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ON AN EXTENSION OF THE LYAPUNOV CRITERION OF STABILITY FOR QUASI-LINEAR SYSTEMS VIA INTEGRAL INEQUALITIES METHODS

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NGUYEN HUU DU

Abstract. In this article, we concern ourselves with a new concept for comparing the stability degree of two dynamical systems. By using the integral inequality method, we give a criterion which allows us to compare the growth rate of two Itô quasi-linear differential equations. It can be viewed as an extension of the Lyapunov criterion to the stochastic case.

1. Introduction

It is known that studying the asymptotic behavior of dynamical systems is important both in theory and in application. Therefore, there are many works dealing with this topic and there exist a large amount of stability criteria for deterministic and stochastic systems. Among these criteria, the characteristic Lyapunov exponent is a powerful tool because it is important for explaining the chaos of the systems under consideration (see [1, 2, 10], etc.). We remark that studying the Lyapunov exponent of a function means comparing its growth rate with the growth rate of the exponential one. However, the class of exponential functions is rather simple and it does not contain much information on the behavior of the function considered. By requirement of technical problems, we have sometimes to replace this class by a larger one, say $\mathcal{C}$, and compare this function with elements of $\mathcal{C}$ in order to know its behavior, especially at $t = \infty$. To realize this we introduce a concept of comparing the behavior of trajectories of solutions of two differential equations as follows: a system is said to be better than a given one in the class $\mathcal{C}$ in view of stability (we will say that this system is more stable than the given one) if, whenever all trajectories of the given system starting from a small neighborhood of the origin 0 belong to the corridor $\{(t,x)\colon t \geq 0, |x| \leq q_t\}$ generated by a positive function $q_t \in \mathcal{C}$, then all trajectories of the second system starting from a suitable neighborhood of 0 must have the same property.

Naturally, this raises a question: when is one system more stable than another? Of course, it is a difficult problem even in the deterministic case. In this article, we deal with a criterion for comparing two quasi-linear systems. This problem is encountered when we investigate a nonlinear stochastic system via its first linear approximation.

The article is organized as follows. Section 2 gives a concept of comparing two differential stochastic equations. In Section 3, we are concerned with the regularity of a

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stochastic linear equation and formulate the main theorem. Section 4 deals with its proof.

Although the result is formulated for the one-dimensional Wiener process (W_t), it remains true for multidimensional processes.

2. Comparison of growth rates of stochastic systems

Let $(\Omega, \mathcal{F}_t, t \geq 0, \mathsf{P})$ be a stochastic basis satisfying the standard conditions (see [7]) and $(W_t, t \geq 0)$ be a Wiener process defined on $(\Omega, \mathcal{F}_t, t \geq 0, \mathsf{P})$. For the sake of simplicity we assume that $(W_t, t \geq 0)$ is a one-dimensional process. We consider a system described by the following stochastic differential equation:

$$\begin{cases} dX_t = a(t, X_t, \omega)\, dt + A(t, X_t, \omega)\, dW_t, \\ X_0 = x \in \mathbf{R}^d, \end{cases} \tag{2.1}$$

where, for all $x \in \mathbf{R}^d$, $(a(t,x))$ and $(A(t,x))$ are two stochastic processes, $\mathcal{F}_t$-adapted, with values in $\mathbf{R}^d$ such that

$$a(t,0) \equiv 0, \qquad A(t,0) \equiv 0 \quad \text{for any } t \geq 0. \tag{2.2}$$

Suppose that for any $x \in \mathbf{R}^d$, equation (2.1) has a unique solution starting from x. Let us recall the classical definition of Lyapunov stability. We write $X_t(x,\omega)$ for the solution of (2.1) satisfying $X_0(x,\omega) = x$. From (2.2), it follows that $X \equiv 0$ is a trivial solution of (2.1).

Definition 2.1 (see [10], p. 206)**.** The trivial solution $X \equiv 0$ of (2.1) is said to be stable if for any $\varepsilon > 0$

$$\lim_{x \to 0} \mathsf{P}\left\{ \sup_{0 \leq t < \infty} |X_t(x,\omega)| \leq \varepsilon \right\} = 1. \tag{2.3}$$

However, this definition gives no information when the solution $X(t,x)$ is unbounded or converges rather fast to 0. We remark that, in fact, considering the stability of the trivial solution means we compare its trajectories with the constant functions because the relation $\sup_{0 \leq t < \infty} |X_t(x,\omega)| \leq \varepsilon$ implies $|X_t(x,\omega)| \leq \varepsilon(t)$ for any $t > 0$, where $\varepsilon(t) = \varepsilon$ for all $t > 0$. This suggests that it is necessary to choose functions varying in time to get more information on the growth rate of solutions. Or, equivalently, instead of using only one, we can use a family of neighborhoods of 0 depending continuously on t and study conditions under which the solutions of (2.1) always belong to this family. Moreover, with the help of this family, we can compare the growth rates of two systems as explained below.

