## On an Asymptotic Integral

By L. C. Hso<br>(Receiced 6th June 1950.)

1. This note gives an asymptotic evaluation of an integral of the form

$$
\begin{equation*}
I_{n}=\int_{a}^{b}\left\{f_{n}(x)\right\}^{n} g(x) d x \tag{1}
\end{equation*}
$$

as $n$ tends to infinity, where $\left\{f_{n}(x)\right\}$ is a sequence of real-valued functions. The theorem to be established is a natural extension of B. Levi's generalised Laplace-Darboux theorem (1, 341-5l); it gives a rule for evaluating a wider class of asymptotic integrals.

In what follows $(x, n) \rightarrow(\xi, \infty)$ denotes that $x, n$ tend to $\xi, \infty$ respectively and independently. For instance, $\phi(x, n) \rightarrow 0$ as $(x, n) \rightarrow$ $(\xi, \infty)$ means that for any given positive $\epsilon$ there exist positive numbers $\delta$ and $N$ such that $|\phi(x, n)|<\epsilon w h e n e v e r|x-\xi|<\delta$ and $n>N$.

Theorem. Let $f_{n}(x)[n=1,2,3, \ldots]$ and $g(x)$ be real functions such that
(A) $g(x)$ and each $f_{n}(x)$ are integrable over ( $a, b$ );
(B) there is a number $A$, independent of $x$ and $n$, such that

$$
\left|f_{n}(x)\right|<A,|g(x)|<A ;
$$

(C) $f_{n}(x)$ altains a positive absolute maximum at $x=\xi_{n}$;
(D) for each posilive number $d$, there is a posilive number $\delta$ (depending on d but nol on $n$ ) such that $\left|f_{n}(x)\right| \leqq f_{n}\left(\xi_{n}\right)-\delta$ whenever $\left|x-\xi_{n}\right| \geqq d$ and $a \leqq x \leqq b ;$
(E) as $n$ tends to infinity, $\xi_{n}$ tends to $\xi$, where $a<\xi<b$;
(F) $g(x)$ is continuous at $x=\xi$ with $g(\xi) \neq 0$;
( C$)$ there are positive cons!ants $h$ and $k$ such that

$$
\begin{equation*}
\lim _{(x, n) \rightarrow(\xi, \infty)}\left|f_{n}(x)-f_{n}\left(\xi_{n}\right)\right| /\left|x-\xi_{n}\right|^{h}=k \tag{ㅇ}
\end{equation*}
$$

Then for $n$ large

$$
\begin{equation*}
I_{n} \sim\left(2_{i}^{\prime} h\right) \Gamma(1 / h)\left\{f_{n}\left(\xi_{n}\right)\right\}^{n} g(\xi)\left\{f_{n}\left(\xi_{n}\right) / n k\right\}_{\}}^{1 / h} . \tag{3}
\end{equation*}
$$

2. The essential step in the proof is to determine a suitable small interval containing the variable point $x=\xi_{n}$ and then let $n$ tend to infinity in order to get the dominant asymptotic value. Write

$$
\begin{equation*}
f_{n}(x)-f_{n}\left(\xi_{n}\right)=-k\left|x-\xi_{n}\right|^{h}\left\{1+R\left(x, \xi_{n}\right)\right\} \tag{4}
\end{equation*}
$$

so that by condition $(G), R\left(x, \xi_{n}\right) \rightarrow 0$ as $(x, n) \rightarrow(\xi, \infty)$. We now have

$$
\begin{align*}
\log f_{n}(x) & =\log f_{n}\left(\xi_{n}\right)+\log \left\{1-k\left|x-\xi_{n}\right|^{h}\left\{f_{n}\left(\xi_{n}^{\prime}\right\}^{-1}(1+R)\right\}\right. \\
& =\log f_{n}\left(\xi_{n}\right)-k\left|x-\xi_{n}\right|^{h}\left\{f_{n}\left(\xi_{n}\right)\right\}^{-1}\left(1+R+R^{\prime}\right) \tag{5}
\end{align*}
$$

say, ${ }^{1}$ where $R^{\prime}=O\left(\left|x-\xi_{n}\right|^{h}\right)$ so that $R^{\prime} \rightarrow 0$ as $(x, n) \rightarrow(\xi, \infty)$. Denote $R+R^{\prime}$ by $\theta\left(x, \xi_{n}\right)$. Then, given any small positive nnmber $\epsilon$, there are positive numbers $\Delta$ and $M$ such that $|\theta|<\epsilon$ whenever $|x-\xi|<2 \Delta$ and $n>M$. We may take $M$ so large that $\left|\xi-\xi_{n}\right|<\Delta$ whenever $n>M$. Thus $\left|x-\xi_{n}\right|<\Delta$ implies $\left|x-\xi_{n}\right|<2 \Delta$. We may also assume $a \leqq \xi_{n}-\Delta, \xi_{n}+\Delta \leqq b$ for $n>M$.

