asymptotic and Monte Carlo p-values are given within parentheses and square brackets, respectively.

^bDegrees of freedom associated with the J -statistics.

Table 4. Unrestricted Model Using Different Instruments, Sweden

Results are based on monthly data from January 1984 to August 1992 (providing 102 observations in total) for the annualized one month Swedish Treasury Bill. The following system is estimated by the generalized method of moments:

$$r_{t+\Delta t} - r_t = (\alpha + \beta r_t)\Delta t + \epsilon_{t+\Delta t},$$

$$E_t[\epsilon_{t+\Delta t}] = 0 \text{ and } E_t[\epsilon_{t+\Delta t}^2] = \sigma^2 r_t^{2\gamma} \Delta t.$$

Instruments	α	β	σ	γ	J -statistic ^a	d.f. ^b
$\{1, r_t\}$	0.1964 (0.0620) [0.0924]	-1.7098 (0.5749) [0.8373]	0.4000 (0.4133) [0.8277]	1.1544 (0.4794) [0.4983]		
$\{1, r_t, r_{t-1}\}$	0.1727 (0.0581) [0.0865]	-1.4883 (0.5413) [0.7670]	0.3994 (0.4258) [0.7388]	1.1691 (0.4929) [0.5041]	1.2942 (0.5236) [0.5480]	2
$\{1, r_t, r_{t-1}, r_{t-2}\}$	0.1633 (0.0549) [0.0895]	-1.4294 (0.5110) [0.8086]	0.3914 (0.4538) [0.7500]	1.1898 (0.5326) [0.5088]	2.7439 (0.6016) [0.6320]	4
$\{1, r_t, r_t r_{t-1}\}$	0.1694 (0.0591) [0.0878]	-1.4692 (0.5496) [0.7950]	0.4792 (0.5471) [0.7708]	1.2704 (0.5295) [0.5058]	2.5120 (0.2848) [0.3150]	2
$\{1, r_t, r_{t-1}, r_t r_{t-1}\}$	0.1883 (0.0570) [0.0890]	-1.6807 (0.5291) [0.8133]	0.3069 (0.4291) [0.9721]	1.1161 (0.6484) [0.5626]	6.8883 (0.1419) [0.1510]	4

Asymptotic and Monte Carlo standard errors for the coefficients are given within parentheses and square brackets, respectively.

^aThe J -statistics are the tests for overidentifying restrictions. Asymptotic and Monte Carlo p-values are given within parentheses and square brackets, respectively.

^bDegrees of freedom associated with the J -statistics.

Table 5. Unrestricted Model Using Different Instruments, Denmark

Results are based on monthly data from January 1976 to December 1991 (providing 191 observations in total) for the Danish annualized one month interest rate. The following system is estimated by the generalized method of moments:

$$r_{t+\Delta t} - r_t = (\alpha + \beta r_t) \Delta t + \epsilon_{t+\Delta t},$$

$$E_t[\epsilon_{t+\Delta t}] = 0 \text{ and } E_t[\epsilon_{t+\Delta t}^2] = \sigma^2 r_t^{2\gamma} \Delta t.$$

Instruments	α	β	σ	γ	J -statistic ^a	d.f. ^b
$\{1, r_t\}$	0.0990 (0.0390) [0.0474]	-0.8155 (0.3458) [0.4238]	0.2600 (0.1120) [0.2417]	0.9703 (0.2226) [0.2910]		
$\{1, r_t, r_{t-1}\}$	0.0913 (0.0391) [0.0467]	-0.7548 (0.3452) [0.4154]	0.2968 (0.1511) [0.3147]	1.0768 (0.2603) [0.3082]	3.5382 (0.1705) [0.1900]	2
$\{1, r_t, r_{t-1}, r_{t-2}\}$	0.0794 (0.0377) [0.0442]	-0.6522 (0.3315) [0.3997]	0.3241 (0.1673) [0.2693]	1.1406 (0.2676) [0.3176]	4.3138 (0.3652) [0.4090]	4
$\{1, r_t, r_t r_{t-1}\}$	0.0911 (0.0378) [0.0486]	-0.7764 (0.3408) [0.4427]	0.3631 (0.1841) [0.2802]	1.1922 (0.2617) [0.3132]	2.8762 (0.2374) [0.2370]	2
$\{1, r_t, r_{t-1}, r_t r_{t-1}\}$	0.0618 (0.0323) [0.0428]	-0.5314 (0.2973) [0.4028]	0.3178 (0.1725) [0.2035]	1.1457 (0.2794) [0.2920]	5.3816 (0.2503) [0.3060]	4

Asymptotic and Monte Carlo standard errors for the coefficients are given within parentheses and square brackets, respectively.

