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1. Introduction. This paper is concerned with the numerical solution of a boundary value problem for the simple form of the equation of heat conduction, where the boundary is not completely specified in advance, but must be determined as part of the solution. Specifically, it is required to find functions $x(t)$ and $u(x, t)$ such that:

$$
\begin{aligned}
&\text{(a)} && u_{xx}(x, t) = u_t(x, t) && (0 < x < x(t),\ t > 0).\\
&\text{(b)} && u_x(0, t) = -1 && (t > 0).\\
(1.1)\quad &\text{(c)} && u(x, t) = 0 && (x \geqq x(t),\ t \geqq 0).\\
&\text{(d)} && x(0) = 0.\\
&\text{(e)} && \dot{x}(t) = -u_x(x(t), t) && (t > 0).^1
\end{aligned}
$$

The problem has a simple physical interpretation, as given by Evans, Isaacson and Macdonald [6]. Consider a bar of length L, made of a substance which undergoes a change in crystalline structure at a certain critical temperature, which we denote by T_0. Assume that this change involves a latent heat of recrystallization, and that the cross-section of the bar does not vary along its length. Let the bar be preheated in such a manner that its initial temperature is T_0 throughout, and so that it is originally in the crystalline form corresponding to the lower energy state. If a constant heat source is applied at one end of the bar, recrystallization will occur, and a boundary line will be propagated along the bar separating the recrystallized segment and that portion which remains in its original state. After an appropriate choice of units for temperature, time, heat, and length, the motion of the interface and the temperature of the bar at any time $t \geqq 0$ will satisfy (1.1), as long as $x(t)$ is less than the length of the bar. For the sake of convenience, we may assume $L = \infty$, for if the bar should ever be completely recrystallized, the question of finding its temperature reduces to a classical, linear, heat flow problem.

Problems such as this, involving the solution of a parabolic equation subject to an "extra" boundary condition (1.1e) which defines the position of the unknown boundary, have been treated in the recent literature under the name of "Stefan problems" [1, 2, 3, 4, 5, 6, 11, 12]. Evans [5], Rubin-

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[1] Throughout this paper, whenever a derivative is taken at the boundary $x(t)$, it is to be interpreted as a derivative from the left.

stein [11], and Sestini [12], have demonstrated the existence of a solution of (1.1) by means of an iterative process based on equation (6.4) derived below. Douglas and Gallie [3] have demonstrated existence for all t such that

$$0 \leqq x(t) < 1,$$

by means of a numerical approach based on an implicit finite-difference analogue for (1.1a). Crank [1], Ehrlich [4], and Landau [9] have proposed finite-difference methods for the solution of more general Stefan problems, but have not given proofs of convergence. Douglas [2] has demonstrated the uniqueness of the solution for a generalization of (1.1).

In the present work, an explicit numerical analogue for (1.1) is formulated. It is shown that the mesh functions so defined converge to $x(t)$, and $u(x, t)$, the solutions of (1.1), in $0 \leqq t < \infty$, and that the convergence is uniform in every finite interval. The existence of the solution is not assumed beforehand; rather, it is shown to be a consequence of the convergence of the mesh functions and an existence theorem for a classical boundary value. It is also shown that the explicit finite-difference algorithm, when applied to a classical problem related to (1.1) defines mesh functions which converge to the solution as the mesh size tends to zero.

To the author's knowledge, the only previous proof of convergence of a numerical integration of a Stefan problem is that of Douglas and Gallie.

2. Some preliminary results. In the following, a function $u(x, t)$ will be said to be regular in a domain D if u_t and u_{xx} are continuous in D.

The question of convergence of the numerical solution of the Stefan problem will be reduced to the same question for a classical boundary value problem, which we now describe.

Let $x(t)$ be given so that

$$(2.1)\quad \begin{aligned} &\text{(a)} \quad \begin{cases} x(0) = 0, \\ x(t) > 0 \end{cases} && (t > 0). \\ &\text{(b)} \quad 0 \leqq x(t_2) - x(t_1) \leqq t_2 - t_1 && (t_2 > t_1 \geqq 0). \end{aligned}$$

LEMMA 1. *If $x(t)$ satisfies (2.1), there exists a unique function $u(x, t)$ such that*

$$(2.2)\quad \begin{aligned} &\text{(a)} \quad u_t(x, t) = u_{xx}(x, t), && (0 < x < x(t), t > 0). \\ &\text{(b)} \quad u_x(0, t) = -1 && (t > 0). \\ &\text{(c)} \quad u(x, t) = 0, && (x \geqq x(t), t \geqq 0), \end{aligned}$$

and u is regular in

$$R = \{(x, t) \mid 0 < x < x(t), \quad t > 0\}.$$

and continuous in the closed first quadrant $x \geqq 0$, $t \geqq 0$.

This is a consequence of a general theorem stated in [7].

In order to study the solution of (2.2), it is useful to examine the function $v(x, t)$ which satisfies

$$(2.3) \quad \begin{array}{lll} \text{(a)} & v_t(x, t) = v_{xx}(x, t), & (-x(t) < x < x(t), t > 0). \\ \text{(b)} & v(\pm x(t), t) = x(t) & (t \geqq 0), \end{array}$$

and is regular in the domain $|x| < x(t)$, $t > 0$. The existence and uniqueness of $v(x, t)$ follows from (2.1) and the classical theorem given in [7].

LEMMA 2. *Let $u(x, t)$ and $v(x, t)$ be the solutions of* (2.2) *and* (2.3) *respectively. Then*

(2.4)
(a) $u(x, t) = v(x, t) - x$
(b) $0 \leqq u(x, t) \leqq x(t) - x$
(c) *$u(x, t)$ has continuous derivatives of all orders in $0 \leqq x < x(t)$, $t > 0$.*

Proof. (a) $v(-x, t)$ also satisfies (2.3) and is regular between the two boundary curves. By uniqueness, $v(-x, t) = v(x, t)$ and therefore $v_x(0, t) = 0$. The statement (a) now follows immediately from the uniqueness theorem for (2.2), since $v(x, t) - x$ is a regular solution of (2.2).

