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ON AN EXTERIOR INITIAL BOUNDARY VALUE PROBLEM FOR NAVIER-STOKES EQUATIONS

By

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Dedicated to Professor Kyûya Masuda on the occasion of his 60th birthday

1. Introduction.

1.1. Problem. In this paper we study the asymptotic behaviour of small solutions of the stationary problem of the three-dimensional Navier-Stokes equations:

$$\begin{cases} -\Delta \mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla \mathfrak{q} = \mathbf{f}, \ \ \nabla\cdot\mathbf{w} = 0 & \text{for } x \in \Omega, \\ \mathbf{w} = \mathbf{g} & \text{for } x \in \partial\Omega, \\ \lim_{|x|\to\infty} \mathbf{w}(x) = \mathbf{u}_\infty & \end{cases} \tag{SP}$$

where $\mathbf{u}_\infty$ is a nonzero constant three-dimensional row vector and Ω is an exterior domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Also, we discuss the stability property of the solutions of (SP) with respect to small L_3-perturbation. To be more precise, let us consider the nonstationary problem :

$$\begin{cases} \mathbf{v}_t - \Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla \mathfrak{p} = \mathbf{f}, \ \ \nabla\cdot\mathbf{v} = 0 & \text{for } t>0,\ x \in \Omega, \\ \mathbf{v} = \mathbf{g} & \text{for } t>0,\ x \in \partial\Omega, \\ \mathbf{v}(0,x) = \mathbf{a}(x) & \text{for } x \in \Omega, \\ \lim_{|x|\to\infty} \mathbf{v}(t,x) = \mathbf{u}_\infty \qquad {}^\forall t > 0. & \end{cases} \tag{NS}$$

Inserting $\mathbf{v}(t,x) = \mathbf{w}(x) + \mathbf{u}(t,x)$, $\mathbf{a}(x) = \mathbf{w}(x) + \mathbf{b}(x)$ and $\mathfrak{p}(t,x) = \mathfrak{q}(x) + \mathfrak{r}(t,x)$ into (NS), we obtain the equations governing the perturbation $\mathbf{u}$:

$$\begin{cases} \left.\begin{matrix} \mathbf{u}_t - \Delta \mathbf{u} + (\mathbf{w}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\mathfrak{r} = \mathbf{0} \\ \nabla\cdot\mathbf{u} = 0 \end{matrix}\right\} & \text{for } t>0,\ x\in\Omega, \\ \mathbf{u}(t,x) = \mathbf{0} & \text{for } t>0,\ x\in\partial\Omega, \\ \mathbf{u}(0,x) = \mathbf{b}(x) & \text{for } x\in\Omega, \end{cases} \tag{P}$$

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We shall prove the existence and asymptotic behaviour globally in time of solutions of (P) when $|\mathbf{u}_\infty|$ and the L_3-norm of $\mathbf{b}$ are very small.

The notation in (SP) and (NS) is the usual one of vector analysis explained below more precisely in the paragraph of notation. Three-dimensional row vectors of functions are denoted with bold-face letters, for example, $\mathbf{w} = \mathbf{w}(x) = {}^{\mathrm{T}}(w_1(x), w_2(x), w_3(x))$ where ${}^{\mathrm{T}}M$ means the transposed M. The solution $\mathbf{w}(x)$ of (SP) can be interpreted as the velocity field of a steady motion of an incompressible fluid in position $x = (x_1, x_2, x_3) \in \Omega$ with an external force $\mathbf{f} = \mathbf{f}(x)$ and a prescribed velocity field $\mathbf{g} = \mathbf{g}(x)$ at the boundary $\partial\Omega$, and the scalar function $\mathfrak{q} = \mathfrak{q}(x)$ is then the associated pressure, where we adopt a coordinate frame fixed to a moving rigid body $\mathcal{O}$ which is identified with a bounded domain in $\mathbb{R}^3$ in the viscous incompressible fluid that occupies the region $\Omega = \mathbb{R}^3 - \bar{\mathcal{O}}$. The solutions $\mathbf{v} = \mathbf{v}(t,x) = {}^{\mathrm{T}}(v_1(t,x), v_2(t,x), v_3(t,x))$ and $\mathfrak{p} = \mathfrak{p}(t,x)$, a scalar function, of (NS) also can be interpreted as the velocity field and its associated pressure of the time-dependent motion of a viscous incompressible fluid in position $x \in \Omega$ at time $t > 0$ with an initial velocity field $\mathbf{a} = \mathbf{a}(x)$ as well as the same external force $\mathbf{f} = \mathbf{f}(x)$ and prescribed velocity $\mathbf{g} = \mathbf{g}(x)$ at $\partial\Omega$ as in (SP).

It is well known that without smallness assumptions, present day analysis yields only a locally in time unique solution of (NS) in the three-dimensional case, while Leray [33] and Hopf [26] proved the existence of square-integrable weak solutions for arbitrary square-integrable initial velocity, whose uniqueness is still unknown.

The first general study of (SP) for arbitrary prescribed data is due to Leray [32]. He proved the existence of smooth solutions of (SP) with a finite Dirichlet integral. But, the solutions obtained by Leray did not provide much qualitative information about the solutions. In particular, nothing was proven about the asymptotic structure of the wake behind the body $\mathcal{O}$. Finn [12] to [16] has studied (SP) within the class of solutions, termed by him physically reasonable, which tend to a limit at infinity like $|x|^{-1/2-\epsilon}$ for some $\epsilon > 0$. For small data he proved both existence and uniqueness within this class. In fact, his solutions satisfy the following estimate :

$$|\mathbf{w}(x) - \mathbf{u}_\infty| \leqq C\,|x|^{-1} \text{ as } |x| \to \infty \text{ and } \nabla\mathbf{w} \in L_3(\Omega) \tag{PR}$$

where C is a constant. Furthermore, his solutions exhibit paraboloidal wake region behind the body $\mathcal{O}$.

Finn has conjectured [17] that for sufficiently small data, physically reasonable solutions are attainable. Namely, the problem is to find a solution $\mathbf{u}(t,x)$ of (P) such that $\mathbf{u}(t,x) \to \mathbf{0}$, that is, $\mathbf{v}(t,x) - \mathbf{w}(x) \to \mathbf{0}$ as $t \to \infty$. This is called a stability problem.

The stability problem was first solved by Heywood [23, 24] in the L_2 framework. Roughly speaking, he proved that if the L_2-norm of $\mathbf{b}(x)$ is very small and if $C < 1/2$, C being the constant in (PR) above, then there exists a unique solution $\mathbf{u}(t,x)$ of (P) satisfying the convergence property :

$$\int_\Omega |\nabla(\mathbf{u}(t,x) - \mathbf{w}(x))|^2\,dx \to 0 \quad \text{and} \quad \int_{\substack{x\in\Omega \\ |x|\leqq R}} |\mathbf{u}(t,x) - \mathbf{w}(x)|^2\,dx \to 0$$

as $t \to \infty$ where R is any positive number. His result was sharpened, in particular with respect to the rate of the convergence, by Masuda [35], Heywood himself [25], Miyakawa

[36], and Maremonti [34] (cf. further references cited therein). But, as Finn showed in [14], if $\mathbf{w}(x)$ is a physically reasonable solution and if the force exerted to the body $\mathcal{O}$ by the flow does not vanish, then $\mathbf{w}(x) - \mathbf{u}_\infty$ is not square-integrable over Ω. Therefore, it seems reasonable to seek a solution of the problem (P) in a class of functions that are not square-integrable over Ω for each time $t > 0$.

In this direction, Kato [29] solved the problem (NS) in the L_n-framework when $\Omega = \mathbb{R}^n$ ($n \geqq 2$), $\mathbf{u}_\infty = \mathbf{0}$ and the L_n-norm of $\mathbf{a}$ is very small. He employed various L_p-norms and L_p–L_q estimates for the semigroup generated by the Stokes operator. Iwashita [28] extended Kato's result to the case that $\Omega \neq \mathbb{R}^n$ ($n \geqq 3$), $\mathbf{u}_\infty = \mathbf{0}$ and that the L_n-norm of $\mathbf{a}$ is also very small. The main point of Iwashita's work was to obtain L_p–L_q estimates of the semigroup generated by the Stokes operator in Ω with zero Dirichlet boundary condition. Since the zero vector $\mathbf{0}$ is a trivial solution to (SP) when $\mathbf{u}_\infty = \mathbf{0}$, expressing the Kato and Iwashita results in other words, we can say that the trivial solution is stable by the small L_n-perturbation.

Recently, when $\mathbf{u}_\infty = \mathbf{0}$ and $\Omega \subset \mathbb{R}^n$ ($n \geqq 3$), Borchers and Miyakawa [5] and Kozono and Yamazaki [31] proved the stability of nontrivial physically reasonable solutions by the small weak L_n-perturbation. Namely, they proved that if the L_n weak norm of $\mathbf{b}$ is very small, then (NS) admits a unique solution $\mathbf{v}(t,x)$ that converges to $\mathbf{w}(x)$ in the L_n weak space with a suitable rate with respect to t as $t \to \infty$. Since the physically reasonable solutions of (SP) belong to the L_n weak space when $\mathbf{u}_\infty = \mathbf{0}$ (cf. (PR)), the stability problem was, therefore, settled in the case where $\mathbf{u}_\infty = \mathbf{0}$ and $n \geqq 3$.

On the other hand, the case where $\mathbf{u}_\infty \neq \mathbf{0}$ has been studied relatively seldom compared with the case where $\mathbf{u}_\infty = \mathbf{0}$ (cf. except for papers cited above, Oseen [38], Babenko [1], Bemelman [2], Faxén [11], Farwig [8, 9], Galdi [19]). In particular, the stability has been proved only in the L_2-framework. This paper is devoted to the study of the stability problem of physically reasonable solutions with respect to small L_3-perturbation in the three-dimensional exterior domain when $\mathbf{u}_\infty$ is a nonzero constant vector. In fact, since $\mathbf{w}(x) - \mathbf{u}_\infty$ belongs to L_3-space when $\mathbf{u}_\infty \neq \mathbf{0}$, which will be proved in Theorem 1.1 below, the stability theorem with respect to the L_3-perturbation is meaningful. As a corollary of our stability theorem, we also prove a unique existence theorem of small strong solutions of (NS) in the L_3-framework when $\mathbf{f} = \mathbf{g} = \mathbf{0}$ and $\mathbf{u}_\infty \neq \mathbf{0}$, which is an extension of the Kato and Iwashita results to the case where $\mathbf{u}_\infty \neq \mathbf{0}$. Moreover, we shall prove that our solutions tend to Kato and Iwashita solutions when $\mathbf{u}_\infty \to \mathbf{0}$ even in the L_∞-space.

1.2. Notation. To state main results, first we outline at this point our notation. The dot $\cdot$ denotes the usual inner product of three-dimensional row vectors. (a_{ij}) means the 3×3 matrix whose i^{th} column and j^{th} row component is a_{ij}. As usual, the subscript t means partial differentiation with respect to t, and moreover we put

$$\partial_t = \partial/\partial t, \ \ \partial_j = \partial/\partial x_j, \ \ \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2,$$
$$\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}, \ \ \alpha = (\alpha_1, \alpha_2, \alpha_3), \ \ |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

For three-dimensional row vector-valued functions $\mathbf{u} = {}^{\mathrm{T}}(u_1, u_2, u_3)$, $\mathbf{v} = {}^{\mathrm{T}}(v_1, v_2, v_3)$

and a scalar-valued function u we put

$$\partial_x^m u = (\partial_x^\alpha u, |\alpha| = m),\ \ \overline{\partial}_x^m u = (\partial_x^\alpha u, |\alpha| \le m),\ \ \partial_t^j\partial_x^\alpha \mathbf{u} = {}^{\mathrm{T}}(\partial_t^j\partial_x^\alpha u_1, \partial_t^j\partial_x^\alpha u_2, \partial_t^j\partial_x^\alpha u_3),$$
$$\partial_x^m \mathbf{u} = (\partial_x^\alpha \mathbf{u}, |\alpha| = m),\ \ \overline{\partial}_x^m \mathbf{u} = (\partial_x^\alpha \mathbf{u}, |\alpha| \le m),\ \ \nabla\mathbf{u} = (\partial_j u_i),$$
$$\Delta\mathbf{u} = {}^{\mathrm{T}}(\Delta u_1, \Delta u_2, \Delta u_3),\ \ (\mathbf{u}\cdot\nabla)\mathbf{v} = {}^{\mathrm{T}}(\sum_{j=1}^3 u_j\partial_j v_1, \sum_{j=1}^3 u_j\partial_j v_2, \sum_{j=1}^3 u_j\partial_j v_3),$$
$$\nabla\cdot\mathbf{u} = \sum_{j=1}^3 \partial_j u_j, \nabla u = {}^{\mathrm{T}}(\partial_1 u, \partial_2 u, \partial_3 u), \nabla u : \nabla\mathbf{v} = {}^{\mathrm{T}}(\nabla u\cdot\nabla v_1, \nabla u\cdot\nabla v_2, \nabla u\cdot\nabla v_3).$$

To denote the special sets, we use the following symbols:

$$B_b = \{x \in \mathbb{R}^3 \mid |x| \leqq b\},\ \ G_b = \{x \in \mathbb{R}^3 \mid |x| \geqq b\},\ \ D_b = \{x \in \mathbb{R}^3 \mid b-1 \leqq |x| \leqq b\},$$
$$S_b = \{x \in \mathbb{R}^3 \mid |x| = b\},\quad \Omega_b = \Omega\cap B_b,\quad \partial\Omega_b = \partial\Omega\cup S_b.$$

Let b_0 be a fixed number such that $B_{b_0} \supset \mathcal{O}$. Sobolev spaces of vector-valued functions are used, as well as of scalar-valued functions. If D is any domain in $\mathbb{R}^3$, $L_p(D)$ denotes the usual L_p-space of scalar functions on D and $\|\cdot\|_{p,D}$ its usual norm. Moreover, we put

$$\|\mathbf{u}\|_{p,D} = \left(\sum_{j=1}^3 \|u_j\|_{p,D}^p\right)^{1/p} \quad (1 \leqq p < \infty),\quad \|\mathbf{u}\|_{\infty,D} = \max_{j=1,2,3}\|u_j\|_{\infty,D},$$
$$\|u\|_{p,m,D} = \|\overline{\partial}_x^m u\|_{p,D},\ \ \|\mathbf{u}\|_{p,m,D} = \|\overline{\partial}_x^m \mathbf{u}\|_{p,D},\ \ (\mathbf{u},\mathbf{v})_D = \int_D \mathbf{u}(x)\cdot\overline{\mathbf{v}(x)}dx.$$

For simplicity, we shall use the following abbreviation: $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$, $\|\cdot\|_p = \|\cdot\|_{p,\Omega}$, $\|\cdot\|_{p,m} = \|\cdot\|_{p,m,\Omega}$, $|\cdot|_p = \|\cdot\|_{p,\mathbb{R}^3}$, $|\cdot|_{p,m} = \|\cdot\|_{p,m,\mathbb{R}^3}$. $\mathcal{D}'$ denotes the set of all distributions on $\mathbb{R}^3$, $\mathcal{S}'$ the set of all tempered distributions on $\mathbb{R}^3$ and $C_0^\infty(D)$ the set of all functions of $C^\infty(\mathbb{R}^3)$ whose support is contained in D. Moreover, we put

$$\begin{aligned}
&L_{p,b}(D) = \{u \in L_p(D) \mid u(x) = 0\ {}^\forall x \notin B_b\},\\
&W_{p,loc}^m(\mathbb{R}^3) = \{u \in \mathcal{S}' \mid \partial_x^\alpha u \in L_p(B_b)\ {}^\forall\alpha : |\alpha| \le m \text{ and } {}^\forall b > 0\},\\
&W_{p,loc}^m(D) = \{u \mid {}^\exists U \in W_{p,loc}^m(\mathbb{R}^3) \text{ such that } u = U \text{ on } D\},\\
&L_{p,loc}(D) = W_{p,loc}^0(D),\\
&W_p^m(D) = \{u \in W_{p,loc}^m(D) \mid \|u\|_{p,m,D} < \infty\},\\
&\dot{W}_p^m(D) = \text{the completion of } C_0^\infty(D) \text{ with respect to } \|\cdot\|_{p,m,D},\\
&\hat{W}_p^m(D) = \{u \in W_{p,loc}^m(D) \mid \|\partial_x^m u\|_{p,D} < \infty\}.
\end{aligned}$$