^aThe J -statistics are the tests for overidentifying restrictions. Asymptotic and Monte Carlo p -values are given within parentheses and square brackets, respectively.

^bDegrees of freedom associated with the J -statistics.

**Table 6. Estimating the Danish Interest Rate Process
with a Structural Change**

Results are based on monthly data from January 1976 to December 1991 (providing 191 observations in total) for the Danish annualized one month interest rate. The following system is estimated by the generalized method of moments:

$$r_{t+\Delta t} - r_t = ((\alpha + d_t\delta_1) + (\beta + d_t\delta_2)r_t)\Delta t + \epsilon_{t+\Delta t},$$

$$E_t[\epsilon_{t+\Delta t}] = 0 \text{ and } E_t[\epsilon_{t+\Delta t}^2] = (\sigma + d_t\delta_3)^2 r_t^{2(\gamma+d_t\delta_4)} \Delta t.$$

Model	α	β	σ	γ	δ_1	δ_2
Unrestricted	0.1529 (0.0623) [0.0678]	-1.1596 (0.4739) [0.5386]	0.0808 (0.0460) [0.0702]	0.3409 (0.2967) [0.3086]	0.0072 (0.1028) [0.1045]	-0.4381 (0.9647) [0.9409]
Brennan-Schwartz	0.1347 (0.0616) [0.0695]	-1.0744 (0.4723) [0.5886]	0.2683 (0.0203) [0.0179]	1	0.0279 (0.0988) [0.1164]	-0.5509 (0.9168) [1.0871]

Table 6 (continued)

δ_3	δ_4	J -statistic ^a	d.f. ^b	D -statistic ^c	d.f. ^d
0.1147 (0.4594) [1.2632]	0.7788 (1.0445) [0.7855]			14.1143 (0.0069) [0.0260]	4
-0.1201 (0.0257) [0.0258]	0	3.3898 (0.1836) [0.2030]	2	19.5212 (0.0002) [0.0120]	3

Asymptotic and Monte Carlo standard errors for the coefficients are given within parentheses and square brackets, respectively.

^aThe J -statistics are the tests for overidentifying restrictions. Asymptotic and Monte Carlo p-values are given within parentheses and square brackets, respectively.

^bDegrees of freedom associated with the J -statistics.

^cThe D -statistics are for pair-wise testing. Asymptotic and Monte Carlo p-values are given within parentheses and square brackets, respectively.

^dDegrees of freedom associated with the D -statistics.

**Table 7. Comparison of Simulated and Empirical
Interest Rates and Interest Rate Changes**

Results are based on monthly data for Swedish and Danish annualized one month interest rates. The sample covers the periods from January 1984 to August 1992 for Sweden and January 1976 to December 1991 for Denmark.

	r_t		Δr_t	
	Empirical	Simulated	Empirical	Simulated
Sweden				
Mean	11.4513	11.4485 (0.6546)	0.0053	0.0025 (0.0190)
Std Dev	1.8766	1.6822 (0.3510)	1.0065	1.0001 (0.1012)
Min	8.2900	8.1444 (0.6641)	-3.4200	-2.9026 (0.6135)
Max	16.3400	16.0799 (1.7013)	3.5400	2.3950 (0.4237)
Skewness	0.4879	0.4284 (0.3826)	0.1260	-0.2776 (0.2656)
Kurtosis ^a	-0.2971	1.0161 (0.7906)	2.6828	0.2723 (0.7115)

Table 7 (continued)

	r_t		Δr_t	
	Empirical	Simulated	Empirical	Simulated
Denmark, Regime I^b				
Mean	13.2346	12.2374 (0.8507)	-0.0050	0.0219 (0.0158)
Std Dev	2.6931	2.1213 (0.4775)	1.2018	0.9847 (0.0980)
Min	8.1859	8.1734 (0.7210)	-3.1071	-2.9194 (0.6672)
Max	20.7891	17.9613 (2.1725)	4.5495	2.5100 (0.4807)
Skewness	0.3803	0.4039 (0.4048)	0.2205	-0.2315 (0.2530)
Kurtosis ^a	-0.3367	-0.1056 (0.7684)	1.4100	0.3256 (0.7305)
Denmark, Regime II^c				
Mean	9.8598	10.2161 (0.4237)	0.0215	-0.0322 (0.0348)
Std Dev	0.8986	0.9463 (0.3226)	0.4441	0.4629 (0.0506)
Min	7.9744	8.5532 (0.4490)	-1.0695	-1.2326 (0.3162)
Max	12.1124	12.8050 (1.5811)	1.1442	1.0051 (0.1834)
Skewness	-0.1080	0.5409 (0.0001)	-0.0431	-0.1905 (0.3320)
Kurtosis ^a	-0.1846	0.3647 (0.7684)	-0.2478	0.0578 (0.7558)

