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On an exotic Lagrangian torus in $\mathbb{C}P^2$

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Abstract

We find a non-displaceable Lagrangian torus fiber in a semi-toric system which is superheavy with respect to a certain symplectic quasi-state. The proof employs both 4-dimensional techniques and those from symplectic field theory. In particular, our result implies Lagrangian $\mathbb{R}P^2$ is not a stem in $\mathbb{C}P^2$, answering a question of Entov and Polterovich.

1. Introduction

The primary goal of this paper is to understand a toric degeneration model of $\mathbb{C}P^2$. Our $\mathbb{C}P^2$ model should be considered as a $\mathbb{Z}_2$-equivariant version of the one used in [FOOO12] for $S^2 \times S^2$. However, we take a slightly different point of view, based on symplectic cuts on cotangent bundles of manifolds with periodic geodesics. This degeneration gives a genuine torus action on an open part of $\mathbb{C}P^2$, which results in an interesting family of Lagrangian tori. In particular, such degenerated torus action still gives a moment polytope. For $S^2 \times S^2$, the polytope reads $P_{S^2\times S^2} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geqslant 0, x_1 + 2x_2 \leqslant 2\} \subset \mathbb{R}^2$ (Figure 1, see [FOOO12]). In the case of $\mathbb{C}P^2$, one similarly have a toric action on an open set, which gives a moment polytope as in Figure 2, and can be described as $P_{\mathbb{C}P^2} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geqslant 0, x_1 + 4x_2 \leqslant 4\}$.

In [FOOO12], Fukaya–Oh–Ohta–Ono considered the Floer theory of smooth fibers in the toric degeneration model of $S^2 \times S^2$, proving by bulk deformation that there are uncountably many non-displaceable fibers in Figure 1. In view of Albers–Frauenfelder's result [AF08], we may interpret this result as stating that only non-displaceable torus fibers below the 'monotone level' in the semi-toric system survive the symplectic cut along a level set of T^*S^2. This implies that the anti-diagonal of $S^2 \times S^2$ is not a stem (see § 2 for the definition of a stem), answering a question raised by Entov and Polterovich in [EntP03]. This was also proved independently by several other authors [EP10, CS10]. In [EP10] it was mentioned that Wehrheim also has an unpublished note on this problem.

From Fukaya–Oh–Ohta–Ono's calculation on $S^2 \times S^2$, we expect the similar picture of $\mathbb{C}P^2$ (Figure 2) also contains uncountably many non-displaceable fibers. This would correspond to an easy adaption of Albers–Frauenfelder's result to $T^*\mathbb{R}P^2$, by considering the $\mathbb{Z}_2$-involution induced by antipodal map on S^2.

In this paper we find one smooth non-displaceable monotone torus fiber in the moment polytope of $\mathbb{C}P^2$ described above, and prove that it is superheavy with respect to some symplectic quasi-state. In particular, we prove the following theorem.

Theorem 1.1. *There is a smooth monotone Lagrangian torus fiber in Figure 2, which is superheavy with respect to a certain symplectic quasi-state. In particular, it is stably non-displaceable.*

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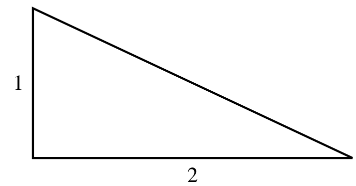

FIGURE 1. $S^2 \times S^2$.

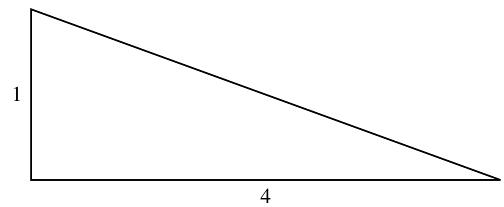

FIGURE 2. $\mathbb{C}P^2$.

The limitation to such a monotone fiber is due to the difficulty of using $\mathbb{R}P^2$ as a bulk to deform our Floer cohomology; compared to the case of [FOOO12], where the bulk is chosen to be an embedded S^2, $\mathbb{R}P^2$ only obtains a non-trivial $\mathbb{Z}_2$-class. But our calculation shows it is essential that we do not use a coefficient ring of characteristic 2 (see § 5); the bulk of $\mathbb{R}P^2$ does not improve our situation in a straightforward way. The author does not know whether this is only technical, although this continuous family is still conceivably non-displaceable (they are, at least, not displaced by any currently known approach). In any case, our computation suffices to show the following.

COROLLARY 1.2. *$\mathbb{R}P^2 \subset \mathbb{C}P^2$ is not a stem. (See the definition of stems from [EntP09].)*

Proof. By [EntP09, Theorem 1.6], a stem is superheavy with respect to any symplectic quasi-states (quasi-morphisms). Since we constructed a quasi-state (and a quasi-morphism) such that our monotone torus is superheavy and the torus is disjoint from $\mathbb{R}P^2$, $\mathbb{R}P^2$ cannot be superheavy with respect to this particular quasi-state (and quasi-morphism), and thus is not a stem. □

This answers the question of Entov and Polterovich [EntP06, Question 9.2] regarding the case of $\mathbb{C}P^2$.

Remark 1.3. It seems possible that our exotic monotone Lagrangian is in fact the Chekanov torus in $\mathbb{C}P^2$. In particular, they both bound four families of disks of Maslov index 2. It would be nice if one could identify the two geometrically, provided the guess is true. Nonetheless, even if such an identification holds, our calculation still gives new information: we would have an identification of a Chekanov torus with a semi-toric fiber and have showed the superheaviness of it. Moreover, our approach is adaptable to more complicated toric degeneration, which will be a topic of upcoming works.

Remark 1.4. Since we work with a monotone Lagrangian torus, our approach does not depend on the deep virtual techniques developed in [FOOO09]. Even for non-monotone torus fibers in our semi-toric picture, we could still avoid virtual techniques since we are in the realm of dimension 4, and hence the geometric genericity arguments as in [FOOO12] are sufficient for our purpose.

The only interesting technical point is that we must use coefficients with characteristic zero, see further discussions in § 5.

Updates. After more than a year since this work was originally submitted in 2012, there have been two updates in this direction. First, Renato Vianna also detected certain exotic Lagrangian tori in $\mathbb{C}P^2$ [Via13]. His tori can be discerned from the one presented here by the families of holomorphic disks with Maslov number 2 that they bound. It would be interesting to compare the two different objects. Much more recently, Oakley and Usher constructed a proof in [OU13] showing that the torus described in the present article is Hamiltonian isotopic to the Chekanov torus. They also generalized the construction to higher dimensions.

2. Preliminaries

The current section summarizes part of the Lagrangian Floer theory developed by Fukaya–Oh–Ohta–Ono [FOOO09, FOOO10a, FOOO11a] etc., as well as the theory of symplectic quasi-states developed by Entov and Polterovich in a series of their works [EntP09, EntP03, EntP06], etc. The aim of this section is to recall basic notions and main framework results of these theories for our applications, as well as for the convenience of readers. Therefore, our scope is rather restricted and will not provide a thorough account of the whole theory. For details and proofs, one is referred to the above-mentioned works. Many of our discussions of Lagrangian Floer theory follow the lines of [FOOO12], especially its arXiv version.

2.1 Lagrangian Floer theory via potential function

Let (M, ω) be a smooth symplectic manifold and $L \subset M$ a relatively spin Lagrangian. This means that L is orientable and that the second Stiefel–Whitney class $w_2(L)$ is in the image of the restriction map $H^2(M, \mathbb{Z}_2) \to H^2(L, \mathbb{Z}_2)$. We first describe the moduli spaces under consideration. Let $J \in \mathcal{J}_\omega$, the space of compatible almost complex structures, and $\beta \in H_2(M, L; \mathbb{Z})$. We denote by $\mathcal{M}^{\text{main}}_{k+1,l}(\beta; M, L; J)$ the space of J-holomorphic bordered stable maps in class β with $k+1$ boundary marked points and l interior marked points. Here, we require the boundary marked points to be ordered counter-clockwise. When no confusion is likely to occur, we will suppress M, L or J.

One of the fundamental results in [FOOO09] shows that one has a Kuranishi structure on $\mathcal{M}^{\text{main}}_{k+1,l}(\beta, L)$ so that the evaluation maps at the ith boundary marked point (jth interior marked point, respectively)

$$ev_i : \mathcal{M}^{\text{main}}_{k+1,l}(\beta, L) \to L,$$

and

$$ev_j^+ : \mathcal{M}^{\text{main}}_{k+1,l}(\beta, L) \to M$$

are weakly submersive (see [FOOO09] for the definition of weakly submersive Kuranishi maps). For given smooth singular simplices $(f_i : P_i \to L)$ of L and $(g_j : Q_j \to M)$ of M, one can also define the fiber product in the sense of Kuranishi structure:

$$\mathcal{M}^{\text{main}}_{k+1,l}(\beta; L; \mathbf{Q}, \mathbf{P}) := \mathcal{M}^{\text{main}}_{k+1,l}(\beta; L)_{(ev_1^+, \ldots, ev_l^+, ev_1, \ldots, ev_k) \times (g_1, \ldots, g_l, f_1, \ldots, f_k)} \left(\prod_{j=1}^{l} Q_j \times \prod_{i=1}^{k} P_i \right).$$

The virtual fundamental chain associated to this moduli space,

$$ev_0 : \mathcal{M}^{\text{main}}_{k+1,l}(\beta; L; \mathbf{Q}, \mathbf{P}) \to L$$

as a singular chain, is defined in [FOOO09] via techniques of virtual perturbations.

We consider the universal Novikov rings:

$$\Lambda = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lambda_i \leqslant \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty \right\},$$
$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geqslant 0}, \lambda_i \leqslant \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty \right\}.$$

Here T is a formal variable. Consider a valuation which assigns $\sigma_T(\sum a_i T^{\lambda_i}) = \lambda_1$ and let $\sigma_T(0) = +\infty$. This induces an $\mathbb{R}$-filtration on Λ and Λ_0, and thus a non-Archimedean topology. Note that $\Lambda_0 \subset \Lambda$, and Λ_0 has a maximal ideal Λ_+ consisting of elements with $\lambda_i > 0$ for all i. The absence of e-variable will reduce the grading of Floer cohomology groups to $\mathbb{Z}_2$, but this is irrelevant to our applications.

The heart of Fukaya–Oh–Ohta–Ono's work is to define a filtered A_∞-structure on $C^*(L; \Lambda_0)$ for a Lagrangian $L \subset M$ and define Floer cohomology by deformations of (weak) bounding cochains. However, we will not mention the explicit constructions here, both because they are far beyond our scope, and because there are plenty of comprehensive references and surveys available in the literature. (For an incomplete list: [FOOO09, FOOO10a, FOOO11a, FOOO10b], etc.) Instead we will adopt a most economic approach towards the applications in mind, by recalling a package made available by the deep theory; namely, the computations on Lagrangian Floer cohomology via potential functions.