If we now use (5), if we make the change of variable

$$
\begin{equation*}
n k\left\{f_{n}\left(\xi_{n}\right)\right\}^{-1}\left|x-\xi_{n}\right|^{h}=t \tag{6}
\end{equation*}
$$

and if we denote $\theta\left(x, \xi_{n}\right)$ by $\theta_{1}(t)$ fcr $\xi_{n} \leqq x \leqq \xi_{n}+\Delta$ and by $\theta_{2}(t)$, for $\xi_{n}-\Delta \leqq x \leqq \xi_{n}$, we obtain

$$
\begin{align*}
J_{n} & \left.\equiv \int_{\xi_{n}-\Delta}^{\xi_{n}+\Delta}\left\{\frac{f_{n}(x)}{f_{n}\left(\xi_{n}\right)}\right\}^{n} \int \frac{n k}{\left(f_{n}\left(\xi_{n}\right)\right.}\right\}^{1 / h} d x \\
& =\left\{\frac{n k}{f_{n}\left(\xi_{n}\right)}\right\}^{1 / h} \int_{\xi_{n}-\Delta}^{\xi_{n}+\Delta} \exp \left[-n k(1+\theta)\left\{\int^{n}\left(\xi_{n}\right)\right\}^{-1}\left|x-\xi_{n}\right|^{h}\right] d x \\
& \left.=\frac{1}{h} \int_{0}^{T}\left[e^{-t\left[1+\theta_{1}(t]\right.}+e^{-\left\{\left[1+\theta_{2}(t)\right]\right.}\right]\right]^{1 / h-1} d t \tag{7}
\end{align*}
$$

where $T=n k\left\{f_{n}\left(\xi_{n}\right)\right\}^{-1} \Delta^{h}$.
Now suppose that $\epsilon$ and $\Delta$ are fixed. Then $T$, which is not less than $n k \cdot \Delta^{h} / R$, tends to infinity with $n$. Since $\left|\theta_{2}\right|<\epsilon$ and $\left|\theta_{2}\right|<\epsilon$, we see from (7) that
$\frac{2}{\bar{h}}\left(\frac{1}{1+\epsilon}\right)^{1 / h} \int_{0}^{T(1+\epsilon)} e^{-u} u^{1 / h-1} d u \leqq J_{n} \leqq \frac{2}{h}\left(1-\frac{1}{1-\epsilon}\right)^{1 / h} \int_{0}^{T(1-\epsilon)} e^{-u} u^{1 / h-1} d u$.
If we now let $n$ tend to infinity, we find that

$$
\begin{equation*}
\frac{2}{h}\left(\frac{1}{1+\epsilon}\right)^{1 / h} \Gamma\left(\frac{1}{h}\right) \leqq \lim J_{n} \leqq \varlimsup J_{n} \leqq \frac{9}{\bar{h}}\left(\frac{1}{1-\epsilon}\right)^{1 / h} \Gamma\left(\frac{1}{h}\right) \tag{8}
\end{equation*}
$$

[^0]Let us now consider the integral

$$
\begin{equation*}
J_{n}^{*} \equiv \int_{n-\Delta}^{\xi_{n}+\Delta}\left(\frac{f_{n}(x)}{f_{n}\left(\xi_{n}\right)}\right\}^{n}\left\{\frac{n k}{J_{n}\left(\xi_{n}\right)}\right)^{1 / h} g(x) d x . \tag{9}
\end{equation*}
$$

Since $g(x)$ is continuous at $x=\xi$, we may assume that $\Delta$ is chosen so that ${ }^{1}$

$$
g(\xi)(1-\epsilon) \leqq g(x) \leqq g(\xi)(1+\epsilon)
$$

whenever $\left|x-\xi_{n}\right|<\Delta$. Thus from ( 8 ) and (9) we may infer that $\frac{2(1-\epsilon)}{\bar{h}(1+\epsilon)} \Gamma \Gamma\left(\frac{1}{\bar{h}}\right) g(\xi) \leqq \underline{\lim } J_{n}^{*} \leqq \overline{\lim } J_{n}^{*} \leqq \frac{2(1+\epsilon)}{h(1-\epsilon)^{1 / 2}} \Gamma\left(\frac{1}{h}\right) g(\xi)$.

On the other hand, by hypothesis ( $D$ ), there is a positive number $\delta$ (independent of $n$ ) such that $\left|f_{n}(x)\right| \leqq f_{n}\left(\xi_{n}\right)-\delta$ whenever $\left|x-\xi_{n}\right| \geqq \Delta$ and $a \leqq x \leqq b$. Hence for these values of $x$

$$
\left|\frac{f_{n}(x)}{f_{n}\left(\xi_{n}\right)}\right| \leqq\left|\frac{f_{n}\left(\xi_{n}\right)-\delta}{f_{n}\left(\xi_{n}\right)}\right| \leqq\left|1-\frac{\delta}{A}\right|=\rho
$$