Standard deviations from the simulations are given within parentheses.

^aDegree of excess kurtosis based on the normal distribution.

^bRegime I refers to the period January 1976 to July 1985.

^cRegime II refers to the period August 1985 to December 1991.

Figure 1a. Swedish one month
interest rate, level

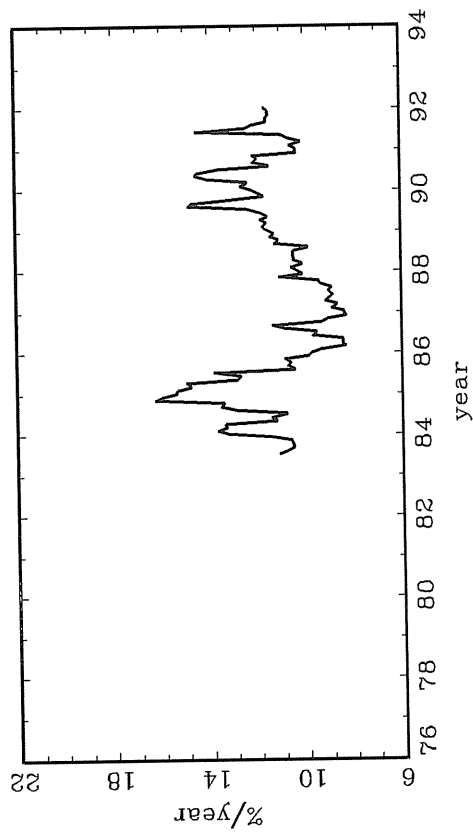


Figure 2a. Danish one month
interest rate, level

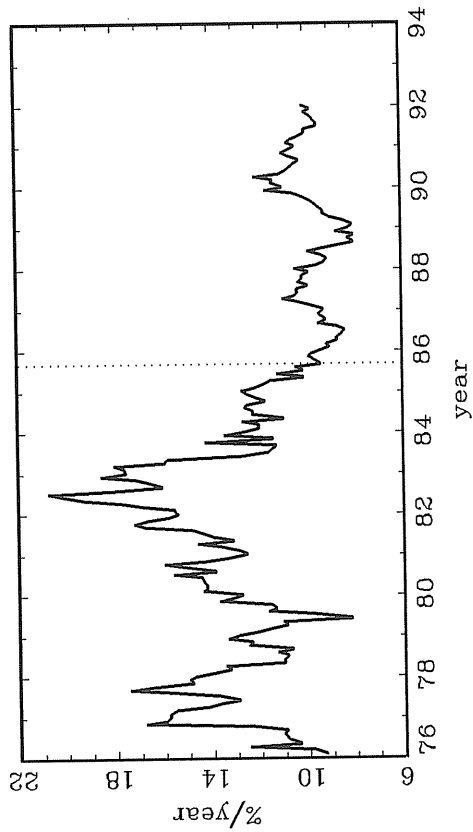


Figure 1b. Swedish one month
interest rate, first difference

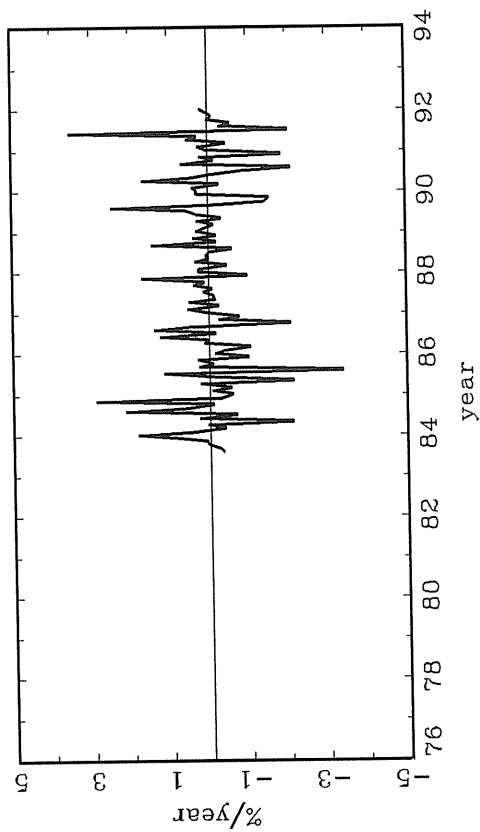


Figure 2b. Danish one month
interest rate, first difference