Together with equation (2.1), we consider another one

$$\begin{cases} dY_t = b(t, Y_t, \omega)\, dt + B(t, Y_t, \omega)\, dW_t, \\ Y_0 = y \in \mathbf{R}^d, \end{cases} \tag{2.4}$$

where $(b(t,x))$ and $(B(t,x))$ satisfy the same conditions as $(a(t,x))$ and $(A(t,x))$, i.e.,

$$b(t,0) \equiv 0, \qquad B(t,0) \equiv 0 \quad \text{for all } t > 0.$$

We denote by $Y_t(y)$ the unique solution of (2.4) starting from y. Let $\mathcal{C}$ be the class of all positive, continuous functions from $[0,\infty)$ to $\mathbf{R}_+$ and $\mathcal{M}$ be a subset of $\mathcal{C}$.

Definition 2.2. The trivial solution $X \equiv 0$ of system (2.1) is said to be more stable than the solution $Y \equiv 0$ of system (2.4) in the comparison class $\mathcal{M}$ if and only if for any $q \in \mathcal{M}$, the relation

$$\lim_{y \to 0} \mathsf{P}\{|Y_t(y)| \leq q_t \text{ for any } t \geq 0\} = 1 \tag{2.5}$$

implies that

$$\lim_{x\to 0} \mathsf{P}\{|X_t(x)| \le q_t \text{ for any } t \ge 0\} = 1. \tag{2.6}$$

Definition 2.2 is an extension of the classical definition of stability. Indeed, it is easy to prove the following theorem

Theorem 2.3 (see [14]). *System* (2.1) *is stable in the sense of* (2.3) *if and only if it is more stable than the trivial system*

$$\dot{Y} = 0, \qquad Y_0 \in \mathbf{R}^d, \tag{2.7}$$

on the class $\mathcal{C}$.

Next, we also give a definition of comparison similar to the definition of asymptotical stability.

Definition 2.4. System (2.1) is said to be really more stable than (2.4) if condition (2.5) implies that there exists a $q^* \in \mathcal{M}$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \frac{q_t^*}{q_t} < 0$$

and

$$\lim_{x\to 0} \mathsf{P}\{|X_t(x)| \le q_t^* \text{ for all } t \ge 0\} = 1. \tag{2.8}$$

Example. Both systems

$$\ddot{X} - 2\dot{X} + 2X = 0, \tag{A}$$

$$dY = \frac{3}{2} Y_t\,dt + Y_t\,dW_t \tag{B}$$

are unstable. The system (A) has the solution $X_t(x_1, x_2) = e^t(x_1 \cos t + x_2 \sin t)$, and by Itô's formula we see that $Y_t(y) = y \exp\{t + W_t\}$. It is easy to see that (A) is more stable than (B) in the class $\mathcal{C}$. Indeed, if $q \in \mathcal{C}$ is such that

$$\lim_{y\to 0} \mathsf{P}\{|Y_t(y)| \le q_t \text{ for all } t \ge 0\} = 1,$$

then from $W_t \simeq N(0,t)$ we get $q_t \ge c \exp\{t + \alpha\sqrt{t}\}$ for some suitable c and α. Hence, $|X_t(x_1, x_2)| \le q_t$ for any $t > 0$ provided $|x_1| + |x_2|$ is sufficiently small, i.e., we have (2.6).

3. Comparison of quasi-linear systems

For any function $f\colon [0,\infty) \to \mathbf{R}^n$, the number

$$\lambda[f] := \lim_{t\to\infty} \frac{1}{t} \ln \|f(t)\|$$

is called the Lyapunov exponent of the function f. We recall the Lyapunov criterion of asymptotic stability for the first approximation. If the linear part

$$\dot{\xi} = A(t)\xi, \qquad \xi \in \mathbf{R}^d, \tag{3.1}$$

is regular (see [6], p. 165) and its top Lyapunov exponent is negative, then the trivial solution of the perturbed system

$$\dot{X} = A(t)X + f(t, X), \qquad X_0 \in \mathbf{R}^d, \tag{3.2}$$

is asymptotically stable, provided $|f(t,x)| \le k|x|^m$, $m > 1$.

Naturally, one wants to generalize this result to stochastic systems described by Itô's differential equations. As mentioned before, the main difficulty we encounter is that the

Wiener process (W_t) has unbounded variations. Then we cannot use directly the method described in [6]. Recently, N. D. Cong in [4] has shown that if the equation

$$\begin{cases} dZ_t = A_t Z_t, \\ Z_0 = z \in \mathbf{R}^d, \end{cases} \tag{3.3}$$

is regular then the linear equation

$$\begin{cases} dZ_t = A_t Z_t + B_t Z_t \, dW_t, \\ Z_0 = z \in \mathbf{R}^d, \end{cases} \tag{3.4}$$

where A_t and B_t are two functions with values in $d \times d$-matrices, satisfying

$$|A_t| \leq A, \qquad |B_t| \leq B \quad \text{for any } t > 0, \tag{3.5}$$

with two certain constant matrices A and B, is also regular. Suppose that $Z_t(z)$ is the solution of (3.3) satisfying $Z_0(z) = z$. We denote by $\lambda[z]$ the Lyapunov exponent of $Z_t(z)$ defined by

$$\lambda[z] = \limsup_{t \to \infty} \frac{1}{t} \ln |Z_t(z)|.$$

In the case the limit exists (instead of lim sup), we say that $Z_t(z)$ has an exact exponent.