say, where $0<\rho<1$. It now follows that, with $\Delta$ fixed,
$J_{n}^{* *}=\left(\int_{a}^{\xi_{n}-\Delta}+\int_{\xi_{n}+\Delta}^{b}\right)\left\{\frac{f_{n}(x)}{\left(f_{n}\left(\xi_{n}\right)\right.}\right)^{n}\left\{\frac{n k}{f_{n}\left(\xi_{n}\right)}\right\}^{1 / h} g(x) d x=O\left(\rho^{n} n^{1 / h}\right) \rightarrow 0$
as $n$ tends to infinity. We may therefore replace $J_{n}{ }^{*}$ by $\left(J_{n}{ }^{*}+J_{n}{ }^{* *}\right)$ in equation (10). Since $\epsilon$ is arbitrary, and since ( $J_{n}{ }^{*}+J_{n}{ }^{* *}$ ) does not depend on $\epsilon$, it now follows that

$$
\lim _{n \rightarrow \infty}\left(J_{n}^{*}+J_{n}^{* *}\right)=\frac{9}{h} \Gamma\left(\frac{1}{h}\right) g(\xi) .
$$

This is equivalent to (3), and our theorem is established.
3. Concrete examples are easily found for illustrating the use of the formula (3). A simple example is, for $n \rightarrow \infty$, $n^{1 / s} \int_{-1 / 2}^{1 / 2}\left(1+\frac{1}{n}-\left|x-\frac{1}{V^{\prime}}\right|^{s}\right)^{n} \sin ^{-1}(1-|x|) d x \sim(1 / s) \Gamma(1 / s) \pi e$, where $s>0$ and $0 \leqq \sin ^{-1} y \leqq \frac{\pi}{2}(0 \leqq y \leqq 1)$. As consequences of our theorem we now mention two important cases as follows:

[^1]I. Levi's case. If $f_{n}(x)=f(x)$, then $\xi_{n}=\xi$ and the equation becomes
\[

$$
\begin{equation*}
\lim _{x \rightarrow \xi}|f(x)-f(\xi)| /|x-\xi|^{n}=k \tag{2}
\end{equation*}
$$

\]

In this case we have

$$
\begin{equation*}
\int_{a}^{b}(f(x))^{n} g(x) d x \sim \frac{2}{h} \Gamma\left(\frac{1}{h}\right)(f(\xi))^{n} g(\xi)\left(\frac{f(\xi)}{n k}\right)^{1 / h} \tag{3}
\end{equation*}
$$

II. Laplace-Darboux case. In the case of Levi, if $f(x)$ is continuous together with its derivatives $f^{\prime}(x), f^{\prime \prime}(x)$ so that $f^{\prime}(\xi)=0, f^{\prime \prime}(\xi)<0$, and ( $\left.\because\right)^{\prime}$ is true for $h=2, k=-\frac{1}{2} f^{\prime \prime}(\xi)$, then (3)' reduces to the classical asymptotic formula of Laplace and Darboux ([2], [3], [4]) :

$$
\begin{equation*}
\int_{a}^{b}(f(x))^{n} g(x) d x \sim(f(\xi))^{n+1 / 2} g(\xi)\left(\frac{-2 \pi}{n f^{\prime \prime}(\xi)}\right)^{1 / 2} \tag{3}
\end{equation*}
$$

4. Two remarks are worthy of mention. (i) In general the constants $h$ and $k$ may always be determined by means of the following equation

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \lim _{x \rightarrow \xi_{n}}\left|f_{n}(x)-f_{n}\left(\xi_{n}\right)\right| /\left|x-\xi_{n}\right|^{n}=k \tag{2}
\end{equation*}
$$

For it is easily seen that (2)* is implied by (2), in view of (4). In the case when $\xi_{n}$ is a constant, ( $\mathbf{I}^{*}{ }^{*}$ and (2)' are equivalent. (ii) If our hypothesis ( $F$ ) is replaced by
$(F)^{*} g(x) \epsilon L(a, b)$ and $g(x)$ possesses limits on both sides of $x=\xi_{n}$ ( $n=1,2,3, \ldots$ ),
then by almost the same treatment as used before we casily obtain

$$
I_{n} \sim(1 / h)\left(g\left(\xi_{n}-\right)+g\left(\xi_{n}+\right)\right)\left(f_{n}\left(\xi_{n}\right)\right)^{n}\left(f_{n}\left(\xi_{n}\right) / n k\right)^{1 / h} \Gamma(1 / h)
$$

## REFERENCES.

[1] Beppo Levi, Publ. Inst., Math. Univ. Nac. Litoral, 6 (1946).
[2] P. S. Laplace, Oeuvres, t. 7, p. 89. Paris (1886).
[3] G. Darboux, Jour. de Math. (3), 4 (1878).
[4] G. Pólya and G. Szegö, Aufgaben und Lehrsälze aus der Analysis, Bd. 1, s. 78 (1925).

Kina's College,
Aberdeen.


[^0]:    ${ }^{1}$ Here we use the fact that $f_{n}\left(\xi_{n}\right)$ cannot tend to zero as $n$ tends to infinity. This follows from condition (D).

[^1]:    ${ }^{1}$ There are slight changes here if $g(\xi)$ is negative; $g(\xi)$ is not zero by hypothesis (F).