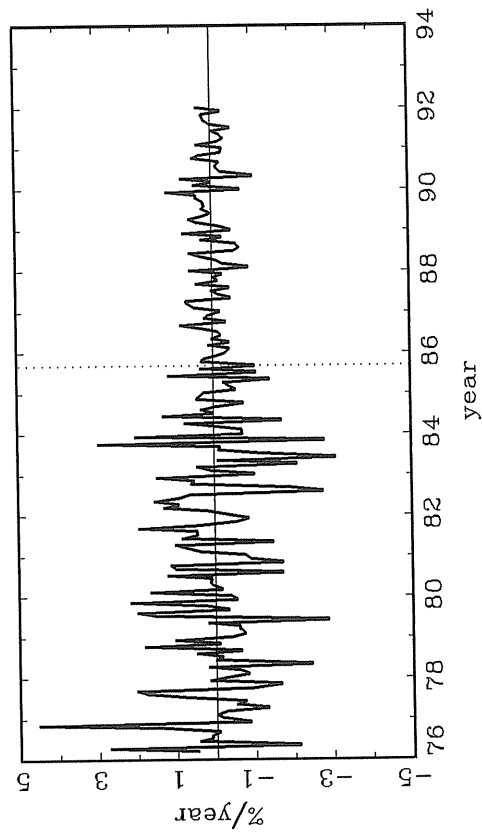


Figure 3a. Simulated and Empirical
Interest Rates, Sweden

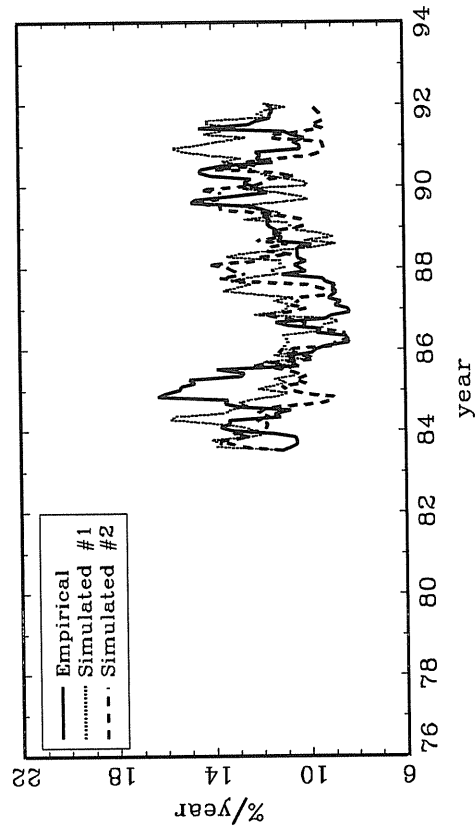


Figure 3b. Simulated and Empirical
Interest Rate Changes, Sweden

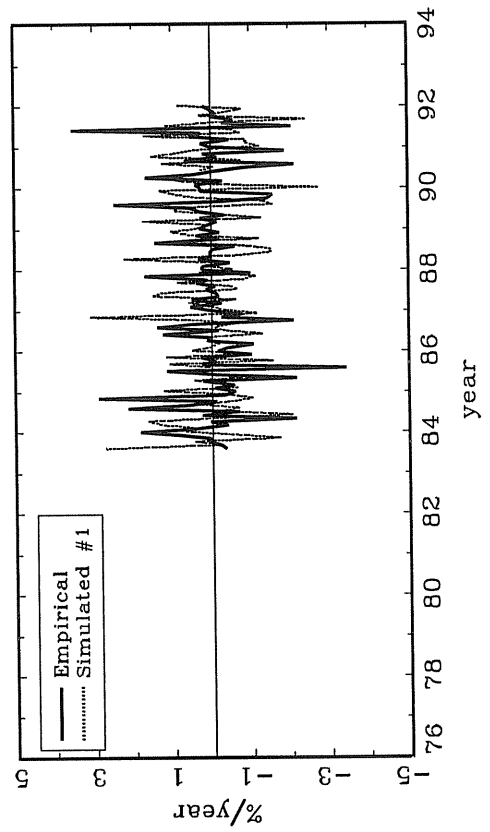


Figure 4a. Simulated and Empirical
Interest Rates, Denmark

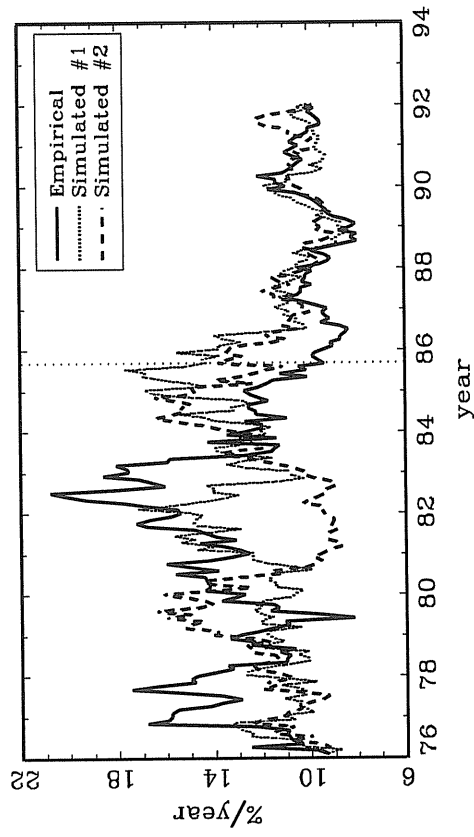


Figure 4b. Simulated and Empirical
Interest Rate Changes, Denmark

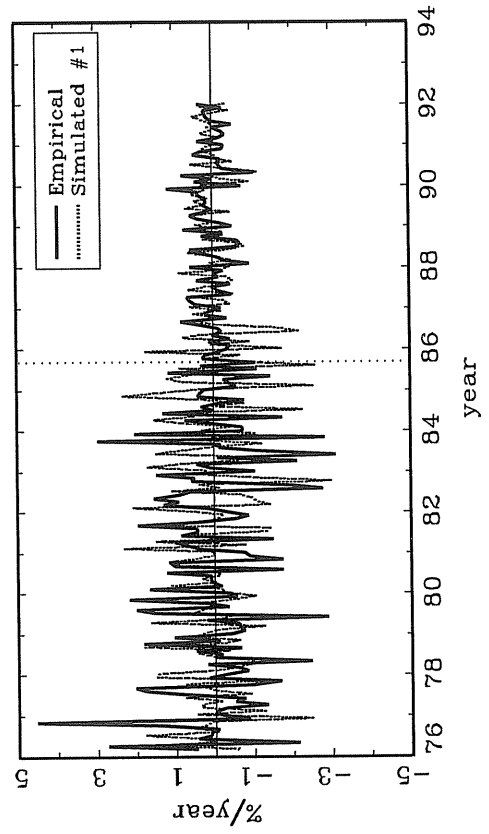


Figure 5a. J-statistic, CIR SR
Sweden

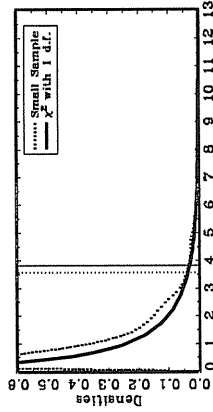


Figure 5b. J-statistic, BS
Sweden

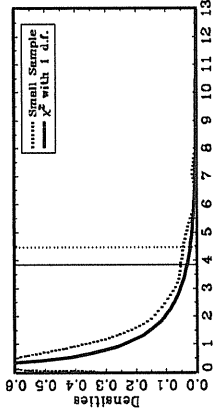