To denote function spaces of three-dimensional row vector-valued functions, we use the blackboard bold letters. For example,

$$\mathbb{L}_q(D) = \{\mathbf{u} = {}^{\mathrm{T}}(u_1, u_2, u_3) \mid u_j \in L_q(D), j = 1,2,3\}.$$

Likewise for $\mathbb{C}_0^\infty(D)$, $\mathbb{L}_{p,b}(D)$, $\mathbb{W}^m_{p,loc}(D)$, $\mathbb{L}_{p,loc}(D)$, $\mathbb{W}^m_p(D)$, $\dot{\mathbb{W}}^m_p(D)$ and $\hat{\mathbb{W}}^m_p(D)$. Moreover, we put

$$\mathbb{J}_p(D) = \text{the completion in } \mathbb{L}_p(D) \text{ of the set } \{\mathbf{u} \in \mathbb{C}_0^\infty(D) \mid \nabla\cdot\mathbf{u} = 0 \text{ in } D\},$$
$$\mathbb{G}_p(D) = \{\nabla p \mid p \in \hat{W}^1_p(D)\},$$
$$\mathbb{W}^m_{p,d}(\partial\Omega) = \{\mathbf{g} \in \mathbb{W}^m_p(\Omega) \mid \mathbf{g}(x) = \mathbf{0} \text{ for } |x| \geqq b_0+1,\ \int_{\partial\Omega} \nu(x)\cdot\mathbf{g}(x)\,d\Gamma = 0\},$$

where $d\Gamma$ is the surface element of $\partial\Omega$ and $\nu(x) = {}^{\mathrm{T}}(\nu_1(x), \nu_2(x), \nu_3(x))$ is the unit outer normal to $\partial\Omega$. According to Fujiwara and Morimoto [18] and Miyakawa [36], the Banach space $\mathbb{L}_p(D)$ admits the Helmholtz decomposition: $\mathbb{L}_p(D) = \mathbb{J}_p(D) \oplus \mathbb{G}_p(D)$, where $\oplus$ denotes the direct sum. Let $\mathbb{P}_D$ be a continuous projection from $\mathbb{L}_p(D)$ onto $\mathbb{J}_p(D)$. The Stokes operator $\mathbb{A}_D$ and the Oseen operator $\mathbb{O}_D(\mathbf{u}_\infty)$ are defined by the relations: $\mathbb{A}_D = -\mathbb{P}_D\Delta$ and $\mathbb{O}_D(\mathbf{u}_\infty) = \mathbb{A}_D + \mathbb{P}_D(\mathbf{u}_\infty\cdot\nabla)$ with the same domain: $\mathcal{D}_p(\mathbb{A}_D) = \mathcal{D}_p(\mathbb{O}_D(\mathbf{u}_\infty)) = \mathbb{J}_p(D) \cap \dot{\mathbb{W}}^1_p(D) \cap \mathbb{W}^2_p(D)$. Note that $\mathbb{O}_D(\mathbf{0}) = \mathbb{A}_D$. For simplicity, we write $\mathbb{P} = \mathbb{P}_\Omega$, $\mathbb{A} = \mathbb{A}_\Omega$ and $\mathbb{O}(\mathbf{u}_\infty) = \mathbb{O}_\Omega(\mathbf{u}_\infty)$. To denote various constants we use the same letter C. By $C_{A,B,\ldots}$ we denote a constant depending on the quantities $A, B, \cdots$. C and $C_{A,B,\ldots}$ will change from line to line. Let $\mathbb{C}$ denote the set of all complex numbers. For two Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y with norm $\|\cdot\|_{\mathcal{L}(X,Y)}$, $\mathcal{B}(I,X)$ the set of all X–valued bounded continuous functions on I and $C(I,X)$ the set of all X–valued continuous functions on I. Finally, $e^{-\mathbb{O}(\mathbf{u}_\infty)t} = T_{\mathbf{u}_\infty}(t)$ denotes the analytic semigroup on $\mathbb{J}_p(\Omega)$ generated by $\mathbb{O}(\mathbf{u}_\infty)$, the existence of which is proved by Miyakawa [36].

1.3. Main results. Now, we shall state our main results. We start with an existence theorem of small solutions to (SP).

THEOREM 1.1. Let $3 < p < \infty$ and let δ and β be any numbers such that $0 < \delta < 1/4$ and $0 < \delta < \beta < 1-\delta$. Let $\mathbf{f} \in \mathbb{L}_\infty(\Omega)$ and $\mathbf{g} \in \mathbb{W}^2_{p,\delta}(\partial\Omega)$. Then, there exists a constant ϵ, $0 < \epsilon \leqq 1$, depending on p, δ and β but independent of $\mathbf{u}_\infty$ such that if $0 < |\mathbf{u}_\infty| \leqq \epsilon$ and $\ll \mathbf{f} \gg_{2\delta} + \|\mathbf{g}\|_{p,2} \leqq \epsilon\,|\mathbf{u}_\infty|^{\beta+\delta}$, then the problem (SP) admits solution $\mathbf{w}$ and $\mathfrak{p}$ possessing the estimate :

$$\|\mathbf{w}-\mathbf{u}_\infty\|_{p,2} + |||\mathbf{w}-\mathbf{u}_\infty|||_\delta + \|\mathfrak{p}\|_{p,1} \leqq |\mathbf{u}_\infty|^\beta, \tag{1.1}$$

where

$$\ll \mathbf{u} \gg_{2\delta} = \sup_{x\in\Omega}(1+|x|)^{5/2}(1+s_{\mathbf{u}_\infty}(x))^{1/2+2\delta}|\mathbf{u}(x)|, \tag{1.2}$$

$$|||\mathbf{u}|||_\delta = \sup_{x\in\Omega}(1+|x|)(1+s_{\mathbf{u}_\infty}(x))^\delta|\mathbf{u}(x)| + \sup_{x\in\Omega}(1+|x|)^{3/2}(1+s_{\mathbf{u}_\infty}(x))^{1/2+\delta}|\nabla\mathbf{u}(x)|, \tag{1.3}$$

$$s_{\mathbf{u}_\infty}(x) = |x| - {}^{\mathrm{T}}x\cdot\mathbf{u}_\infty/|\mathbf{u}_\infty|. \tag{1.4}$$

REMARK 1.2. The similar result was obtained recently by Novotny and Padula [37] in the compressible viscous fluid case. From works due to Finn [12] to [16] and Farwig

[8, 9], the dependence of solutions on $\mathbf{u}_\infty$ is not clear. Since such dependence plays an important role to solve the stability problem, we shall prove Theorem 1.1 in this paper.

REMARK 1.3. The estimate (1.1) represents the wake region behind $\mathcal{O}$. Moreover, by (1.1), $\mathbf{w}-\mathbf{u}_\infty \in \mathbb{L}_3(\Omega)$ and $\nabla\mathbf{w} \in \mathbb{L}_{3/2}(\Omega)$. In fact,

$$\begin{aligned} \|\mathbf{w}-\mathbf{u}_\infty\|_3 &\leqq \left[2\pi\int_0^\infty \frac{dr}{(1+r)^3 r^\delta}\int_0^\pi \frac{\sin\theta\, d\theta}{(1-\cos\theta)^\delta}\right]^{1/3} |\mathbf{u}_\infty|^\beta, \\ \|\nabla\mathbf{w}\|_{3/2} &\leqq \left[2\pi\int_0^\infty \frac{dr}{(1+r)^{9/4} r^{(3+\delta)/4}}\int_0^\pi \frac{\sin\theta\, d\theta}{(1-\cos\theta)^{(3+\delta)/4}}\right]^{2/3} |\mathbf{u}_\infty|^\beta. \end{aligned} \tag{1.5}$$

Now, we shall state our stability theorem, that is, the existence of solutions of (P) globally in time. According to the approach due to Kato [29], instead of (P), we consider the integral equation. Namely, in view of (1.1), if we write $(\mathbf{w}\cdot\nabla)\mathbf{u} = (\mathbf{u}_\infty\cdot\nabla)\mathbf{u} + ((\mathbf{w}-\mathbf{u}_\infty))\cdot\nabla\mathbf{u}$ in (P) and if we apply the projection $\mathbb{P}$ to the resulting formula, the first formula in (P) is reduced to

$$\mathbf{u}_t + \mathbb{O}(\mathbf{u}_\infty)\,\mathbf{u} = -\mathbb{P}\,[\mathcal{L}[\mathbf{w}]\mathbf{u} + \mathcal{N}[\mathbf{u}]], \tag{1.6}$$

where

$$\mathcal{L}[\mathbf{w}]\mathbf{u} = ((\mathbf{w}-\mathbf{u}_\infty)\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{w}, \tag{1.7}$$

$$\mathcal{N}[\mathbf{u}] = (\mathbf{u}\cdot\nabla)\mathbf{u}. \tag{1.8}$$

Then, applying Duhamel's principle to (1.6), we have the integral equation

$$\mathbf{u}(t) = T_{\mathbf{u}_\infty}(t)\mathbf{b} - \int_0^t T_{\mathbf{u}_\infty}(t-s)\,\mathbb{P}\,[\mathcal{L}[\mathbf{w}]\,\mathbf{u}(s) + \mathcal{N}[\mathbf{u}(s)]]\,ds. \tag{1.9}$$

Instead of (P), we shall solve (1.9).

THEOREM 1.4. Let $3 < p < \infty$ and let δ and β be the same as in Theorem 1.1. In addition, we assume that $0 < \delta < \min(1/6, 4/p)$. Let $\mathbf{f} \in \mathbb{L}_\infty(\Omega)$, $\mathbf{g} \in \mathbb{W}^2_{p,d}(\partial\Omega)$ and $\mathbf{b}(x) \in \mathbb{J}_3(\Omega)$. Then, there exists an $\epsilon > 0$, $0 < \epsilon \leqq 1$, depending only on p, β, and δ essentially such that if $0 < |\mathbf{u}_\infty| \leqq \epsilon$, $\ll \mathbf{f} \gg_{2\delta} + \|\mathbf{g}\|_{p,2} \leqq \epsilon\,|\mathbf{u}_\infty|^{\beta+\delta}$ and $\|\mathbf{b}\|_3 \leqq \epsilon$, then the problem (1.9) admits a unique solution $\mathbf{u} \in \mathcal{B}([0,\infty), \mathbb{J}_3(\Omega))$ possessing the following properties:

$$[\mathbf{u}]_{3,0,t} + [\mathbf{u}]_{p,\mu(p),t} + [\nabla\mathbf{u}]_{3,1/2,t} \leqq \sqrt{\epsilon}, \tag{1.10}$$

$$\lim_{t\to 0+}\left[\|\mathbf{u}(t,\cdot)-\mathbf{b}\|_3 + [\mathbf{u}]_{p,\mu(p),t} + [\nabla\mathbf{u}]_{3,1/2,t}\right] = 0. \tag{1.11}$$

Here and hereafter, we put

$$[\mathbf{z}]_{p,\rho,t} = \sup_{0<s<t} s^\rho \|\mathbf{z}(s,\cdot)\|_p, \tag{1.12}$$

$$\mu(p) = \frac{3}{2}\left(\frac{1}{3}-\frac{1}{p}\right) = \frac{1}{2}-\frac{3}{2p} \quad \text{for } p \geqq 3. \tag{1.13}$$

Moreover, we have the relations:

$$\begin{aligned} &[\mathbf{u}]_{q,\mu(q),t} \leqq C_q \left(\epsilon + \epsilon^{1/2+\beta}\right), \quad 3 < q < \infty, \\ &\|\mathbf{u}(t,\cdot)\|_\infty \leqq C_m \left(\epsilon + \epsilon^{1/2+\beta}\right) \left(t^{-1/2} + t^{-(1-3/2m)}\right), \end{aligned} \tag{1.14}$$

for any $t > 0$ where m is a number such that $3 < m < p$.

Finally, we consider the convergence of solutions of (NS) as $|\mathbf{u}_\infty| \to 0$ in the case that $\mathbf{f}(x) = \mathbf{g}(x)$.

THEOREM 1.5. Let us consider the problem (NS) in the case that $\mathbf{f}(x) = \mathbf{g}(x) = \mathbf{0}$. Let $0 < \beta < 1$ and let $\mathbf{a}(x) = \mathbf{u}_\infty + \mathbf{b}(x)$ be an initial velocity. Then, there exists an ϵ, $0 < \epsilon \leqq 1$, depending on β but independent of $\mathbf{u}_\infty$ and $\mathbf{b}$ such that if $|\mathbf{u}_\infty| \leqq \epsilon$, $\mathbf{b} \in \mathbb{J}_3(\Omega)$ and $\|\mathbf{b}\|_3 \leqq \epsilon$, then (NS) admits a unique solution $\mathbf{v}_{\mathbf{u}_\infty}(t,x)$ with suitable pressure part $\mathfrak{p}_{\mathbf{u}_\infty}(t,x)$ such that $\mathbf{u}(t,x) = \mathbf{v}_{\mathbf{u}_\infty}(t,x) - \mathbf{u}_\infty \in \mathcal{B}([0,\infty), \mathbb{J}_3(\Omega))$ and (1.10), (1.11) and (1.14) hold for the present $\mathbf{u}$ with suitable constants C_q and C_m independent of ϵ, β and $\mathbf{u}_\infty$. Moreover, we have the following convergence property:

$$\begin{aligned} &\|\mathbf{v}_{\mathbf{u}_\infty}(t,\cdot) - \mathbf{u}_\infty - \mathbf{v}_{\mathbf{0}}(t,\cdot)\|_q \leqq C_q \left(t^{-\mu(q)} + t^{3/2q}\right) |\mathbf{u}_\infty|^\beta \quad 3 \leqq {}^\forall q < \infty, \\ &\|\mathbf{v}_{\mathbf{u}_\infty}(t,\cdot) - \mathbf{v}_{\mathbf{0}}(t,\cdot)\|_\infty \leqq C_m \left(t^{(1-3/2m)} + 1\right) |\mathbf{u}_\infty|^\beta, \\ &\|\nabla(\mathbf{v}_{\mathbf{u}_\infty}(t,\cdot) - \mathbf{v}_{\mathbf{u}_\infty}(t,\cdot))\|_3 \leqq C \left(t^{-1/2} + 1\right) |\mathbf{u}_\infty|^\beta \end{aligned} \tag{1.15}$$

for any $t > 0$ where m is a constant > 3.

2. Preparation for the latter sections.

In this section, we shall discuss some basic facts which will be used in the latter sections. Throughout this section, D denotes a bounded domain in $\mathbb{R}^3$ with smooth boundary ∂D. We start with a proposition concerning inequalities of Poincaré's type and an extension of functions.

PROPOSITION 2.1. Let $1 < p < \infty$. (1) Then, the following two relations hold:

$$\|v\|_{p,D} \leqq C_D \left(\|\nabla v\|_{p,D} + \left| \int_D v(x)\,dx \right| \right) \quad {}^\forall v \in W_p^1(D), \tag{2.1}$$

$$\|v\|_{p,D} \leqq C_D \|\nabla v\|_{p,D} \quad {}^\forall v \in \dot{W}_p^1(D). \tag{2.2}$$

(2) Let m be an integer $\geqq 0$. Then, for any $u \in W_p^m(D)$, there exists a $v \in W_p^m(\mathbb{R}^3)$ such that $u = v$ in D and $|v|_{p,m} \leqq C_{p,m,D} \|u\|_{p,m,D}$, where $C_{p,m,D}$ is a constant independent of u and v.