(b) We give a proof due to Evans [5]. From the maximum-minimum theorem [10] for parabolic equations, and the monotonicity of $x(t)$, it follows that

$$0 \leqq v(x, t) \leqq x(t), \qquad |x| \leqq x(t).$$

Therefore, from (a)

$$-x \leqq u(x, t) \leqq x(t) - x, \qquad 0 \leqq x \leqq x(t),$$

and we need only show that $u(x, t) \geqq 0$. Let (x_0, t_0) be such that $u(x_0, t_0) < 0$. Then by (2.2c) and the maximum-minimum theorem, there exists t_1 with $0 < t_1 \leqq t_0$ such that $u(0, t_1)$ is the minimum of $u(x, t)$ in $0 \leqq x \leqq x(\tau)$, $0 \leqq \tau \leqq t_0$. But then

$$u_x(0, t_1) = \lim_{x \downarrow 0} \frac{u(x, t_1) - u(0, t_1)}{x} \geqq 0,$$

which contradicts (2.2b).

(c) If a function is a regular solution of the heat equation in a domain

D, then all of its derivatives exist in that domain [8]. Hence, $v(x, t)$ has derivatives of all orders in $-x(t) < 0 < x(t), t > 0$.

But

$$\frac{\partial^n u(x,t)}{\partial x^n} = \frac{\partial^n v(x,t)}{\partial x^n} \qquad (0 \leqq x < x(t)),$$

for $n > 1$. (Note that no implication is made concerning the existence or continuity of derivatives at the boundary $x = x(t)$.)

COROLLARY. *If $u(x, t)$ is the solution of (2.2), then*

$$0 \leqq u_t(x, t) \leqq 1, \qquad 0 \leqq x < x(t), \qquad t > 0. \tag{2.5}$$

Consequently, $u_x(x, t)$ is uniformly continuous in x in the closed interval $0 \leqq x \leqq x(t)$ for every $t > 0$.

Proof. Let $\Delta t > 0$, fixed but arbitrary, and let

$$Q(x,t) = \frac{v(x, t + \Delta t) - v(x,t)}{\Delta t}; \qquad -x(t) \leqq x \leqq x(t),$$

where $v(x, t)$ is the solution of (2.3). Then $Q(x, t)$ is continuous in $-x(t) \leqq x \leqq x(t)$, $t \geqq 0$, and satisfies the heat equation in the interior of this set. Also, by (2.4a) and (2.2c),

$$Q[\pm x(t), t] = \frac{u[x(t), t + \Delta t]}{\Delta t}, \qquad t \geqq 0$$

Therefore, by (2.4b) and (2.1)

$$0 \leqq Q[\pm x(t), t] \leqq \frac{x(t + \Delta t) - x(t)}{\Delta t} \leqq 1, \qquad t \geqq 0.$$

and it follows from the maximum principle that

$$0 \leqq Q(x, t) \leqq 1, \qquad |x| \leqq x(t), \qquad t \geqq 0.$$

But from the definition of $Q(x, t)$, and the fact that $v_t(x, t) = u_t(x, t)$ for $0 \leqq x < x(t)$, (2.5) follows by letting Δt tend to zero. The statement concerning u_x follows from (2.5), the mean value theorem, and (2.2a).

3. The difference equations. In this section we will define an explicit numerical procedure which will be applied to the Stefan problem (1.1), and to the classical problem (2.2), where a boundary curve $x(t)$ is given beforehand satisfying (2.1). It is convenient to consider the latter problem first.

Let $1 > h > 0$ and $k = \lambda h^2$, where λ is a constant such that

$$0 < \lambda \leqq \tfrac{1}{2}. \tag{3.1}$$

The quantities h and k will be taken to be increments in x and t, respectively. Denote

$$x(nk) = x(n) \qquad (n = 0, 1, 2, \cdots),$$

and let

$$M_n = [x(n)/h] - 1, \tag{3.2}$$

where $[z]$ is the greatest integer in z. Also, define

$$P_n = \frac{x(n) - M_n h}{h}.$$

Then

$$1 \leqq P_n < 2. \tag{3.3}$$

Let $\mathfrak{M}(h)$ be the set of mesh points

$$\mathfrak{M}(h) = \left\{(mh, nk) \;\middle|\; \begin{matrix} m = 0, 1, & \cdots, & M_n \\ n = 0, 1, 2, & \cdots. & \end{matrix}\right\}.$$

We will be considering functions of two variables which are defined on points of $\mathfrak{M}(h)$. For a function $u(x, t)$, it will be convenient to denote its values on mesh points by

$$u(mh, nk) = u(m, n).$$

Now let $x(t)$ be given, satisfying (2.1), and let $f(t)$ and $g(t)$ be functions such that a solution $w(x, t)$ exists for the boundary value problem:

$$\begin{aligned} &\text{(a)} \qquad w_t(x, t) = w_{xx}(x, t), && (0 < x < x(t)).\\ (3.4) \quad &\text{(b)} \qquad w_x(0, t) = f(t), && (0 < t \leqq T).\\ &\text{(c)} \qquad w(x(t), t) = g(t), && (0 \leqq t \leqq T). \end{aligned}$$