It is known that (see [12]) the spectrum (i.e., the set of all finite Lyapunov exponents) of (3.3) consists of d nonrandom constants, namely

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d \tag{3.6}$$

(taking all multiplicities into account).

The regularity of (3.3) (see [6], p. 139) means that there exists a normal fundamental matrix Z_t of the solution of (3.3) such that the k-column vector of Z_t takes the exact Lyapunov exponent λ_k. Set

$$\Phi_t = Z_t \exp\{-\Lambda t\}, \qquad t \geq 0, \ \Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_d\}.$$

Then it is easy to see [4] that

$$\lambda[\Phi] = \lambda[\Phi^{-1}] = 0. \tag{3.7}$$

Our main result is the following

Theorem 3.1. *Suppose that* (3.3) *is regular and is really more stable than the system*

$$\begin{cases} dY_t = a(t, Y_t) \, dt + \sigma(t, Y_t) \, dW_t, \\ Y_0 = y \in \mathbf{R}^d, \end{cases} \tag{3.8}$$

in the class $\mathcal{C}$. Then the perturbed system

$$\begin{cases} dX_t = \big[A_t X_t + f(t, X_t)\big] + \big[B_t X_t + g(t, X_t)\big] \, dW_t, \\ X_t = x \in \mathbf{R}^d, \end{cases} \tag{3.9}$$

with locally Lipschitz functions $f(t,x)$ and $g(t,x)$ satisfying the condition

$$|f(t,x)| \leq k \min\left\{|x|^\alpha, |x|^\beta\right\}, \qquad |g(t,x)| \leq k \min\left\{|x|^\alpha, |x|^\beta\right\}, \tag{3.10}$$

where $\alpha > 1 > \beta > 0$, will be more stable than (3.8) *in the class $\mathcal{C}$.*

4. Proof of the main result

We divide the proof of Theorem 3.1 into several steps. We first investigate a property of stochastic integral with respect to a Wiener process W_t.

Lemma 4.1. *Let ϕ_t be a stochastic process, $\mathcal{F}_t$-progressively measurable and such that*

$$\mathsf{P}\left\{\int_0^T \phi_t^2\,dt < \infty\right\} = 1 \quad \textit{for all } T > 0.$$

Then there exists a random variable $\eta = \eta(\phi)$ such that

$$\left|\int_0^t \phi_s\,dW_s\right| \le \eta\sqrt{m(t)\big(|\ln m(t)| + 1\big)}, \tag{4.1}$$

where $m(t) = \int_0^t \phi_s^2\,ds$. Furthermore, the distribution of $\eta(\phi)$ is the same for every process ϕ_t.

Proof. To simplify notation, we set

$$M(t) = \int_0^t \phi_s\,dW_s, \qquad m(t) = \int_0^t \phi_s^2\,ds.$$

We define a family of stopping times $\tau(t)$ by

$$\tau(t) = \begin{cases} \inf\{s\colon m(s) > t\}, & \\ \infty & \text{if } t \ge m(\infty) = \lim_{t\uparrow\infty} m(t). \end{cases}$$

From Theorem 7.2′ in [9, p. 92], it follows that on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathsf{P}})$ of $(\Omega, \mathcal{F}, \mathsf{P})$, there exists an $\widetilde{\mathcal{F}}$-Brownian motion $\mu(t)$ such that $\mu(t) = M(\tau(t))$, $t \in [0, \infty)$. Consequently, we can represent $M(t)$ by an $\widetilde{\mathcal{F}}$-Brownian motion $\mu(t)$ and the stopping times $m(t)$, i.e.,

$$\int_0^t f(s,\omega)\,dW_s = \mu(m(t)).$$

On the other hand, since $\mu(t))$ is a Brownian motion, we have, in view of the law of the iterated logarithm, that

$$\limsup_{t\to\infty} \frac{|\mu(t)|}{\sqrt{2t\ln\ln t}} = 1, \qquad \limsup_{t\to 0} \frac{|\mu(t)|}{\sqrt{2t\ln\ln t}} = 1 \quad \text{a.s.}$$

