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On an Extension of Green's Relation and a Structure of Semigroup

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Abstract

This paper contains an attempt to discuss some properties of relations induced by translations, including distribution of idempotents. We will begin with start two equivalence relations, give some results to be used frequently and show a distribution of idempotents. In fact it will be shown in "1.Introduction and Preliminaries" that the two relations are relations including Green's relations. Using the relations, we will discuss idempotents which behave as left or right identities in equivalence classes induced by the relations. In the last section, as an application of the properties of idempotent and a semigroup extended by translations shown here, some special class of semigroups, abundand semigroups, will be discussed, which is closely related with the distribution of idempotents.

1 Introduction and Preliminaries

Throughout this paper, $\Psi_L(S)$ will denote the semigroup of all left translations of a semigroup S, and $\Psi_R(S)$ will denote the semigroup of all right translations of S.

If S is any semigroup and $a, b \in S$, we say that $x \prec_L y$ if and only if for any $\psi, \phi \in \Psi_L(S), \psi(x) = \phi(x)$ implies $\psi(y) = \phi(y)$, and $x \prec_R y$ if and only if for any $\mu, \gamma \in \Psi_R(S), \mu(x) = \gamma(x)$ implies $\mu(y) = \gamma(y)$. Let us also define two equivalence relations Π_L and Π_R as follows: $x\Pi_L y$ if and only if $x \prec_L y$ and $y \prec_L x$, and $x\Pi_R y$ if and only if $x \prec_R y$ and $y \prec_R x$.

We define Π_H as the intersection of Π_L and Π_R, and Π_D as the union of Π_L and Π_R.

Now $S = \bigcup_{a\in S} \Pi_L(a)$, $S = \bigcup_{b\in S} \Pi_R(b)$, and $S = \bigcup_{a,b\in S} \Pi_H(a, b)$ stand for the partitions induced by the equivalence relations Π_L, Π_R and Π_H, and $\Pi_L(a)$, $\Pi_R(b)$ and $\Pi_H(a, b)$ will be called $\Pi_L - class$ including a, $\Pi_R - class$ including b and $\Pi_H - class$ including a and b, respectively.

Now we can define two semigroups, $T_R = S \cup \Psi_R(S)$ and $T_L = S \cup \Psi_L(S)$ with S as right and left ideals, respectively.

(1) Let x, y be any elements of S and ψ, ϕ be any elements of $\Psi_R(S)$, and the operation $\odot$ on T_R will be defined as follows: $x \odot y = x \cdot y, x \odot \psi = \psi(x), \psi \odot x = \psi \circ r_x, \psi \odot \phi = \psi \circ \phi,$ where $\cdot$ is the operation on the semigroup S, $\psi \circ \phi$ is defined by $\psi \circ \phi(z) = \phi(\psi(z))$ and r_x is the right translation on S defined by $r_x(z) = z \cdot x$ for all $z \in S$.
(2) Let x, y be any elements of S and μ, γ be any elements of $\Psi_L(S)$, and the operation $\odot$ on T_L will be defined as follows: $x \odot y = x \cdot y, x \odot \mu = l_x \circ \mu, \mu \odot x = \mu(x), \mu \odot \gamma = mu \circ \gamma,$ where $\cdot$ is the operation on the semigroup S $\mu \circ \gamma$ is defined by $\mu \circ \gamma(z) = \mu(\gamma(z))$ and l_x is the left translation on S defined by $l_x(z) = x \cdot z$ for all $z \in S$.

To show that T_R and T_L are semigroups, it is necessary that the following eight equations hold for all $x, y \in S$ and $f, g \in \Psi_R(S)$ and $\Psi_L(S)$, respectively.

(1) $(x \odot y) \odot z = x \odot (y \odot z)$
(2) $(x \odot f) \odot g = x \odot (f \odot g)$
(3) $(x \odot f) \odot y = x \odot (f \odot y)$
(4) $(x \odot y) \odot g = x \odot (y \odot g)$
(5) $(f \odot g) \odot h = f \odot (g \odot h)$
(6) $(f \odot x) \odot y = f \odot (x \odot y)$
(7) $(f \odot g) \odot x = f \odot (g \odot x)$
(8) $(f \odot x) \odot g = f \odot (x \odot g)$