Assume furthermore that f and g are such that

$$\left| \frac{\partial^r w}{\partial x^r}(x, t) \right| \leqq K \qquad (r = 1, 2, 3, 4) \tag{3.5}$$

in the closed set

$$R_T = \{(x, t) \mid 0 \leqq x \leqq x(t), 0 \leqq t \leqq T\}.$$

Let $n_0 = n_0(h)$ be the first integer such that $M_{n_0} = 3$. Then for $n \geqq n_0$, the following relationships obtain between values of $w(x, t)$ on mesh points of $\mathfrak{M}(h)$ for which $(n + 1)k \leqq T$:

$$
\begin{aligned}
&\text{(a)}\quad w(0, n+1) = w(1, n+1) - hf(n+1) + K\cdot 0(h^2)\\
&\text{(b)}\quad w(m, n+1) = \lambda w(m-1, n) + (1-2\lambda)w(m, n)\\
&\qquad + \lambda w(m+1, n) + K\cdot 0(h^4) \qquad (m = 1, 2, \cdots, M_n - 1).\\
&\text{(c)}\quad w(M_n, n+1) = \frac{2\lambda}{1+P_n} w(M_n - 1, n) + \left(1 - \frac{2\lambda}{P_n}\right) w(M_n, n)\\
&\qquad + \frac{2\lambda g(n)}{(1+P_n)P_n} + K\cdot 0(h^3).
\end{aligned} \tag{3.6}
$$

Also, if $M_{n+1} = 1 + M_n$, which will occur if

$$x(n) \leqq (M_n + 2)h \leqq x(n+1),$$

then

$$
\begin{aligned}
w(M_{n+1}, n+1) &= \frac{P_{n+1}}{1+P_{n+1}} w(M_n, n+1)\\
&\qquad + \frac{g(n+1)}{1+P_{n+1}} + K\cdot 0(h^2).
\end{aligned} \tag{3.6d}
$$

In each equation, the quantities $0(h^2)$, $0(h^3)$, and $0(h^4)$ are independent of K.

A detailed derivation of these equations is given in [13], and consequently only a brief explanation need be given here. Equation (3.6a) is a discrete analogue for (3.4b), while (3.6b) follows from (3.4a) for the case of equally spaced values of the variable x. Equation (3.6c) also approximates (3.4a) with unequal spacings in x, and it makes use of the boundary condition (3.4c). The last relation, (3.6d), is simply linear interpolation for $w(M_{n+1}, n+1)$ based on $w(M_n, n+1)$ and the boundary value $g(n+1)$. From (2.1), the fact that h is less than unity, and (3.1), it follows that either

$$M_{n+1} = M_n$$

or

$$M_{n+1} = 1 + M_n .$$

Since that portion of (3.6) which is homogeneous in $w(m, n)$ will be used several times throughout the remainder of the paper, it will be useful to formulate it in a more convenient manner. Let $\tilde{S}$ be a $(1 + M_n)$-dimensional vector

$$\tilde{S} = \begin{bmatrix} s(0) \\ s(1) \\ \vdots \\ s(M_n) \end{bmatrix}$$

and $\tilde{H}(n)$ be the operator on $\tilde{S}$ which produces the $(1 + M_{n+1})$-dimensional vector

$$\tilde{R} = \tilde{H}(n)\tilde{S},$$

with

$$\begin{aligned}
&\text{(a)} \quad r(0) = r(1),\\
&\text{(b)} \quad r(m) = \lambda s(m-1) + (1-2\lambda)s(m) + \lambda s(m+1)\\
&\qquad\qquad (m = 1, 2, \cdots M_n - 1),\\
&\text{(c)} \quad r(M_n) = \frac{2\lambda}{1+P_n} s(M_n - 1) + \left(1 - \frac{2\lambda}{P_n}\right) s(M_n),
\end{aligned} \tag{3.7}$$

and if $M_{n+1} = 1 + M_n$, then $r(M_{n+1})$ is given by

$$\text{(d)} \quad r(M_{n+1}) = \frac{P_{n+1}}{1 + P_{n+1}} r(M_n)$$

Clearly $\tilde{H}(n)$ is a linear operator dependent on the sequence $\{x(n)\}$ and the mesh size h. A fundamental property of $\tilde{H}(n)$ is the following:

LEMMA 3. *If the norm ($\| \ \|$) of a vector is taken to be the maximum of the absolute values of its components, then*

$$\| \tilde{R} \| = \| \tilde{H}(n)\tilde{S} \| \leqq \| \tilde{S} \|, \tag{3.8}$$

or in words, $\tilde{H}(n)$ is "stable". Furthermore, if all components of $\tilde{S}$ are nonnegative, so are those of $\tilde{R}$.

Proof. The assertion follows from the fact that the coefficients in (3.7) are all nonnegative, and the sum of the coefficients in each equation does not exceed unity. These properties are consequences of (3.1) and (3.3).

In terms of the operator $\tilde{H}(n)$, we can write (3.6) as

$$\tilde{W}(n+1) = \tilde{H}(n)\tilde{W}(n) + \tilde{S}(n+1),$$

where

$$\tilde{W}(n) = \begin{bmatrix} w(0, n) \\ w(1, n) \\ \vdots \\ w(M_n, n) \end{bmatrix} \tag{3.9}$$

and $\tilde{S}(n+1)$ is a $(1 + M_{n+1})$-dimensional vector whose components can be ascertained by inspection of (3.6).

The statements concerning the size of the error terms in (3.6) are not directly applicable to the classical problem (2.2) or to the Stefan problem. For the former, although the existence of a solution is guaranteed by Lemma 1, there is no reason to expect that it has four bounded derivatives in R_T for any $T > 0$. Obviously, nothing can be said about the latter at this time, since we have not yet shown that a solution exists. Nevertheless, these estimates will be used to advantage in §5, in a situation where the hypothesis of four bounded derivatives will automatically be satisfied. For the present, we will make use of the transformations $\{\tilde{H}(n)\}$ to formulate numerical methods for (1.1) and (2.2).