Proof. See [19, II.4] for (1) and [19, II.2] for (2).

In order to state the so-called Bogovskii's lemma, Proposition 2.2 below, we introduce the space $\dot{W}_{p,a}^m(D)$ in the following manner:

$$\dot{W}_{p,a}^m(D) = \{u \in \dot{W}_p^m(D) \mid \int_D u(x)\,dx = 0\}. \tag{2.3}$$

PROPOSITION 2.2. Let $1 < p < \infty$ and let m be an integer $\geqq 0$. Then, there exists a $\mathbb{B} \in \mathcal{L}(\dot{W}^m_{p,a}(D), \dot{\mathbb{W}}^{m+1}_p(D))$ such that $\nabla \cdot \mathbb{B}[f] = f$ in D.

Proof. See Bogovskii [3, 4] (also Giga and Sohr [22, Lemma 2.1] and Iwashita [28, Proposition 2.5], Galdi [19, III.3]).

To use a cut–off technique, we use the following proposition which is easily proved by using Propositions 2.1 and 2.2 (cf. Kobayashi and Shibata [30, Proposition 2.4]).

PROPOSITION 2.3. Let $1 < p < \infty$ and $b > b_0$. Set $G = \Omega$, Ω_{b+1} or $\mathbb{R}^3$. Let m be an integer $\geqq 1$ and let φ be a function of $C^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| \leqq b-1$ and $\varphi(x) = 0$ for $|x| \geqq b$. If $\mathbf{u} \in \mathbb{W}^m_{p,loc}(G)$, $\nabla \cdot \mathbf{u} = 0$ in G and $\mathbf{u} = 0$ on $\partial\Omega$ when $G = \Omega$ or Ω_{b+1}, then $(\nabla\varphi) \cdot \mathbf{u} \in \dot{W}^m_{p,a}(D_b)$. As a result, $\mathbb{B}[(\nabla\varphi)\cdot\mathbf{u}] \in \dot{\mathbb{W}}^{m+1}_p(D_b)$, $\nabla \cdot \mathbb{B}[(\nabla\varphi)\cdot\mathbf{u}] = (\nabla\varphi)\cdot\mathbf{u}$ and $|\mathbb{B}[(\nabla\varphi)\cdot\mathbf{u}]|_{p,m+1} \leqq C_{p,m,\varphi,b}\|\mathbf{u}\|_{p,m,D_b}$.

The following proposition is concerned with the regularity of the projection $\mathbb{P}_G$ for $G = D$ or $G = \Omega$.

PROPOSITION 2.4. Let $1 < p < \infty$ and let m be an integer $\geqq 0$. Set $G = D$ or $G = \Omega$. Then, $\mathbb{P}_G \in \mathcal{L}(\mathbb{W}^m_p(G), \mathbb{W}^m_p(G) \cap \mathbb{J}_p(G))$.

Proof. See Giga and Miyakawa [21] for $G = D$ and Giga and Sohr [22] for $G = \Omega$.

We shall quote a Cattabriga theorem of a unique existence of solutions to the following equation:

$$-\Delta\,\mathbf{u} + \nabla\,\mathfrak{p} = \mathbf{f},\ \ \nabla\cdot\mathbf{u} = f \text{ in } D,\ \ \mathbf{u} = \mathbf{0} \text{ on } \partial D. \tag{2.4}$$

PROPOSITION 2.5. Let $1 < p < \infty$ and let m be an integer $\geqq 0$. Put

$$W^m_{p,a}(D) = \{\, f \in W^m_p(D) \mid \int_D f(x)\,dx = 0\,\}. \tag{2.5}$$

Then, for any $\mathbf{f} \in \mathbb{W}^m_p(D)$ and $f \in W^{m+1}_{p,a}(D)$, there exists a unique $\mathbf{u} \in \mathbb{W}^{m+2}_p(D)$ which together with some $\mathfrak{p} \in W^{m+1}_p(D)$ solves (2.4); $\mathfrak{p}$ is unique up to an additive constant. Moreover, the following estimate is valid:

$$\|\mathbf{u}\|_{p,m+2,D} + \|\nabla\mathfrak{p}\|_{p,m+1,D} \leqq C_{p,m,D}\,\{\|\mathbf{f}\|_{p,m,D} + \|f\|_{p,m+1,D}\}. \tag{2.6}$$

Proof. See Cattabriga [6], Galdi and Simader [20], and Farwig and Sohr [10].

Finally, we shall discuss a unique existence of solutions to the following equation:

$$-\Delta\,\mathbf{u} + (\mathbf{u}_\infty\cdot\nabla)\mathbf{u} + \nabla\,\mathfrak{p} = \mathbf{f},\ \ \nabla\cdot\mathbf{u} = f \text{ in } D,\ \ \mathbf{u} = \mathbf{0} \text{ on } \partial D, \tag{2.7}$$

with a side condition:

$$\int_D \mathfrak{p}(x)\,dx = c. \tag{2.8}$$

PROPOSITION 2.6. Let $1 < p < \infty$ and let m be an integer $\geqq 0$. Set

$$\begin{aligned} \mathcal{W}^m_p(D) &= \mathbb{W}^m_p(D) \times W^{p+1}_{p,a}(D) \times \mathbb{C}, \\ |||(\mathbf{f}, f, c)|||_{p,m,D} &= \|\mathbf{f}\|_{p,m,D} + \|f\|_{p,m+1,D} + |c|. \end{aligned} \tag{2.9}$$

Then, there exist $\mathbb{L}_{\mathbf{u}_\infty,D} \in \mathcal{L}(\mathcal{W}_p^m(D), \mathbb{W}_p^{m+2}(D))$ and $\mathfrak{l}_{\mathbf{u}_\infty,D} \in \mathcal{L}(\mathcal{W}_p^m(D), W_p^{m+1}(D))$ such that $\mathbf{u} = \mathbb{L}_{\mathbf{u}_\infty,D}(\mathbf{f}, f, c)$ and $\mathfrak{p} = \mathfrak{l}_{\mathbf{u}_\infty,D}(\mathbf{f}, f, c)$ solve the problem (2.7) and (2.8) uniquely. Moreover, for any $\sigma > 0$ we have the relation

$$\begin{aligned} \|(\mathbb{L}_{\mathbf{u}_\infty,D} - \kappa\,\mathbb{L}_{\mathbf{u}'_\infty,D})(\mathbf{f}, f, c)\|_{p,m+2,D} + \|(\mathfrak{l}_{\mathbf{u}_\infty,D} - \kappa\,\mathfrak{l}_{\mathbf{u}'_\infty,D})(\mathbf{f}, f, c)\|_{p,m+1,D} \\ \leqq C_{p,m,D,\sigma}(1 - \kappa + \kappa\,|\mathbf{u}_\infty - \mathbf{u}'_\infty|)\,|||(\mathbf{f}, f, c)|||_{p,m,D} \end{aligned} \tag{2.10}$$

provided that $|\mathbf{u}_\infty|$, $|\mathbf{u}'_\infty| \leqq \sigma$ where $\kappa = 0$ and 1.

Proof. First, we consider the solvability of (2.4) with side condition (2.8). Let $\mathbf{u}$ and $\mathfrak{p}$ be solutions to (2.4) and set

$$d = |D|^{-1}\left(c - \int_D \mathfrak{p}(x)\,dx\right) \text{ and } \mathfrak{q}(x) = \mathfrak{p}(x) + d.$$

Then, $\mathbf{u}$ and $\mathfrak{q}$ satisfy (2.8) as well as (2.4). The uniqueness of solutions to the problem (2.4) and (2.8) follows from Proposition 2.5 and (2.1). Therefore, in view of Proposition 2.5 and (2.1) we can define the solution operators $\mathbb{M} \in \mathcal{L}(\mathcal{W}_p^m(D), \mathbb{W}_p^{m+2}(D))$ and $\mathfrak{m} \in \mathcal{L}(\mathcal{W}_p^m(D), W_p^{m+1}(D))$ such that if we set $\mathbf{u} = \mathbb{M}(\mathbf{f}, f, c)$ and $\mathfrak{p} = \mathfrak{m}(\mathbf{f}, f, c)$ then $\mathbf{u}$ and $\mathfrak{p}$ satisfy (2.4) and (2.8).

Now, we apply $\mathbb{M}$ and $\mathfrak{m}$ to (2.7), and then

$$\begin{cases} \left.\begin{aligned} -\Delta\,\mathbb{M}(\mathbf{f}, f, c) + (\mathbf{u}_\infty\cdot\nabla)\mathbb{M}(\mathbf{f}, f, c) + \nabla\,\mathfrak{m}(\mathbf{f}, f, c) \\ = \mathbf{f} + (\mathbf{u}_\infty\cdot\nabla)\,\mathbb{M}(\mathbf{f}, f, c) \\ \nabla\cdot\mathbb{M}(\mathbf{f}, f, c) = f \qquad\qquad \end{aligned}\right\} \text{ in } D, \\ \mathbb{M}(\mathbf{f}, f, c) = \mathbf{0} \text{ on } \partial D, \quad \displaystyle\int_D \mathfrak{m}(\mathbf{f}, f, c)\,dx = c. \end{cases} \tag{2.11}$$

If we define the operator $S_{\mathbf{u}_\infty} \in \mathcal{L}(\mathcal{W}_p^m(D))$ by the relation

$$S_{\mathbf{u}_\infty}(\mathbf{f}, f, c) = ((\mathbf{u}_\infty\cdot\nabla)\,\mathbb{M}(\mathbf{f}, f, c), 0, 0),$$

then $S_{\mathbf{u}_\infty}$ is a compact operator, because $(\mathbf{u}_\infty\cdot\nabla)\,\mathbb{M}(\mathbf{f}, f, c)$ belongs to $\mathbb{W}_p^{m+1}(D)$ which is compactly imbedded into $\mathbb{W}_p^m(D)$. Let us prove that $\mathbb{I}+S_{\mathbf{u}_\infty}$ has a bounded inverse for each $\mathbf{u}_\infty \in \mathbb{R}^3$. In view of Fredholm's alternative theorem, it suffices to show that $\mathbb{I}+S_{\mathbf{u}_\infty}$ is injective. Let us pick up $(\mathbf{f}, f, c) \in \mathcal{W}_p^m(D)$ such that $(\mathbb{I} + S_{\mathbf{u}_\infty})(\mathbf{f}, f, c) = (\mathbf{0}, 0, 0)$, that is, $\mathbf{f} + (\mathbf{u}_\infty\cdot\nabla)\,\mathbb{M}(\mathbf{f}, f, c) = \mathbf{0}$, $f = c = 0$. Set $\mathbf{u} = \mathbb{M}(\mathbf{f}, 0, 0)$ and $\mathfrak{p} = \mathfrak{m}(\mathbf{f}, 0, 0)$, and then by (2.11) $\mathbf{u}$ and $\mathfrak{p}$ satisfy the relations

$$-\Delta\,\mathbf{u} + (\mathbf{u}_\infty\cdot\nabla)\,\mathbf{u} + \nabla\,\mathfrak{p} = \mathbf{0}, \ \ \nabla\cdot\mathbf{u} = 0 \text{ in } D, \ \ \mathbf{u} = \mathbf{0} \text{ on } \partial D, \quad \int_D \mathfrak{p}(x)\,dx = 0. \tag{2.12}$$

In view of Proposition 2.5 by the boot-strap argument we see that $\mathbf{u}$ and $\mathfrak{p}$ are sufficiently smooth, and then the multiplication of the first equation in (2.12) by $\mathbf{u}$ and the integration by parts imply that $\|\nabla\mathbf{u}\|_{2,D}^2 = 0$, and hence $\mathbf{u} = \mathbf{0}$, because of the Dirichlet condition and (2.2). Using the equation again, we see that $\nabla\mathfrak{p} = \mathbf{0}$ which together with $\int_D \mathfrak{p}(x)\,dx = 0$ and (2.1) implies that $\mathfrak{p} = 0$. Thus, for each $\mathbf{u}_\infty \in \mathbb{R}^3$, $\mathbb{I} + S_{\mathbf{u}_\infty}$ has its inverse

$(\mathbb{I}+S_{\mathbf{u}_\infty})^{-1} \in \mathcal{L}(\mathcal{W}_p^m(D))$. Since $(S_{\mathbf{u}_\infty} - S_{\mathbf{u}'_\infty})(\mathbf{f}, f, c) = ((\mathbf{u}_\infty - \mathbf{u}'_\infty)\cdot\nabla\,\mathbb{M}(\mathbf{f}, f, c), 0, 0)$, we have

$$(\mathbb{I}+S_{\mathbf{u}'_\infty})^{-1} = \sum_{j=0}^{\infty} \left[(\mathbb{I}+S_{\mathbf{u}_\infty})^{-1}(S_{\mathbf{u}_\infty} - S_{\mathbf{u}'_\infty})\right]^j \,(\mathbb{I}+S_{\mathbf{u}_\infty})^{-1}$$

provided that

$$\|(\mathbb{I}+S_{\mathbf{u}_\infty})^{-1}\|_{\mathcal{L}(\mathcal{W}_p^m(D))}\,\|\mathbb{M}\|_{\mathcal{L}(\mathcal{W}_p^m(D),\mathbb{W}_p^{m+2}(D))}\,|\mathbf{u}_\infty - \mathbf{u}'_\infty| \leqq \frac{1}{2},$$

which implies that $(\mathbb{I}+S_{\mathbf{u}_\infty})^{-1}$ is continuous with respect to $\mathbf{u}_\infty \in \mathbb{R}^3$. Then it follows easily that for any compact set $K \subset \mathbb{R}^3$ there exists a constant $C_K > 0$ such that

$$\|(\mathbb{I}+S_{\mathbf{u}_\infty})^{-1}\|_{\mathcal{L}(\mathcal{W}_p^m(D))} \leqq C_K, \quad {}^\forall\,\mathbf{u}_\infty \in K.$$

If we set $\mathbb{L}_{\mathbf{u}_\infty,D} = \mathbb{M}\,(\mathbb{I}+S_{\mathbf{u}_\infty})^{-1}$ and $\mathfrak{l}_{\mathbf{u}_\infty,D} = \mathfrak{m}(\mathbb{I}+S_{\mathbf{u}_\infty})^{-1}$, then we see easily that $\mathbb{L}_{\mathbf{u}_\infty,D}$ and $\mathfrak{l}_{\mathbf{u}_\infty,D}$ satisfy the required property, except for (2.10) with $\kappa = 1$. But, since

$$\begin{aligned}(\mathbb{L}_{\mathbf{u}_\infty,D} - \mathbb{L}_{\mathbf{u}'_\infty,D},\, \mathfrak{l}_{\mathbf{u}_\infty,D} - \mathfrak{l}_{\mathbf{u}'_\infty,D})(\mathbf{f}, f, c)\\ = (\mathbb{L}_{\mathbf{u}_\infty,D},\, \mathfrak{l}_{\mathbf{u}_\infty,D})((\mathbf{u}_\infty - \mathbf{u}'_\infty)\cdot\nabla\,\mathbb{L}_{\mathbf{u}'_\infty,D}(\mathbf{f}, f, c), 0, 0),\end{aligned}$$

the estimate (2.10) with $\kappa = 1$ also follows immediately from (2.10) with $\kappa = 0$. This completes the proof of the proposition.

3. L_p solutions of the Oseen equation. In this section, we shall discuss L_p solutions of the following equation:

$$-\Delta\,\mathbf{u} + (\mathbf{u}_\infty\cdot\nabla)\,\mathbf{u} + \nabla\,\mathfrak{p} = \mathbf{f},\ \ \nabla\cdot\mathbf{u} = 0 \ \text{ in } \Omega,\ \ \mathbf{u} = \mathbf{g} \ \text{ on } \partial\Omega. \tag{3.1}$$

The goal of this section is to prove the following theorem.