Therefore, the random variable η defined by

$$\eta := \sup_{0<t<\infty} \frac{|\mu(t)|}{\sqrt{t\big(|\ln t| + 1\big)}}$$

is finite, i.e., $\mathsf{P}\{\eta < \infty\}$ and the distribution of η do not depend on the process (ϕ_t). The definition of η implies that

$$|\mu(t)| < \eta\sqrt{t\big(|\ln t| + 1\big)}.$$

Hence,

$$\left|\int_0^t \phi_s(\omega)\,dW_s\right| \le \eta\sqrt{m(t)\big(|\ln m(t)| + 1\big)}.$$

Lemma 1 is proved. □

Lemma 4.2. *Under the hypotheses of Theorem* 3.1*, the growth rate of the solutions of* (3.9) *is less than the growth rate of an exponential function. That is, there exist two random variables M and N such that for any solution X_t of* (3.9) *satisfying $X_0 = x$ we have*

$$X_t \leq |x| M \exp\{Nt\} \qquad \text{for any } t > 0.$$

Furthermore, the distribution of M is independent of x.

Proof. By Itô's formula we have

$$\begin{aligned} d\ln|X_t| &= \left(\frac{X_t'(A_t X_t + f(t,X_t)) + \frac{1}{2}|B_t X_t + g(t,X_t)|^2}{|X_t|^2} - \left[\frac{X_t'(B_t X_t + g(t,X_t))}{|X_t|^2} \right]^2 \right) dt \\ &\quad + \frac{X_t'(B_t X_t + g(t,X_t))}{|X_t|^2}\, dW_t, \end{aligned}$$

where X' denotes the transpose vector of X.

It follows from (3.10) that

$$\left| \frac{y'(A_t y + f(t,y)) + \frac{1}{2}|B_t y + g(t,y)|^2}{|y|^2} \right| \leq |A_t| + |B_t|^2 + k + k^2$$

and

$$\frac{\left| y'(B_t y + g(t,y)) \right|}{|y|^2} \leq |B_t| + k$$

for any $y \in \mathbf{R}^d \setminus \{0\}$. On the other hand, by (4.1) we have

$$\begin{aligned} &\left| \int_0^t \frac{X_s'(B_s X_s + g(s,X_s))}{|X_s|^2}\, dW_s \right| \\ &\quad \leq \eta \sqrt{ \int_0^t \frac{|X_s'(B_s X_s + g(s,X_s))|^2}{|X_s|^4}\, ds \left[\left| \ln \int_0^t \frac{|X_s'(B_s X_s + g(s,X_s))|^2}{|X_s|^4}\, ds \right| + 1 \right] } \\ &\quad \leq \eta \sqrt{ \left(\int_0^t \big(|B_s| + k\big)^2 ds \right) \left(\left| \ln \int_0^t \big(|B_s| + k\big)^2 ds \right| + 1 \right) } \\ &\quad \leq \eta \sqrt{ (B+k)^2 t \big[\, |\ln (B+k)^2 t| + 1 \big] } \end{aligned}$$

(since $x(|\ln x| + 1)$ is increasing).

Hence,

$$\ln|X_t| - \ln|x| \leq \big(A + B^2 + k + k^2\big)t + \eta \sqrt{(B+k)^2 t \big[\, |\ln(B+k)^2 t| + 1 \big]},$$

or

$$|X_t| \leq \exp\left\{ \big(A + B^2 + k + k^2\big)t + \eta \sqrt{(B+k)^2 t \big[\, |\ln(B+k)^2 t| + 1 \big]} \right\}.$$

Putting

$$N = A + B + k + k^2 + 1, \qquad M = \sup_{t \geq 0} \exp\left\{ \eta \sqrt{(B+k)^2 t [|\ln(B+k)^2 t| + 1]} - t \right\} < \infty$$

we get

$$|X_t| \leq |x| M \exp\{Nt\}, \qquad t \geq 0.$$

It is easy to see that the distribution of M is independent of x. The proof is complete. □

Corollary 4.3. *For any fixed $T < \infty$ and $\varepsilon > 0$ we have*

$$\lim_{x \to 0} \mathsf{P}\left\{ \sup_{0 \leq t \leq T} |X_t(x)| > \varepsilon \right\} = 1.$$

Proof. The result follows from the fact that M has the same distribution for every $x \in \mathbf{R}^d$. □

We now turn to the main step of the proof of Theorem 3.1. Suppose that $q \in \mathcal{C}$ is such that

$$\lim_{y\to 0} \mathsf{P}\{|Y_t(y)| < q_t \text{ for any } t \geq 0\} = 1.$$

Since (3.3) is really more stable than (3.8), there exists a $q^* \in \mathcal{C}$ such that

$$\limsup_{t\to\infty} t^{-1} \ln(q_t^*/q_t) = -\epsilon < 0$$

and

$$\lim_{z\to 0} \mathsf{P}\left\{|Z_t(z)| \leq q_t^* \text{ for any } t \geq 0\right\} = 1. \tag{4.2}$$

We have to show that

$$\lim_{x\to 0} \mathsf{P}\{|X_t(x)| \leq q_t \text{ for any } t \geq 0\} = 1. \tag{4.3}$$