It will be shown that equations (1) to (8) hold on T_R as follows:
Let $f = \psi \in \Psi_R$ and $g = \phi \in \Psi_R$.
(1) and (5):Since S and $\Psi_R(S)$ are semigroups, it is obvious that $(x \odot y) \odot z = x \odot (y \odot z)$ and $(f \odot g) \odot h = f \odot (g \odot h)$.
(2):$(x \odot \psi) \odot \phi = \psi(x) \odot \phi = \phi(\psi(x)) = \psi \circ \phi(x) = x \odot (\psi \odot \phi)$
(3): $(x \odot \psi) \odot y = \psi(x) \odot y = \psi(x) \cdot y$ and $x \odot (\phi \odot y) = x \odot (\psi \circ r_y) = r_y(\psi(x)) = \psi(x) \cdot y$.
(4):$(x \odot y) \odot \phi = \phi(x \cdot y) = x \cdot \phi(y)$ and $x \odot (y \odot \phi) = x \odot \phi(y) = x \cdot \phi(y)$.
(6):$(\psi \odot x) \odot y = (\psi \circ r_x) \odot y = (\psi \circ r_x) \cdot y = \psi \circ r_x \circ r_y$ and $\psi \odot (x \odot y) = \psi \odot (x \cdot y) = \psi \circ r_{x \cdot y}$.
Since $r_{x \cdot y}(z) = z \cdot (x \cdot y) = (z \cdot x) \cdot y = r_y(r_x(z)) = (r_x \circ r_y)(z)$, $\psi \circ r_{x \cdot y} = \psi \circ r_x \circ r_y$.
(7):$(\psi \odot \phi) \odot x = (\psi \circ \phi) \odot x = \psi \circ \phi \circ k_x = \psi \circ (\phi \circ r_x) = \psi \circ (\phi \odot x) = \psi \odot (\phi \odot x)$.
(8):$(\psi \odot x) \odot \phi = (\psi \circ r_x) \odot \phi = (\psi \circ r_x) \circ \phi$ and $\psi \odot (x \odot \phi) = \psi \odot \phi(x) = \psi \circ k_{\phi(x)}$. Since for any $z \in S$, $((\psi \circ r_x) \circ \phi)(z) = \phi((\psi \circ r_x)(z)) = \phi(r_x(\psi(z))) = \phi(\psi(z) \cdot x) = \psi(z) \cdot \phi(x)$ and $(\psi \circ r_{\phi(x)})(z) = r_{\phi(x)}(\psi(z)) = \psi(z) \cdot \phi(x)$, we have $(\psi \circ r_x) \circ \phi = \psi \circ r_{\phi(x)}$.
It is also shown that equations (1) to (8) hold on T_L similarily.

In fact, from the definition of the operation on T_R, $x \odot y = x \cdot y \in S$ and $x \odot \psi = \psi(x) \in S$ for any $x, y \in S$ and $\psi \in \Psi_R(S)$ imply that S is a right ideal of T_R. $\psi \odot x = \psi \circ r_x \in \Psi_R$ and $\psi \odot \phi = \psi \circ \phi \in \Psi_R$ for any $x \in S$ and $\psi, \phi \in \Psi_R$ imply that Ψ_R is a right ideal of T_R.
It is shown that S is a left ideal of T_L and Ψ_L is a left ideal of T_L.

We begin with a brief list of some basic results without proof, which will be used throughout the paper.

Lemma 1.

(1) If $x \in \Pi_R(y)$, then $x \odot t \in \Pi_R y \odot t$ for all $t \in T_R$;
(2) If $x \in \Pi_L(y)$, then $t \odot x \in \Pi_L t \odot y$ for all $t \in T_L$.

Lemma 2.

(1) $x \prec_L xu$ for all $x, u \in S$;
(2) $x \prec_R vx$ for all $x, v \in S$.

It is also easily shown that $\mathcal{R} \subseteq \Pi_L \subseteq R^*$, $\mathcal{L} \subseteq \Pi_R \subseteq L^*$ and $\mathcal{H} \subseteq \Pi_H \subseteq H^*$ for the Green's relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$, and the relations R^*, L^* and H^* defined by J. Fountain and others. They called that a semigroup in which each $R^* - class$ and each $L^* - class$ contains an idempotent is an abundant semigroup. In particular, if a semigroup is regular, then $\mathcal{R} = \Pi_L = R^*$, $\mathcal{L} = \Pi_R = L^*$ and $H = \Pi_H = H^*$.