A *potential function* $\mathfrak{PO}^L$ is a Λ_+-valued function defined on the set of *weak bounding cochains* of L, denoted as $\widehat{\mathcal{M}}_{\text{weak}}(L)$. For any $b \in \widehat{\mathcal{M}}_{\text{weak}}(L)$, one may associate a Floer cohomology group $HF^*(L, b)$ for the pair (L, b). We do not define the weak bounding cochains in general; however, according to [FOOO12, Theorem A.2], by passing to the canonical model, $\widehat{\mathcal{M}}_{\text{weak}}(L)$ can be identified with $H^1(L; \Lambda_0)/H^1(L; 2\pi\sqrt{-1}\mathbb{Z})$ for any monotone Lagrangian submanifolds with minimal Maslov number equal to 2. Hence, in the rest of this paper, $\widehat{\mathcal{M}}_{\text{weak}}(L)$ will refer to this particular set, and the potential function can be written as

$$\mathfrak{PO}^L : H^1(L; \Lambda_0)/H^1(L; 2\pi\sqrt{-1}\mathbb{Z}) \to \Lambda_+. \tag{2.1}$$

With the monotonicity assumption above, one may compute $\mathfrak{PO}$ explicitly as in [FOOO12, Theorems A.1 and A.2]. Choose a basis $\{e_i\}_{i=1}^n$ for $H^1(L; \mathbb{Z})$ and represent $b = \sum_{i=1}^n x_i e_i$ for $b \in H^1(L; \Lambda_0)$ and $x_i \in \Lambda_0$. Then the potential function is written as

$$\mathfrak{PO}^L(b) = \sum_{\mu(\beta)=2} ev_0^*([\mathcal{M}_1(L; J, \beta)]) T^{\omega(\beta)/2\pi} \exp(b(\partial\beta)). \tag{2.2}$$

Here $[\partial\beta] \in H_1(L; \mathbb{Z})$, hence $b(\partial\beta) \in \Lambda_0$. Writing in coordinates, the potential function can be regarded as a function from $(\Lambda_0/2\pi\sqrt{-1}\mathbb{Z})^n$ to Λ_+. A change of coordinate $y_i = e^{x_i}$ transforms the function in (2.2) into the form commonly used in the literature (and thus changing the domain from $(\Lambda_0/2\pi\sqrt{-1}\mathbb{Z})^n$):

$$\begin{aligned} \mathfrak{PO}^L &: (\Lambda_0 \backslash \Lambda_+)^n \to \Lambda_+, \\ (y_1, \dots, y_n) &\mapsto \sum_{\mu(\beta)=2} ev_0^*([\mathcal{M}_1(L; J, \beta)]) T^{\omega(\beta)/2\pi} \prod_{i=1}^n y_i^{l_i}, \end{aligned} \tag{2.3}$$

where $\partial\beta = \sum_{i=1}^n l_i e_i^*$, for $\{e_i^*\}_{i=1}^n$ a dual basis in $H_1(L; \mathbb{Z})$. The following result manifests the importance of potential functions.

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Theorem 2.1 [FOOO12, Theorem 2.3]. *Let L be a Lagrangian torus in (M,ω). Suppose that $H^1(L;\Lambda_0)/H^1(L;2\pi\sqrt{-1}\mathbb{Z}) \subset \widehat{\mathcal{M}}_{\text{weak}}(L)$ and $b \in H^1(L;\Lambda_0)$ is a critical point of the potential function $\mathfrak{PO}^L$ of L. Then we have*

$$HF^*(L,b) \cong H^*(L;\Lambda_0).$$

In particular, L is non-displaceable.

Remark 2.2. Results stated in this section hold valid for Lagrangians satisfying [FOOO12, Condition 6.1], that is, if any non-empty moduli space of holomorphic disks has Maslov index $\geqslant 2$. For monotone Lagrangians, there is also an alternative approach developed by Biran and Cornea [BC09] via pearl complexes, which was used in Vianna's work [Via13] and that of many others. If one is willing to go further into virtual perturbation theories, it is shown in [FOOO10a] that the results holds true for toric fibers.

2.2 Symplectic quasi-states and Lagrangian Floer theory

In this section we briefly review the theory of symplectic quasi-states developed by Entov and Polterovich. A *symplectic quasi-state* is a functional $\zeta : C^\infty(M) \to \mathbb{R}$ satisfying the following axioms for $H, K \in C^\infty(M)$ and $\lambda \in \mathbb{R}$:

(i) (Normalization). $\zeta(1) = 1$;
(ii) (Monotonicity). If $H \leqslant K$, then $\zeta(H) \leqslant \zeta(K)$;
(iii) (Quasi-linearity). If $\{H, K\} = 0$, then $\zeta(H + \lambda K) = \zeta(H) + \lambda\zeta(K)$;
(iv) (Vanishing). If $\mathrm{supp}(H)$ is displaceable, then $\zeta(H) = 0$;
(v) (Symplectic invariance). $\zeta(H) = \zeta(H \circ f)$ for $f \in \mathrm{Symp}_0(M)$.

Given a symplectic quasi-state ζ and a subset $S \subset M$, S is called ζ-*heavy* if

$$\zeta(F) \geqslant \inf_{x\in S} F(x), \quad \text{for all } F \in C^\infty(M),$$

and ζ-*superheavy* if

$$\zeta(F) \leqslant \sup_{x\in S} F(x), \quad \text{for all } F \in C^\infty(M).$$

One of the basic properties of these subsets proved in [EntP03] is that a ζ-superheavy subset is always ζ-heavy, and a ζ-heavy set is stably non-displaceable (this is a notion strictly stronger than non-displaceability). Let $V \subset C^\infty(M)$ be a finite-dimensional linear subspace spanned by pairwisely Poisson-commuting functions. Let $\Psi : M \to V^*$ be the moment map defined by $\langle \Psi(x), F\rangle = F(x)$ for $F \in V$. A non-empty fiber of this moment map is called a *stem* if the rest of the fibers are all displaceable. The following theorem was essentially proved in [EntP03, Theorem 1.6].

Theorem 2.3 [EntP09]. *A stem is a superheavy subset with respect to arbitrary symplectic quasi-states.*

In general, the existence of symplectic quasi-states is already an intriguing question. In [EntP08] it is shown that, given a direct sum decomposition of $QH_{2n}(M) = \mathbb{F} \oplus QH'$, where $\mathbb{F}$ is a field, one may associate a symplectic quasi-state ζ_e to the unit element $e \in \mathbb{F}$.

The relations between symplectic quasi-states and Lagrangian Floer theory are established by the *i-operator* (sometimes also referred to in the literature as the open–closed string maps or the Albers map). The version of i-operator we need involves the deformation by a weak bounding cochain b, thus it will be denoted by

$$i_b^*: QH^*(M) \to HF(L, b).$$

The concrete definition of i_b^* was given in [FOOO09], and we refer interested readers there for details (see also [BC09] for a similar operator in the context of pearl complexes). The key property of i_b^* we need is given in the following proposition.

Proposition 2.4. *i_b^* is a ring homomorphism which sends the unit of $QH^*(M)$ to that of $HF(L, b)$.*

This fact was shown in [FOOO09, 7.4.2–7.4.6], which passes to the so-called *canonical model* of $C^*(L; \Lambda_0)$ and involved deep algebraic techniques in filtered A_∞ algebras; therefore it is beyond the scope of the present paper.

With this understood, our proof will rely on the following key results.

Theorem 2.5 [FOOO11b, Theorem 18.8]. *Let L be a relatively spin Lagrangian submanifold of M, $b \in \widehat{\mathcal{M}}_{\text{weak}}(L)$ be a weak bounding cochain, and $e \in QH^*(M; \Lambda)$.*

(1) *If $e \cup e = e$ and $i_b^*(e) \neq 0$, then L is ζ_e-heavy.*

(2) *If $QH^*(M; \Lambda) = \Lambda \oplus Q$ is a direct factor decomposition as a ring, and e comes from a unit of the factor Λ which satisfies $i_b^*(e) \neq 0$, then L is ζ_e-superheavy.*

Corollary 2.6. *Suppose $QH^*(M; \Lambda) = \bigoplus_{i=1}^n \Lambda e_i$ as a ring, for $e_i \in QH^*(M; \Lambda)$ being a series of idempotents (in particular QH^* is semi-simple). If $HF^*((L, b); \Lambda) \neq 0$, then L is superheavy for a certain symplectic quasi-state ζ_{e_k}, $1 \leqslant k \leqslant n$.*

Proof. This is implicit from the proof of [FOOO11b, Theorem 23.4]. Since i_b^* sends the unit to the unit, at least one of the idempotents e_k has non-vanishing image. From Theorem 2.5, L is superheavy. □

Combining Corollary 2.6, Theorem 2.1 and (2.3), provided we have a semi-simple quantum cohomology ring for the ambient manifold M, to show a monotone Lagrangian torus is superheavy with respect to certain symplectic quasi-state, it suffices to compute the contribution of each moduli space of holomorphic disks of Maslov index 2 and find the critical points for the potential function, which will be the topic of subsequent sections.

3. A semi-toric system of $\mathbb{C}P^2$

3.1 Description of the system

We first briefly recall the semi-toric model for $S^2 \times S^2$ following the idea of [EntP09, Sei98]. Write $S^2 \times S^2$ as

$$\{x_1^2 + y_1^2 + z_1^2 = 1\} \times \{x_2^2 + y_2^2 + z_2^2 = 1\} \subset \mathbb{R}^3 \times \mathbb{R}^3.$$

Let

$$\widetilde{F}(x_1, y_1, z_1; x_2, y_2, z_2) = z_1 + z_2,$$

$$\widetilde{G}(x_1, y_1, z_1; x_2, y_2, z_2) = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^2};$$

then

$$\Phi_{S^2 \times S^2} = (F, G) := (\tfrac{1}{2}(\widetilde{F} + \widetilde{G}), \tfrac{1}{2}(2 - \widetilde{G})) : S^2 \times S^2 \to \mathbb{R}^2$$

defines a Hamiltonian system. $\widetilde{G}$ is not integrable when it equals 0, that is, at the anti-diagonal $\bar{\Delta}$. This Hamiltonian system gives a moment polytope as in Figure 1 up to a rescale of the symplectic form, with a singularity at $(0,1)$ representing a Lagrangian sphere. In classical terms, this is in fact a moment polytope for $S^2\times S^2\backslash\bar{\Delta}$, where any tubular neighborhood $N(\bar{\Delta})$ of $\bar{\Delta}$ will be mapped into to a neighborhood of $(1,0)$.

Another useful point of view is to consider S^2 equipped with the standard round metric, which induces a metric on its cotangent bundle. $S^2\times S^2$ is obtained from T^*S^2 by a symplectic cut at the hypersurface

$$M_1=\{p\in T^*S^2:|p|=1\},$$

that is, we collapse the orbits formed by the geodesic flow. The circle action on this hypersurface is exactly the unit-speed geodesic flow we use for cutting. See [Ler95] for details of the construction of symplectic cuts. This way we also identify naturally cotangent vectors of length $\leqslant 1$ in T^*S^2 with $S^2\times S^2\backslash\Delta$. In this perspective, we may describe the $\mathbb{T}^2$-action induced by Φ in a geometric way. Consider the rotation of S^2 along an axis. The cotangent map of this rotation generates the circle action $\tau_{\widetilde{F}}$ on the whole T^*S^2. Another circle action $\tau_{\widetilde{G}}$ is generated by the unit geodesic flow on the complement of the zero section (we already used it for symplectic cut above). Both $\tau_{\widetilde{F}}$ and $\tau_{\widetilde{G}}$ descend under the symplectic cut and commute, thus inducing a genuine $\mathbb{T}^2$-action on $S^2\times S^2\backslash\bar{\Delta}$.