By using the transform

$$U_t = \Phi_t^{-1} X_t, \tag{4.4}$$

it is easy to see that

$$dU_t = \left\{\Lambda U_t + \Phi_t^{-1} f(t, \Phi_t U_t) - \Phi_t^{-1} B_t g(t, \Phi_t U_t)\right\} dt + \Phi^{-1} g(t, \Phi_t U_t)\, dW_t.$$

To simplify notation we set

$$\overline{f}(t,u) = \operatorname{col}\big(\overline{f}_1(t,u), \overline{f}_2(t,u), \dots, \overline{f}_d(t,u)\big) = \Phi_t^{-1}\big[f(t,\Phi_t u) - B_t g(t, \Phi_t u)\big],$$
$$\overline{g}(t,u) = \operatorname{col}\big(\overline{g}_1(t,u), \overline{g}_2(t,u), \dots, \overline{g}_d(t,u)\big) = \Phi_1^{-1} g(t, \Phi_t u).$$

Then,

$$dU_t = \left[\Lambda U_t + \overline{f}(t, U_t)\right] dt + \overline{g}(t, U_t)\, dW_t.$$

Note that $\lambda[\Phi] = \lambda[\Phi^{-1}] = 0$ (see (3.7)); by (3.10) it is easy to see that

$$|\overline{f}(t,u)| \leq k_t \min\left\{|u|^\alpha, |u|^\beta\right\}, \qquad |\overline{g}(t,u)| \leq k_t \min\left\{|u|^\alpha, |u|^\beta\right\}, \tag{4.5}$$

where $k\colon [0,\infty) \to \mathbf{R}^+$ is a positive function satisfying

$$\lambda[k] = \limsup \frac{1}{t} \ln k_t = 0. \tag{4.6}$$

Applying Itô's formula to the process $|U_t|^2$ we obtain

$$d|U_t|^2 = \left[2U_t'\Lambda U_t + 2U_t'\overline{f}(t,U_t) + |\overline{g}(t,U_t)|^2\right] dt + 2U_t'\overline{g}(t,U_t)\, dW_t.$$

Therefore, for any $\lambda_d \leq \lambda$ we have

$$e^{-2\lambda t}|U_t|^2 = |U_0|^2 + 2\int_0^t U_s'(\Lambda - \lambda I)U_s\, ds + \int_0^t e^{-2\lambda s}\big(2U_s'\overline{f}(s,U_s) + |\overline{g}(s,U_s)|^2\big)\, ds$$
$$+ 2\int_0^t e^{-2\lambda s} U_s'\overline{g}(s,U_s)\, dW_s.$$

Since $\lambda \geq \lambda_d \geq \lambda_i$ for any $i = 1, 2, \dots, d$ we have $U_s'(\Lambda - \lambda I)U_s \leq 0$. Hence

$$\begin{aligned} e^{-2\lambda t}|U_t|^2 \leq |U_0|^2 + \int_0^t e^{-2\lambda s}\big(2U_s'\overline{f}(s,U_s) + |\overline{g}(s,U_s)|^2\big)\, ds \\ + 2\int_0^t e^{-2\lambda s} U_s'\overline{g}(s,U_s)\, dW_s. \end{aligned} \tag{4.7}$$

Let $h(t) = \max\{t^{\varepsilon_1}, t_2^{\varepsilon}\}$, where $0 \le \varepsilon_1 < \frac{1}{2} < \varepsilon_2$. It is clear that $h(\cdot)$ is an increasing function and there exists a constant $c = c(\varepsilon_1, \varepsilon_2)$ such that

$$\sqrt{t(|\ln t| + 1)} \le ch(t), \qquad t > 0. \tag{4.8}$$

Therefore, (4.1) and (4.8) imply

$$\left| \int_0^t e^{-2\lambda s} U_s' \overline{g}(s, U_s)\, dW_s \right| \le c\eta h\bigg(\int_0^t e^{-4\lambda s} \big| U_s' \overline{g}(s, U_s) \big|^2\, ds \bigg).$$

Hence, from (4.7) we obtain

$$\begin{aligned} e^{-2\lambda t}|U_t|^2 \le |U_0|^2 &+ \int_0^t e^{-2\lambda s}\big(2U_s'\overline{f}(s, U_s) + |\overline{g}(s, U_s)|^2\big)\, ds \\ &+ 2c\eta h\bigg(\int_0^t e^{-4\lambda s} \big| U_s'\overline{g}(s, U_s)\big|^2\, ds \bigg). \end{aligned} \tag{4.9}$$

By using (4.5) we have

$$|u'\overline{f}(s, u)| \le k_s \min\left\{|u|^{1+\alpha}, |u|^{1+\beta}\right\}, \qquad |u'\overline{g}(s, u)| \le k_s \min\left\{|u|^{1+\alpha}, |u|^{1+\beta}\right\}$$
$$|\overline{g}(s, u)|^2 = |\overline{g}(s, u)| \cdot |\overline{g}(s, u)| \le k_s^2 \, \min\left\{|u|^{1+\alpha}, |u|^{1+\beta}\right\}.$$

