# On an extension of a quadratic transformation formula due to Kummer 

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Received December 27, 2008; accepted June 5, 2009

Abstract. The aim of this research note is to prove the following new transformation formula

$$
\left.\begin{array}{l}
(1-x)^{-2 a}{ }_{3} F_{2}\left[\begin{array}{rc}
a, a+\frac{1}{2}, d+1 & \\
c+\frac{3}{2}, & d
\end{array} \quad ; \frac{x^{2}}{(1-x)^{2}}\right.
\end{array}\right] .
$$

valid for $|x|<\frac{1}{2}$ and if $|x|=\frac{1}{2}$, then $\operatorname{Re}(c-2 a)>0$, where $A=\sqrt{16 d^{2}-16 c d-8 d+1}$. For $d=c+\frac{1}{2}$, we get quadratic transformations due to Kummer. The result is derived with the help of the generalized Gauss's summation theorem available in the literature.
AMS subject classifications: 33C05, 33D15
Key words: quadratic transformation formula, Kummer transformation, Gauss summation theorem

## 1. Introduction and results required

We start with the following very interesting and useful quadratic transformation formula due to Kummer [3] or [5, eq. 1, p. 65] viz.

$$
(1-x)^{-2 a}{ }_{2} F_{1}\left[\begin{array}{c}
a, a+\frac{1}{2},  \tag{1}\\
c+\frac{1}{2},
\end{array} ; \frac{x^{2}}{(1-x)^{2}}\right]={ }_{2} F_{1}\left[\begin{array}{cc}
2 a, & c, \\
2 c,
\end{array}\right]
$$

[^0]The aim of this research note is to provide an extension of (1) by employing the following summation formula [4, eq. 10, p. 534]

$$
{ }_{3} F_{2}\left[\begin{array}{ccc}
f, a, c+1 &  \tag{2}\\
b, & c & ; 1
\end{array}\right]=\frac{(c-a)(\alpha-f)}{c} \frac{\Gamma(b) \Gamma(b-a-f-1)}{\Gamma(b-a) \Gamma(b-f)}
$$

provided $\operatorname{Re}(b-a-f)>1, c \neq 0,-1,-2, \ldots$ and $\alpha$ is given by

$$
\alpha=\frac{c(1+a-b)}{a-c} .
$$

## 2. Main result

The extension of the Kummer's quadratic transformation formula (1) to be established is given by the following theorem.

Theorem 1. For $|x|<\frac{1}{2}$ or if $|x|=\frac{1}{2}$, then $\operatorname{Re}(c-2 a)>0$, we have for $d \neq 0,-1$, $-2, \ldots$

$$
\left.\begin{array}{l}
(1-x)^{-2 a}{ }_{3} F_{2}\left[\begin{array}{r}
a, a+\frac{1}{2}, d+1 \\
c+\frac{3}{2}, \\
\quad d
\end{array} ; \frac{x^{2}}{(x-1)^{2}}\right. \tag{3}
\end{array}\right] .
$$

where

$$
\begin{equation*}
A=\sqrt{16 d^{2}-16 c d-8 d+1} \tag{4}
\end{equation*}
$$

Proof. In order to derive (3), we proceed as follows. Denoting the left-hand side of (3) by $S(x)$, we have

$$
\left.\begin{array}{rl}
S(x) & =(1-x)^{-2 a}{ }_{3} F_{2}\left[\begin{array}{c}
a, a+\frac{1}{2}, d+1 \\
c+\frac{3}{2}, \\
\end{array} \quad d \quad \frac{x^{2}}{(1-x)^{2}}\right.
\end{array}\right] .
$$

Using now Bailey's transform of the double series, the appropriate Pochhammer
symbol transformation formula and summing up the resulting series, we get

$$
\begin{aligned}
S(x) & =\sum_{n=0}^{\infty} \sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(a)_{m}\left(a+\frac{1}{2}\right)_{m}(d+1)_{m}(2 a+2 m)_{n-2 m}}{\left(c+\frac{3}{2}\right)_{m}(d)_{m} m!(n-2 m)!} x^{n} \\
& =\sum_{n=0}^{\infty} \frac{(2 a)_{n}}{n!} x^{n} \sum_{m=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{\left(-\frac{1}{2} n\right)_{m}\left(-\frac{1}{2} n+\frac{1}{2}\right)_{m}(d+1)_{m}}{\left(c+\frac{3}{2}\right)_{m}(d)_{m} m!} \\
& =\sum_{n=0}^{\infty} \frac{(2 a)_{n}}{n!} x^{n}{ }_{3} F_{2}\left[\begin{array}{c}
-\frac{1}{2} n,-\frac{1}{2} n+\frac{1}{2}, d+1 \\
c+\frac{3}{2}, \\
;
\end{array}\right] .
\end{aligned}
$$

The inner ${ }_{3} F_{2}$ can be evaluated using (2) by taking $f=-\frac{1}{2} n, a=-\frac{1}{2} n+\frac{1}{2}, b=c+\frac{3}{2}$ and $c=d$. After simplification we get

$$
\begin{aligned}
S(x) & =\sum_{n=0}^{\infty} \frac{(2 a)_{n}(c)_{n} 2^{n} x^{n}}{(2 c+2)_{n} n!} \frac{\left(2 d+\frac{1}{2} A+\frac{1}{2}\right)_{n}\left(2 d-\frac{1}{2} A+\frac{1}{2}\right)_{n}}{\left(2 d+\frac{1}{2} A-\frac{1}{2}\right)_{n}\left(2 d-\frac{1}{2} A-\frac{1}{2}\right)_{n}} \\
& ={ }_{4} F_{3}\left[\begin{array}{c}
2 a, \quad c, \quad 2 d+\frac{1}{2} A+\frac{1}{2}, 2 d-\frac{1}{2} A+\frac{1}{2} \\
2 c+2,2 d+\frac{1}{2} A-\frac{1}{2}, 2 d-\frac{1}{2} A-\frac{1}{2}
\end{array}\right]
\end{aligned}
$$

where $A$ is the same as in (4). This completes the proof of (3).
Corollary 1. In (3), if we take $d=c+\frac{1}{2}$, then since $A=1$, we get at once the Kummer's result (1) which was rederived by Bailey [1, p. 243] by employing Gauss's summation theorem [2, eq. 1, p. 2]. Hence (3) can be regarded as an extension of (1).

## Acknowledgment

The authors are highly grateful to the worthy referees for their very useful suggestions which led to a better presentation of this research note.

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