Figure 5c. J-statistic, CIR SR
Denmark

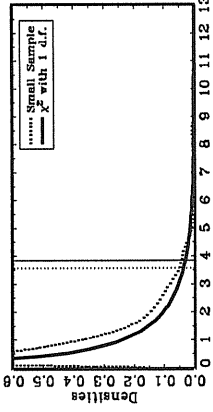


Figure 5d. J-statistic, BS
Denmark

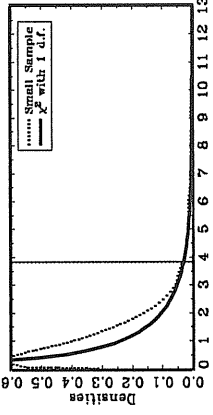


Figure 5e. J-statistic, Instrument Set 1
Sweden

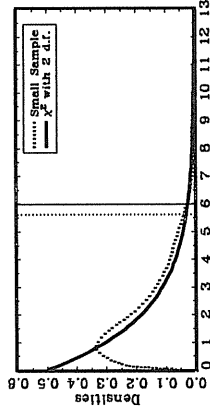


Figure 5f. J-statistic, Instrument Set 2
Sweden

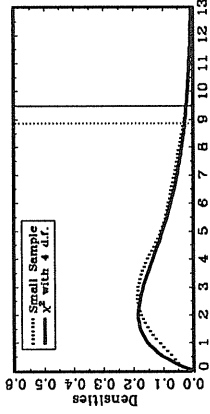


Figure 5g. J-statistic, Instrument Set 3
Sweden

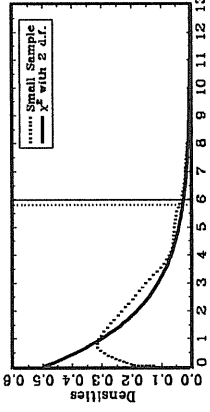


Figure 5h. J-statistic, Instrument Set 4
Sweden

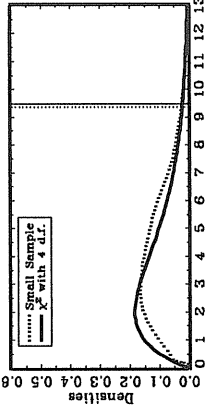


Figure 5i. J-statistic, Instrument Set 1
Denmark