THEOREM 3.1. Let $3 < p < \infty$ and let K be any compact set in $\mathbb{R}^3$. If $\mathbf{f} \in \mathbb{L}_p(\Omega)\cap\mathbb{L}_1(\Omega)$ and $\mathbf{g} \in \mathbb{W}^2_{p,d}(\partial\Omega)$, then the problem (3.1) admits unique solutions $\mathbf{u} \in \mathbb{W}_p^2(\Omega)$ and $\mathfrak{p} \in W_p^1(\Omega)$ satisfying the estimate

$$\|\mathbf{u}\|_{p,2} + \|\mathfrak{p}\|_{p,1} \leqq C_{p,K}\{\,\|\mathbf{f}\|_p + \|\mathbf{f}\|_1 + \|\mathbf{g}\|_{p,2}\,\}, \tag{3.2}$$

for any $\mathbf{u}_\infty \in K$ with some constant $C_{p,K}$ independent of $\mathbf{u}_\infty$, $\mathbf{f}$ and $\mathbf{g}$.

3.1. Basic property of the Oseen fundamental solutions. In this paragraph, we shall discuss the basic property of the fundamental solutions $\chi_{jk}(\mathbf{u}_\infty)(x)$ and $\pi_j(x)$, j, $k = 1, 2, 3$, of the Oseen equation:

$$-\Delta\,\mathbf{w} + (\mathbf{u}_\infty\cdot\nabla)\,\mathbf{w} + \nabla\,\mathfrak{p} = \mathbf{g},\ \ \nabla\cdot\mathbf{w} = 0 \ \text{ in } \mathbb{R}^3. \tag{3.3}$$

Put

$$\begin{aligned}\chi_{jk}(\mathbf{u}_\infty) &= \mathcal{F}^{-1}\left[p_{jk,\mathbf{u}_\infty}(\xi)\right], \quad p_{jk,\mathbf{u}_\infty}(\xi) = \frac{\delta_{jk} - \xi_j\,\xi_k\,|\xi|^{-2}}{|\xi|^2 + i\,\mathbf{u}_\infty\cdot\xi},\\ \pi_j &= \mathcal{F}^{-1}\left[\frac{\xi_j}{i\,|\xi|^2}\right],\end{aligned} \tag{3.4}$$

where $i = \sqrt{-1}$, $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and δ_{jk} is the Kronecker's delta symbol, that is, $\delta_{jj} = 1$ and $\delta_{jk} = 0$ for $j \neq k$. The following formula is well known (cf. Oseen [38], Galdi [19, IV.2 and VII. 3], Kobayashi and Shibata [30]):

$$\begin{cases} \chi_{jk}(\mathbf{u}_\infty)(x) = (\delta_{jk}\,\Delta - \partial_j\partial_k)\,\Xi\,(\sigma)(x), \\ \Xi\,(\sigma)(x) = \dfrac{1}{8\pi\,\sigma}\displaystyle\int_0^{\sigma s_{\mathbf{u}_\infty}(x)} \frac{1-e^{-\alpha}}{\alpha}\,d\alpha, \quad \sigma = |\mathbf{u}_\infty|/2 \neq 0, \end{cases} \tag{3.5}$$

$$\chi_{jk}(\mathbf{0})(x) = \frac{1}{8\pi\,|x|}\left(\delta_{jk} + \frac{x_j\,x_k}{|x|^2}\right), \tag{3.6}$$

$$\pi_j(x) = \frac{x_j}{4\pi\,|x|^3}, \tag{3.7}$$

where $s_{\mathbf{u}_\infty}(x)$ is the same as in Theorem 1.1.

LEMMA 3.2. Assume that $\mathbf{u}_\infty \neq \mathbf{0}$ and let $\chi_{jk}(\mathbf{u}_\infty)$, $s_{\mathbf{u}_\infty}$ and σ be the same as in (3.5). Then, for any $\delta\colon 0 \leqq \delta \leqq 1$ there exists $C_\delta > 0$ independent of $\mathbf{u}_\infty$ such that

$$\begin{aligned} |\chi_{jk}(\mathbf{u}_\infty)(x)| &\leqq \frac{C_\delta}{(\sigma\, s_{\mathbf{u}_\infty}(x))^\delta\,|x|}, \\ |\nabla\,\chi_{jk}(\mathbf{u}_\infty)(x)| &\leqq \frac{C_\delta}{(\sigma\, s_{\mathbf{u}_\infty}(x))^\delta\, s_{\mathbf{u}_\infty}(x)^{1/2}\,|x|^{3/2}}, \\ |\nabla\,\chi_{jk}(\mathbf{u}_\infty)(x)| &\leqq \frac{C_\delta}{(\sigma\, s_{\mathbf{u}_\infty}(x))^\delta}\left[\frac{\sigma^{1/2}}{|x|^{3/2}} + \frac{1}{|x|^2}\right]. \end{aligned} \tag{3.8}$$

Proof. See Oseen [38], Galdi [19, VII.3], and also Kobayashi and Shibata [30].

LEMMA 3.3. Let $3 < p < \infty$ and $\sigma_0 > 0$. Assume that $|\mathbf{u}_\infty| \leqq \sigma_0$. Put

$$\chi(\mathbf{u}_\infty) * \mathbf{f} = {}^{\mathrm{T}}\!\left(\sum_{j=1}^3 \chi_{1j} * f_j, \sum_{j=1}^3 \chi_{2j} * f_j, \sum_{j=1}^3 \chi_{3j} * f_j\right), \quad \pi * \mathbf{f} = \sum_{j=1}^3 \pi_j * f_j$$

for $\mathbf{f} = {}^{\mathrm{T}}(f_1, f_2, f_3)$ where the asterisk $*$ stands for the convolution. If $\mathbf{f} \in \mathbb{L}_p(\mathbb{R}^3) \cap \mathbb{L}_1(\mathbb{R}^3)$, then $\chi(\mathbf{u}_\infty) * \mathbf{f} \in \mathbb{W}_p^2(\mathbb{R}^3)$ and $\pi * \mathbf{f} \in W_p^1(\mathbb{R}^3)$; moreover,

$$|\chi(\mathbf{u}_\infty) * \mathbf{f}|_{p,2} + |\pi * \mathbf{f}|_{p,1} \leqq C_{p,\sigma_0}(|\mathbf{f}|_p + |\mathbf{f}|_1), \tag{3.9}$$

$$\|\chi(\mathbf{u}_\infty) * \mathbf{f} - \chi(\mathbf{u}'_\infty) * \mathbf{f}\|_{p,2,B_b} \leqq C_{p,b}|\mathbf{u}_\infty - \mathbf{u}'_\infty|^{1/2}\,(|\mathbf{f}|_p + |\mathbf{f}|_1). \tag{3.10}$$

Proof. Let $\varphi^0(\xi)$ be a function of $C^\infty(\mathbb{R}^3)$ such that $0 \leqq \varphi^0 \leqq 1$, $\varphi^0(\xi) = 1$ for $|\xi| \leqq 1$ and $\varphi^0(\xi) = 0$ for $|\xi| \geqq 2$ and put $\varphi^\infty(\xi) = 1 - \varphi^0(\xi)$. Set

$$\chi_{jk}^N(\mathbf{u}_\infty) = \mathcal{F}^{-1}\left[\varphi^N(\xi)\,p_{jk,\mathbf{u}_\infty}(\xi)\right], \quad \pi_j^N = \mathcal{F}^{-1}\left[\varphi^N(\xi)\,\xi_j(i\,|\xi|^2)^{-1}\right] \tag{3.11}$$

for $N = 0$ and ∞. To handle with $\chi_{jk}^\infty(\mathbf{u}_\infty)$ and π_j^∞, we use the following theorem concerning the L_p boundedness of the Fourier multiplier.

PROPOSITION 3.4. (cf. Hörmander [27, Theorem 7.9.5]) Let $1 < p < \infty$ and let $k(\xi) \in C^\infty(\mathbb{R}^3 - \{\mathbf{0}\})$ satisfy the condition $|\partial_\xi^\alpha k(\xi)| \leqq M\,|\xi|^{-|\alpha|}$ for $|\xi| \leqq 2$ and $\xi \in \mathbb{R}^3 - \{\mathbf{0}\}$ with some constant $M > 0$. Then,

$$|\mathcal{F}^{-1}[k\,\hat{u}]|_p \leqq C_p\,M\,|u|_p, \quad {}^\forall u \in L_p(\mathbb{R}^3)$$

where C_p is a constant independent of M, and u and $\hat{u}$ denote the Fourier transforms of u.

Since $\varphi^\infty(\xi) = 0$ for $|\xi| \leqq 1$, by Proposition 3.4 we see easily that

$$\begin{aligned} &|\partial_x^2\,\chi_{jk}(\mathbf{u}_\infty) * f|_p + |\chi_{jk}^\infty(\mathbf{u}_\infty) * f|_{p,2} + |\pi_j * f|_{p,1} \leqq C\,M\,|f|_p, \\ &|\chi_{jk}^\infty(\mathbf{u}_\infty) * f - \chi_{jk}^\infty(\mathbf{u}'_\infty) * f|_{p,2} \leqq C\,M\,|\mathbf{u}_\infty - \mathbf{u}'_\infty|\,|f|_p. \end{aligned} \tag{3.12}$$

In order to handle with χ_{jk}^∞ and π_j^∞, we need the following lemma.

LEMMA 3.5. Let $\chi_{jk}^\infty(\mathbf{u}_\infty)$ and $s_{\mathbf{u}_\infty}$ be the same as in (3.11) and (1.4), respectively. Then, we have the following relations:

$$|\chi_{jk}^0(\mathbf{u}_\infty)(x)| \leqq C(1 + |x|)^{-1}, \tag{3.13}$$

$$|\nabla\chi_{jk}^0(\mathbf{u}_\infty)(x)| \leqq C(1 + |\mathbf{u}_\infty|^{1/2})(1 + s_{\mathbf{u}_\infty}(x))^{-1/2}(1 + |x|)^{-3/2}, \tag{3.14}$$

$$|\partial_x^\alpha \pi_j^0(x)| \leqq C(1 + |x|)^{-(2+|\alpha|)} \quad {}^\forall\alpha, \tag{3.15}$$

where we have put $s_{\mathbf{0}}(x) = |x|$.

Postponing the proof of Lemma 3.5, we continue the proof of Lemma 3.3. When $3 < p < \infty$, by (3.13) to (3.15) we see easily that

$$|\chi_{jk}^0(\mathbf{u}_\infty) * f|_{p,1} + |\pi_j * f|_{p,1} \leqq (|\chi_{jk}^0(\mathbf{u}_\infty)|_{p,1} + |\pi_j|_{p,1})|f|_1 \leqq C\,|f|_1. \tag{3.16}$$

Since

$$\begin{aligned} &|\varphi^0(\xi)\left(p_{jk,\mathbf{u}_\infty}(\xi) - p_{jk,\mathbf{u}'_\infty}(\xi)\right)| \\ &\leqq C\,\varphi^0(\xi)\left(\frac{|(\mathbf{u}_\infty - \mathbf{u}'_\infty)\cdot\xi|}{||\xi|^2 + i\,\mathbf{u}_\infty\cdot\xi|\,||\xi|^2 + i\,\mathbf{u}'_\infty\cdot\xi|}\right)^{1/2}\left(\frac{1}{|\xi|^2}\right)^{1/2} \leqq C\,\varphi^0(\xi)\frac{|\mathbf{u}_\infty - \mathbf{u}'_\infty|^{1/2}}{|\xi|^{5/2}}, \end{aligned}$$

we have

$$|\chi_{jk}^0(\mathbf{u}_\infty) * f - \chi_{jk}^0(\mathbf{u}'_\infty) * f|_{\infty,1} \leqq C|\mathbf{u}_\infty - \mathbf{u}'_\infty|^{1/2}\int_{|\xi|\leqq 2}|\xi|^{-5/2}\,d\xi\,|f|_1. \tag{3.17}$$

Combining (3.12), (3.16) and (3.17), we have Lemma 3.3.

A proof of Lemma 3.5. We shall prove only (3.14) in the case that $\mathbf{u}_\infty \neq 0$, because other assertions will also be proved in a similar manner. Since $\chi_{jk}^0(\mathbf{u}_\infty) = \chi_{jk}(\mathbf{u}_\infty) * \widehat{\varphi^0}$ and since $1 + s_{\mathbf{u}_\infty}(x) \leqq 1 + s_{\mathbf{u}_\infty}(x - y) + s_{\mathbf{u}_\infty}(y)$, by Lemma 3.2

$$\begin{aligned} (1 + s_{\mathbf{u}_\infty}(x))^{1/2}|\nabla\chi_{jk}^0(\mathbf{u}_\infty)(x)| &\leqq C\left\{\int_{\mathbb{R}^3}\frac{|\widehat{\varphi^0}(x-y)|}{|y|^{3/2}}\,dy\right. \\ &\quad\left. + C\int_{\mathbb{R}^3}(1 + s_{\mathbf{u}_\infty}(x-y))^{1/2}|\widehat{\varphi^0}(x-y)|\left[\frac{|\mathbf{u}_\infty|^{1/2}}{|y|^{3/2}} + \frac{1}{|y|^2}\right]dy\right\} \\ &\leqq C\int_{\mathbb{R}^3}\frac{1 + |\mathbf{u}_\infty|^{1/2}}{(1 + |x-y|)^4}\left[\frac{1}{|y|^{3/2}} + \frac{1}{|y|^2}\right]dy \end{aligned}$$

where we have used the facts that $s_{\mathbf{u}_\infty}(x-y) \leqq 2|x-y|$ and that $\widehat{\varphi^0}$ is rapidly decreasing. Observing that

$$\int_{|y|\leqq(1+|x|)/2} \frac{dy}{(1+|x-y|)^4\,|y|^q} \leqq C\left(\frac{2}{1+|x|}\right)^4\left(\frac{1+|x|}{2}\right)^{3-q},$$
$$\int_{|y|\geqq(1+|x|)/2} \frac{dy}{(1+|x-y|)^4\,|y|^q} \leqq \left(\frac{2}{1+|x|}\right)^q \int_{\mathbb{R}^3} \frac{dy}{(1+|y|)^4}$$

for $0 < q < 3$, we have (3.14).

