PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, July 1972

ON AN EXPLICIT CHARACTERIZATION OF SPHERICAL CURVES

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ABSTRACT. It will be proved that the "explicit characterization" of spherical curves recently obtained by S. Breuer and D. Gottlieb (Proc. Amer. Math. Soc. **27** (1971), pp. 126–127) is, without any precondition on the curvature and torsion, a necessary and sufficient condition for a curve to be a spherical curve. The proof is based on an earlier result of the present author on spherical curves (Monatsh. Math. **67** (1963), pp. 363–365).

1. **Introduction.** The curves to be considered here are curves in a Euclidean 3-space with an equation of the form $\boldsymbol{x}=\boldsymbol{x}(s)$, $s \in [0, L]$, where s is the arc length and the vector function $\boldsymbol{x}(s)$ is of class C^4. For such a curve, the following facts are well known:

(i) The nonnegative curvature (C^2-function) k_1 defined on $[0, L]$ is unique.

(ii) If k_1 is nowhere zero, the torsion (C^1-function) k_2 defined on $[0, L]$ is unique except for a sign.

(iii) These two functions k_1, k_2 completely determine the shape and size of the curve.

In books on elementary differential geometry (see for example [**2**, p. 32]), the condition for a curve to be a spherical curve, i.e. for it to lie on a sphere, is usually given in the form

$$[k_2^{-1}(k_1^{-1})']' + k_2 k_1^{-1} = 0, \tag{1.1}$$

where the prime denotes differentiation with respect to the arc length s.

Clearly, condition (1.1) has a meaning only if k_1 and k_2 are nowhere zero, and it is only under this precondition that (1.1) is a necessary and sufficient condition for a curve to be a spherical curve. When "cleared of fractions", (1.1) takes the form

$$(-k_1 k_1'' + 2k_1'^2)k_2 + k_1 k_1' k_2' + k_1^2 k_2^3 = 0. \tag{1.2}$$

Received by the editors May 27, 1971.

AMS 1970 *subject classifications*. Primary 53A05.

Key words and phrases. Spherical curve, curvature, torsion, nowhere zero.

[1] The author wishes to thank his colleague Dr. Y. H. Au-Yeung for his help during the preparation of this paper.

Although condition (1.2) has a meaning whether or not k_1 and k_2 are nowhere zero, it is neither a necessary condition nor a sufficient condition for a curve to be a spherical curve unless k_1 and k_2 are nowhere zero [**5**, Theorem 5.2].

A condition for a curve to be a spherical curve which is both necessary and sufficient and which holds without any precondition on k_1 or k_2 has been given by the author [**4**] in the following form:

THEOREM 1.1. *A C^4-curve $\boldsymbol{x}=\boldsymbol{x}(s)$, $s \in [0, L]$, parametrized by its arc length s, is a spherical curve if and only if*

(i) *its curvature k_1 is nowhere zero* (*then its torsion k_2 is defined*), *and*

(ii) *there exists a C^1-function f defined on* $[0, L]$ *such that*

$$fk_2 = (k_1^{-1})', \qquad f' + k_2k_1^{-1} = 0. \tag{1.3}$$

Moreover, a curve satisfying this condition lies on a sphere of radius $(k_1^{-2}+f^2)^{1/2}$ (*which is of course a constant*).

Quite recently, S. Breuer and D. Gottlieb [**1**], using a result of theirs on differential equations, derived from (1.1) the following "explicit characterization" of spherical curves:

$$k_1^{-1}(s) = A \cos\int_0^s k_2\, ds + B \sin\int_0^s k_2\, ds, \tag{1.4}$$

where A and B are constants. Since (1.4) is deduced from (1.1), we can hardly expect that it would be a necessary and a sufficient condition for a curve to be a spherical curve without the precondition that k_1 and k_2 are nowhere zero. It is therefore a pleasant surprise that this should turn out to be the case, as we shall show by proving the following theorem.

THEOREM 1.2. *A C^4-curve $\boldsymbol{x}=\boldsymbol{x}(s)$, $s \in [0, L]$, parametrized by its arc length s, with curvature k_1 and torsion k_2 is a spherical curve if and only if*

$$\left(A \cos\int_0^s k_2\, ds + B \sin\int_0^s k_2\, ds\right) k_1(s) = 1, \tag{1.5}$$

where A, B are constants. Moreover, a curve satisfying condition (1.5) *lies on a sphere of radius* $(A^2+B^2)^{1/2}$.

2. Proof of Theorem 1.2.

SUFFICIENCY. It suffices to show that if a curve satisfies condition (1.5), then it satisfies the conditions in Theorem 1.1. First, (1.5) implies that k_1 is nowhere zero. Next, writing (1.5) as

$$A \cos\int_0^s k_2\, ds + B \sin\int_0^s k_2\, ds = k_1^{-1}(s)$$

and differentiating, we get

$$\left(-A \sin\int_0^s k_2\, ds + B \cos\int_0^s k_2\, ds\right) k_2(s) = (k_1^{-1})'(s).$$

It can now be verified that the function f defined by

$$f(s) = -A \sin\int_0^s k_2\, ds + B \cos\int_0^s k_2\, ds$$

satisfies condition (1.3). This completes the proof of the sufficiency of condition (1.5).

NECESSITY. Suppose that $\boldsymbol{x}=\boldsymbol{x}(s)$, $s \in [0, L]$, is a spherical curve. Then the conditions in Theorem 1.1 are satisfied. Let us define a C^2-function θ and two C^1-functions g and h on $[0, L]$ by

$$\theta(s) = \int_0^s k_2(s)\, ds, \tag{2.1}$$

$$\begin{aligned} g(s) &= k_1^{-1}(s) \cos\theta(s) - f(s) \sin\theta(s), \\ h(s) &= k_1^{-1}(s) \sin\theta(s) + f(s) \cos\theta(s). \end{aligned} \tag{2.2}$$

If we differentiate equations (2.2) with respect to s and take account of (2.1) and (1.3), we find that g' and h' are both identically zero. Therefore, $g(s)=A$, $h(s)=B$, where A, B are constants. Now substituting these in (2.2) and solving the resulting equations for $k_1^{-1}(s)$, we get

$$k_1^{-1}(s) = A \cos\theta(s) + B \sin\theta(s),$$

which is (1.5). This proves the necessity of condition (1.5).

Finally, to prove the last assertion in Theorem 1.2, we note from the first part of this section that, for a curve satisfying the condition $k_1^{-1}=A\cos\theta+B\sin\theta$, the function $f=-A\sin\theta+B\cos\theta$ satisfies condition (1.3) in Theorem 1.1. Therefore, by Theorem 1.1, this curve lies on a sphere with radius

$$[(k_1^{-1})^2 + f^2]^{1/2} = (A^2 + B^2)^{1/2}.$$

The proof of Theorem 1.2 is now complete.

3. **A remark.** Condition (1.4) was originally derived from (1.1) by S. Breuer and D. Gottlieb [**1**] by using a rather profound result of theirs on differential equations. But it can also be derived very simply as follows.

As in (2.1), we let $\theta(s)=\int_0^s k_2\, ds$. Then the differential equation (1.1) can be rewritten as

$$d^2k_1^{-1}/d\theta^2 + k_1^{-1} = 0, \tag{3.1}$$

the solution of which is $k_1^{-1} = A \cos \theta + B \sin \theta$, where A, B are constants. This proves (1.4).

It is interesting to note that C. E. Weatherburn [**3**, p. 25, Exercise 3] came very close to obtaining (1.4), but he stopped at (3.1).

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