We consider first the case where the boundary curve $x(t)$ is given in advance. For each $h > 0$, we define a function $v(x, t, h)$ in $0 \leqq x \leqq x(t)$, $t > 0$. The function will be given on mesh points by the following difference equations, and intermediate values may be thought of as being obtained by linear interpolation, and the condition that

$$(3.10) \qquad v(x, t, h) = 0, \qquad t \geqq 0, \qquad x \geqq x(t).$$

Let

$$(3.11) \quad v(x, t, h) = x(t) - x, \qquad 0 \leqq x \leqq x(t), \qquad 0 \leqq t \leqq n_0 k.$$

For $n \geqq n_0$, let the mesh point values be defined inductively by:

$$(3.12) \qquad \tilde{V}(n+1) = \tilde{H}(n)\tilde{V}(n) + \tilde{D},$$

where $\tilde{V}(n)$ is defined in analogy with $\tilde{W}(n)$ in (3.9) and

$$\tilde{D} = \begin{bmatrix} h \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus, $v(m, n)$ satisfies recurrence relations of the form (3.6) with $K = g(n) = 0$ and $f(n) = -1$.

For the Stefan problem, it is necessary to compute an approximate boundary curve $x(t, h)$. Let

$$(3.13) \qquad x(t, h) = t, \qquad 0 \leqq t \leqq n_0 k,$$

and for $n \geqq n_0$, let

$$(3.14) \qquad x(n+1) = x(n) + k\,\frac{v(M_n, n)}{hP_n},$$

with $v(x, t, h)$ defined as before by (3.10, 3.11, 3.12). Since the operator $\tilde{H}(n)$ depends only on $x(n+1)$, while the latter depends only on $x(t, h)$ and $v(x, t, h)$ for $0 \leqq t \leqq nk$, the computational process is well defined, provided $x(t, h)$ satisfies (2.1). It will be shown in §4 that this is the case. Of course, for the Stefan problem, $\tilde{H}(n)$ depends on $v(m, r)$ for $r \leqq n$, since $x(n)$ is a function of these values. Consequently, $\{\tilde{H}(n)\}$ is no longer a linear sequence of operators, but Lemma 3 still holds, since it follows from the properties of the coefficients in (3.7).

4. Properties of the mesh functions. As a preliminary to the convergence proof, we establish some results concerning the behavior of the mesh functions.

For convenience, define the difference quotients $\Delta v(m, n)$ and $\Delta x(n)$ by:

$$\Delta v(m, n) = \frac{v(m, n+1) - v(m, n)}{k} \qquad (m = 0, 1, \cdots, M_n; n \geqq n_0) \tag{4.1}$$

$$\Delta x(n) = \frac{x(n+1) - x(n)}{k} \qquad (n = 0, 1, 2, \cdots). \tag{4.2}$$

Also let

$$P'_{n+1} = P_n + \frac{x(n+1) - x(n)}{k}.$$

From (3.12), and the definition of $\tilde{H}(n)$,

$$\text{(a)}\ \Delta v(m, n) = \frac{v(m-1, n) - 2v(m, n) + v(m+1, n)}{h^2} \qquad (1 \leqq m \leqq M_n - 1). \tag{4.3}$$

$$\text{(b)}\ \Delta v(M_n, n) = \frac{2}{h^2}\left[\frac{v(M_n - 1, n)}{1 + P_n} - \frac{v(M_n, n)}{P_n}\right].$$

Also, if $x(t)$ is not given beforehand,

$$\Delta x(n) = \frac{v(M_n, n)}{hP_n}. \tag{4.4}$$

We now state the recurrence relations which are satisfied by $\Delta v(m, n)$.

LEMMA 4. Let $\Delta x(r)$ satisfy

$$0 \leqq \Delta x(r) \leqq 1 \qquad (r = n_0, n_0 + 1, \cdots, n).$$

Then if $M_{n+1} = M_n$, the difference quotients $\Delta v(m, n+1)$ are obtained by means of the following formulas:

(4.5)

$$\text{(a)}\quad \Delta v(0, n+1) = \Delta v(1, n+1)$$

$$\text{(b)}\quad \Delta v(m, n+1) = \lambda \Delta v(m-1, n) + (1 - 2\lambda)\Delta v(m, n) + \lambda \Delta v(m+1, n) \qquad (m = 1, 2, \cdots, M_n - 1).$$

$$\text{(c)}\quad \left[1 + \frac{\lambda h \Delta x(n)}{1 + P_n}\right]\Delta v(M_n, n+1) = \frac{2\lambda}{1 + P_n}\Delta v(M_n - 1, n) + \left(1 - \frac{2\lambda}{P_n}\right)\Delta v(M_n, n) + \frac{2\lambda \Delta x(n)}{(1 + P_n)P_n}\frac{v(M_n, n+1)}{hP'_{n+1}}.$$

If $M_{n+1} = 1 + M_n$, then equations (4.5a) and (4.5b) still hold, but $\Delta v(M_n, n + 1)$ is obtained by multiplying the right hand side of (4.5c) by the factor $(1 + P'_{n+1})/2$ and

$$\Delta v(M_{n+1}, n + 1) = 0. \tag{4.5d}$$

The derivation of these relations is a straightforward matter, and the details are given in [13].

This lemma has as a consequence the following important theorem:

THEOREM 1. *If $x(t)$ is given, satisfying (2.1), and $v(m, n)$ is defined by means of the difference equations (3.12) for $n > n_0$, and by (3.11) for $n = n_0$, then for $n \geqq n_0$:*

$$\begin{aligned}
&\text{(a)} \quad v(m, n) \geqq 0 \qquad (m = 0, 1, \cdots, M_n).\\
&\text{(b)} \quad \Delta v(m, n) \geqq 0 \qquad (m = 0, 1, \cdots, M_n).\\
(4.6) \quad &\text{(c)} \quad 0 \leqq v(m, n) - v(m + 1, n) \leqq h \qquad (m = 0, 1, \cdots, M_n - 1).\\
&\text{(d)} \quad v(m, n) \leqq x(n) - mh \qquad (m = 0, 1, \cdots, M_n).
\end{aligned}$$

Furthermore, if $x(t)$ is not given beforehand, but is defined by (3.13) and (3.14), then the above assertions still hold for $v(m, n)$, and

$$0 \leqq \Delta x(n) \leqq 1, \qquad (n = 0, 1, 2, \cdots). \tag{4.7}$$

Proof. We proceed by induction on n. The conclusions of the theorem are obviously true for $n = n_0$. Assume only (4.6a) and (4.6b) are true the nth time interval. We will show first that (4.6c, d) and (4.7) follow from this assumption.