We proceed to the case for $\mathbb{C}P^2$. Consider the $\mathbb{Z}_2$-action on T^*S^2 induced by the antipodal map on the zero section. It is readily seen that the symplectic cut at the level set M_1 is also $\mathbb{Z}_2$-equivariant, so we may well quotient out this $\mathbb{Z}_2$ action first and then perform the symplectic cut. This is equivalent to performing the symplectic cut on $T^*\mathbb{R}P^2$, which results in a symplectic $\mathbb{C}P^2$. In summary we have the following commutative diagram, which is equivariant with respect to the action of $\tau_{\tilde{F}}$ and $\tau_{\tilde{G}}$.

$$\begin{tikzcd}
T^*_{\leqslant 1}S^2 \arrow[r, hook] \arrow[d, "\pi"] & S^2\times S^2 \arrow[d, "\iota"] \\
T^*_{\leqslant 1}\mathbb{R}P^2 \arrow[r, hook] & \mathbb{C}P^2
\end{tikzcd}$$

Here π is the 2-to-1 cover over $T^*\mathbb{R}P^2$ and $T^*_{\leqslant 1}S^2$ and $T^*_{\leqslant 1}\mathbb{R}P^2$ denotes cotangent vectors of length $\leqslant 1$, and ι is the standard 2-fold branched cover from $S^2\times S^2$ to $\mathbb{C}P^2$, branching along the diagonal. Notice now both $\tau_{\tilde{F}}$ and $\tau_{\tilde{G}}$ are $\mathbb{Z}_2$-equivariant under the deck transformation, therefore, the above 2-fold cover induces two commuting circle action on $\mathbb{C}P^2\backslash\mathbb{R}P^2$. However, since the $\mathbb{Z}_2$-action halves the length of each geodesic, to get a time-1 periodic flow, the Hamiltonian function generating the circle action descended from $\tau_{\tilde{G}}$ should be the descendant of $\frac{1}{2}\tilde{G}$ on $S^2\times S^2$. The end result after appropriate reparameterizations is a toric model for $\mathbb{C}P^2\backslash\mathbb{R}P^2$ with moment polytope as in Figure 2. Note from the reasoning regarding $\tilde{G}$, after the reparameterization the line area of $\mathbb{C}P^2$ is 2 (if the line class area had been 1, the sizes in Figure 2 would have been $\frac{1}{2}$ by 2). Similar to the case of $S^2\times S^2$, $(1,0)$ indeed represents the standard Lagrangian $\mathbb{R}P^2\subset\mathbb{C}P^2$. We will denote this semi-toric moment map as $\Phi_{\mathbb{C}P^2}$.

3.2 Symplectic cutting $\mathbb{C}P^2$

The main ingredient of our proof, following an idea of the arXiv version of [FOOO12], is to split $\mathbb{C}P^2$ into two pieces and glue the holomorphic curves. The splitting we use is described as follows. We continue to regard $\mathbb{C}P^2$ as a result of cutting along M_1 in $T^*\mathbb{R}P^2$. Consider $M_\epsilon=\{|p|=\epsilon\}\subset T^*\mathbb{R}P^2\hookrightarrow\mathbb{C}P^2$. A further symplectic cut along M_ϵ results in two pieces, and we examine this cutting in slightly more detail.

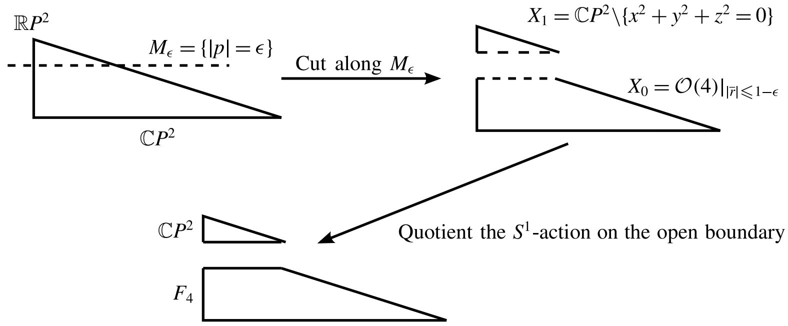

FIGURE 3. Cutting along M_ϵ.

Let X_0, X_1 be the two components of $\mathbb{C}P^2 \backslash M_\epsilon$, where X_1 contains the original $\mathbb{R}P^2$. Their closures, denoted X_0' and X_1', respectively, have a boundary being the lens space $L(4, 1)$ equipped with the standard contact form (the one coming from S^3 quotiented by a $\mathbb{Z}_4$-action). Therefore a neighborhood of the boundary admits a local S^1-action.

As is constructed in [Ler95], by quotienting such an action on $\partial X_1'$ and gluing back to X_1', one completes the symplectic cutting and this operation results in $X_1'' := (\mathbb{C}P^2, 2\epsilon\omega_0)$. Denote by $H \in H_2(\mathbb{C}P^2, \mathbb{Z})$ the homology class of a line; then $X_1'' \backslash X_1$ is an embedded symplectic divisor in X_1'' of class $2H$, which we called the *cut locus* or *cut divisor*. Following the same procedure for the other piece X_0 leads to a minimal symplectic 4-manifold (see for example [Dor13, Lemma 1.1]), along with a symplectic sphere of self-intersection $(+4)$ inherited from the quadric $Q := \{x^2 + y^2 + z^2 = 0\}$ in the original $\mathbb{C}P^2$. Moreover, it contains a symplectic sphere of self-intersection (-4) as the cut locus, from which we see that X_0'' is indeed the symplectic fourth Hirzebruch surface F_4 (meaning a symplectic $S^2 \times S^2$ with (-4)-divisor) by McDuff's famous classification of rational and ruled manifolds [McD90].

We also want to examine such a cut process from the other side of M_1. Biran's decomposition theorem for $\mathbb{C}P^2$ [Bir01] implies that $\mathbb{C}P^2 \backslash \mathbb{R}P^2$ is indeed a symplectic disk bundle $\mathcal{O}(4)$ over a sphere, where the zero section has symplectic area 4, and the symplectic form is given by $\pi^*\omega_\Sigma + d(\bar{r}^2\alpha)$. Here π is the projection to the zero section, ω_Σ a standard symplectic form on the sphere up to a rescale, $\bar{r}$ the radial coordinate of the fiber and α a connection form of the circle bundle associated to $\mathcal{O}(4)$. Then the fiber class has at most symplectic area 1, and the total space can be identified symplectically with $\mathbb{C}P^2 \backslash \mathbb{R}P^2$ with the standard symplectic form.

In this case X_0' is identified with $\{|\bar{r}| \leqslant 1 - \epsilon\} \subset \mathcal{O}(4)$ and the geodesic flow in $T^*\mathbb{R}P^2$ is identified with the action of the one obtained by multiplying $e^{i\theta}$ in each fiber. Therefore, one may perform a symplectic cut along $|\bar{r}| = 1 - \epsilon$ for $1 \gg \epsilon > 0$, the resulting manifold is again the symplectic F_4 as above, where the form is compatible with the standard (integrable) complex structure obtained as $P(\mathcal{O} \oplus \mathcal{O}(4))$. To summarize, we have the following lemma (see also Figure 3).

LEMMA 3.1. *Consider $\mathbb{C}P^2$ as a consequence of a symplectic cut along the contact type hypersurface $\{|p| = 1\} \subset T^*\mathbb{R}P^2$. Then a further symplectic cut along $\{|p| = \epsilon\}$ results in a $\mathbb{C}P^2$ with rescaled symplectic form, as well as a symplectic fourth Hirzebruch surface whose zero section has symplectic area equal to 2. Moreover, the symplectic $\mathbb{C}P^2$ comes naturally with a symplectic quadric as the cut locus, and F_4 with a (-4)-sphere as the cut locus.*

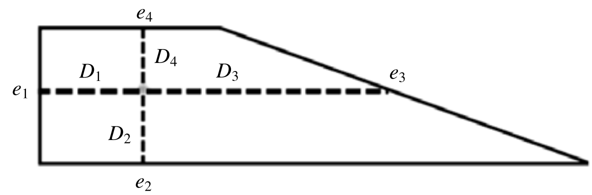

FIGURE 4. $H_2(F_4, L; \mathbb{Z})$.

Remark 3.2. Discussions above seem to be well known. For a dual perspective via symplectic fiber sum, one is referred to, for example, [Dor13]. In particular, the above cutting can be seen as a reverse procedure of symplectic rational blow-down of the (-4)-sphere in the symplectic F_4.

3.3 Second homology classes of $\mathbb{C}P^2$ with boundary on a semi-toric fiber

From § 3.1, we have obtained a desired family of Lagrangian tori as semi-toric fibers in $\mathbb{C}P^2$. From now on L will denote one of the semi-toric fibers parameterized by $\mathbb{R}^2$-coordinates in Figure 2. Our next task is to understand $H_2(\mathbb{C}P^2, L)$. In our exposition we will use *Borel–Moore homology*, for which one can find a comprehensive treatment in [CG97]. However, it is instructive to point out that in this article we only consider Borel–Moore homology for symplectic manifolds with cylindrical ends. In such cases, Borel–Moore homology is simply a convenient terminology, equivalent to *usual homology relative to cylindrical ends*. When we consider Borel–Moore homology classes relative to a Lagrangian L, this means the homology relative to both cylindrical ends and L.

From the usual long exact sequence for relative homology, one easily sees that $H_2(\mathbb{C}P^2, L; \mathbb{Z})$ has rank 3. Again split $\mathbb{C}P^2$ along M_ϵ into a copy of F_4 and $(\mathbb{C}P^2, \epsilon\omega_{\mathrm{std}})$ as in § 3.2, while keeping $L \subset F_4$ by choosing ϵ small enough.

From the classification theorem of homology classes in [CO06], one obtains eight homology classes of interests, marked as $[D_i]$ and $[e_i]$, $i = 1, 2, 3, 4$ in Figure 4. In the figure, e_i are the $\mathbb{T}^2$-equivariant divisors and, as relative cycles, D_i denote the images of J_0-holomorphic disks which intersect e_j exactly δ_{ij} times counting multiplicity. For ease of drawing we did not draw D_3 perpendicular to e_3, but it is understood in the way Cho and Oh describe in [CO06].

Out of these eight classes, one has a basis of $H_2(F_4, L; \mathbb{Z})$ consisting of $[e_1]$, $[e_2]$, $[D_1]$ and $[D_2]$. Other classes have relations

$$[e_3] = [e_1], \quad [e_4] = [e_2] - 4[e_1], \tag{3.1}$$

$$[D_1] + [D_3] + 4[D_2] = [e_2], \quad [D_2] + [D_4] = [e_1]. \tag{3.2}$$

One way of checking these relations is to use Poincaré pairings and gluing chains with opposite boundaries on L. Notice that there is a natural homomorphism to the Borel–Moore homology of $F_4 \backslash e_4$:

$$\iota : H_2(F_4, L; \mathbb{Z}) \to H_2^{BM}(F_4 \backslash e_4, L; \mathbb{Z}).$$

Here ι is indeed the composition of two homomorphisms: the first from $H_2(F_4, L; \mathbb{Z})$ to $H_2(F_4, L \cup \mathcal{N}(e_4); \mathbb{Z})$ by quotienting a tubular neighborhood $\mathcal{N}(e_4)$ of e_4, and the second from the excision isomorphism $H_2(F_4, L \cup \mathcal{N}(e_4); \mathbb{Z})$ to $H_2(F_4 \backslash \mathcal{N}'(e_4), L \cup \mathcal{C}; \mathbb{Z})$, where $\mathcal{N}'(e_4)$ is a smaller tubular neighborhood contained in $\mathcal{N}(e_4)$, and $\mathcal{C} = \mathcal{N}(e_4) \backslash \mathcal{N}'(e_4)$ is a cylindrical end in $F_4 \backslash \mathcal{N}(e_4)$. The latter is by definition a Borel–Moore homology relative to L, which by homotopy invariance of usual relative homology is exactly what is on the target of ι, since $F_4 \backslash e_4$ is homotopy equivalent to $F_4 \backslash \mathcal{N}(e_4)$. In this special case it can be intuitively understood as restricting a singular chain to the portion that is inside $F_4 \backslash e_4$.