2 Idempotent As a Local Identity

Let S be an semigroup, then we have the following lemmas which show that idempotent behaves as left or right identity elements in $\Pi_L - class$ or $\Pi_R - class$, respectively.

Lemma 3. Let e be any element of S, then

(1) e is an idempotent if and only if $e \cdot t = t$ for all $t \in \Pi_L(e)$;
(2) e is an idempotent if and only if $s \cdot e = s$ for all $s \in \Pi_R(e)$.

Proof. (1) It is trivial that $e \cdot e = e$, since $e \in \Pi_L(e)$. Conversely, assume that e is an idempotent, that is, $e \cdot e = e$, then from the definition of the inner left translation, $e \cdot e = f_e(e) = e = I(e)$ implies that $e \cdot t = f_e(t) = I(t) = t$ for all $t \in \Pi_L(e)$, where I is the identity mapping and f_e is the inner left translation.
(2) is shown similarly. Q.E.D.

Lemma 4. Let e be any idempotent of S , then

(1) For any element $t \in T_L$ and any element $s \in \Pi_L(e)$, there exist an element $u \in \Pi_L(t \odot s))$ such that $t \odot s = u \cdot s$;
(2) For any element $t \in T_R$ and any element $s \in \Pi_R(e)$, there exists an element $v \in \Pi_R(s \odot t)$ such that $s \odot t = s \cdot t$.

Proof. (1) It is sufficient to show that for any $\psi \in \Psi_L(S)$ on S and any $s \in \Pi_L(e)$, $\psi(s) = u \cdot s$ for some $u \in \Pi_L(\psi(s))$. It follows from the fact that $\psi(s) = \psi(es) = \psi(e) \cdot s$ for any element $s \in \Pi_L(e)$ (from Lemma 3) and $u = \psi(e) \in \Pi_L(\psi(s))$ (from Lemma 2).
(2) is shown similarly. Q.E.D.

Lemma 5. Let e and f be any idempotents of S.

(1) If $e \in \Pi_L(f)$ then e is an inverse of f;
(2) If $e \in \Pi_R(f)$ then e is an inverse of f.

Proof. (1) We have $e \cdot f \cdot e = e \cdot (f \cdot e) = e \cdot e = e$ from Lemma 3 and that f is an idempotent in $\Pi_L(e)(= \Pi_L(f))$. Similarly it is shown that $f \cdot e \cdot f = f \cdot (e \cdot f) = f \cdot f = f$. Q.E.D.

From above lemmas, it is shown that each $\Pi_H - class$ contains a unique idempotent if it has.

Theorem 1. For any elements $a, b \in S$, the $\Pi_H - class, \Pi_L(a) \cap \Pi_R(b)$, cannot have more than one idempotent.

Proof. Assume that e and f be idempotents in a $\Pi_H - class$, $\Pi_L(a) \cap \Pi_R(b)$, that is, $e, f \in \Pi_L(f) \cap \Pi_R(f) = \Pi_L(e) \cap \Pi_R(e)$. Then $f = f \cdot e \cdot f = f \cdot (e \cdot f) = e \cdot f = e$ from Lemma 3. Q.E.D.

Lemma 6. For any elements $a, b \in S$, if $\Pi_R(a) \cap \Pi_L(b)$ contains an idempotent then $ab \in \Pi_L(a) \cap \Pi_R(b)$.

Proof. Assume that e is an idempotent in a $\Pi_H - class$ such that $e \in \Pi_R(a) \cap \Pi_L(b)$, that is, $a \in \Pi_R(e)$ and $b \in \Pi_L(e)$. Then from Lemma 2 and Lemma 3, we have that $r_b(a) \in \Pi_R(r_b(e))$ for the inner right translation r_b such that $r_b(a) = a \cdot b$ and $r_b(e) = e \cdot b = b$, since $e \in \Pi_L(b)$. Thus $a \cdot b \in \Pi_R(b)$. Similarly, we have that $l_a(b) \in \Pi_L(l_a(e))$ for the inner left translation l_a such that $l_a(b) = a \cdot b$ and $l_a(e) = a \cdot e = a$, since $e \in \Pi_R(a)$. Thus $a \cdot b \in \Pi_L(a)$. Q.E.D.

Lemma 7. For any element $a \in S$, the following conditions are equivalent:

(1) $\Pi_R(a)$ contains an idempotent;
(2) For elemnt $t \in T_R$, there exist an element $u \in \Pi_R(a \odot t)$ such that $a \odot t = a \cdot u$.