From (3.12), (4.3), and (4.6b),

$$\begin{aligned}
h &= v(0, n) - v(1, n) \geqq v(1, n) - v(2, n) \geqq \cdots\\
&\geqq v(M_n - 1) - v(M_n, n) \geqq \frac{v(M_n, n)}{P_n}.
\end{aligned}$$

This is the proof of (4.6c). The statement (4.6d) follows obviously from this, since

$$\begin{aligned}
v(m, n) &= [v(m, n) - v(m + 1, n)] + [v(m + 1, n) - v(m + 2, n)] + \cdots\\
&\quad + [v(M_n - 1, n) - v(M_n, n)] + v(M_n, n)\\
&\leqq (M_n - m)h + hP_n\\
&= x(n) - mh.
\end{aligned}$$

For the Stefan problem, (4.7) follows from (3.14), (4.6a), and (4.6d) with

$m = M_n$. This yields the important fact that the boundary curve $x(t, h)$ has the properties (2.1).

To see that $v(m, n + 1) \geqq 0$ for $m = 0, 1, \cdots, M_{n+1}$, we need only use the induction assumption (4.6a) and the second conclusion in Lemma 3. The fact that $\Delta v(m, n + 1) \geqq 0$ for $m = 0, 1, \cdots M_{n+1}$ follows from the nonnegativity of the coefficients in the recurrence relations of Lemma 4, and the induction assumption (4.6b). This completes the induction.

It is now possible to derive an important relationship between the functions $x(t, h)$ and $v(x, t, h)$, which are proposed as numerical approximations to the solution of the Stefan problem (1.1).

THEOREM 2. *Let $x(t, h)$ and $v(x, t, h)$ be the functions proposed in §3 for the solution of the Stefan problem* (1.1). *Then for every t of the form $t = nk$ such that $t > n_0 k$, the following relationship obtains between $x(t, h)$ and $v(x, t, h)$:*

$$x(t, h) = t[1 + 0(h)] - \int_0^{x(t,h)} v(x, t, h)\, dx, \tag{4.8}$$

where the quantity $0(h)$ is independent of t.

Proof. It is meaningful to speak of the integral of $v(x, t, h)$, since this function was extended between mesh points by linear interpolation. We first derive a discrete analogue of (4.8), relating the values of $x(t, h)$ and $v(x, t, h)$ on mesh points. From the relations (4.3):

$$hk \sum_{m=1}^{M_n} v(m, n) = \frac{k}{h} \sum_{m=1}^{M_n - 1} [v(m - 1, n) - 2v(m, n) + v(m + 1, n)] + \frac{2k}{h}\left[\frac{v(M_n - 1, n)}{1 + P_n} - \frac{v(M_n, n)}{P_n}\right].$$

Using the fact that

$$\frac{v(0, n) - v(1, n)}{h} = 1,$$

and (4.4), this equation can be written

$$kh \sum_{m=1}^{M_n} \Delta v(m, n) = k - k\Delta x(n) - \frac{(P_n - 1)hk}{2} \Delta v(M_n, n).$$

Now let $N > n_0$ and sum both sides of the last equation from $n = n_0$ to $n = N$ to obtain:

$$kh \sum_{n=n_0}^{N} \sum_{m=1}^{M_n} \Delta v(m, n) = (N + 1)k - x(N + 1) - n_0 k + x(n_0) - \frac{hk}{2} \sum_{n=n_0}^{N} (P_n - 1)\Delta v(M_n, n). \tag{4.9}$$

By interchanging the order of summation on the left, we obtain

$$kh\sum_{m=1}^{M_N} \sum_{n=n_m}^{N} \Delta v(m, n) = \sum_{m=1}^{M_N} v(m, N+1)h - \sum_{m=1}^{M_N} v(m, n_m)h, \tag{4.10}$$

where for $m = 1, 2$, the quantity $v(m, n_m)$ is the initial value given by (3.11) and for $m > 2$, it is the interpolated value given by (3.7d). Therefore, from (4.6d)

$$\sum_{m=1}^{M_N} v(m, n_m)h \leqq h\sum_{m=1}^{M_N} (3h) = 3M_N h^2. \tag{4.11}$$

By the definition of n_0,

$$0 \leqq x(n_0) - n_0 k < k. \tag{4.12}$$

Also, the last sum on the right of (4.9) is dominated by

$$Q = \frac{hk}{2} \sum_{n=n_0}^{N} \Delta v(M_n, n),$$

which can be written as

$$Q = \frac{h}{2} \sum_{m=2}^{M_N} \sigma_m,$$

where σ_m is the growth of $v(m, n)$ during the time interval for which

$$(m+1)h \leqq x(n) \leqq (m+2)h.$$

Therefore

$$\sigma_m < 3h,$$

by (4.6d), and we have

$$Q \leqq \tfrac{3}{2} M_N h^2. \tag{4.13}$$

By substituting (4.10) into (4.9) and applying the estimates (4.11), (4.12), (4.13), we obtain

$$| x(N+1) - (N+1)k + \sum_{m=1}^{M_N} v(m, N+1)h | \leqq k + \tfrac{5}{2} M_N h^2$$

which, after the substitution $t = (N+1)k$ takes the form:

$$\left| x(t, h) - t + \sum_{m=1}^{M_N} hv(mh, t, h) \right| \leqq k + \frac{5h}{2} x(t, h).$$

But, by Theorem 1, the function $v(x, t, h)$ satisfies a Lipschitz condition in x, independent of t. Hence

$$\int_0^{x(t,h)} v(x, t, h)\, dx = \sum_{m=1}^{M_N} v(mh, t, h)h + x(t, h)0(h),$$

where $0(h)$ is independent of t. Since $x(t, h) < t$ by Theorem 1, the proof of Theorem 2 is now complete.