3.2. A construction of a parametrix. In this paragraph, we shall construct a parametrix of the problem

$$-\Delta\,\mathbf{u} + (\mathbf{u}_\infty\cdot\nabla)\,\mathbf{u} + \nabla\,\mathfrak{p} = \mathbf{f},\ \ \nabla\cdot\mathbf{u} = f \text{ in } \Omega,\ \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega. \tag{3.18}$$

For notational simplicity, we set

$$\mathcal{K}_p(\Omega) = \mathbb{L}_{p,b_0+4}(\Omega)\times\{f\in W^1_p(\Omega)\mid \int_\Omega f(x)\,dx = 0 \text{ and } f(x) = 0 \text{ for } |x|\geqq b_0+1\},$$

where b_0 is the same number as in paragraph 1.2. Moreover, put $b = b_0+4$ and let φ be a function of $C^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x|\leqq b-2$ and $\varphi(x) = 0$ for $|x|\geqq b-1$. Let $3<p<\infty$, $\Pi_b\,\mathbf{f}$ denote the restriction of $\mathbf{f}$ to Ω_b and set $\mathbf{f}_0(x) = \mathbf{f}(x)$ for $x\in\Omega$ and $\mathbf{f}_0(x) = \mathbf{0}$ for $x\notin\Omega$. Assume that $(\mathbf{f}, f)\in\mathcal{K}_p(\Omega)$. A parametrix will be constructed by a compact perturbation of the operators $R_0(\mathbf{u}_\infty)$ and $\mathfrak{p}(\mathbf{u}_\infty)$ defined as follows:

$$\begin{aligned} R_0(\mathbf{u}_\infty)(\mathbf{f}, f) &= (1-\varphi)\,(\chi(\mathbf{u}_\infty)*\mathbf{f}_0) + \varphi\,\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f}, f) + R_1(\mathbf{u}_\infty)(\mathbf{f}, f),\\ \mathfrak{p}(\mathbf{u}_\infty)(\mathbf{f}, f) &= (1-\varphi)\,(\pi*\mathbf{f}_0) + \varphi\,\mathfrak{l}_{\mathbf{u}_\infty}(\mathbf{f}, f), \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} R_1(\mathbf{u}_\infty) &= \mathbb{B}[(\nabla\varphi)\cdot(\chi(\mathbf{u}_\infty)*\mathbf{f})] - \mathbb{B}[(\nabla\varphi)\cdot(\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f}, f))],\\ \mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f}, f) &= \mathbb{L}_{\mathbf{u}_\infty,\Omega_b}(\Pi_b\,\mathbf{f}, \Pi_b\,f, \int_{B_b}\pi*\mathbf{f}_0\,dx),\\ \mathfrak{l}_{\mathbf{u}_\infty}(\mathbf{f}, f) &= \mathfrak{l}_{\mathbf{u}_\infty,\Omega_b}(\Pi_b\,\mathbf{f}, \Pi_b\,f, \int_{B_b}\pi*\mathbf{f}_0\,dx), \end{aligned}$$

and $\mathbb{L}_{\mathbf{u}_\infty,\Omega_b}$ and $\mathfrak{l}_{\mathbf{u}_\infty,\Omega_b}$ are the same as in Proposition 2.6 with $D = \Omega_b$. Since $\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f}, f) = \mathbf{0}$ on $\partial\Omega$, $\nabla\cdot\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f}, f) = \Pi_b\,f = f$ in Ω_b and $\varphi = 1$ on $\operatorname{supp} f\subset B_{b_0+1}$, we have

$$\begin{aligned} \int_{D_{b-1}} (\nabla\varphi)\cdot\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f}, f)\,dx &= \int_{\Omega_b}\nabla\cdot[\varphi\,\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f}, f)]\,dx - \int_{\Omega_b}\varphi\,(\nabla\cdot\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f}, f))\,dx\\ &= \int_{\partial\Omega}\nu\cdot\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f}, f)\,d\Gamma - \int_\Omega f\,dx = 0, \end{aligned}$$

and hence $(\nabla\varphi)\cdot\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f},f)\in W^2_{p,a}(D_{b-1})$ (cf. (2.5)). By Propositions 2.2 and 2.3 we see that $R_1(\mathbf{u}_\infty)$ is well defined and that $R_1(\mathbf{u}_\infty)\in\mathcal{L}(\mathcal{K}_p(\Omega),\dot{\mathbb{W}}^3_p(D_{b-1}))$. Let K be any compact set in $\mathbb{R}^3$ and $\mathbf{u}_\infty\in K$. By Lemma 3.3 and Proposition 2.6 we have

$$\|R_0(\mathbf{u}_\infty)(\mathbf{f},f)\|_{p,2}+\|\mathfrak{p}(\mathbf{u}_\infty)(\mathbf{f},f)\|_{p,1}\leqq C_{K,b}\left(\|\mathbf{f}\|_p+\|f\|_{p,1}\right), \tag{3.20}$$

$$(-\Delta+(\mathbf{u}_\infty\cdot\nabla))\,R_0(\mathbf{u}_\infty)(\mathbf{f},f)+\nabla\,\mathfrak{p}(\mathbf{u}_\infty)(\mathbf{f},f)=\mathbf{f}+S_{\mathbf{u}_\infty}(\mathbf{f},f)\text{ in }\Omega, \tag{3.21}$$

$$\nabla\cdot R_0(\mathbf{u}_\infty)(\mathbf{f},f)=f\text{ in }\Omega,\qquad R_0(\mathbf{u}_\infty)(\mathbf{f},f)=\mathbf{0}\text{ on }\partial\Omega \tag{3.22}$$

where

$$\begin{aligned}S_{\mathbf{u}_\infty}(\mathbf{f},f)=&-2(\nabla\varphi):(\chi(\mathbf{u}_\infty)*\mathbf{f}_0)-(\Delta\varphi)(\chi(\mathbf{u}_\infty)*\mathbf{f}_0)\\&+2(\nabla\varphi):(\nabla\,\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f},f))+(\Delta\varphi)\,\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f},f)\\&-((\mathbf{u}_\infty\cdot\nabla)\varphi)(\chi(\mathbf{u}_\infty)*\mathbf{f}_0)+((\mathbf{u}_\infty\cdot\nabla)\varphi)(\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f},f))\\&+(-\Delta+(\mathbf{u}_\infty\cdot\nabla))\,R_1(\mathbf{u}_\infty)(\mathbf{f},f)-(\nabla\varphi)(\pi*\mathbf{f}_0)+(\nabla\varphi)(\mathfrak{l}_{\mathbf{u}_\infty}(\mathbf{f},f)).\end{aligned}$$

Note that $S_{\mathbf{u}_\infty}(\mathbf{f},f)\in\mathbb{W}^1_p(\Omega)$ and that supp $S_{\mathbf{u}_\infty}(\mathbf{f},f)\subset D_{b-1}$, and hence if we put $\mathcal{J}_{\mathbf{u}_\infty}(\mathbf{f},f)=(S_{\mathbf{u}_\infty}(\mathbf{f},f),0)$, then $\mathcal{J}_{\mathbf{u}_\infty}$ is a compact operator from $\mathcal{K}_p(\Omega)$ into itself. Our task is to show the existence of the inverse operator $(\mathbb{I}+\mathcal{J}_{\mathbf{u}_\infty})^{-1}$ of $\mathbb{I}+\mathcal{J}_{\mathbf{u}_\infty}$. In order to do this, the following lemma is a key.

LEMMA 3.6. Let $1<p<\infty$. If $\mathbf{u}\in\hat{\mathbb{W}}^2_p(\Omega)$ and $\mathfrak{p}\in\hat{W}^1_p(\Omega)$ satisfy the homogeneous equation

$$-\Delta\,\mathbf{u}+(\mathbf{u}_\infty\cdot\nabla)\,\mathbf{u}+\nabla\mathfrak{p}=\mathbf{0},\;\;\nabla\cdot\mathbf{u}=0\text{ in }\Omega,\;\mathbf{u}=\mathbf{0}\text{ on }\partial\Omega, \tag{3.23}$$

and the growth order condition

$$\lim_{R\to\infty}R^{-3}\int_{R\leqq|x|\leqq 2R}(|\mathbf{u}(x)|^p+|\mathfrak{p}(x)|^p)\,dx=0, \tag{3.24}$$

then $\mathbf{u}(x)=\mathbf{0}$ and $\mathfrak{p}(x)=0$.

Proof. See Iwashita [28] and Kobayashi and Shibata [30].

LEMMA 3.7. Let $1<p<\infty$. Then, for each $\mathbf{u}_\infty\in\mathbb{R}^3$, $\mathbb{I}+\mathcal{J}_{\mathbf{u}_\infty}$ has its inverse $(\mathbb{I}+\mathcal{J}_{\mathbf{u}_\infty})^{-1}\in\mathcal{L}(\mathcal{K}_p(\Omega))$.

Proof. Since $\mathcal{J}_{\mathbf{u}_\infty}$ is compact, in view of Fredholm's alternative theorem it suffices to show that $\mathbb{I}+\mathcal{J}_{\mathbf{u}_\infty}$ is injective, and hence let us pick up $(\mathbf{f},f)\in\mathcal{K}_p(\Omega)$ such that $(\mathbb{I}+\mathcal{J}_{\mathbf{u}_\infty})(\mathbf{f},f)=(\mathbf{0},0)$, that is, $f=0$ and $\mathbf{f}+S_{\mathbf{u}_\infty}(\mathbf{f},0)=\mathbf{0}$. Put $\mathbf{u}=R_0(\mathbf{u}_\infty)(\mathbf{f},0)$ and $\mathfrak{p}=\mathfrak{p}(\mathbf{u}_\infty)(\mathbf{f},0)$. By (3.20) to (3.22), we see that $\mathbf{u}$ and $\mathfrak{p}$ satisfy the condition in Lemma 3.6, and hence $\mathbf{u}=\mathbf{0}$ and $\mathfrak{p}=0$. That is,

$$R_0(\mathbf{u}_\infty)(\mathbf{f},0)=\mathbf{0}\text{ and }\mathfrak{p}(\mathbf{u}_\infty)(\mathbf{f},0)=0\text{ in }\Omega. \tag{3.25}$$

Since $\varphi(x)=1$ for $|x|\leqq b-2$ and $1-\varphi(x)=1$ for $|x|\geqq b-1$ and since supp $R_1(\mathbf{u}_\infty)(\mathbf{f},0)\subset D_{b-1}$, by (3.25) we see that

$$\begin{aligned}&\chi(\mathbf{u}_\infty)*\mathbf{f}_0=\mathbf{0}\text{ and }\quad\pi*\mathbf{f}_0=0\quad\text{for }|x|\geqq b-1,\\&\mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f},0)=\mathbf{0}\text{ and }\mathfrak{p}_{\mathbf{u}_\infty}(\mathbf{f},0)=0\text{ for }|x|\leqq b-2.\end{aligned} \tag{3.26}$$

Put $\mathbf{z} = \mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f}, 0)$ for $x \in \Omega_b$ and $\mathbf{z} = \mathbf{0}$ for $x \in \mathcal{O}$ and $\mathfrak{q} = \mathfrak{l}_{\mathbf{u}_\infty}(\mathbf{f}, 0)$ for $x \in \Omega_b$ and $\mathfrak{q} = 0$ for $x \in \mathcal{O}$. By (2.7) and (2.8) we have

$$-\Delta\,\mathbf{z} + (\mathbf{u}_\infty \cdot \nabla)\,\mathbf{z} + \nabla\mathfrak{q} = \mathbf{f}_0, \;\; \nabla \cdot \mathbf{z} = \mathbf{0} \text{ in } B_b,\, \mathbf{z} = \mathbf{0} \text{ on } S_b,$$
$$\int_{B_b} \mathfrak{q}\,dx = \int_{\Omega_b} \mathfrak{l}_{\mathbf{u}_\infty}(\mathbf{f}, 0)\,dx = \int_{B_b} \pi * \mathbf{f}\,dx.$$

In view of (3.26), $\chi(\mathbf{u}_\infty) * \mathbf{f}_0 - \mathbf{z}$ and $\pi * \mathbf{f}_0 - \mathfrak{q}$ satisfy (2.7) and (2.8) with $\mathbf{f} = \mathbf{0}$, $f = 0$, $c = 0$ and $D = B_b$, and hence by Proposition 2.6, we have

$$\chi(\mathbf{u}_\infty) * \mathbf{f}_0 = \mathbb{L}_{\mathbf{u}_\infty}(\mathbf{f}, 0) \text{ and } \pi * \mathbf{f}_0 = \mathfrak{l}_{\mathbf{u}_\infty}(\mathbf{f}, 0) \text{ in } \Omega_b. \tag{3.27}$$

In particular, $R_1(\mathbf{u}_\infty)(\mathbf{f}, 0) = \mathbf{0}$ in Ω, because supp $\nabla\varphi \subset D_{b-1} \subset \Omega_b$. Then, combining (3.25) to (3.27), we see easily that $\chi(\mathbf{u}_\infty) * \mathbf{f}_0 = \mathbf{0}$ and $\pi * \mathbf{f}_0 = 0$ in Ω, and hence $\mathbf{f} = \mathbf{0}$, which completes the proof of the lemma.

LEMMA 3.8. Let $3 < p < \infty$. Then, for any compact set $K \subset \mathbb{R}^3$, there exists a constant $M_{K,p} > 0$ such that $\|(\mathbb{I} + \mathcal{J}_{\mathbf{u}_\infty})^{-1}\|_{\mathcal{L}(\mathcal{K}_p(\Omega))} \leqq M_{K,p}$ provided that $\mathbf{u}_\infty \in K$.

Proof. By (2.10) with $\kappa = 1$ and (3.10), $\|\mathcal{J}_{\mathbf{u}_\infty} - \mathcal{J}_{\mathbf{u}'_\infty}\|_{\mathcal{L}(\mathcal{K}_p(\Omega))} \leqq C_{K,p}|\mathbf{u}_\infty - \mathbf{u}'_\infty|^{1/2}$. Since

$$(\mathbb{I} + \mathcal{J}_{\mathbf{u}'_\infty})^{-1} = \left\{\sum_{j=0}^{\infty} \left[(\mathbb{I} + \mathcal{J}_{\mathbf{u}_\infty})^{-1}(\mathcal{J}_{\mathbf{u}_\infty} - \mathcal{J}_{\mathbf{u}'_\infty})\right]^j\right\}(\mathbb{I} + \mathcal{J}_{\mathbf{u}_\infty})^{-1}$$

provided that

$$C_{K,p}\|(\mathbb{I} + \mathcal{J}_{\mathbf{u}_\infty})^{-1}\|_{\mathcal{L}(\mathcal{K}_p(\Omega))}|\mathbf{u}_\infty - \mathbf{u}'_\infty|^{1/2} \leqq 1/2,$$

by Lemma 3.7 and the compactness of K we have the lemma immediately, so that the proof is completed.

By (3.19), (3.21) and Lemmas 3.6 and 3.8, we see that when $\mathbf{f} \in \mathbb{L}_{p,b}(\Omega)$ and $\mathbf{g} \in \mathbb{W}^2_{p,d}(\partial\Omega)$, the problem (3.1) admits a unique solution $\mathbf{u}$ and $\mathfrak{p}$ of the form

$$\begin{aligned} \mathbf{u} &= \mathbf{g} + R_0(\mathbf{u}_\infty)(\mathbb{I} + \mathcal{J}_{\mathbf{u}_\infty})^{-1}(\mathbf{f} - (-\Delta + (\mathbf{u}_\infty \cdot \nabla))\,\mathbf{g}, -\nabla \cdot \mathbf{g}), \\ \mathfrak{p} &= \mathfrak{p}(\mathbf{u}_\infty)(\mathbb{I} + \mathcal{J}_{\mathbf{u}_\infty})^{-1}(\mathbf{f} - (-\Delta + (\mathbf{u}_\infty \cdot \nabla))\,\mathbf{g}, -\nabla \cdot \mathbf{g}), \end{aligned} \tag{3.28}$$

which satisfy the estimate

$$\|\mathbf{u}\|_{p,2} + \|\mathfrak{p}\|_{p,1} \leqq C_p(\|\mathbf{f}\|_p + \|\mathbf{g}\|_{p,2}). \tag{3.29}$$

In (3.29), the constant C_p depends on K but is independent of $\mathbf{u}_\infty \in K$ whenever K is any compact set in $\mathbb{R}^3$.