ι is surjective with kernel $[e_4]$, so the classes in $H_2^{BM}(F_4\backslash e_4, L; \mathbb{Z})$ can still be represented by e_i, $i = 1, 2, 3$ and D_j, $j = 1, 2, 3, 4$ appropriately punctured with the same relations as in (3.1), (3.2). These facts can be easily seen from the duality between the Borel–Moore homology and the usual cohomology.

On the other side of the cutting, which is $\mathbb{C}P^2\backslash Q$, where $Q = \{x^2+y^2+z^2 = 0\}$ is the standard quadric, the second Borel–Moore homology contains only a 2-torsion: this is indeed the relative homology group $H_2(\mathbb{C}P^2, Q; \mathbb{Z})$. We will only consider Borel–Moore cycles with asymptotics equal to a union of certain Reeb orbits of $\partial^\infty(\mathbb{C}P^2\backslash Q) = L(4, 1)$. Regard Borel–Moore cycles with $2k$ punctures (number of Reeb orbits) at infinity as equivalent, and denote such equivalence classes by kH'. Note that k already contains information of the homology classes: cycles in k_1H' and k_2H' represent the same Borel–Moore classes in $H_2(\mathbb{C}P^2\backslash Q, \partial^\infty(\mathbb{C}P^2\backslash Q))$ if and only if $k_1 - k_2 \equiv 0 \bmod 2$, but the relative Chern number will depend on the actual equivalence classes instead of solely the Borel–Moore classes. See § 3.4.

Relations between classes in $X_0 = F_4\backslash e_4$ and $X_1 = \mathbb{C}P^2\backslash Q$

To describe the relations between classes in the two pieces, we first fix a basis of $H_2^{BM}(F_4\backslash e_4, L; \mathbb{Z})$ consisting of $\{\iota[e_2], \iota[D_1], \iota[D_2], \iota[D_4]\}$. When no possible confusion occurs, we will simply suppress ι by abuse of notation. Notice that cycles in kH' in X_1 have $2k$ punctures counting multiplicity, which matches with cycles with coefficient $2k$ in the D_4-component in X_0. Of particular interest is that by matching a cycle $C_{H'} \subset X_1$ in class H' with a 2-cycle of class $2[D_4]$ with correct asymptotics, one obtains a relative cycle in $\mathbb{C}P^2$ with boundary on L. The class in $H_2(\mathbb{C}P^2, L; \mathbb{Z})$ represented by such a cycle is denoted by $[D_4'] = 2[D_4]\#[H']$.

To understand $[D_4']$ more explicitly, notice that $\partial[D_4'] = 2\partial[D_4] = -2\partial[D_2] \in H_1(L; \mathbb{Z})$. Therefore, one may match a cycle in class $[D_4']$ with one in $2[D_2]$ to obtain a closed cycle in $\mathbb{C}P^2$. Such a cycle intersects e_2 positively twice counting multiplicities, and therefore represents nothing but the line class in projective space $H \in H_2(\mathbb{C}P^2; \mathbb{Z})$. In summary, we deduce that

$$[H']\#2[D_4]\#2[D_2] = [D_4']\#2[D_2] = H \in H_2(\mathbb{C}P^2; \mathbb{Z}). \tag{3.3}$$

Classes $[e_1]$ and $[e_3]$ do not extend naturally to closed classes as in $H_2(\mathbb{C}P^2)$. However, using the same method as for $[D_4]$, taking them twice caps cycles in H' of X_1. Therefore we also have

$$[H']\#2[e_1] = [H']\#2[e_3] = H \in H_2(\mathbb{C}P^2; \mathbb{Z}). \tag{3.4}$$

These gluing relations will play an important role later. It is also readily seen that $\{H, [D_1], [D_2]\}$ forms a basis of $H_2(\mathbb{C}P^2, L; \mathbb{Z})$, where $F_4\backslash e_4$ is (symplectically) embedded to $\mathbb{C}P^2$ in a canonical way, thus inducing a natural inclusion of Borel–Moore 2-cycles.

3.4 Computation of the relative Chern numbers and Conley–Zehnder indices

We now compute the Maslov indices for $H_2(\mathbb{C}P^2, L)$ by understanding the relative Chern classes and Conley–Zehnder indices involved. A technical reason for our case being slightly more complicated than the case of a Lagrangian S^2 is that there is no natural splitting of $T(T^*\mathbb{R}P^2)$. This is caused by the non-orientability of $\mathbb{R}P^2$ (to compare the case of Lagrangian S^2, see for example [Hin04, Eva10b, LW12]). However, we will use a trivialization of the splitting surface M_ϵ which seems even more natural and convenient in the (semi-)toric context. From § 3.2, we remind the reader that X_i, $i = 0, 1$ denotes the two components of the complement of $M_\epsilon \subset \mathbb{C}P^2$ and X_i' represents their closures.

As we already saw, there is an S^1-action on $\partial X_i' = M_\epsilon$ for both $i = 0, 1$. In the toric picture of $\mathbb{C}P^2\backslash\mathbb{R}P^2$, such an S^1-action induces a vector field on M_ϵ which is dual to $\partial/\partial x_2$ in the moment polytope. This action induces a natural trivialization of the contact distribution over its own orbits. We will call such a trivialization Φ and use it to compute the Conley–Zehnder indices and first Chern numbers. For the definitions of these two invariants one is referred to [EGH00], or [Eva10b, Hin04].

By definition, the Poincaré return map with respect to such a trivialization is always identity, therefore,

$$\mu_{\mathrm{CZ}}^{\Phi} \equiv 0. \tag{3.5}$$

We will pursue the first Chern number for (Borel–Moore) classes described in § 3.3 in the rest of the section.

We start with X_0. As always, we assume the Lagrangian torus fiber is contained in this side. Consider again the $\mathcal{O}(4)$ disk bundle as in § 3.2, from which we cut along another hypersurface $M_{\epsilon/2} = \{r = 1 - \epsilon/2\}$ to obtain a symplectic fourth Hirzebruch surface $\overline{X}_0$. One may also equip it with a compatible toric complex structure. The anti-canonical divisor is defined by the equivariant divisors on the boundary of the moment polytope; therefore, the anti-canonical line bundle $\bigwedge^2 T\overline{X}_0$ admits an equivariant section ξ vanishing exactly on the boundary equivariant divisors with order 1.

Embed X_0 equivariantly into $\overline{X}_0$. Take any cycle $u : \Sigma \to X_0$ with boundary on a torus fiber L and asymptotics being Reeb orbits of ∂X_0 with additional assumption that it intersects the toric boundary of $\overline{X}_0$ transversally. It has boundary Maslov index zero if we take the trivialization induced by the torus action near L. Assume that u intersects transversally with the equivariant divisors. The pull-back $u^* \bigwedge^2(TX_0, J)$ thus comes naturally with a section $u^*\xi$ which vanishes at the inverse image of $u(\Sigma) \cap \bigcup_{i=1}^4 e_i$ with order ± 1 depending on the intersection form. $u^*\xi$ is clearly equivariant with the S^1-action on ∂X_0 and the torus boundary thus agreeing with the trivialization there. This observation immediately computes the following:

$$c_1^{\Phi}(D_1) = c_1^{\Phi}(D_2) = c_1^{\Phi}(D_3) = 1, \quad c_1^{\Phi}(D_4) = 0. \tag{3.6}$$

We would like to remind the reader that D_4 here represents a Borel–Moore homology class of X_0, thus has no intersection with the relevant equivariant divisors ($X_0 \cong F_4 \backslash e_4$). Notice also that the first Chern number of e_2 is independent of the choice of trivializations. From (3.1) and (3.2) we may compute the rest of the Chern numbers, summarized as follows:

$$c_1^{\Phi}(e_1) = c_1^{\Phi}(e_3) = 1, \quad c_1^{\Phi}(e_2) = 6. \tag{3.7}$$

For the relative Chern classes in X_1, we again focus on cycles with asymptotics equalling copies of S^1-orbits on M_ϵ. Note that, when counting multiplicity, there are always an even number of S^1-orbits since simple orbits represent a non-trivial element in $\pi_1(T^*\mathbb{R}P^2)$. The class kH' has $2k$ asymptotics, which can be capped by $2k[D_4]\#2k[D_2]$ to form a closed cycle in $\mathbb{C}P^2$ from (3.2). Such a class intersects positively with e_2 at $2k$ points, thus itself being the class kH in $\mathbb{C}P^2$. From our computation in X_0, we see that

$$c_1^{\Phi}(kH') = 3k - 2k = k > 0. \tag{3.8}$$

4. Classification of Maslov 2 disks

4.1 A quick review of symplectic field theory and neck-stretching

In this section we collect basic definitions and facts from symplectic field theory (SFT), especially the part of neck-stretching, mostly for the reader's convenience and to fix notation.

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For more details, we refer interested readers to [EGH00, BEHWZ03] and other expositions such as [Eva10b, Hin04, LW12].

Given a closed symplectic manifold (M,ω), we call (N,α) a *contact type hypersurface* if there is a neighborhood V of N such that V is diffeomorphic to $(-\epsilon,\epsilon)\times N$, and ∂_s is a Liouville vector field in V, that is, $\mathcal{L}_{\partial_s}\omega=\omega$. Here s is the coordinate of the first component of V. In this case, $\alpha=i_{\partial_s}\omega$ is a contact form, of which the contact distribution is denoted by ξ, and the Reeb flow is denoted by R.

An almost complex structure $J\in\mathcal{J}_\omega$ is called *adjusted* if the following conditions hold in V:

(i) $J|_\xi=\widetilde{J}$ is independent of s;

(ii) $J(\partial_s)=R$.

We now consider a deformation of a given adjusted almost complex structure J. Let $V_t=[-t-\epsilon, t+\epsilon]$ and $\beta_t:V_t\to[-\epsilon,\epsilon]$ be a strictly increasing function with $\beta_t(s)=s+t$ on $[-t-\epsilon,-t-\epsilon/2]$ and $\beta_t(s)=s-t$ on $[t+\epsilon/2,t+\epsilon]$. Define a smooth embedding $f_t:V_t\times N\hookrightarrow M$ by

$$f_t(s,m)=(\beta_t(s),m).$$

Let $\bar{J}_t$ be the $\partial/\partial s$-invariant almost complex structure on $V_t\times N$ such that $\bar{J}_t(\partial/\partial s)=R$ and $\bar{J}_t|_\xi=J|_\xi$. Glue the almost complex manifold $(M\backslash f_t(V_t\times N),J)$ to $(V_t\times N,\bar{J}_t)$ via f_t to obtain the family of almost complex structures J_t on M.

Notice that each J_t agrees with J away from the collar $(-\epsilon,\epsilon)\times N$. And on this collar, it agrees with J on ξ. Suppose N is separating, and let $M\backslash N=W\cup U$, where W has a concave boundary and U a convex boundary. When $t\to\infty$, the neck-stretching process results in an almost complex structure J_∞ on the union of symplectic completions $\overline{W}=(-\infty,0]\times N\cup W$ of W and $\overline{U}=U\cup[0,+\infty)$ of U. On the cylindrical ends, we require $J|_\infty(\partial_s)=R$ and $J|_\infty=J|_\xi$ similar to the definition of $\overline{J}_t$. In the exact same way, we define J_∞ on $SN=((-\infty,+\infty)\times N,d(e^t\alpha))$, the symplectization of N.