Putting $\overline{k}_s = \max\left(k_t, k_t^2\right)$ and using (4.9) we have

$$\begin{aligned} e^{-2\lambda t}|U_t|^2 = |U_0|^2 &+ 3\int_0^t e^{-2\lambda s}\overline{k}_s \min\left\{|U_s|^{1+\alpha}, |U_s|^{1+\beta}\right\}\, ds \\ &+ 2c\eta h\bigg(\int_0^t e^{-4\lambda s}\overline{k}_s \min\left\{|U_s|^{2(1+\alpha)}, |U_s|^{2(1+\beta)}\right\}\, ds \bigg). \end{aligned} \tag{4.10}$$

We consider two cases:

Case 1: $\lambda_d < 0$. Taking $\lambda = \lambda_d$, from (4.10) we obtain

$$e^{-2\lambda_d t}|U_t|^2 \le |U_0|^2 + 3\int_0^t e^{-2\lambda_d s}\overline{k}_s|U_s|^{1+\alpha}\, ds + 2c\eta h\bigg(\int_0^t e^{-4\lambda_d s}\overline{k}_s |U_s|^{2(1+\alpha)}\, ds \bigg).$$

By putting $v_t = |U_t|^2 e^{-2\lambda_d t}$ it is easy to see that

$$v_t \le v_0 + 3\int_0^t e^{\gamma s}\overline{k}_s v_s^{(1+\alpha)/2}\, ds + 2c\eta h\bigg(\int_0^t e^{2\gamma s}\overline{k}_s v_s^{1+\alpha}\, ds \bigg),$$

where $\gamma = (\alpha - 1)\lambda_d < 0$.

Since $\limsup_{t\to\infty} t^{-1}\ln \overline{k}_t = 0$ (see (4.6)) and $\gamma < 0$, we have

$$\int_0^\infty e^{\gamma s}\overline{k}_s\, ds := \delta < \infty.$$

The Cauchy–Schwarz inequality leads to

$$\int_0^t e^{\gamma s}\overline{k}_s v_s^{(1+\alpha)/2}\, ds \le \sqrt{\int_0^t e^{\gamma s}\overline{k}_s\, ds \int_0^t e^{\gamma s}\overline{k}_s v_s^{1+\alpha}\, ds} \le \sqrt{\delta}\sqrt{\int_0^t e^{\gamma s}\overline{k}_s v_s^{1+\alpha}\, ds}.$$

On the other hand, since $\gamma < 0$, we have $e^{2\gamma s} < e^{\gamma s}$ for any $s > 0$. Hence

$$v_t \le v_0 + 3\sqrt{\delta}\sqrt{\int_0^t e^{\gamma s}\overline{k}_s v_s^{1+\alpha}\, ds} + 2c\eta h\bigg(\int_0^t e^{\gamma s}\overline{k}_s v_s^{1+\alpha}\, ds \bigg).$$

It is obvious that $\sqrt{t} \le h(t)$ for any $t \ge 0$. We have

$$v_t \le v_0 + (3\sqrt{\delta} + 2c\eta)h\bigg(\int_0^t e^{\gamma s}\overline{k}_s v_s^{1+\alpha}\, ds \bigg). \tag{4.11}$$

Set $H(t,u) = Nh(u)$, $F(t,v) = e^{\gamma t}\overline{k}_s v^{1+\alpha}$, and $N = 3\sqrt{\delta} + 2c\eta$. Then (4.11) becomes

$$v_t \le v_0 + H\biggl(t, \int_0^t F(s, v_s)\, ds\biggr).$$

Suppose that ϕ_t is the solution of the equation

$$\dot{\phi}_t = e^{\gamma t}\overline{k}_t\bigl[v_0 + Nh(\phi_t)\bigr]^{1+\alpha}, \qquad \phi_0 = 0; \tag{4.12}$$

then from Theorem 6.11 in [13, §2.6, p. 111] we have

$$v_t \le v_0 + Nh(\phi_t). \tag{4.13}$$

We now estimate the solution ϕ_t of (4.12). It is easy to see that

$$\dot{\phi}_t \le 2^\alpha e^{\gamma t}\overline{k}_t\bigl[v_0^{1+\alpha} + N^{1+\alpha}h^{1+\alpha}(\phi_t)\bigr].$$

Hence, by putting $M = 2^\alpha N^{1+\alpha}$, $\Delta = 2^\alpha \delta$ we have

$$\phi_t \le \Delta v_0^{1+\alpha} + M\int_0^t e^{\gamma s}\overline{k}_s h^{1+\alpha}(\phi_s)\, ds.$$