COROLLARY. *For every* $t = nk$ *with* $n \geqq n_0$,

$$x(t, h)\left[1 + \frac{x(t, h)}{2}\right] \geqq t[1 + 0(h)]. \tag{4.14}$$

Proof. By Theorem 1,

$$0 \leqq v(x, t, h) \leqq x(t, h) - x.$$

Hence

$$x(t, h) + \int_0^{x(t,h)} v(x, t, h)\, dx \leqq x(t, h) + \int_0^{x(t,h)} [x(t, h) - x]\, dx$$

$$= x(t, h)\left[1 + \frac{x(t, h)}{2}\right].$$

To obtain (4.14), substitute this in (4.8).

We will now state a lemma which will be useful later.

LEMMA 5. Let $x(t)$ and $y(t)$ be two functions satisfying (2.1), and $v(x, t, h)$ and $u(x, t, h)$ be the mesh functions defined (for the same $h > 0$), for $x(t)$ and $y(t)$ respectively, by means of the difference equations (3.10), (3.11), (3.12). Then if

$$| x(t) - y(t) | \leqq \epsilon, \tag{4.15}$$

in $0 \leqq t \leqq T$, it follows that

$$| u(x, t, h) - v(x, t, h) | \leqq \epsilon + 4h. \tag{4.16}$$

in $0 \leqq x < \infty$, $0 \leqq t \leqq T$.

The proof of this lemma is given in [13]. It makes use of the stability of the linear operators defined in §3, and of (4.6d). For an analogous theorem concerning the boundary value problem (2.2), the reader is referred to Douglas and Gallie [3].

5. Convergence of the mesh functions for the classical problem. In this section we will demonstrate the convergence of a numerical solution for (3.4), under the strong assumption (3.5). We will then remove this assumption and obtain a similar convergence theorem for the less general problem (2.2). In each case it is assumed that $x(t)$ satisfies (2.1).

For (3.4), assume that in some way, appropriate values $z(m, n_0)$ are defined for $m = 0, 1, 2$. Then define $z(m, n)$ on mesh points of $\mathfrak{M}(h)$ by requiring that $z(m, n)$ satisfy (3.6) with K taken to be zero. Then we have the following theorem:

THEOREM 3. *Under the hypotheses* (3.5), (5.1) $\quad | z(x, t, h) - w(x, t) | \leqq \sigma + KT \cdot 0(h),\ (x, t) \in R_T \cap \mathfrak{M}(h),$

where

$$\sigma = \max_m | z(m, n_0) - w(m, n_0) |.$$

Proof. Let

$$e(m, n) = w(m, n) - z(m, n).$$

By subtracting each of the equations defining $z(m, n)$ from its counterpart in (3.6), we obtain

$$\tilde{E}(n + 1) = \tilde{H}(n)\tilde{E}(n) + \tilde{T}(n + 1); n_0 \leqq n \leqq \left[\frac{T}{k}\right],$$

where

$$\tilde{T}(n + 1) = K \begin{bmatrix} 0(h^2) \\ 0(h^4) \\ \vdots \\ 0(h^4) \\ 0(h^3) \end{bmatrix}$$

if $M_{n+1} = M_n$, and

$$\tilde{T}(n + 1) = K \begin{bmatrix} 0(h^2) \\ 0(h^4) \\ \vdots \\ 0(h^4) \\ 0(h^3) \\ 0(h^2) \end{bmatrix}$$

if $M_{n+1} = 1 + M_n$. In both these vectors $0(h^4)$ occurs $M_n - 1$ times.

Since the operators $\tilde{H}(n)$ for a given $x(t)$ are linear, we may think of $\tilde{E}(n)$ as the superposition of four vectors

$$\tilde{E}(n) = \textstyle\sum_{i=1}^{4} \tilde{E}_i(n),$$

where

$$\| \tilde{E}_i(n_0) \| = \delta_{i4}\sigma \qquad (i = 1, 2, 3, 4) \tag{5.2}$$

and

$$\tilde{E}_i(n + 1) = \tilde{H}(n)\tilde{E}_i(n) + \tilde{T}_i(n + 1) \quad (n_0 \leqq n \leqq [T/k]) \tag{5.3}$$

with

$$\tilde{T}_i(n+1) = K \begin{bmatrix} \delta_{i1}\, 0(h^2) \\ \delta_{i2}\, 0(h^4) \\ \vdots \\ \delta_{i2}\, 0(h^4) \\ \delta_{i2}\, 0(h^3) \end{bmatrix},$$

if $M_{n+1} = M_n$ and

$$\tilde{T}_i(n+1) = K \begin{bmatrix} \delta_{i1}\, 0(h^2) \\ \delta_{i2}\, 0(h^4) \\ \vdots \\ \delta_{i2}\, 0(h^4) \\ \delta_{i2}\, 0(h^3) \\ \delta_{i3}\, 0(h^2) \end{bmatrix}$$

if $M_{n+1} = 1 + M_n$. Here again $0(h^4)$ occurs $M_n - 1$ times in both cases. The symbol δ_{ik} is the Kronecker delta.

It is convenient to estimate the norms of the vectors $E_i(n)$ separately, making use of the stability of the operators as stated in Lemma 3.