3.3. A proof of Theorem 3.1. In the course of the proof, let K be any compact set in $\mathbb{R}^3$ and $\mathbf{u}_\infty \in K$. Set $\mathbf{f}_0(x) = \mathbf{f}(x)$ for $x \in \Omega$ and $\mathbf{f}_0(x) = \mathbf{0}$ for $x \in \Omega$, and let $\psi(x) \in C^\infty(\mathbb{R}^3)$ such that $\psi(x) = 1$ for $|x| \leqq b_0 + 2$ and $\psi(x) = 0$ for $|x| \geqq b_0 + 3$. Put

$$\begin{aligned} \mathbf{v} &= (1 - \psi)\,\chi(\mathbf{u}_\infty) * \mathbf{f}_0 + \mathbb{B}[(\nabla\psi) \cdot (\chi(\mathbf{u}_\infty) * \mathbf{f}_0)], \\ \mathfrak{q} &= (1 - \psi)\,\pi * \mathbf{f}_0. \end{aligned} \tag{3.30}$$

By Proposition 2.3 and (3.9), we have

$$\|\mathbf{v}\|_{p,2} + \|\mathfrak{q}\|_{p,1} \leqq C_{K,p}\{\|\mathbf{f}\|_p + \|\mathbf{f}\|_1\}, \tag{3.31}$$
$$\nabla \cdot \mathbf{v} = \mathbf{0} \text{ in } \Omega,\ \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega, \tag{3.32}$$
$$(-\Delta + (\mathbf{u}_\infty \cdot \nabla))\mathbf{v} + \nabla \mathfrak{q} = (1-\psi)\mathbf{f} + \mathbf{h} \text{ in } \Omega, \tag{3.33}$$

where

$$\begin{aligned}\mathbf{h} = {} & 2(\nabla\psi)\colon \nabla\chi(\mathbf{u}_\infty) * \mathbf{f}_0 + (\Delta\psi)\chi(\mathbf{u}_\infty) * \mathbf{f}_0 - ((\mathbf{u}_\infty \cdot \nabla)\psi)\chi(\mathbf{u}_\infty) * \mathbf{f}_0 \\ & + (-\Delta + (\mathbf{u}_\infty \cdot \nabla))\,\mathbb{B}[(\nabla\psi)\cdot(\chi(\mathbf{u}_\infty) * \mathbf{f}_0)] - (\nabla\psi)\pi * \mathbf{f}_0.\end{aligned} \tag{3.34}$$

By Proposition 2.3 and (3.9) we have also

$$\operatorname{supp}\mathbf{h} \subset D_{b_0+3}, \quad \|\mathbf{h}\|_p \leqq C_{K,p}(\|\mathbf{f}\|_p + \|\mathbf{f}\|_1). \tag{3.35}$$

Now, we put

$$\begin{aligned}\mathbf{w} &= \mathbf{g} + R_0(\mathbf{u}_\infty)(\mathbb{I} + \mathcal{J}_{\mathbf{u}_\infty})^{-1}(\psi\mathbf{f} - \mathbf{h} - (-\Delta + (\mathbf{u}_\infty \cdot \nabla))\mathbf{g}, -\nabla \cdot \mathbf{g}), \\ \mathfrak{r} &= \mathfrak{p}(\mathbf{u}_\infty)(\mathbb{I} + \mathcal{J}_{\mathbf{u}_\infty})^{-1}(\psi\mathbf{f} - \mathbf{h} - (-\Delta + (\mathbf{u}_\infty \cdot \nabla))\mathbf{g}, -\nabla \cdot \mathbf{g}).\end{aligned} \tag{3.36}$$

Then, by (3.28)

$$(-\Delta + (\mathbf{u}_\infty \cdot \nabla))\mathbf{w} + \nabla\mathfrak{r} = \psi\mathbf{f} - \mathbf{h},\ \nabla \cdot \mathbf{w} = 0 \text{ in } \Omega,\ \mathbf{w} = \mathbf{g} \text{ on } \partial\Omega, \tag{3.37}$$

and moreover by (3.35) and (3.29)

$$\|\mathbf{w}\|_{p,2} + \|\mathfrak{r}\|_{p,1} \leqq C_{K,p}(\|\mathbf{f}\|_p + \|\mathbf{f}\|_1 + \|\mathbf{g}\|_{p,2}). \tag{3.38}$$

If we put $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\mathfrak{p} = \mathfrak{q} + \mathfrak{r}$, then combining (3.31), (3.32), (3.33), (3.37), and (3.38), we see that $\mathbf{u}$ and $\mathfrak{p}$ solve (3.1) uniquely and satisfy (3.2), which completes the proof of Theorem 3.1.

4. On an existence theorem of solutions to a stationary problem; A proof of Theorem 1.1. In this section we shall prove Thoerem 1.1 by the usual contraction mapping principle. To this end, the following theorem is the key of our argument.

THEOREM 4.1. Let $3 < p < \infty$ and $0 < \delta < 1/4$. Let $\ll \cdot \gg_{2\delta}$ and $|||\cdot|||_\delta$ be the same as in Theorem 1.1. Assume that $0 < |\mathbf{u}_\infty| \leqq 1$. If $\ll \mathbf{f} \gg_{2\delta} < \infty$ and $\mathbf{g} \in \mathbb{W}^2_{p,d}(\partial\Omega)$, then the problem (3.1) admits unique solutions $\mathbf{u} \in \mathbb{W}^2_p(\Omega)$ and $\mathfrak{p} \in W^1_p(\Omega)$ such that

$$\|\mathbf{u}\|_{p,2} + \|\mathfrak{p}\|_{p,1} + |||\mathbf{u}|||_\delta \leqq C_{p,\delta}|\mathbf{u}_\infty|^{-\delta}\{\ll \mathbf{f} \gg_{2\delta} + \|\mathbf{g}\|_{p,2}\}. \tag{4.1}$$

Postponing the proof of Theorem 4.1, we shall prove Theorem 1.1. Put $\mathbf{w} = \mathbf{u}_\infty + \mathbf{v}$, and then the problem (SP) is reduced to the following problem:

$$\begin{aligned} -\Delta\mathbf{v} + (\mathbf{u}_\infty \cdot \nabla)\,\mathbf{v} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} + \nabla\mathfrak{q} = \mathbf{f},\ \nabla \cdot \mathbf{v} = \mathbf{0} &\text{ in } \Omega, \\ \mathbf{v} = -\mathbf{u}_\infty + \mathbf{g} \qquad &\text{on } \partial\Omega.\end{aligned} \tag{4.2}$$

Let $\psi(x)$ be a function of $C_0^\infty(\mathbb{R}^3)$ such that $\psi(x) = 1$ for $|x| \leqq b_0$ and $\psi(x) = 0$ for $|x| \geqq b_0 + 1$, and then $\mathbf{u}_\infty \psi(x) \in \mathbb{W}^2_{p,d}(\partial\Omega)$, because supp $(\mathbf{u}_\infty \psi) \subset B_{b_0+1}$, $\mathbf{u}_\infty \psi = \mathbf{u}_\infty$ on $\partial\Omega$, and moreover

$$\int_{\partial\Omega} \nu \cdot (\mathbf{u}_\infty \psi)\, d\Gamma = \int_{\partial\Omega} \nu \cdot \mathbf{u}_\infty\, d\Gamma = 0. \tag{4.3}$$

In fact,

$$\int_{D_{b_0+1}} \nabla \cdot (\mathbf{u}_\infty \psi)\, dx = \int_{B_{b_0+1}} \nabla \cdot (\mathbf{u}_\infty \psi)\, dx = \int_{S_{b_0+1}} \frac{x}{|x|} \cdot (\mathbf{u}_\infty \psi)\, d\sigma = 0,$$

where $d\sigma$ is the surface element of S_{b_0+1}, and hence by Proposition 2.3, $\mathbf{a} = \mathbf{u}_\infty \psi - \mathbb{B}[\nabla \cdot (\mathbf{u}_\infty \psi)]$ satisfies the relations: $\nabla \cdot \mathbf{a} = 0$ in $\mathbb{R}^3$ and $\mathbf{a} = \mathbf{u}_\infty$ on $\partial\Omega$. In particular, by integration by parts

$$0 = \int_{\mathcal{O}} \nabla \cdot \mathbf{a}\, dx = -\int_{\partial\Omega} \nu \cdot \mathbf{a}\, d\Gamma = -\int_{\partial\Omega} \nu \cdot \mathbf{u}_\infty\, d\Gamma,$$

which shows (4.3).

Let us introduce the invariant set $\mathcal{I}$ as follows:

$$\begin{aligned} \mathcal{I} = \{(\mathbf{y}, \mathfrak{p}) \in & \mathbb{W}^2_p(\Omega) \times W^1_p(\Omega) \mid \mathbf{y} = -\mathbf{u}_\infty + \mathbf{g} \text{ on } \partial\Omega, \\ & \|(\mathbf{y}, \mathfrak{p})\|_{\mathcal{I}} = \|\mathbf{y}\|_{p,2} + \|\mathfrak{p}\|_{p,1} + |||\mathbf{y}|||_\delta \leqq |\mathbf{u}_\infty|^\beta/2\,\}. \end{aligned}$$

Given $(\mathbf{y}, \mathfrak{p}) \in \mathcal{I}$, let $\mathbf{z}$ and $\mathfrak{q}$ denote solutions of the equations

$$\begin{aligned} -\Delta \mathbf{z} + (\mathbf{u}_\infty \cdot \nabla)\,\mathbf{z} + \nabla \mathfrak{q} = \mathbf{f} - (\mathbf{y} \cdot \nabla)\,\mathbf{y},\ \ \nabla \cdot \mathbf{z} = \mathbf{0} & \text{ in } \Omega, \\ \mathbf{z} = -\mathbf{u}_\infty + \mathbf{g} \qquad & \text{ on } \partial\Omega. \end{aligned}$$

Observe that

$$\begin{aligned} \ll \mathbf{f} - (\mathbf{y} \cdot \nabla)\,\mathbf{y} \gg_{2\delta} & \leqq \ll \mathbf{f} \gg_{2\delta} + |||\mathbf{y}|||_\delta^2 \leqq \epsilon\,|\mathbf{u}_\infty|^{\delta+\beta} + |\mathbf{u}_\infty|^{2\beta}/4, \\ \|-\mathbf{u}_\infty \psi + \mathbf{g}\|_{p,2} & \leqq |\psi|_{p,2}|\mathbf{u}_\infty| + \epsilon\,|\mathbf{u}_\infty|^{\delta+\beta}, \end{aligned}$$

and hence by (4.1)

$$\begin{aligned} \|(\mathbf{z}, \mathfrak{q})\|_{\mathcal{I}} & \leqq C_{p,\delta}|\mathbf{u}_\infty|^{-\delta}\{2\epsilon|\mathbf{u}_\infty|^{\delta+\beta} + |\mathbf{u}_\infty|^{2\beta}/4 + |\psi|_{p,2}|\mathbf{u}_\infty|\} \\ & \leqq C_{p,\delta}(2\epsilon + |\mathbf{u}_\infty|^{\beta-\delta}/4 + |\psi|_{p,2}|\mathbf{u}_\infty|^{1-\beta})|\mathbf{u}_\infty|^\beta. \end{aligned}$$

If we choose $\epsilon > 0$ so small that

$$C_{p,\delta}(2\epsilon + \epsilon^{\beta-\delta}/4 + |\psi|_{p,2}\epsilon^{1-\beta}) \leqq 1/2, \tag{A.1}$$

we have $\|(\mathbf{z}, \mathfrak{q})\|_{\mathcal{I}} \leqq |\mathbf{u}_\infty|^\beta/2$. Therefore, if we define the map G by the relation $G(\mathbf{y}, \mathfrak{p}) = (\mathbf{z}, \mathfrak{q})$, then G maps $\mathcal{I}$ into itself. Let $(\mathbf{y}_j, \mathfrak{p}_j) \in \mathcal{I}$, $j = 1, 2$. Since

$$\begin{aligned} \ll (\mathbf{y}_1 \cdot \nabla)\mathbf{y}_1 - (\mathbf{y}_2 \cdot \nabla)\mathbf{y}_2 \gg_{2\delta} & \leqq (|||\mathbf{y}_1|||_\delta + |||\mathbf{y}_2|||_\delta)|||\mathbf{y}_1 - \mathbf{y}_2|||_\delta \\ & \leqq |\mathbf{u}_\infty|^\beta |||\mathbf{y}_1 - \mathbf{y}_2|||_\delta, \end{aligned}$$

by (4.1) we have

$$\|G(\mathbf{y}_1,\mathfrak{p}_1)-G(\mathbf{y}_2,\mathfrak{p}_2)\|_{\mathcal{I}} \leqq C_{p,\delta}|\mathbf{u}_\infty|^{\beta-\delta}\|\mathbf{y}_1-\mathbf{y}_2\|_\delta$$
$$\leqq C_{p,\delta}|\mathbf{u}_\infty|^{\beta-\delta}\|(\mathbf{y}_1,\mathfrak{p}_1)-(\mathbf{y}_2,\mathfrak{p}_2)\|_{\mathcal{I}}.$$

If we choose $\epsilon > 0$ so small that

$$C_{p,\delta}\,\epsilon^{\beta-\delta} \leqq 1/2, \tag{A.2}$$

then G is a contraction map of $\mathcal{I}$, and therefore there exists a unique fixed point $(\mathbf{v},\mathfrak{q}) \in \mathcal{I}$. Obviously, if we put $\mathbf{w} = \mathbf{u}_\infty + \mathbf{v}$, then $\mathbf{w}$ and $\mathfrak{q}$ solve (SP) and satisfy (1.1), which completes the proof of Theorem 1.1.

Now, we shall prove Theorem 4.1, below. First of all, we note that

$$\|\mathbf{f}\|_p + \|\mathbf{f}\|_1 \leqq 2 \ll \mathbf{f} \gg_{2\delta} \int_{\mathbb{R}^3} \frac{dx}{(1+|x|)^{5/2}(1+s_{\mathbf{u}_\infty}(x))^{1/2+2\delta}}.$$

In the course of the proof, we always assume that $|\mathbf{u}_\infty| \leqq 1$. Also, we use the polar coordinate system

$$y_1 = r\cos\theta,\ \ y_2 = r\sin\theta\cos\psi,\ \ y_3 = r\sin\theta\sin\psi \tag{4.4}$$

for $0 \leqq \theta \leqq \pi$, $0 \leqq \psi \leqq 2\pi$ and $0 \leqq r < \infty$. Let S be an orthogonal matrix such that $S\mathbf{u}_\infty = |\mathbf{u}_\infty|^{\mathrm{T}}(1,0,0)$ and put $s(y) = |y| - y_1$. By a change of variable: $y = Sx$,

$$|x| = |y| = r \text{ and } s_{\mathbf{u}_\infty}(x) = s(y) = r(1-\cos\theta). \tag{4.5}$$

In particular, using the assumption $\delta < 1/4$, we have

$$\int_{\mathbb{R}^3} \frac{dx}{(1+|x|)^{5/2}(1+s_{\mathbf{u}_\infty}(x))^{1/2+2\delta}} = 2\pi\int_0^\infty \frac{dr}{(1+r)^{5/2}r^{1/2+2\delta}}\int_0^\pi \frac{\sin\theta\,d\theta}{(1-\cos\theta)^{1/2+2\delta}}$$

which implies that

$$\|\mathbf{f}\|_p + \|\mathbf{f}\|_1 + \|\mathbf{g}\|_{p,2} \leqq C_\delta \ll \mathbf{f} \gg_{2\delta} + \|\mathbf{g}\|_{p,2} \tag{4.6}$$

with some constant C_δ independent of $\mathbf{u}_\infty$. By Theorem 3.1, the problem (3.1) admits unique solutions $\mathbf{u}$ and $\mathfrak{p}$ that satisfy the estimate

$$\|\mathbf{u}\|_{p,2} + \|\mathfrak{p}\|_{p,1} \leqq C_{p,\delta}(\ll \mathbf{f} \gg_{2\delta} + \|\mathbf{g}\|_{p,2}),$$

which together with Sobolev's inequality implies that

$$\|\mathbf{u}\|_{\infty,1} \leqq C_p\|\mathbf{u}\|_{p,2} \leqq C_{p,\delta}(\ll \mathbf{f} \gg_{2\delta} + \|\mathbf{g}\|_{p,2}), \tag{4.7}$$

and then it suffices to prove that

$$(1+s_{\mathbf{u}_\infty}(x))^\delta|\mathbf{u}(x)| \leqq C_{p,\delta}|\mathbf{u}_\infty|^{-\delta}(\ll \mathbf{f} \gg_{2\delta} + \|\mathbf{g}\|_{p,2})|x|^{-1}, \tag{4.8}$$
$$(1+s_{\mathbf{u}_\infty}(x))^{1/2+\delta}|\nabla\,\mathbf{u}(x)| \leqq C_{p,\delta}|\mathbf{u}_\infty|^{-\delta}(\ll \mathbf{f} \gg_{2\delta} + \|\mathbf{g}\|_{p,2})|x|^{-3/2} \tag{4.9}$$

for $|x| \geqq b_0+4$. Recall that $\mathbf{u} = \mathbf{v}+\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are the same as in (3.30) and (3.36), respectively. When $|x| \geqq b_0+4$, we have $\mathbf{v} = \chi(\mathbf{u}_\infty)*\mathbf{f}_0$ and $\mathbf{w} = \chi(\mathbf{u}_\infty)*\mathcal{M}_{\mathbf{u}_\infty}(\mathbf{k}, -\nabla\cdot\mathbf{g})$, where $\mathbf{k} = \psi\mathbf{f}-\mathbf{h}-(-\Delta+(\mathbf{u}_\infty\cdot\nabla))\mathbf{g}$ (cf. (3.36)) and $\mathcal{M}_{\mathbf{u}_\infty}(\mathbf{k}, -\nabla\cdot\mathbf{g})$ is the zero extension to the whole space $\mathbb{R}^3$ of the first component of $(\mathbb{I} + \mathcal{J}_{\mathbf{u}_\infty})^{-1}(\mathbf{k}, -\nabla\cdot\mathbf{g})$. By Lemma 3.8, (3.35) and (4.6), we see that

$$\operatorname{supp}\,[\mathcal{M}_{\mathbf{u}_\infty}(\mathbf{k}, -\nabla\cdot\mathbf{g})]_0 \subset B_{b_0+3}, \tag{4.10}$$

$$|[\mathcal{M}_{\mathbf{u}_\infty}(\mathbf{k}, -\nabla\cdot\mathbf{g})]_0|_1 \leqq C_{p,b_0,\delta}(\ll \mathbf{f} \gg_{2\delta} + \|\mathbf{g}\|_{p,2}). \tag{4.11}$$

In order to show that (4.8) and (4.9) hold for $\chi(\mathbf{u}_\infty) * [\mathcal{M}_{\mathbf{u}_\infty}(\mathbf{k}, -\nabla\cdot\mathbf{g})]_0$, it suffices to prove the following lemma.