We choose $0 < \varepsilon_1 < 1/2 < \varepsilon_2$ such that $\varepsilon_1(1+\alpha) := \gamma_1 > 1$ and $\varepsilon_2(1+\alpha) := \gamma_2 > 1$. Set

$$G(u) = \int_1^u \frac{dx}{h^{1+\alpha}(x)}.$$

We have

$$G(u) = \begin{cases} \frac{1}{1-\gamma_1}u^{1-\gamma_1} - \frac{1}{1-\gamma_1} & \text{if } 0 < x < 1,\\ \frac{1}{1-\gamma_2}u^{1-\gamma_2} - \frac{1}{1-\gamma_2} & \text{if } 1 \le x. \end{cases}$$

It is easy to see that $\lim_{u\to\infty} G(u) = (\gamma_2 - 1)^{-1} > 0$ and $\lim_{u\to 0} G(u) = -\infty$.

By applying Bihari's inequality (see [6], p. 110) we obtain

$$\phi_t \le G^{-1}\biggl\{G\left(\Delta v_0^{1+\alpha}\right) + M\int_0^t e^{\gamma s}\overline{k}_s\, ds\biggr\} \tag{4.14}$$

for any t satisfying the following relation:

$$G\left(\Delta v_0^{1+\alpha}\right) + M\int_0^t e^{\gamma s}\overline{k}_s\, ds < \frac{1}{\gamma_2 - 1}.$$

Summing up, we get

$$|X_t|^2 \le |\Phi_t|^2 e^{2\lambda_d t}\biggl\{|X_0| + Nh\biggl(G^{-1}\biggl[G(\Delta|X_0|^{1+\alpha}) + M\int_0^t e^{\gamma s}\overline{k}_s\, ds\biggr]\biggr)\biggr\}. \tag{4.15}$$

We prove that there exists a random variable $A > 0$ such that

$$\mathsf{P}\bigl\{|\Phi_t|e^{\lambda_d t} \le Aq_t \text{ for any } t \ge 0\bigr\} = 1. \tag{4.16}$$

Indeed, the relation $\limsup_{t\to\infty} t^{-1}\ln(q_t^*)/q_t = -\epsilon < 0$ implies that there is a constant $c > 0$ such that $q_t^* \le ce^{-(\epsilon/2)t}q_t$ for any $t > 0$. Moreover, from (4.2) it follows that $\lambda_d \le \lambda[q^*]$. Let $Z_t = (Z_t^{(1)}, \dots, Z_t^{(d)})$ where $Z_t^{(k)}$ is the kth column of Z_t. Denote $\mathcal{K} = \{k\colon \lambda[Z_t^{(k)}] = \lambda_d\}$. Since Z_t is normal, for t sufficiently large ($t > t_0$ say) we have $\|Z_t\| = \max_{k\in\mathcal{K}} \|Z_t^{(k)}\|$. Therefore,

$$\|\Phi_t\| = \max_{0\le k\le d}\Bigl\|Z_t^{(k)}e^{-\lambda_k t}\Bigr\| \le \max_{k\in\mathcal{K}}\Bigl\|Z_t^{(k)}e^{-\lambda_d t + (\epsilon/2)t}\Bigr\| \le q_t^* e^{-\lambda_d t + (\epsilon/2)t} \le cq_t e^{-\lambda_d t}$$

for $t > t_0$. It remains only to put

$$A = \max\biggl\{c, \max_{0<t<t_0}\|\Phi_t\|e^{\lambda_d t}/q_t\biggr\}$$

to get the result.

On the other hand, since $\lim_{X_0\to 0} G(\Delta|X_0|^{1+\alpha}) = -\infty$ and the distribution of η (and hence the distribution of M and N) is the same for every X_0, for any $\varepsilon > 0$ we have

$$\lim_{X_0\to 0} \mathsf{P}\left\{G^{-1}\left[G(\Delta|X_0|^{1+\alpha}) + M\int_0^t e^{\gamma s}\overline{k}_s\, ds\right] \geq \varepsilon\right\} = 0$$

uniformly in t. Therefore,

$$\lim_{X_0\to 0} \mathsf{P}\left\{|X_0| + Nh\left(G^{-1}\left[G(\Delta|X_0|^{1+\alpha}) + M\int_0^t e^{\gamma s}\overline{k}_s\, ds\right]\right) \geq \frac{1}{A}\right\} = 0.$$

Hence,

$$\lim_{X_0\to 0} \mathsf{P}\big\{|X_t| \leq q_t, t \geq 0\big\} = 1,$$

i.e., we get (4.3). This means that (3.9) is more stable than (3.8).