Let $M_\infty=\overline{W}\cup SN\cup\overline{U}$ and J_∞ be the almost complex structure defined above. Let Σ be a Riemann surface with nodes. A *level-k holomorphic building* consists of the following data.

(i) (Level) A labelling of the components of $\Sigma\backslash\{\text{nodes}\}$ by integers $\{1,\dots,k\}$ which are the *levels*. Two components sharing a node differ at most by 1 in levels. Let Σ_r be the union of the components of $\Sigma\backslash\{\text{nodes}\}$ with label r.

(ii) (Asymptotic matching) Finite energy holomorphic curves $v_1:\Sigma_1\to U$, $v_r:\Sigma_r\to SN$, $2\leqslant r\leqslant k-1$, $v_k:\Sigma_k\to W$. Any node shared by Σ_l and Σ_{l+1} for $1\leqslant l\leqslant k-1$ is a positive puncture for v_l and a negative puncture for v_{l+1} asymptotic to the same Reeb orbit γ. v_l should also extend continuously across each node within Σ_l.

Now for a given stretching family $\{J_{t_i}\}$ as previously described, as well as J_{t_i}-curves $u_i:S\to(M,J_{t_i})$, we define the Gromov–Hofer convergence as follows.

A sequence of J_{t_i}-curves $u_i:S\to(M,J_{t_i})$ is said to be *convergent to a level-k holomorphic building* v in Gromov–Hofer's sense, using the above notation, if there is a sequence of maps $\phi_i:S\to\Sigma$, and for each i, there is a sequence of $k-2$ real numbers t_i^r, $r=2,\dots,k-1$, such that:

(i) (domain) ϕ_i are locally biholomorphic except that they may collapse circles in S to nodes of Σ;

(ii) (map) the sequences $u_i\circ\phi_i^{-1}:\Sigma_1\to U$, $u_i\circ\phi_i^{-1}+t_i^r:\Sigma_r\to SH$, $2\leqslant r\leqslant k-1$, and $u_i\circ\phi_i^{-1}:\Sigma_k\to W$ converge in C^∞-topology to corresponding maps v_r on compact sets of Σ_r.

Now the celebrated compactness result in SFT reads as follows.

THEOREM 4.1 [Eva10a, Theorem 5.32], [BEHWZ03, Theorem 10.3]. *If u_i has a fixed homology class, there is a subsequence t_{i_m} of t_i such that $u_{t_{i_m}}$ converges to a level-k holomorphic building in the Gromov–Hofer sense.*

This theorem as stated appeared in [Hin04, Eva10b]. It should be considered an immediate consequence of [BEHWZ03, Theorem 10.3], but some clarifications might be appropriate. The original formulation in [BEHWZ03] involves the finiteness of the following symplectic energy $E(u)$ defined in [BEHWZ03, § 9.2]:

$$\begin{gathered} E(u) = E_\omega(u) + E_\lambda(u), \\ E_\omega(u) = \int_{S_1} u^*\omega + \int_{S_2} u^* p_N^* \omega, \\ E_\lambda(F) = \sup \int_{S_2} (\varphi \circ a)\, da \wedge u^*\lambda. \end{gathered}$$

Here, $u : S \to (M, J_{t_i})$ is a J_{t_i}-holomorphic map as before, $S_1 = S\backslash S_2, S_2 = u^{-1}(V)$ and $p_N : (-\epsilon, \epsilon) \times N$ is the projection to the factor N (considered as $\{0\} \times N \subset (-\epsilon, \epsilon) \times N = V$). We will suppress the dependence on t_i since it is irrelevant.

The finiteness of $E(u)$ can be implied by the restriction of fixed homology class, justified as follows. First of all, notice $E_\omega(u) \leqslant C\omega(u)$ for some constant C depending only on ϵ. It suffices to compare the two sides in the portion S_2. In our coordinates, $\omega = d(e^t\lambda)$, so

$$\begin{aligned} \int_{S_2} u^* p_N^* \omega &= \int_{S_2} u^*(dt \wedge \lambda + d\lambda) \\ &\leqslant e^\epsilon \int_{S_2} u^*(e^t dt \wedge \lambda + e^t d\lambda) \\ &= C \int u^* d(e^t\lambda) = C\omega(u). \end{aligned}$$

One then notices this is the only relevant finite energy condition from [BEHWZ03, Lemma 9.2], which asserts $\widetilde{C} \cdot E_\omega(u) \geqslant E(u)$ for some constant $\widetilde{C}$ again depending only on ϵ. The upshot is that the finiteness of ω-area is sufficient to guarantee the Gromov–Hofer compactness as claimed in Theorem 4.1; for this reason we will not mention the energy condition in the rest of our paper.

The definitions and statements above hold true for bordered stable maps with no extra complications, as long as the Lagrangian boundary does not intersect the contact type boundary N. Since the choice of almost complex structure will play an important role in subsequent sections, we would like to specify a special class of adjusted almost complex structures for later applications.

We end this section by drawing the reader's attention to a special class of adjusted almost complex structures particularly suited for our purpose, which can easily be generalized to any toric manifolds or semi-toric situations similar to the case treated here. Denote by e_1', e_2' and e_3' the pre-images of the three edges of $\Phi_{\mathbb{C}P^2}$, numbering in a coherent way as in F_4 in § 3.3.

DEFINITION 4.2. We say $J \in \mathcal{J}^\epsilon_{\mathrm{tadj}}$, the space of *compatible toric adjusted almost complex structures*, if J is compactible with $\omega_{\mathrm{std}}|_{X_0}$ and adjusted to the hypersurface $M_\epsilon = \Phi^{-1}_{\mathbb{C}P^2}(\{x_2 = 1 - \epsilon\})$ while e_1', e_2' and e_3' are J-holomorphic in $X_0 = \Phi^{-1}_{\mathbb{C}P^2}(\{x_2 < 1 - \epsilon\})$. Moreover, J is invariant under the circle action generated by Reeb flow in a neighborhood of M_ϵ.

It is not hard to see that $\mathcal{J}^\epsilon_{\mathrm{tad}\,j}$ is non-empty. Note that e_1', e_3' intersects M_ϵ transversely, and they are foliated by simple orbits of the circle action. Moreover, the Liouville vector field

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near M_ϵ is invariant under the circle action, and is tangent to e'_1 and e'_3 in a neighborhood of M_ϵ. Therefore, one only needs to define the almost complex structure to be adjusted, whose restriction to the contact distribution is invariant under the circle action, then extend to the rest of X_0 in an $\omega_{\mathrm{std}}|_{X_0}$-compatible way so that $e'_i \cap X_0$ are holomorphic for $i = 1, 2, 3$.

We would like to point out that one can still achieve transversality within $\mathcal{J}^{\epsilon}_{\mathrm{tad}\, j}$ because no (punctured) holomorphic curves lie entirely in the region where we fixed the almost complex structure, with the exceptions of e'_i, which are clearly regular in their own right (see Wendl's automatic transversality in § 4.2.3). Moreover, the space of such almost complex structures is contractible, because it is just the space of sections of a bundle with contractible fibers with prescribed values on a closed set.

4.2 Contributions of holomorphic disks of Maslov index 2

In this section we will compute terms involved in equation (2.3) by studying the evaluation of several moduli spaces. We first study the configurations of limits under neck-stretching of holomorphic disks of Maslov index 2, then study all possible cases of resulting holomorphic buildings.

Here we fix some more notation convenient for our exposition. For a Borel–Moore class B, we consider the moduli space of holomorphic disks punctured at an interior point and with one marked point on the boundary, which we denote by $\mathcal{M}^k_1(B; M, J)$ if the interior puncture is asymptotic to k times of a simple Reeb orbit. We also consider the evaluation maps

$$ev^i : \mathcal{M}^k_1(B; M, J) \to \mathcal{N},$$

where $\mathcal{N}$ is the Morse–Bott manifold in which the interior puncture lies. When no confusion is likely to occur, we sometimes suppress M and J.

4.2.1 *Neck-stretching of holomorphic disks.* Given $J \in \mathcal{J}^{\epsilon}_{\mathrm{tad}\, j}$, we may perform neck-stretching described in 4.1 along M_ϵ, and denote $J^+ := J_\infty|_{X_0}$, $J^- := J_\infty|_{X_1}$. We would like to clarify the choice of M_ϵ first. Given a Lagrangian torus fiber, we can always choose $\epsilon \ll 1$ so that L lies inside X_0, and we only consider the neck-stretching along this fixed hypersurface. Although such a fixed stretching data cannot compute the superpotential for all Lagrangian fiber in the semi-toric picture, this is irrelevant; we are interested in non-displaceability and superheaviness of a fixed fiber instead of a theory of family Lagrangians. Hence we remain in the usual neck-stretching argument set-up.

Recall that X_0 can be compactified to F_4 by collapsing the circle action on the boundary. Under this operation, the asymptotic boundary of X_0 collapses to the edge e_4, and (part of) e'_i gives rise to e_i in F_4 for $i = 1, 2, 3$.

Let L be an arbitrary torus fiber in X_0 we first note the following lemma.

Lemma 4.3. *For $J \in \mathcal{J}^{\epsilon}_{\mathrm{tad}\, j}$, suppose one has an irreducible J^+-holomorphic curve C with finite energy, possibly with boundary on L and punctures on $\partial^\infty(X_0)$. Assume C has finite ω-energy. Then it has its class in the positive span of $\{[D_i]\}_{i=1}^4$. In particular, the relative Chern number $c_1^\Phi(C) \geqslant 0$, and equality holds if and only if $[C] = k[D_4]$ for some $k \in \mathbb{Z}^{\geqslant 0}$.*

Proof. We use the compactification $(\widetilde{X}_0, \widetilde{J})$ from Lemma 4.7, of which the proof is independent of our actual neck-stretching analysis.

From this compactification (by collapsing circle actions on $\partial^\infty X_0$) we can obtain a punctured holomorphic curve in an almost complex manifold $(\widetilde{X}_0, \widetilde{J})$ simply by embedding C into $X_0 \hookrightarrow \widetilde{X}_0$. From the finite energy assumption and the compatibility of $\widetilde{\omega}$ and $\widetilde{J}$, the image of C in $(\widetilde{X}_0, \widetilde{J})$

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can be compactified by removal of singularities, to a closed $\widetilde{J}$-holomorphic curve with boundary on L. Since Σ is a $\widetilde{J}$-holomorphic curve identified with e_4 in $\widetilde{X}_0$, and the rest of the e_i are also naturally identified with compactification's toric boundary of X_0 (again from collapsing the circle action), the first assertion is a direct consequence of positivity of intersection. The second assertion follows from the calculation of § 3.4. □

We are now ready to prove the following lemma.

LEMMA 4.4. *For $J \in \mathcal{J}^{\epsilon}_{\mathrm{tad}\,j}$, the equation of the potential function (2.3) has at most four terms of contributions coming from $[D_1]$, $[D_2]$, $[D_3]$ and $[D_4']$, i.e. $\mathcal{M}^k_1(L; J, \beta)$ is non-empty only when β is one of these four classes.*

Proof. Given $J \in \mathcal{J}^{\epsilon}_{\mathrm{tad}\,j}$, by neck-stretching we obtain a family of almost complex structure J_t. Consider a homology class A which admits J_{t_i}-holomorphic disks with Maslov index 2 for a sequence $t_i \nearrow \infty$, $i \in \mathbb{Z}^+$. By the compactness Theorem 4.1, it converges to a holomorphic building. We then have one of the following cases.

Case 1: the X_1-part of the holomorphic building is empty. Since X_0 is symplectomorphic to $F_4 \backslash e_4$, (3.6) and Lemma 4.3 implies D_1, D_2, D_3 are the only possibilities, otherwise the Maslov index must exceed 2.