LEMMA 4.2. Let $b > 0$, $\mathbf{g} \in \mathbb{L}_{1,b}(\mathbb{R}^3)$ and $0 < |\mathbf{u}_\infty| \leqq 1$. Then, for $|x| \geqq b+1$ we have the following relations:

$$|\chi(\mathbf{u}_\infty) * \mathbf{g}(x)| \leqq C_{\delta,b}|\mathbf{u}_\infty|^{-\delta}(1 + s_{\mathbf{u}_\infty}(x))^{-\delta}|x|^{-1}|\mathbf{g}|_1,$$
$$|\nabla\chi(\mathbf{u}_\infty) * \mathbf{g}(x)| \leqq C_{\delta,b}|\mathbf{u}_\infty|^{-\delta}(1 + s_{\mathbf{u}_\infty}(x))^{-(1/2+\delta)}|x|^{-3/2}|\mathbf{g}|_1.$$

Proof. The argument is the same, so that we shall prove only the second estimate, below. Since $1 + s_{\mathbf{u}_\infty}(x) \leqq 1 + s_{\mathbf{u}_\infty}(x-y) + s_{\mathbf{u}_\infty}(y)$ and since $s_{\mathbf{u}_\infty}(y) \leqq 2b$ and $|x-y| \geqq |x|/(b+1)$ when $|x| \geqq b+1$ and $|y| \leqq b$, by (3.8) we have

$$\begin{aligned}
(1 + s_{\mathbf{u}_\infty}(x))^{1/2+\delta}|\nabla\chi(\mathbf{u}_\infty) * \mathbf{g}(x)| &\leqq \frac{2^{1/2+\delta}C_\delta}{|\mathbf{u}_\infty|^\delta}\int_{\mathbb{R}^3}\frac{|\mathbf{g}(y)|\,dy}{|x-y|^{3/2}} \\
+ 2^{1/2+\delta}C_0\int_{\mathbb{R}^3}&\left[\frac{|\mathbf{u}_\infty|^{1/2}}{|x-y|^{3/2}} + \frac{1}{|x-y|^2}\right](1 + s_{\mathbf{u}_\infty}(y))^{1/2+\delta}|\mathbf{g}(y)|\,dy \\
\leqq \frac{C_{\delta,b}}{|\mathbf{u}_\infty|^\delta}\int_{\mathbb{R}^3}&\left\{\frac{1}{|x-y|^{3/2}} + \frac{1}{|x-y|^2}\right\}|\mathbf{g}(y)|\,dy \leqq \frac{C_{\delta,b}|\mathbf{g}|_1}{|\mathbf{u}_\infty|^\delta|x|^{3/2}},
\end{aligned}$$

which shows the second inequality of the lemma.

In particular, by (4.10), (4.11) and Lemma 4.2 we have

$$\begin{aligned}
(1 + s_{\mathbf{u}_\infty}(x))^\delta|x||\mathbf{w}(x)| &+ (1 + s_{\mathbf{u}_\infty}(x))^{1/2+\delta}|x|^{3/2}|\nabla\mathbf{w}(x)| \\
&\leqq C_{\delta,p}|\mathbf{u}_\infty|^{-\delta}(\ll \mathbf{f} \gg_{2\delta} + \|\mathbf{g}\|_{p,2}) \qquad \text{for } |x| \geqq b_0+4.
\end{aligned} \tag{4.12}$$

In order to show that (4.8) and (4.9) also hold for $\chi(\mathbf{u}_\infty) * \mathbf{f}_0$, we use the following lemma.

LEMMA 4.3. Let $0 < \delta < 1/4$. Let $\mathbf{g} \in \mathbb{L}_\infty(\mathbb{R}^3)$ and assume that

$$< \mathbf{g} >_{2\delta} = \sup_{x\in\mathbb{R}^3}(1 + |x|)^{5/2}(1 + s_{\mathbf{u}_\infty}(x))^{1/2+2\delta}|\mathbf{g}(x)| < \infty. \tag{4.13}$$

Then, for $|x| \geqq 1$ we have the relations

$$|\chi(\mathbf{u}_\infty) * \mathbf{g}(x)| \leqq C_\delta|\mathbf{u}_\infty|^{-\delta}(1 + s_{\mathbf{u}_\infty}(x))^{-\delta}|x|^{-1}, \tag{4.14}$$

$$|\nabla\chi(\mathbf{u}_\infty) * \mathbf{g}(x)| \leqq C_\delta|\mathbf{u}_\infty|^{-\delta}(1 + s_{\mathbf{u}_\infty}(x))^{-(1/2+\delta)}|x|^{-3/2}. \tag{4.15}$$

Obviously, applying Lemma 4.3 to $\chi(\mathbf{u}_\infty) * \mathbf{f}_0$, and combining the resulting estimate and (4.12) implies (4.8) and (4.9), and hence we can complete the proof of Theorem 4.1. Therefore, we shall prove Lemma 4.3, below. Although Farwig [8, 9] proved Lemma 4.3 essentially by refining the argument due to Finn [12], in order to make the paper self-contained as much as possible, we shall give a proof of Lemma 4.3. Our argument is a little bit different from the argument due to Finn and Farwig in the case of the gradient estimate. Since $1 + s_{\mathbf{u}_\infty}(x) \leqq 1 + s_{\mathbf{u}_\infty}(x-y) + s_{\mathbf{u}_\infty}(y)$, by (3.8) and (4.5)

$$\begin{aligned}
&(1 + s_{\mathbf{u}_\infty}(x))^\delta |\chi(\mathbf{u}_\infty) * \mathbf{g}(x)| \\
&\quad\leqq 2^\delta < \mathbf{g} >_{2\delta} \left(\frac{C_\delta}{|\mathbf{u}_\infty|^\delta} + C_0\right) \int_{\mathbb{R}^3} \frac{dy}{|y|(1+|x-y|)^{5/2}(1+s_{\mathbf{u}_\infty}(x-y))^{1/2+\delta}} \\
&\quad\leqq 2^\delta < \mathbf{g} >_{2\delta} \left(\frac{C_\delta}{|\mathbf{u}_\infty|^\delta} + C_0\right) \int_{\mathbb{R}^3} \frac{dy}{|y|(1+|Sx-y|)^{5/2}(1+s(Sx-y))^{1/2+\delta}}.
\end{aligned}$$

Using the assumption $1/2 + \delta < 1$ and (4.4) and (4.5), we have

$$\begin{aligned}
\int_{|y|\geqq(|x|+1)/2} \frac{dy}{|y|(1+|Sx-y|)^{5/2}(1+s(Sx-y))^{1/2+\delta}} &\leqq \frac{2}{1+|x|}\int_{\mathbb{R}^3} \frac{dy}{(1+|y|)^{5/2}s(y)^{1/2+\delta}} \\
&= \frac{4\pi\,\beta_{1/2+\delta}}{1+|x|}\int_0^\infty \frac{r^2\,dr}{(1+r)^{5/2}r^{1/2+\delta}}.
\end{aligned}$$

Here and hereafter, we write

$$\beta_q = \int_0^\pi \frac{\sin\theta\, d\theta}{(1-\cos\theta)^q} = \frac{2^{1-q}}{1-q}.$$

Since $1 + |Sx - y| \geqq (1 + |x|)/2$ when $|y| \leqq (1 + |x|)/2$, by Hölder's inequality, (4.4) and (4.5) we have

$$\begin{aligned}
&\int_{|y|\leqq(|x|+1)/2} \frac{dy}{|y|(1+|Sx-y|)^{5/2}(1+s(Sx-y))^{1/2+\delta}} \\
&\leqq \left(\frac{2}{1+|x|}\right)^{5/2}\left(\int_{|y|\leqq(|x|+1)/2} |y|^{-8/3}\,dy\right)^{3/8}\left(\int_{|y|\leqq 3(|x|+1)/2} s(y)^{-4/5}\,dy\right)^{5/8} \leqq \frac{C}{1+|x|}.
\end{aligned}$$

Combining these estimations implies (4.14).

In order to show (4.15), we observe that

$$|\nabla\chi(\mathbf{u}_\infty) * \mathbf{g}(x)| \leqq < \mathbf{g} >_{2\delta} \int_{\mathbb{R}^3} \frac{|\nabla\chi(\mathbf{u}_\infty)(y)|\,dy}{(1+|x-y|)^{5/2}(1+s_{\mathbf{u}_\infty}(x-y))^{1/2+2\delta}}. \tag{4.16}$$

Since $|x - y| \geqq |y|/2$ when $|y| \geqq 2|x|$, by (3.8) we have

$$\begin{aligned}
&(1+s_{\mathbf{u}_\infty}(x))^{1/2+\delta} \int_{|y|\geqq 2|x|} \frac{|\nabla\chi(\mathbf{u}_\infty)(y)|\,dy}{(1+|x-y|)^{5/2}(1+s_{\mathbf{u}_\infty}(x-y))^{1/2+2\delta}} \\
&\leqq \frac{C_\delta}{|\mathbf{u}_\infty|^\delta}\left(\frac{1}{2|x|}\right)^{3/2} \int_{\mathbb{R}^3} \frac{dy}{(1+|y|)^{5/2}(1+s(y))^{1/2+2\delta}} + C_0 \int_{|y|\geqq 2|x|} \frac{dy}{|y|^4\, s(y)^{1/2}} \qquad (4.17) \\
&\leqq \frac{C_\delta}{|\mathbf{u}_\infty|^\delta\,|x|^{3/2}},
\end{aligned}$$

because

$$\int_{|y|\geqq 2|x|} \frac{dy}{|y|^4\, s(y)^{1/2}} = 2\pi\,\beta_{1/2} \int_{2|x|}^{\infty} \frac{dr}{r^{5/2}} = \frac{4\pi}{5}\left(\frac{1}{|x|}\right)^{3/2}.$$

Since $|x-y| \geqq |x|/2$ when $|y| \leqq 1/2$ and $|x| \geqq 1$, by (3.8) we have

$$\begin{aligned}
&(1+s_{\mathbf{u}_\infty}(x))^{1/2+\delta} \int_{|y|\leqq 1/2} \frac{|\nabla\chi(\mathbf{u}_\infty)(y)|\,dy}{(1+|x-y|)^{5/2}(1+s_{\mathbf{u}_\infty}(x-y))^{1/2+2\delta}} \\
&\leqq \left(\frac{2}{|x|}\right)^{5/2} \left\{ \frac{C_\delta}{|\mathbf{u}_\infty|^\delta} \int_{|y|\leqq 1/2} \frac{dy}{|y|^{3/2}} + C_0 \int_{|y|\leqq 1/2} \frac{dy}{|y|^{3/2}\, s(y)^{1/2}} \right\} \qquad (4.18)\\
&\leqq \frac{C_\delta}{|\mathbf{u}_\infty|^\delta\, |x|^{3/2}}.
\end{aligned}$$

Since

$$|\nabla\chi(\mathbf{u}_\infty)(y)| \leqq \frac{C_\delta}{|\mathbf{u}_\infty|^\delta (1+|y|)^{3/2}(1+s_{\mathbf{u}_\infty}(y))^{1/2+\delta}},$$

for $|y| \geqq 1/2$ as follows from (3.8) and the fact that $0 < |\mathbf{u}_\infty| \leqq 1$, by changing the variable : $x-y=z$ when $|y| \leqq |x|/2$ we have

$$\int_{1/2\leqq|y|\leqq 2|x|} \frac{|\nabla\chi(\mathbf{u}_\infty)(y)|\,dy}{(1+|x-y|)^{5/2}(1+s_{\mathbf{u}_\infty}(x-y))^{1/2+2\delta}} \leqq \frac{C_\delta}{|\mathbf{u}_\infty|^\delta} \sum_{p=0}^{1} \int_\omega h_p(Sx,y)\,dy \qquad (4.19)$$

where $\omega = \{y \in \mathbb{R}^3 \mid |x|/2 \leqq |y| \leqq 2|x|\}$ and

$$h_p(x,y) = \frac{1}{(1+|y|)^{3/2+p}(1+s(y))^{1/2+(1+p)\delta}(1+|x-y|)^{5/2-p}(1+s(x-y))^{1/2+(2-p)\delta}}.$$

In view of (4.16) to (4.19), in order to show (4.15) it now suffices to prove that

$$\int_\omega h_p(x,y)\,dy \leqq \frac{C_\delta}{|x|^{3/2}(1+s(x))^{1/2+\delta}}, \quad |x| \geqq 1,\ \ p=0,1, \qquad (4.20)$$

because $s(Sx) = s_{\mathbf{u}_\infty}(x)$. Since

$$\begin{aligned}
&\int_\omega \frac{dy}{(1+s(x-y))^{1/2+(2-p)\delta}(1+|x-y|)^{5/2-p}} \\
&\leqq \int_{|y|\leqq 3|x|} \frac{dy}{(1+s(y))^{1/2+(2-p)\delta}(1+|y|)^{5/2-p}} \qquad (4.21)\\
&\leqq 2\pi\beta_{1/2+(2-p)\delta} \int_0^{3|x|} \frac{r^2\,dr}{(1+r)^{5/2-p}\, r^{1/2+(2-p)\delta}} \\
&\leqq C_\delta \max(1, |x|^{p-\delta}) \qquad \text{for } |x| \geqq 1,
\end{aligned}$$

when $s(x) \leqq 1$ and $|x| \geqq 1$ we have

$$\int_\omega h_p(x,y)dy \leqq \frac{C_\delta}{|x|^{3/2+p}} \max(1, |x|^{p-\delta}) \leqq \frac{C_\delta}{|x|^{3/2}(1+s(x))^{1/2+\delta}}. \tag{4.22}$$

Therefore, we assume that $|x| \geqq 1$ and $s(x) \geqq 1$, below. Let ξ and η be numbers such that $0 \leqq \xi \leqq \pi$, $0 \leqq \eta \leqq 2\pi$ and

$$x_1 = |x|\cos\xi, \quad x_2 = |x|\sin\xi\cos\eta, \quad x_3 = |x|\sin\xi\sin\eta.$$