$\tilde{E}_1(n)$: From the mean value theorem and the fact that all coefficients in $H(n)$ are nonnegative, it follows that

$$\| \tilde{E}_1(n) \| \leqq \| \tilde{A}(n) \|, \qquad (n_0 \leqq n \leqq [T/k]).$$

where

$$\tilde{A}(n_0) = 0 \tag{5.4}$$

and

$$\tilde{A}(n+1) = \tilde{H}(n)\tilde{A}(n) + \begin{bmatrix} \frac{Kh^2}{2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (n_0 < n \leqq [T/k]).$$

But by dividing both sides of the last equation by $Kh/2$, it can be seen from (3.11), (3.12), (4.6d), and (5.4) that

$$\| \tilde{A}(n) \| \leqq Khx(n)/2.$$

Therefore

$$\| \tilde{E}_1(n) \| \leqq TKh/2 \qquad (0 \leqq n \leqq [T/k]).$$

$\tilde{E}_2(n)$: From (5.2) and the repeated application of (5.3) for $i = 2$, it follows that

$$\| \tilde{E}_2(n) \| \leqq \sum_{k=n_0+1}^{n} \| \tilde{T}_2(k) \|$$
$$\leqq nK \cdot 0(h^3) < TK \cdot 0(h) \qquad (n_0 < n < [T/k]).$$

$\tilde{E}_3(n)$: The components of $\tilde{T}_3(n)$ are all zero except for values of n where interpolation is necessary. Now, for those values of n under consideration, the number of times that this occurs does not exceed the ratio $x(T)/h$, which is by assumption less than T/h. Hence,

$$\| \tilde{E}_3(n) \| \leqq \sum_{k=n_0+1}^{n} \| \tilde{T}_3(k) \| \leqq KT \cdot 0(h).$$

$\tilde{E}_4(n)$: Since the error vector $\tilde{T}_4(n)$ vanishes for all n,

$$\| \tilde{E}_4(n) \| \leqq \| \tilde{E}_4(n_0) \| = \sigma.$$

These estimates can be combined to yield (5.1).

THEOREM 4. *Let $x(t)$ be given, satisfying (2.1), and let $u(x, t)$ be the solution of (2.2), and $v(x, t, h)$ be defined for the curve $x(t)$ by (3.11), (3.12). Then for any $T > 0$ and $\epsilon > 0$, there exists a quantity $h_0 > 0$ such that $h \leqq h_0$ implies:*

$$| v(x, t, h) - u(x, t) | \leqq \epsilon$$

on all mesh points of $\mathfrak{M}(h) \cap R_T$.

Proof. Let

$$y_\epsilon(t) = x(t) - \epsilon/4; t \geqq t_\epsilon,$$

where

$$t_\epsilon = \sup \{t \mid x(t) \leqq \epsilon/4\}.$$

Then by Lemma 2, statement (2.4c), $u(x, t)$ has four uniformly bounded x-derivatives in $0 \leqq x \leqq y_\epsilon(t)$, $t_\epsilon \leqq t \leqq T$. Let K_ϵ be this common bound, and

$$g_\epsilon(t) = u(y_\epsilon(t), t),$$
$$f_\epsilon(t) = -1$$

for $t \geqq t_\epsilon$. Take n_1 to be the first integer such that $y(n_1 k) \geqq 3h$, and define a mesh function $v_1(m, n)$ in

$$R_\epsilon(T) = \{0 \leqq x \leqq y_\epsilon(t), t_\epsilon \leqq t \leqq T\}$$

by

$$v_1(m, n_1) = v(m, n_1)$$

for $m = 0, 1, 2$ and for $n \geqq n_1$ by the recurrence formulas (3.6) with

$$f(n) = -1,$$
$$g(n) = g_\epsilon(n),$$
$$K = 0,$$

and $x(t)$ replaced by $y_\epsilon(t)$. Then since the hypotheses of Theorem 3 are satisfied,

(5.5) $$| v_1(m, n) - u(m, n) | \leqq K_\epsilon T \cdot 0(h)$$

in $R_\epsilon(T) \cap \mathfrak{M}(h)$.

Now define $v_2(m, n)$ in $R_\epsilon(T) \cap \mathfrak{M}(h)$ by means of (3.11), (3.12) with $x(t)$ replaced by $y_\epsilon(t)$. From (2.4b),

$$| g(n) | \leqq \epsilon/4, n = n_1, n_1 + 1, \cdots,$$

and therefore from the stability of the difference operators,

(5.6) $$| v_2(m, n) - v_1(m, n) | \leqq \epsilon/4$$

on all mesh points under consideration. However, if $v_2(m, n)$ is taken to be zero to the right of $y_\epsilon(t)$, it follows from Lemma 5 and (4.6d) that

(5.7) $$| v_2(m, n) - v(m, n) | \leqq 4h + \epsilon/4$$

in $R(T) \cap \mathfrak{M}(h)$. Combining estimates (5.5), (5.6), (5.7), we conclude that

$$| v(m, n) - u(m, n) | < K_\epsilon T \cdot 0(h) + \epsilon/2.$$

To complete the proof, choose h so that

$$K_\epsilon T \cdot 0(h) < \epsilon/2.$$

6. Convergence of the mesh functions for the Stefan problem. In §4 it was shown that for every $h > 0$, the curve $x(t, h)$ computed for the Stefan problem satisfies the conditions (2.1). Hence the family of functions

$$F = \{x(t, h) \mid h > 0\}$$

is equibounded and equiuniformly continuous for all $t \geqq 0$. If G is an infinite subfamily of F, it follows from the theorem of Arzela and Ascoli that there is a sequence $\{x(t, h_r)\}$ of functions in G which converges to a limit $x(t)$ in $t \geqq 0$, and uniformly in every finite interval. Hence, we may choose a sequence $\{h_r\}$ with the following properties:

$$(6.1)\quad \begin{aligned} &\text{(a)}\quad h_r > h_{r+1} > 0 \\ &\text{(b)}\quad \lim_{r\to\infty} h_r = 0 \\ &\text{(c)}\quad h_{r+1} \text{ divides } h_r . \\ &\text{(d)}\quad \text{The sequence } \{x(t,\ h_r)\} \text{ converges to a limit } x(t) \text{ in } t \geqq 0, \\ &\qquad \text{and uniformly in every finite interval.} \end{aligned}$$

LEMMA 6. *The limit function $x(t)$, defined by* (6.1), *has properties* (2.1).

Proof. The Lipschitz condition (2.1b) is obvious, since every approximating function $x(t,\ h_r)$ satisfies the condition. To see that $x(t) > 0$ for $t > 0$, simply let h tend to zero in (4.14).