Case 2: the X_1-part of the holomorphic building is non-empty. Consider the X_1-part of the holomorphic building S_1. Since it must have periodic orbits as asymptotes, it is a Borel–Moore cycle of class kH' for some $k \in \mathbb{Z}^+$. Therefore, $c_1^{\Phi}(S_1) \geqslant 1$ by (3.8), and the equality holds only when $k = 1$. To close up this cycle in $\mathbb{C}P^2$, one must cap S_1 by some cycle in X_0. However, from our computations in § 3.4 and Lemma 4.3 we saw that all classes but multiples of $[D_4]$ have positive first Chern number. Therefore, the only J_∞-holomorphic building with Maslov index 2 consists of a cycle in class H' in X_1 and a holomorphic disk in the class $2[D_4]$ in X_0. The class they form in $H_2(\mathbb{C}P^2, L; \mathbb{Z})$ is $[D_4']$. □

Lemma 4.4 narrows our study down to four classes. Notice the above two lemmata assume no genericity of J. Moreover, from the proof we see that to understand the contributions of $[D_i]$, $i = 1, 2, 3$, it suffices to study the stretching limit. To understand holomorphic disks in $[D_4']$, we need a slightly more detailed description of the limit holomorphic building.

LEMMA 4.5. *When $t \to \infty$, J_t-holomorphic disks in class D_4' converge to a holomorphic building consisting of the following levels, if they exist:*

(1) *the X_1-part is a holomorphic plane in class H' with one asymptotic puncture of multiplicity 2;*
(2) *the symplectization part is a trivial cylinder with one asymptotic puncture of multiplicity 2 on both positive and negative sides;*
(3) *the X_0-part is a holomorphic disk in class $2[D_4]$ with a single puncture of multiplicity 2.*

Proof. In the proof of Lemma 4.4 we already saw that the X_1-part can only be of class H' and that the X_0-part is a cycle in class $2[D_4]$ by counting Maslov indices and numbers of punctures. To see that the X_0-part is a holomorphic disk with a single puncture of multiplicity 2 instead of two simple punctures, notice that otherwise the holomorphic building will be forced to have at least genus 1, since simple orbits cannot be capped by disks on the X_1 side. This verifies (3).

In the symplectization part, since all orbits have the same period, and the positive end has exactly one orbit of multiplicity 2, the negative end also has at most 2 orbits counting

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multiplicities. Again, since simple orbits do not close up in X_1, there must be two negative ends counting multiplicity. Since the λ-energy (see its definition in, for example, [EGH00]) is now zero for the symplectization part, the image of the symplectization part is a trivial cylinder. Since branched covers over the trivial cylinder always create genus in this holomorphic building, we conclude that the symplectization part is indeed an unbranched double cover of the trivial cylinder. This verifies (2), as well as that the X_1-part has exactly one puncture of multiplicity 2. The rest of the assertions in (1) are easy. □

4.2.2 *Contribution of* $[D_i]$, $i = 1, 2, 3$. In this section, we prove the following proposition.

Proposition 4.6. *For generic* $J \in \mathcal{J}^{\epsilon}_{\mathrm{tad}\, j}$, $\deg(ev_{0*}[\mathcal{M}_1([D_i]; J)]) = 1$ *for* $i = 1, 2, 3$.

Proof. Let us perform a neck-stretch on J, so that all disks of $\mathcal{M}_1([D_i]; J_t)$ lie entirely in X_0. Since J is cylindrical near M_ϵ, we have the following claim.

Lemma 4.7. *The pair* $(X_0, J^+ = J_\infty|_{X_0})$ *is biholomorphic to an open set* U *of a closed symplectic manifold with a compatible almost complex structure* $(\widetilde{X}_0, \widetilde{\omega}, \widetilde{J})$, *so that the following holds:*

(i) $(\widetilde{X}_0, \widetilde{\omega})$ *is in fact the result of the symplectic cut constructed in* § 3.1, *i.e.* $\widetilde{X}_0 = X''_0 = F_4$;

(ii) $\Sigma = \widetilde{X}_0 \backslash U$ *is a* $\widetilde{J}$*-divisor.*

Proof. This is simply a translation between the set-up of relative invariants of [LR01] and the one of SFT in the case where Reeb orbits foliate the contact type hypersurface. $\widetilde{X}_0$ as a symplectic manifold comes from collapsing the circle action on ∂X_0, which forms a symplectic divisor Σ. For some small $\delta > 0$, near Σ the symplectic form of $\widetilde{X}_0$ can be written as:

$$\omega = \pi^*\tau_0 + d(r'\lambda'), \tag{4.1}$$

for $\delta > r' > 0$. Here τ_0 is a symplectic form on Σ, r' a radial coordinate; π is the radial projection to Σ, and λ' a connection 1-form (in our case it is also a contact form) on level sets of r', satisfying $d\lambda' = \pi^*\tau_0$. Given any complex structure J on Σ, J can be lifted to the horizontal distributions ξ (i.e. the contact distributions), while the almost complex structure on the whole neighborhood can be defined by further requiring $J(r'\partial_{r'}) = R'$. Here R' is the Hamiltonian flow generated by the local (in our case also global) S^1-action. Conversely, given an almost complex structure satisfying $J(r'\partial_{r'}) = R'$ and invariant under the circle action on $\widetilde{U}\backslash\Sigma$ where $\widetilde{U}$ is a neighborhood of Σ, it has a natural extension to Σ.

On the SFT side, endow a symplectic form written as $d(r\lambda)$ to the collar of $N = \partial X_0$, $1 + \delta \geqslant r > 1$, where λ is the contact form on N. This coordinate can be transformed back to the one in § 4.1 by taking a *log*-function on the cylindrical coordinate. The zero level set there becomes the level set $r = 1$ in the current coordinate. In the current coordinate, the toric adjustedness of J^+ is equivalent to the invariance under both flows of $r\partial_r$ and R, and that $J^+(\partial'_r) = R$, where R are the contact distribution and the Reeb flow, respectively.

Notice the fact that $(N \times (1, 1 + \delta), d(r\lambda))$ is symplectomorphic to $(N \times (0, \delta), d\lambda + d(r\lambda))$ just by shifting the r-coordinate. By choosing τ_0 so that $\pi^*\tau_0 = d\lambda$, the symplectic cut, from the perspective of this coordinate change, is simply to glue a divisor Σ to $(N \times (0, \delta)$, $d\lambda + d(r\lambda))$, then the symplectic form extends naturally. In particular, the shift above provides a symplectic identification of a collar neighborhood of $\partial X_0 \subset X_0$ and Σ of $\widetilde{U}\backslash\Sigma$. Under such an identification, J^+ induces an almost complex structure $\widetilde{J}$ on $\widetilde{U}\backslash\Sigma$, which is invariant under $r'\partial_{r'}$ and the Hamiltonian flow R' by the assumption of toric adjustedness. It is then straightforward to see that $\widetilde{J}$ extends to the cut divisor Σ in the new coordinate. Extending further the identification on $\widetilde{U}$ to a diffeomorphism between $U = \widetilde{X}_0 \backslash \Sigma$ and X_0, we induce $\widetilde{J}$ by J^+ on the whole $\widetilde{X}_0$. $\square$

Given Lemma 4.7 and the removable singularity theorem, we may identify $\mathcal{M}_1([D_i], J_t)$ with the moduli space of holomorphic disks without punctures in F_4 endowed with an toric adjusted almost complex structure $\widetilde{J}$ so that equivariant divisors as e_i are $\widetilde{J}$-holomorphic. A problem arises after the compactification: $\widetilde{J}$ is never generic, in the sense that e_4 has negative Chern number, yet always $\widetilde{J}$-holomorphic. We cannot use Cho–Oh's classifications either, because $\widetilde{J}$ is not toric. However, we can still prove the following lemma.

Lemma 4.8. *The moduli space $\mathcal{M}_1([D_i]; \widetilde{X}_0, \widetilde{J})$ is compact for $i = 1, 2, 3$. Hence when t is large enough,*

$$ev_{0*}([\mathcal{M}_1([D_i]; \mathbb{C}P^2, J_t)]) = ev_{0*}([\mathcal{M}_1([D_i]; \widetilde{X}_0, \widetilde{J})]), \quad i = 1, 2, 3.$$

Proof. To understand the left-hand side, we may consider the problem in the limit and replace the left-hand side by X_0 and J^+. By the same analysis as in case 2 of Lemma 4.4, any leveled curves coming as a limit for $t \to +\infty$ must have an empty X_1-part. Further, Lemma 4.7 identifies such a curve as one on the right-hand side which has no e_4-component. Hence the conclusion follows provided one can prove the $[D_i]$ are indecomposable on the right-hand side. Corresponding classes are also indecomposable on the left-hand side by a similar reasoning. But in our application we will assume L is monotone, so all classes on the left-hand side are clearly indecomposable for the ω-area reason, so we will omit the actual proof.

Take $[D_1]$ as an example, and the rest of the cases are similar. Assume $u : \Sigma \to (\widetilde{X}_0, \widetilde{J})$ is a stable curve in the moduli space of right-hand side with irreducible components $\Sigma_1, \ldots, \Sigma_k$, and the homology classes of all of these components are written in terms of basis $\{[D_i]\}_{i=1}^4$. Let $\Sigma_1 \ldots, \Sigma_l$ be components of e_4, while $[e_4] = [D_1] + [D_3] - 4[D_4]$. Now Lemma 4.3 implies that $[\Sigma_j]$, $j > l$ all lie in the positive cone spanned by $[D_i]$ for $i = 1, 2, 3, 4$. By comparing coefficients of $[D_3]$, we conclude that $l = 0$. It then follows easily that $k = 1$ and $[\Sigma_1] = [D_1]$. Since $[D_1]$ pairs trivially with $[e_4]$, our claim is proved by positivity of intersections. $\square$

Lemma 4.8 implies that $\mathcal{M}_1([D_i]; \widetilde{X}_0, \widetilde{J})$ is in fact compact even with no genericity assumption since the class itself is indecomposable. Therefore, the standard cobordism arguments apply. In particular, one may choose a generic path $\{J_t\}_{t \in [0,1]}$ connecting J_0 and $J_1 = \widetilde{J}$ for J_0 also satisfying that e_i, $i = 1, 2, 3, 4$ are J_0-holomorphic. Recall from [CO06] that there is an integrable complex structure J_0 where $ev_{0*}([\mathcal{M}_1([D_i]; \widetilde{X}_0, J_0)])$ is known to be $[L]$, hence concluding our proof of Proposition 4.6. $\square$

4.2.3 *The contribution of $[D_4']$.* Our goal for this section is to prove the following proposition.

Proposition 4.9. *For a generic choice of $J \in \mathcal{J}^{\epsilon}_{\mathrm{tad}\, j}$,*

$$\deg(ev_{0*}[\overline{\mathcal{M}}_1(D_4'; \mathbb{C}P^2, J)]) = 2.$$

As already explained in previous sections, we only need to consider $J \in \mathcal{J}^{\epsilon}_{\mathrm{tad}\, j}$ with its neck stretched sufficiently long. We first briefly review Wendl's automatic transversality theorem.