Case 2: $\lambda_d \geq 0$. We take a $\sigma > 0$ such that $\lambda_d + 2\sigma < \lambda q$. Using (4.10) with $\lambda = \lambda_d + \sigma$ we also obtain

$$|U_t|^2 \leq e^{2(\lambda_d+\sigma)t}\Bigg[|U_0|^2 + 3\int_0^t e^{-2(\lambda_d+\sigma)s}\overline{k}_s|U_s|^{1+\beta}\, ds + 2c\eta h\bigg(\int_0^t e^{-4(\lambda_d s+\sigma)}\overline{k}_s|U_s|^{2+2\beta}\, ds\bigg)\Bigg].$$

By the same argument as above and putting $v_t = |U_t|^2 e^{-2(\lambda_d+\sigma)t}$ we have

$$v_t \leq v_0 + \big(3\sqrt{\delta} + 2c\eta\big)h\bigg(\int_0^t e^{\gamma s}\overline{k}_s|v|^{1+\beta}\, ds\bigg),$$

where $\gamma = (\beta - 1)(\lambda_d + \sigma) < 0$.

We choose $\varepsilon_1 < \frac{1}{2} < \varepsilon_2$ such that $\gamma_1 = \varepsilon_1(1+\beta) < 1$ and $\gamma_2 = \varepsilon_2(1+\beta) < 1$. In this case, $\lim_{u\to 0} G(u) = (\gamma_1 - 1)^{-1}$ and $\lim_{u\to\infty} G(u) = +\infty$. Therefore, the inverse function G^{-1} is defined on $[(\gamma_1 - 1)^{-1}, +\infty)$. Similarly to (4.15),

$$|X_t|^2 \leq |\Phi_t|^2 e^{2(\lambda_d+\sigma)t}\bigg\{|X_0| + Nh\bigg(G^{-1}\left[G\left(\Delta|X_0|^{1+\beta}\right) + M\int_0^t e^{\gamma s}\overline{k}_s\, ds\right]\bigg)\bigg\}. \tag{4.17}$$

It is easy to see that

$$\lim_{T\to\infty}\left\{\sup_{t>T} e^{-2\sigma}th\bigg(G^{-1}\left[G\left(\Delta|X_0|^{1+\beta}\right) + M\int_0^t e^{\gamma s}\overline{k}_s\, ds\right]\bigg)\right\} = 0 \tag{4.18}$$

for any X_0.

In the same way as in the proof of (4.16), we can find a random variable A satisfying

$$\mathsf{P}\left\{|\Phi_t|e^{(\lambda_d+2\sigma)} \leq Aq_t \text{ for any } t > 0\right\} = 1.$$

In view of (4.17), this implies that

$$|X_t|^2 \leq A^2 q_t^2 e^{-2\sigma t}\bigg\{|X_0| + Nh\bigg(G^{-1}\left[G(\Delta|X_0|^{1+\beta}) + M\int_0^t e^{\gamma s}\overline{k}_s\, ds\right]\bigg)\bigg\}$$

for any $t > 0$.

For $\varepsilon > 0$ fixed, since M and N are equidistributed, it follows from (4.18) that there is a $T_0 > 0$ such that

$$\mathsf{P}\left\{A^2 e^{-2\lambda t}Nh\bigg(G^{-1}\left[G\left(\Delta|X_0|^{1+\beta}\right) + M\int_0^t e^{\gamma s}\overline{k}_s\, ds\right]\bigg) \leq 1 \text{ for all } t \geq T_0\right\} \geq 1 - \frac{\varepsilon}{2}$$

for any $|X_0| \leq 1$.

On the other hand, from Corollary 4.3 it follows that there is $1 > \delta > 0$ such that if $|X_0| < \delta$ then

$$\mathsf{P}\{|X_t| \le q_t, t \le T_0\} \ge 1 - \frac{\varepsilon}{2}.$$

Summing up, for $|X_0| < \delta$ we have

$$\mathsf{P}\{|X_t| \le q_t \text{ for all } t \ge 0\} \ge 1 - \varepsilon,$$

i.e., we get (4.3),

$$\lim_{X_0 \to 0} \mathsf{P}\{|X_t| \le q_t \text{ for all } t \ge 0\} = 1.$$

Theorem 3.1 is proved.

Remark. In comparison with the results in [10, p. 291], we see that Theorem 3.1 allows us to conclude that the growth rate of solutions of system (3.9) does not exceed the growth rate of its linear part (3.3), whereas theorems in [10] lead only to the conclusion that the trivial solution is stable.

Example. Suppose that $a(x)$ and $b(x)$ are two differential functions whose second order derivatives are bounded by a constant k. By Taylor's expansion we have

$$a(x) = Ax + f(x), \qquad b(x) = Bx + g(x), \qquad x \in \mathbf{R}^d,$$

where $|f(x)| \le k|x|^2$ and $|g(x)| \le k|x|^2$. If the top Lyapunov exponent λ_d of the linear system

$$dZ_t = AZ_t\,dt + BZ_t\,dW_t$$

is negative, then there exists a neighbourhood U of 0 such that every solution, starting from U, of the system

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t$$

has an Lyapunov exponent which does not exceed the top Lyapunov exponent λ_d.

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Faculty of Mathematics, Mechanics, and Informatics, Vietnam National University, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam
E-mail address: nhdu2001@yahoo.com

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