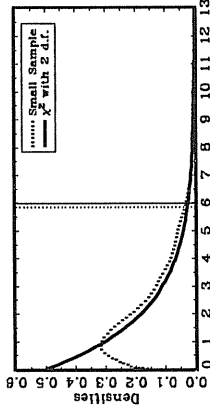


Figure 5j. J-statistic, Instrument Set 2
Denmark

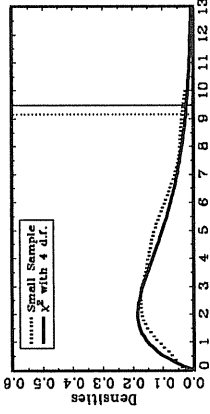


Figure 5k. J-statistic, Instrument Set 3
Denmark

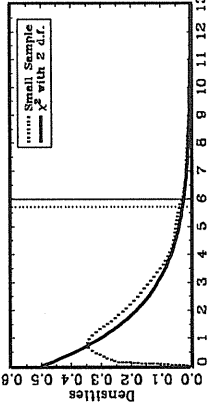


Figure 5l. J-statistic, Instrument Set 4
Denmark

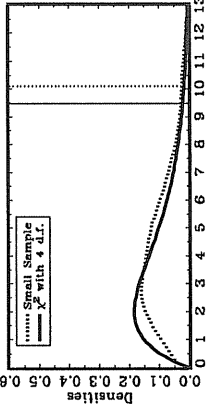


Figure 5m. J-statistic, BS
Dummy Model

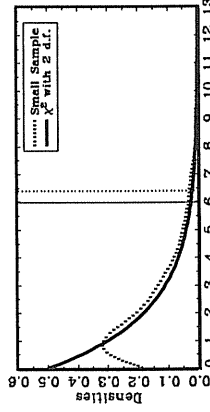


Figure 5n. D-statistic, BS
Dummy Model

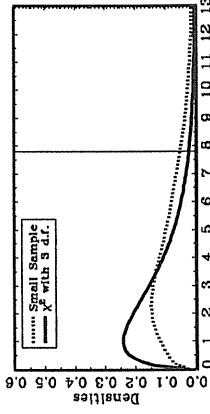


Figure 5o. Wald statistic, BS
Dummy Model

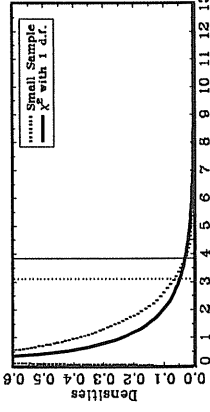


Figure 5p. D-statistic, Unrestricted
Dummy Model