One of the new ingredients of Wendl's theorem is the introduction of the invariant *parity*, defined in [HWZ95], to the formula. Let Y be a symplectic cobordism, where $Y^{\pm}$ are the positive (respectively negative) boundaries. Given a T-periodic orbit γ of $Y^{\pm}$, one has an associated

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asymptotic operator, which takes the form of $\mathbb{A} = -I_0\partial_t - S(t)$ on $L^2(S^1, \mathbb{R}^2)$ by taking a trivialization of the normal bundle. Here I_0 is the standard complex structure on $\mathbb{R}^2$, while $S(t)$ is a continuous family of symmetric matrices. For $\lambda \in \sigma(\mathbb{A})$, one may define a *winding number* $w(\lambda)$ to be the winding number of non-trivial λ-eigenfunction of $\mathbb{A}$. It is proved in [HWZ95] that $w(\lambda)$ is an increasing function of λ which takes every integer value exactly twice. For non-degenerate operators $\mathbb{A}$ (i.e. $0 \notin \sigma(\mathbb{A})$), we define

$$\alpha_+(\mathbb{A}) = \max\{w(\lambda) \mid \lambda \in \sigma(\mathbb{A}), \lambda < 0\},$$
$$\alpha_-(\mathbb{A}) = \min\{w(\lambda) \mid \lambda \in \sigma(\mathbb{A}), \lambda > 0\},$$

and the parity $p(\mathbb{A}) = \alpha_+(\mathbb{A}) - \alpha_-(\mathbb{A}) (\mathrm{mod}\ 2)$. If $\mathbb{A}$ is degenerate, we define $\alpha_\pm(\mathbb{A} \pm \delta)$ and $p(\mathbb{A} \pm \delta)$ for small $\delta > 0$. For a given puncture, the actual perturbation depends on which of $Y^\pm$ it lies on, as well as whether the moduli space we consider constrains the puncture inside a Morse–Bott family. Chris Wendl pointed out to the author that, in our case when the contact type boundary is foliated by a 2-dimensional family of Reeb orbits, since the eigenvalue 0 has multiplicity 2, *either way of perturbation incurs odd parity.* (The reader should also be reminded that the parity is independent of the choice of the trivialization; hence it suffices to look at the trivialization where the Poincaré return map is trivial, that is, the one we chose for $\mu_{\mathrm{CZ}} = 0$.)

Now given a non-constant punctured holomorphic curve $u : \Sigma_g \to Y$, the virtual index is computed as

$$\mathrm{ind}(u) = (n-3)\chi(\Sigma) + 2c_1^\Phi(u) + \sum_{\gamma^+}(\mu_{\mathrm{CZ}}(\gamma^+) + \tfrac{1}{2}\dim(\mathcal{N})) - \sum_{\gamma^-}(\mu_{\mathrm{CZ}}(\gamma^-) - \tfrac{1}{2}\dim(\mathcal{N})).$$

Here $\gamma^\pm$ runs over all positive (respectively negative) punctures, and $\mathcal{N}$ is the Morse–Bott manifold formed by the Reeb orbits.

We now define the *normal Chern number* as:

$$2c_N(u) = \mathrm{ind}(u) - 2 + 2g + \#\Gamma_0 + \#\pi_0(\partial\Sigma_g).$$

Here, Γ_0 denotes the number of punctures of even parities; hence in our applications when the contact type boundary is foliated by 2-dimensional family of Reeb orbits, this term always vanishes.

The last input is the total order of critical points of an almost complex curve u:

$$Z(du) = \sum_{z \in du^{-1}(0) \cap \Sigma} \mathrm{ord}(du; z) + \frac{1}{2} \sum_{z \in du^{-1}(0) \cap \partial\Sigma} \mathrm{ord}(du; z).$$

Having understood these, Wendl's automatic transversality theorem reads as follows.

Theorem 4.10 [Wen10, Theorem 1]. *Suppose* $\dim Y = 4$ *and* $u : (\Sigma, j) \to (Y, J)$ *is a non-constant curve with only Morse–Bott asymptotic orbits. If*

$$\mathrm{ind}(u) > c_N(u) + Z(du),$$

then u is regular.

For the contribution to equation (2.3) of holomorphic disks in class D_4', we consider the gluing problem of $\mathcal{M}^2(H'; X_1, J^-)$ and $\mathcal{M}_1^2(2[D_4]; X_0, J^+)$. Note that this is sufficient by the configuration analysis of the limit holomorphic building in Lemma 4.5. The standard gluing

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argument requires the following conditions:

(1) curves in both $\mathcal{M}^2(H';X_1,J^-)$ and $\mathcal{M}^2_1(2[D_4];X_0,J^+)$ are regular;
(2) $ev^1\times ev^1:\mathcal{M}^2(H';X_1,J^-)\times\mathcal{M}^2_1(2[D_4];X_0,J^+)\to S^2\times S^2$ is transversal to the diagonal $\Delta\subset S^2\times S^2$. Here $S^2=\mathcal{N}$ is exactly the Morse–Bott family parameterizing Reeb orbits on M_ϵ.

One sees that condition (2) is automatic since the first component of the evaluation map is surjective onto S^2. This corresponds to the standard fact in Gromov–Witten theory that, given any compatible almost complex structure J in $\mathbb{C}P^2$, an embedded J-holomorphic conic Σ and a point $p\in\Sigma$, there is a unique J-complex line tangent to Σ at p. For (1) we apply Wendl's automatic transversality in dimension 4.

The virtual index of an irreducible curve $C\in\mathcal{M}^2(2[D_4];X_0,J^+)$ reads:

$$\mathrm{ind}(u)=(2-3)(2-1-1)+0+0-(0-1)=1.$$

The computation also shows that, for generic J, the compactification of this moduli space does not contain irreducible curves with critical points or sphere bubbles since these are codimension 2 phenomena. On the other hand, we can compute $c_N(u)=0$. Therefore, automatic transversality holds for all $C\in\mathcal{M}^2_1(2[D_4];X_0,J^+)$.

To show that disk bubbles do not appear, we again use Lemma 4.7 to identify the moduli space $\mathcal{M}^2_1(2[D_4];X_0,J^+)$ to one on F_4, denoted by $\mathcal{M}^{\mathrm{crit}}_1(2[D_4];e_4,\widetilde{X}_0,L;\widetilde{J})$, where $\widetilde{J}$ is the extended almost complex structure.

Definition 4.11. Let $\mathcal{M}^{\mathrm{crit}}_1(2[D_4];e_4,\widetilde{X}_0,L;\widetilde{J})$ be the moduli space of $\widetilde{J}$-holomorphic disks $u:(D^2,j)\to(\widetilde{X}_0,\widetilde{J})$ which satisfies the following:

- u has an interior marked point x and a boundary marked point y;
- $u(\partial D)\subset L$, $u(x)\in e_4$, $du(x)=0$ with order 1.

Now by collapsing the Reeb orbits on ∂X_0, a stable punctured disk in $\mathcal{M}^2_1(2[D_4];X_0,J^+)$ descends to a stable disk in $\mathcal{M}^{\mathrm{crit}}_1(2[D_4];e_4,\widetilde{X}_0,L;\widetilde{J})$. The order of vanishing of du exactly corresponds to the multiplicity of the asymptotic Reeb orbit.

Lemma 4.12. *Holomorphic disks $u\in\mathcal{M}^{crit}_1(2[D_4];e_4,\widetilde{X}_0,L;\widetilde{J})$ are regular for generic $\widetilde{J}$. Moreover, the moduli space is compact.*

Proof. The argument is taken almost word-for-word from the case of open manifolds. Since u cannot develop critical points other than x for generic choice of $\widetilde{J}$, we may apply Wendl's automatic transversality, Theorem 4.10. We have the Fredholm index:

$$\mathrm{ind}(u)=-1+2\cdot 2=3,$$
$$c_N(u)=\tfrac{1}{2}(3-2+1)=1.$$

Since we have a unique critical point of order 2,

$$3=\mathrm{ind}(u)>c_N(u)+Z(du)=1+1=2,$$

verifying the transversality of $u\in\mathcal{M}^{\mathrm{crit}}_1(2[D_4];e_4,\widetilde{X}_0,L;\widetilde{J})$. We now only need to show that $\partial\mathcal{M}^{\mathrm{crit}}_1(2[D_4];e_4,\widetilde{X}_0,L;\widetilde{J})=\emptyset$. The argument of Lemma 4.8 shows that the only possible type of reducible stable curve $u\in\partial\mathcal{M}^{\mathrm{crit}}_1(2[D_4];e_4,\widetilde{X}_0,L;\widetilde{J})$ consists of a union of 2 disks in class $[D_4]$ (by comparing coefficients of either $[D_1]$ or $[D_3]$ for possible irreducible decompositions). However, given a sequence $u_k\in\mathcal{M}^{\mathrm{crit}}_1(2[D_4];e_4,\widetilde{X}_0,L;\widetilde{J})$ converging to u, $u_k(x)\in e_4$ are always critical values which cannot approach the boundary. If a disk bubble occurs, one of the components inherits such a critical point, and thus has intersection index with e_4 at least 2. But this contradicts the fact that each component is in class $[D_4]$. $\square$

To compute the evaluation map, we may now choose a generic path connecting $\{J^s\}_{s\in[0,1]}$ connecting $J^1 = \widetilde{J}$ and the standard toric complex structure J^0 of F_4 as in [CO06], while requiring that e_i, $i = 1,2,3,4$ are J^s-holomorphic. In view of the arguments in Lemma 4.12, the moduli space $\mathcal{M}_1^{\mathrm{crit}}(2[D_4]; F_4, L; J^s)$ does not develop disk or sphere bubbles. Moreover, $2[D_4]$ does not admit a multiple cover more than 2-fold, whereas these 2-fold covers are in fact curves in $\mathcal{M}_1^{\mathrm{crit}}(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})$ instead of the boundary of the moduli space $\partial\mathcal{M}_1^{\mathrm{crit}}(2[D_4]; e_4, \widetilde{X}_0, L; J^s)$. Therefore, the standard cobordism argument in [MS04] applies. In particular, $ev_{0*}[\mathcal{M}_1^{\mathrm{crit}}(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})] = ev_{0*}[\mathcal{M}_1^{\mathrm{crit}}(2[D_4]; F_4, L; J^0)]$. By Cho–Oh's classification, $\mathcal{M}_1^{\mathrm{crit}}(2[D_4]; F_4, L; J^0)$ consists only of double covers of embedded disks in $\mathcal{M}_1([D_4]; F_4, L; J^0)$ with critical points on intersections with e_4. Therefore, from the identification of $\mathcal{M}_1^2(2[D_4]; X_0, J^+)$ and $\mathcal{M}_1^{\mathrm{crit}}(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})$,

$$\begin{aligned} ev_{0*}[\mathcal{M}_1^2(2[D_4]; X_0, J^+)] &= ev_{0*}[\mathcal{M}_1^{\mathrm{crit}}(2[D_4]; e_4, \widetilde{X}_0, L; \widetilde{J})] \\ &= ev_{0*}[\mathcal{M}_1^{\mathrm{crit}}(2[D_4]; F_4, L; J^0)] = 2[L]. \end{aligned}$$

On the X_1 side, what concerns us is $\mathcal{M}^2(H'; X_1, J^-)$. We already saw from the argument of condition (2) of the standard gluing argument that these curves correspond one-to-one to closed curves in $\mathbb{C}P^2$ of line class which are tangent to the given embedded conic. In particular no bubbling or critical points occurs for these curves. The virtual index of such a curve C_1 is

$$\mathrm{ind}(C_1) = (2-3)(2-1) + 2 + 1 - 0 = 2,$$

and $c_N(C_1) = 0$. This verifies condition (1). Moreover, $ev^1 : \mathcal{M}^2(H; X_1, J^-) \to S^2$ is surjective and of degree 1 from considering the Gromov–Witten invariants of tangent lines on the conic after closing up the orbits on the boundary. Therefore, the standard gluing argument applies and leads to the following commutative diagram.