Proof. (1) $\to$ (2): Let e be an idempotent in $\Pi_R(a)$, then from Lemma 4, we have that for any element $t \in T_R$, there exists an element $u \in \Pi_R(a \odot t)$ such that $a \odot t = a \cdot t$.
(2) $\to$ (1): Let I be the identity translation (which is also in T_R), then there exists an element $u \in \Pi_R(I(a)) = \Pi_R(a)$ such that $I(a) = a \cdot u$. From the fact that $u \in \Pi_R(a), a \cdot u = r_u(a) = I(a)$ implies that $u \cdot u = r_u(u) = I(u) = u$. Thus u is an idempotent in $\Pi_R(a)$. Q.E.D.

Similarly, we also have

Lemma 8. For any element $b \in S$, the following conditions are equivalent:

(1) $\Pi_L(b)$ contains an idempotent;
(2) For any element $t \in T_L$, there exist an element $v \in \Pi_L(t \odot b))$ such that $t \odot b = u \cdot b$.

3 Strictly Abundant Semigroup

From Lemma 4, it will be easily shown that a semigroup is abundant if and only if each $\Pi_R - class$ and each $\Pi_L - class$ contains an idempotent. We will call a semigroup strictly abundant if each $\Pi_H - class$ contains an idempotent. The following theorem is also a direct result from above lemmas, which shows that strictly abundant semigroup is a disjoint union of semigroups.

Lemma 9. Let $\Pi_H(e)$ and $\Pi_H(f)$ are any $\Pi_H - classes$ which contain idempotents e and f.

(1) If the $\Pi_H - classes$, $\Pi_H(e)$ and $\Pi_H(f)$ are included in a same $\Pi_R - class$, then there exists a homomorphism from $\Pi_H(e)$ onto $\Pi_H(f)$;
(2) If the $\Pi_H - classes$, $\Pi_H(e)$ and $\Pi_H(f)$ are included in a same $\Pi_L - class$, then there exists a homomorphism from $\Pi_H(e)$ onto $\Pi_H(f)$.

Proof. (1)The mapping $\rho : \Pi_H(e) \to \Pi_H(f)$ is defined by $\rho(s) = f \cdot s \cdot f$, for $s \in \Pi_H(e)$. Assume that $s, t \in \Pi_H(e)$, then $\rho(s) \cdot \rho(t) = (f \cdot s \cdot f) \cdot (f \cdot t \cdot f) = f \cdot s \cdot f \cdot f \cdot t \cdot f = f \cdot s \cdot f \cdot t \cdot f = f \cdot s \cdot t \cdot f = \rho(s \cdot t)$, since $s \in \Pi_R(f)$. Thus the mapping$\rho : \Pi_H(e) \to \Pi_H(f)$ is a homomorphism. Let t be any element in $\Pi_H(f)$, that is, $t \in \Pi_L(b)$. Since $e \in \Pi_R(e) \cap \Pi_L(t)$, we have $e \cdot t \cdot e = e \cdot t \in \Pi_R(t) \cap \Pi_L(e) = \Pi_R(e) \cap \Pi_L(e) \in \Pi_H(e)$ from Lemma 3 and Lemma 6. And $\rho(e \cdot t \cdot e) = f \cdot e \cdot t \cdot e \cdot f = (f \cdot e) \cdot t \cdot (e \cdot f) = f \cdot t \cdot e = f \cdot (t \cdot e) = f \cdot t = t$.
(2) is similarly shown. Q.E.D.

Corollary 1. For any elements $a, b \in S$, $\Pi_R(a) \cap \Pi_L(b)$ contains an idempotent if and only if $\Pi_R(a) \cap \Pi_L(b)$ is a monoid.

Theorem 2. Let S be any strictly abundant semigroup, the S is a disjoint union of monoids.

Corollary 2. Let $\Pi_H(e)$ and $\Pi_H(f)$ are any $\Pi_H - classes$ which have idempotents, e and f , respectively.

(1) If two $\Pi_H - classes$, $\Pi_H(e)$ and $\Pi_H(f)$ are included in a same $\Pi_R - class$,
then $\Pi_H(e)\Pi_H(f) \subseteq \Pi_H(e)$;
(2) If two $\Pi_H - classes$, $\Pi_H(e)$ and $\Pi_H(f)$ are included in a same $\Pi_L - class$,
then $\Pi_H(e)\Pi_H(f) \subseteq \Pi_H(f)$.

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