Now let $T > 0$, and let $\mathfrak{M}(h_r)$ be the mesh corresponding to h_r, for the curve $x(t)$, as defined in §5. From (6.1c) every mesh point of $\mathfrak{M}(h_r)$ is also a mesh point of $\mathfrak{M}(h_{r+1})$. Let

$$\mathfrak{M} = \bigcup_{r=0}^{\infty} \mathfrak{M}(h_r).$$

Then $\mathfrak{M}$ is dense in $R(T)$, from (6.1b).

As before, let $v(x,\ t,\ h_r)$ be the mesh function which is computed along with the boundary $x(t,\ h_r)$, according to the method given in §3 for the Stefan problem.

THEOREM 5: *If*

$$x(t) = \lim_{r\to\infty} x(t,\ h_r),$$

where $\{h_r\}$ is a sequence of the type (6.1), *and if $u(x,\ t)$ is the solution of* (2.2) *for this boundary, then the sequence $\{v(x,\ t,\ h_r)\}$ converges uniformly to $u(x,\ t)$ on $\mathfrak{M}$.*

Proof. For the limit curve $x(t)$, form the mesh functions $\{u(x,\ t,\ h_r)\}$ according to the definitions (3.11), (3.12). Let $\epsilon > 0$ be given, and choose r_0 so that for $r \geqq r_0$:

$$(6.2)\quad \begin{aligned} &\text{(a)}\quad |\, u(x, t, h_r) - u(x, t)\,| \leqq \frac{\epsilon}{3} \qquad ((x, t) \in \mathfrak{M}(h_r)). \\ &\text{(b)}\quad |\, x(t, h_r) - x(t)\,| \leqq \frac{\epsilon}{3} \qquad (0 \leqq t \leqq T). \\ &\text{(c)}\quad 4h_r \leqq \frac{\epsilon}{3}. \end{aligned}$$

The existence of such an integer r_0 is guaranteed by Theorem 4, and (6.1). But by Lemma 5 and (6.2b, c), we have for $r \geqq r_0$:

$$| v(x, t, h_r) - u(x, t, h_r) | \leqq \frac{\epsilon}{3} + 4h_r \leqq \frac{2\epsilon}{3}$$

on mesh points of $\mathfrak{M}(h_r)$. Combining this with (6.2a), we conclude that for $r \geqq r_0$

$$| v(x, t, h_r) - u(x, t) | \leqq \epsilon$$

on mesh points of $\mathfrak{M}(h_r)$.

THEOREM 6. *Let $x(t) = \lim_{r\to\infty} x(t, h_r)$ and $u(x, t)$ be the solution of (2.2) for this boundary. Then $x(t)$ is differentiable in $t \geqq 0$ and*

$$\text{(6.3)} \quad \begin{aligned} &\text{(a)} \qquad \dot{x}(0) = 1 \\ &\text{(b)} \qquad \dot{x}(t) = -u_x(x(t), t) \qquad (t > 0). \end{aligned}$$

Consequently the pair of functions $x(t)$ and $u(x, t)$ is a solution of the Stefan problem (1.1).

Proof. (6.3a) follows from the fact that for $t > 0$

$$1 \geqq \frac{x(t)}{t} \geqq \frac{1}{1 + x(t)/2}.$$

Simply let $t \to 0$. For (6.3b), we give a proof similar to that of Evans [5]; however, we do not assume, as he did, that $u_t(x, t)$ is continuous in the closed set $0 \leqq x \leqq x(t)$, $t \geqq 0$. From (4.8) and Theorem 5, it follows that

$$x(t) = t - \int_0^{x(t)} u(x, t)\, dx \tag{6.4}$$

for any t which appears in the mesh $\mathfrak{M}$. But these values of t are dense in $0 \leqq t \leqq T$, and both sides of (6.4) are continuous in t; hence it holds for all t in $(0, \infty)$, since T is arbitrary.

For $\Delta t > 0$ it follows from (6.4) that

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = 1 - \int_0^{x(t)} \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t}\, dx - \int_{x(t)}^{x(t+\Delta t)} \frac{u(x, t + \Delta t)}{\Delta t}\, dx \qquad (t > 0).$$

But from (2.4b) and (2.1a), the second term is dominated by Δt. Furthermore, from (2.5), the difference quotient in the first integral on the right tends boundedly to $u_t(x, t) = u_{xx}(x, t)$. Hence, by Lebesgue's bounded convergence theorem, $\dot{x}(t)$ exists (from above) and

$$\dot{x}(t) = 1 - \int_0^{x(t)} u_{xx}(x, t)\, dx \qquad (t > 0).$$

To complete the proof, perform the indicated integration and apply (2.2b). A similar argument applies to the case $\Delta t < 0$.

THEOREM 7. *If h_r is any sequence having properties* (6.1), *then* $\{x(t, h_r)\}$ *and* $\{v(x, t, h_r)\}$ *converge uniformly to* $x(t)$ *and* $u(x, t)$, *the solution of the Stefan problem.*

Proof. From the uniqueness theorem of Douglas [2], the solution of (1.1) whose existence is guaranteed by Theorem 6 is the only one. Suppose there is a sequence $\{h_r\}$ for which $\{x(t, h_r)\}$ does not converge. Then by the Arzela-Ascoli theorem, it would be possible to choose two subsequences $\{h_{1r}\}$ and $\{h_{2r}\}$ such that

$$\lim_{r\to\infty} x(t, h_{1r}) = x_1(t),$$

$$\lim_{r\to\infty} x(t, h_{2r}) = x_2(t),$$

and $x_1(t) \neq x_2(t)$ for some value of t. But by Theorem 6, both x_1 and x_2 would be boundary curves for the Stefan problem, and this would contradict the uniqueness theorem. Since, from Theorem 5, the convergence of $\{x(t, h_r)\}$ implies the convergence of $\{v(x, t, h_r)\}$, the proof is complete.

Condition (6.1c) is not essential for any of these conclusions; it is imposed simply as a matter of convenience in exposition.

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