$$\begin{tikzcd} \mathcal{M}^2(H'; X_1, J^-)\ {}_{ev^1}\times_{ev^1} \mathcal{M}_1^2(2[D_4]; X_0, J^+) \arrow[r, "\text{glue}"] \arrow[dr, "\widetilde{ev}_0"'] & \mathcal{M}_1(D_4'; \mathbb{C}P^2, L) \arrow[d, "ev_0"] \\ & L \end{tikzcd}$$

Here $\widetilde{ev}_0$ is the evaluation of $\mathcal{M}_1^2(2[D_4]; X_0, J^+)$ to the boundary marked points. It then follows that, for t sufficiently large,

$$\deg[ev_0 : \mathcal{M}_1(D_4'; \mathbb{C}P^2, L; J_t) \to L] = 2. \tag{4.2}$$

5. Completion of the proof

To summarize, we have computed evaluation maps of all holomorphic disks of Maslov index 2 of $(\mathbb{C}P^2, J_t)$ for the fiber over $\mathbf{u} = (u_1, u_2)$ when t is sufficiently large, as long as $(u_1, u_2) \in \mathbb{R}^2$ lies in X_0. Note that the ω-area of the four families of Maslov 2 disks are exactly u_1, u_2, $4 - u_1 - u_2$ and $2 - 2u_2$ (see [CO06]). Plugging these inputs into (2.3) we deduce that the potential function of $\mathbb{C}P^2$ in the toric degeneration picture is written as:

$$\mathfrak{PO}^u(y) = T^{u_1} y_1 + T^{u_2} y_2 + T^{4-u_1-4u_2} y_1^{-1} y_2^{-4} + 2T^{2-2u_2} y_2^{-2}.$$

Taking $\mathbf{u} = (\frac{2}{3}, \frac{2}{3})$, the equations of critical points are written as

$$\begin{cases} y_1^2 y_2^4 = 1, \\ 1 - 4y_2^{-5} y_1^{-1} - 4y_2^{-3} = 0. \end{cases} \tag{5.1}$$

We therefore deduce that $y_1 y_2^2 = \pm 1$. When we take it to be 1, we have $y_2^3 = 8$ and thus it clearly has three solutions. This verifies that $\mathfrak{PO}^u$ contains a critical point for some values y_1 and y_2; hence $HF^*(L, b) \neq 0$ for some choice of bounding cochain b. Moreover, $QH_*(\mathbb{C}P^2, \Lambda)$ is semi-simple and decomposes into three direct factors of Λ. Writing $QH^*(\mathbb{C}P^2; \Lambda) = \Lambda[z]/(z^3 - T)$, the idempotents are simply $\frac{1}{3}\epsilon_i^{-2}(z^2 + \epsilon_i z + \epsilon_i^2)$ for $i = 1, 2, 3$. Here ϵ_i are the roots of $x^3 - T = 0$ in Λ. Now Corollary 2.6 implies $L(\mathbf{u})$ is indeed superheavy with respect to some symplectic quasi-state. This concludes our proof of Theorem 1.1.

Remark 5.1. Our example manifests two interesting aspects of the effect of the choice of Novikov rings. Namely, we found three local systems on the exotic monotone fiber, which is different from the case of the calculation of [FOOO12], where the monotone exotic fiber in $S^2 \times S^2$ only has half the number of local systems of the standard monotone fiber, that is, the product of equators.

According to comments due to Kenji Fukaya, combining results from [AFOOO], our computation implies that this single exotic fiber is sufficient to generate a certain Fukaya category with characteristic zero coefficients. However, this fiber is disjoint from $\mathbb{R}P^2$, thus cannot generate any version of Fukaya category with characteristic 2 coefficients (in fact our torus is always a zero object for characteristic 2 Fukaya categories of $\mathbb{C}P^2$). This shows that the choice of the characteristic of coefficient rings could be more than technical.

Remark 5.2. Leonid Polterovich brought up another very interesting question: is it possible to distinguish the symplectic quasi-states/morphisms for the three idempotents of $QH^*(\mathbb{C}P^2; \Lambda)$? These three quasi-states/morphisms are intuitively very closely related; given the Novikov field $\Lambda_{\text{nov}} = \{a_i T^{\lambda_i} : \lambda_i \in \mathbb{Z}, \lim \lambda_i = +\infty\}$, then $\Lambda_{\text{nov}}[z]/(z^3 - T)$ is already a field. However, when we tensor this ring by Λ, the algebraic closure of Λ_{nov}, the resulting ring is only semi-simple, as indicated in our computation. Therefore, the identity in the field $\Lambda_{\text{nov}}[z]/(z^3 - T)$ splits into three idempotents after a purely algebraic procedure; thus, intuitively, the three symplectic quasi-states/morphisms are 'algebraically split' from the original quasi-state/morphism as well.

It was known to Entov and Polterovich [EntP03, EntP06] that S^2 carries a unique symplectic quasi-state. However, the corresponding statement for quasi-morphism is not known even for the spectral quasi-morphisms in this case. For $\mathbb{C}P^n$, $n \geqslant 2$, there are no results available. In general, there is no systematic approach for identifying two symplectic quasi-morphisms/states, or distinguishing them when they are only known to be supported on the same Lagrangian submanifolds.

However, Michael Entov and Leonid Polterovich[1] kindly showed the author that the following theorem indeed holds.

THEOREM 5.3 (Entov–Polterovich). *Assume $e, f \in QH(M)$ are idempotents so that $ef = f$. Then:*

(1) $\mu_e \leqslant \mu_f$, $\zeta_e \geqslant \zeta_f$. *Hence any e-superheavy set is also f-superheavy and any f-heavy set is also e-heavy;*
(2) *assume μ_e is a genuine (and not partial) quasi-morphism; then $\mu_e = \mu_f$ and thus μ_f is a genuine quasi-morphism as well.*

[1] Private communications.

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Here μ_e, μ_f are symplectic quasi-morphisms of corresponding idempotents, while ζ_e and ζ_f are the quasi-states (see their precise definition from [Ent14]). The argument uses mainly algebraic properties of spectral numbers (e.g. those listed in [EntP09, § 3.4]). This is the first, to the author's best knowledge, criterion for two symplectic quasi-morphisms to be identified. It would be very interesting to understand if there is a deeper meaning from this observation.

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References

AFOOO M. Abouzaid, K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Quantum cohomology and split generation in Lagrangian Floer theory*, in preparation.

AF08 P. Albers and U. Frauenfelder, *A nondisplaceable Lagrangian torus in T^*S^2*, Comm. Pure Appl. Math. **61** (2008), 1046–1051.

Bir01 P. Biran, *Lagrangian barriers and symplectic embeddings*, Geom. Funct. Anal. **11** (2001), 407–464.

BC09 P. Biran and O. Cornea, *Rigidity and uniruling for Lagrangian submanifolds*, Geom. Topol. **13** (2009), 2881–2989.

BEHWZ03 F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki and E. Zehnder, *Compactness results in symplectic field theory*, Geom. Topol. **7** (2003), 799–888.

CS10 Y. Chekanov and F. Schlenk, *Notes on monotone Lagrangian twist tori*, Preprint (2010), arXiv:1003.5960.

CO06 C.-H. Cho and Y.-G. Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. **10** (2006), 773–814.

CG97 N. Chriss and V. Ginzburg, *Representation theory and complex geometry* (Birkhäuser, Boston, MA, 1997).

Dor13 J. G. Dorfmeister, *Minimality of symplectic fiber sums along spheres*, Asian J. Math. **17** (2013), 423–442.

EGH00 Y. Eliashberg, A. Givental and H. Hofer, *Introduction to symplectic field theory GAFA 2000 (Tel Aviv, 1999)*, Geom. Funct. Anal (2000), 560–673; Special Volume, Part II.

EP10 Y. Eliashberg and L. Polterovich, *Symplectic quasi-states on the quadric surface and Lagrangian submanifolds*, Preprint (2010), arXiv:1006.2501.

Ent14 M. Entov, *Quasi-morphisms and quasi-states in symplectic topology*, Preprint (2014), arXiv:1404.6408.

EntP03 M. Entov and L. Polterovich, *Calabi quasimorphism and quantum homology*, Int. Math. Res. Not. IMRN **2003** (2003), 1635–1676.

EntP06 M. Entov and L. Polterovich, *Quasi-states and symplectic intersections*, Comment. Math. Helv. **81** (2006), 75–99.

EntP08 M. Entov and L. Polterovich, *Symplectic quasi-states and semi-simplicity of quantum homology*, in *Toric topology*, Contemporary Mathematics, vol. 460 (American Mathematical Society, Providence, RI, 2008), 47–70; English summary.

EntP09 M. Entov and L. Polterovich, *Rigid subsets of symplectic manifolds*, Compos. Math. **145** (2009), 773–826.

Eva10a J. D. Evans, *Lagrangian spheres in Del Pezzo surfaces*, J. Topol. **3** (2010), 181–227.

Eva10b J. D. Evans, *Symplectic topology of some Stein and rational surfaces*, PhD thesis, University of Cambridge (2010).

FOOO09 K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction*, AMS/IP Studies in Advanced Mathematics, vol. 46 (American Mathematical Society, Providence, RI, 2009).

FOOO10a K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory on compact toric manifolds I*, Duke Math. J. **151** (2010), 23–74.

FOOO10b K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory on compact toric manifolds: survey*, Preprint (2010), arXiv:1011.4044.

FOOO11a K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory on compact toric manifolds II: bulk deformations*, Selecta Math. (N.S.) **17** (2011), 609–711.

FOOO11b K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Spectral invariants with bulk, quasimorphisms and Lagrangian Floer theory*, Preprint (2011), arXiv:1105.5123.

FOOO12 K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Toric degeneration and non-displaceable Lagrangian tori in $S^2 \times S^2$*, Int. Math. Res. Not. IMRN **2012** (2012), 2942–2993.

Hin04 R. Hind, *Lagrangian spheres in $S^2 \times S^2$*, Geom. Funct. Anal. **14** (2004), 303–318.

HWZ95 H. Hofer, K. Wysocki and E. Zehnder, *Properties of pseudo-holomorphic curves in symplectisations. II. Embedding controls and algebraic invariants*, Geom. Funct. Anal. **5** (1995), 270–328.

Ler95 E. Lerman, *Symplectic cuts*, Math. Res. Lett. **2** (1995), 247–258.

LR01 A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov–Witten invariants of Calabi–Yau 3-folds*, Invent. Math. **145** (2001), 151–218.

LW12 T.-J. Li and W. Wu, *Lagrangian spheres, symplectic surfaces and the symplectic mapping class group*, Geom. Topol. **16** (2012), 1121–1169.

McD90 D. McDuff, *The structure of rational and ruled symplectic 4-manifolds*, J. Amer. Math. Soc. **3** (1990), 679–712.

MS04 D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, vol. 52 (American Mathematical Society, Providence, RI, 2004).

OU13 J. Oakley and M. Usher, *On certain Lagrangian submanifolds of $S^2 \times S^2$ and $\mathbb{C}P^n$*, Preprint (2013), arXiv:1311.5152.

Sei98 P. Seidel, *Symplectic automorphisms of T^*S^2*, Preprint (1998), arXiv:math/9803084.

Via13 R. Vianna, *On exotic Lagrangian tori in $\mathbb{C}P^2$*, Preprint (2013), arXiv:1305.7512.

Wen10 C. Wendl, *Automatic transversality and orbifolds of punctured holomorphic curves in dimension four*, Comment. Math. Helv. **85** (2010), 347–407.

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