Let ϵ be a very small positive number and $\rho(\theta)$ a function of $C^\infty(\mathbb{R})$ such that $0 \leqq \rho(\theta) \leqq 1$, $\rho(\theta) = 1$ for $\theta \leqq 1/4$ and $\rho(\theta) = 0$ for $\theta \geqq 1/2$. Since

$$(1 - \cos(\xi/4)) \leqq 1 - \cos\xi \leqq 32(1 - \cos(\xi/4)) \quad \text{for } 0 \leqq \xi \leqq \pi, \tag{4.23}$$

we have $(1 - \rho(\theta/\xi))s(y)^{-(1/2+(1+p)\delta)} \leqq C_\delta s(x)^{-(1/2+(1+p)\delta)}$ for $|y| \geqq |x|/2$, and hence by (4.21)

$$\int_\omega (1 - \rho(\theta/\xi))h_p(x,y)dy \leqq \frac{C_\delta \max(1, |x|^{p-\delta})}{|x|^{3/2+p}s(x)^{1/2+(1+p)\delta}} \leqq \frac{C_\delta}{|x|^{3/2}s(x)^{1/2+\delta}}, \tag{4.24}$$

because $s(x) \geqq 1$ and $|x| \geqq 1$. Since

$$\begin{aligned} |x-y|^2 &= |x|^2 + r^2 - 2|x|r(\cos\xi\cos\theta + \sin\xi\sin\theta(\cos\eta\cos\varphi + \sin\eta\sin\varphi)) \\ &= |x|^2 + r^2 - 2|x|r(\cos\xi\cos\theta + \sin\xi\sin\theta\cos(\eta - \varphi)), \end{aligned} \tag{4.25}$$

when $0 \leqq \theta \leqq \xi/2$ we have by (4.23)

$$\begin{aligned} |x-y|^2 &\geqq |x|^2 + r^2 - 2|x|r(\cos\xi\cos\theta + \sin\xi\sin\theta) \\ &= |x|^2 + r^2 - 2|x|r\cos(\xi - \theta) \\ &\geqq |x|^2 + r^2 - 2|x|r\cos(\xi/2) \geqq |x|^2(1 - \cos^2(\xi/2)) \\ &= |x|^2(1 - \cos\xi)/2. \end{aligned} \tag{4.26}$$

When $\epsilon \leqq \xi \leqq \pi$, by (4.26) we have $|x - y| \geqq |x|(1 - \cos\epsilon)^{1/2}/2$, and hence

$$(1 + s(x))^{1/2+\delta} \int_\omega \rho(\theta/\xi)h_p(x,y)dy \leqq \frac{C_\epsilon}{|x|^4} \int_\omega \left\{ \frac{1}{s(x-y)^{1/2}} + \frac{1}{s(y)^{1/2}} \right\} dy \leqq \frac{C_\epsilon}{|x|^{3/2}},$$

which together with (4.24) and (4.22) implies that (4.20) holds for $\epsilon \leqq \xi \leqq \pi$.

Now, let $\epsilon > 0$ be chosen so small that a finite number of inequalities below will hold and we consider the case where $0 < \xi \leqq \epsilon$. Note that $s(x) = 0$ when $\xi = 0$ so that this case is already over. Put $q = 3/2 - p$ and

$$K_j = \int_{G_j} \rho(\theta/\xi)h_p(x,y)dy, \quad j = 1, 2,$$

$$G_1 = \{x \in \mathbb{R}^3 \mid (1 - 4\xi^{1/q})|x| \leqq |y| \leqq (1 + 4\xi^{1/q})|x|\}, \quad G_2 = \omega - G_1.$$

Since

$$\begin{aligned}
&s(x)^{1/2+\delta}K_2\\
&\leqq C_\delta \int_{G_2} \frac{\rho(\theta/\xi)}{(1+|x-y|)^{5/2-p}(1+|y|)^{3/2+p}} \left\{ \frac{1}{s(y)^{1/2}} + \frac{1}{s(x-y)^{1/2+(2-p)\delta}} \right\} dy\\
&\leqq \frac{C_\delta}{|x|^{3/2+p}} \left(\int_{(1+4\xi^{1/q})|x|}^{2|x|} \frac{r^2 dr}{r^{1/2}(r-|x|)^{5/2-p}} \right.\\
&\qquad \left. + \int_{|x|/2}^{(1-4\xi^{1/q})|x|} \frac{r^2 dr}{r^{1/2}(|x|-r)^{5/2-p}} \right) \int_0^{\xi/2} \frac{\sin\theta d\theta}{(1-\cos\theta)^{1/2}}\\
&\quad + \frac{C_\delta}{|x|^{3/2+p}} \int_\omega \frac{dy}{(1+s(x-y))^{1/2+(2-p)\delta}(1+|x-y|)^{5/2-p}},
\end{aligned}$$

if $\epsilon > 0$ is chosen so small that $1-\cos(\xi/2) \leqq \xi^2$ for $0 < \xi \leqq \epsilon$, then by (4.21) and the fact that $(3/2-p)/q = 1$,

$$\begin{aligned}
K_2 &\leqq \frac{C_\delta}{s(x)^{1/2+\delta}} \left\{ \frac{|x|^{3/2}\xi}{|x|^{3/2+p}\left(4\xi^{1/q}|x|\right)^{3/2-p}} + \frac{\max(1,|x|^p)}{|x|^{3/2+p}} \right\}\\
&\leqq \frac{C_\delta}{|x|^{3/2}(1+s(x))^{1/2+\delta}}
\end{aligned} \tag{4.27}$$

because $1 \leqq s(x) \leqq 2|x|$.

Finally, we shall consider the case where $y \in G_1$ and $0 \leqq \theta \leqq \xi/2$. By integration by parts with respect to θ,

$$K_1 \leqq \int_{(1-4\xi^{1/q})|x|}^{(1+4\xi^{1/q})|x|} \int_0^\pi \int_0^{2\pi} \frac{r^2\rho(\theta/\xi)\sin\theta}{r^{2+p}(1-\cos\theta)^{1/2}m_p(x,y)} dr d\theta d\varphi \leqq L_1 + L_2$$

where

$$m_p(x,y) = (1+s(x-y))^{1/2+(2-p)\delta}(1+|x-y|)^{5/2-p},$$

$$L_1 = \frac{C}{|x|^{2+p}} \int_{(1-4\xi^{1/q})|x|}^{(1+4\xi^{1/q})|x|} \int_0^\pi \int_0^{2\pi} \frac{r^2|\rho'(\theta/\xi)|(1-\cos\theta)^{1/2}}{\xi m_p(x,y)} dr d\theta d\varphi,$$

$$L_2 = \frac{C}{|x|^{1+p}} \int_{(1-4\xi^{1/q})|x|}^{(1+4\xi^{1/q})|x|} \int_0^\pi \int_0^{2\pi} r\rho(\theta/\xi)(1-\cos\theta)^{1/2} \left| \frac{\partial}{\partial\theta} m_p(x,y)^{-1} \right| dr d\theta d\varphi.$$

In view of (4.26), we know that

$$|x-y| \geqq s(x)^{1/2}|x|^{1/2}/2 \quad \text{for } 0 \leqq \theta \leqq \xi/2. \tag{4.28}$$

Since we can choose $\epsilon > 0$ so small that

$$\frac{|\rho'(\theta/\xi)|(1-\cos\theta)^{1/2}}{\xi} \leqq \frac{C\sin\theta}{(1-\cos\xi)^{1/2}} \quad \text{for } 0 < \xi \leqq \epsilon,$$

putting $G_3 = \{z \in \mathbb{R}^3 \mid |x|^{1/2}s(x)^{1/2}/2 \leqq |z| \leqq 3|x|\}$, by (4.28) we have

$$\begin{aligned} L_1 &\leqq \frac{C}{|x|^{2+p}(1-\cos\xi)^{1/2}} \int_{G_1} \frac{|\rho'(\theta/\xi)|dy}{|x-y|^{5/2-p}s(x-y)^{1/2+(2-p)\delta}} \\ &\leqq \frac{C}{|x|^{3/2+p}s(x)^{1/2}} \int_{G_3} \frac{dz}{|z|^{5/2-p}s(z)^{1/2+(2-p)\delta}} \\ &\leqq \begin{cases} \dfrac{C_\delta \beta_{1/2+2\delta}}{|x|^{3/2}s(x)^{1/2}} \displaystyle\int_{|x|^{1/2}s(x)^{1/2}/2}^{3|x|} \frac{t^2}{t^{3+2\delta}}dt & \text{for } p = 0 \\ \dfrac{C_\delta \beta_{1/2+\delta}}{|x|^{5/2}s(x)^{1/2}} \displaystyle\int_{|x|^{1/2}s(x)^{1/2}/2}^{3|x|} \frac{t^2}{t^{2+\delta}}dt & \text{for } p = 1 \end{cases} \\ &\leqq \frac{C_\delta}{|x|^{3/2}(1+s(x))^{1/2+\delta}} \end{aligned} \tag{4.29}$$

because $1 \leqq s(x) \leqq 2|x|$. To proceed with the estimation, we put

$$x_1 - y_1 = |x-y|\cos\zeta, \quad x_2 - y_2 = |x-y|\sin\zeta\cos\psi, \quad x_3 - y_3 = |x-y|\sin\zeta\sin\psi,$$

and then

$$\sin^2\zeta \geqq c\begin{cases} (s(x)/|x|)^{1/3} & \text{for } p = 0, \\ 1 & \text{for } p = 1, \end{cases} \tag{4.30}$$

provided that $y \in G_1$ and $0 \leqq \theta \leqq \xi/2$ with a suitably small constant $c > 0$. In fact, choosing $\epsilon > 0$ so small that

$$|x|\sin\xi - r\sin\theta \geqq |x|(\sin\xi - (1+4\xi^{1/q})\sin(\xi/2)) \geqq |x|\xi/4$$

when $r \leqq (1+4\xi^{1/q})|x|$, $0 \leqq \theta \leqq \xi/2$ and $0 \leqq \xi \leqq \epsilon$, we have

$$\begin{aligned} |x-y|^2\sin^2\zeta &= (x_2-y_2)^2 + (x_3-y_3)^2 \\ &= |x|^2\sin^2\xi + r^2\sin^2\theta - 2|x|r\sin\xi\sin\theta\cos(\varphi-\eta) \\ &\geqq |x|^2\sin^2\xi + r^2\sin^2\theta - 2|x|r\sin\xi\sin\theta \\ &= (|x|\sin\xi - r\sin\theta)^2 \\ &\geqq (|x|\xi/4)^2 \quad \text{for } y \in G_1 \text{ and } 0 \leqq \theta \leqq \xi/2. \end{aligned}$$

On the other hand, by (4.25)

$$\begin{aligned} |x-y|^2 &\leqq |x|^2 + r^2 - 2|x|r(\cos\xi\cos\theta - \sin\xi\sin\theta) \\ &= |x|^2 + r^2 - 2|x|r\cos(\xi+\theta) \leqq k(r) \quad \text{for } 0 \leqq \theta \leqq \xi/2 \end{aligned}$$

where $k(r) = |x|^2 + r^2 - 2|x|r\cos(3\xi/2)$. Since $1/q = 2/3$ for $p = 0$ and $1/q = 2$ for $p = 1$, we can choose $\epsilon > 0$ so small that $1 - 4\xi^{1/q} < \cos(3\xi/2) < 1 + 4\xi^{1/q}$, and hence

$$k(r) \leqq \max\left(k((1+4\xi^{1/q})|x|), k((1-4\xi^{1/q})|x|)\right) \leqq C|x|^2 \begin{cases} \xi^{4/3} & \text{for } p=0, \\ \xi^2 & \text{for } p=1, \end{cases}$$

because $k'(|x|\cos(3\xi/2)) = 0$. Since $\xi^2 \geqq c'(1-\cos\xi) = c's(x)/|x|$ with a positive constant c' when $0 \leqq \xi \leqq \epsilon$ and ϵ is small enough, we have (4.30).

According to (4.30), let ζ_0 be a number such that $\sin^2\zeta_0 = c(s(x)/|x|)^{1/3}$ for $p = 0$ and $\sin^2\zeta_0 = c$ for $p = 1$, and then putting

$$y_1 = x_1 - t\cos\zeta, \quad y_2 = x_2 - t\sin\zeta\cos\psi, \quad y_3 = x_3 - t\sin\zeta\sin\psi,$$

by (4.30) and (4.28) we see that

$$\zeta_0 \leqq \zeta \leqq \pi, \quad 0 \leqq \psi \leqq 2\pi, \quad |x|^{1/2}s(x)^{1/2}/2 \leqq t \leqq 3|x| \tag{4.31}$$

when $y \in G_1$ and $0 \leqq \theta \leqq \xi/2$ provided that $\epsilon > 0$ is small enough. By direct calculation,

$$\begin{aligned}\left|\frac{\partial}{\partial\theta}|x-y|\right| &= \frac{1}{|x-y|}\{|(x_1-y_1)r\sin\theta| + |(x_2-y_2)r\cos\theta\cos\varphi| \\ &\qquad + |(x_3-y_3)r\cos\theta\sin\varphi|\} \\ &\leqq r\sin\theta + 2\sqrt{2}rs(x-y)^{1/2}/|x-y|^{1/2}, \\ \left|\frac{\partial}{\partial\theta}s(x-y)\right| &= \left|\frac{\partial}{\partial\theta}(|x-y| - (x_1 - r\cos\theta))\right| \\ &\leqq 2r\sin\theta + 2\sqrt{2}rs(x-y)^{1/2}/|x-y|^{1/2}\end{aligned}$$

because $|z_2|, |z_3| \leqq \sqrt{2}s(z)^{1/2}|z|^{1/2}$ which follows from the fact that $s(z) = (z_2^2+z_3^2)/(|z|+z_1) \geqq (z_2^2+z_3^2)/(2|z|)$. Thus,

$$\begin{aligned}\left|\frac{\partial}{\partial\theta}m_p(x,y)^{-1}\right| &\leqq C_\delta\left\{\frac{\left|\frac{\partial}{\partial\theta}s(x-y)\right|}{(1+s(x-y))^{3/2+(2-p)\delta}(1+|x-y|)^{5/2-p}}\right. \\ &\qquad \left. + \frac{\left|\frac{\partial}{\partial\theta}|x-y|\right|}{(1+s(x-y))^{1/2+(2-p)\delta}\,(1+|x-y|)^{7/2-p}}\right\} \\ &\leqq C_\delta\left\{\frac{r\sin\theta}{|x-y|^{5/2-p}s(x-y)^{3/2+(2-p)\delta}} + \frac{r}{|x-y|^{3-p}s(x-y)^{1+(2-p)\delta}}\right\},\end{aligned}$$

which, inserted into the definition of L_2, implies that $L_2 \leqq C_\delta(M_1+M_2)$ where

$$\begin{aligned}M_1 &= \frac{s(x)^{1/2}}{|x|^{3/2+p}}\int_{G_1}\frac{\rho(\theta/\xi)dy}{s(x-y)^{3/2+(2-p)\delta}|x-y|^{5/2-p}}, \\ M_2 &= \frac{1}{|x|^{1+p}}\int_{G_1}\frac{\rho(\theta/\xi)dy}{s(x-y)^{1+(2-p)\delta}|x-y|^{3-p}},\end{aligned}$$

because $(1-\cos\theta)^{1/2} \leqq \sin\theta$ and $(1-\cos\theta)^{1/2} \leqq (1-\cos\xi)^{1/2} \leqq (s(x)/|x|)^{1/2}$ when $0 \leqq \theta \leqq \xi \leqq \pi/2$. Using the change of variable: $z = x - y$ and (4.31), we have

$$\begin{aligned}
M_1 &\leqq \frac{s(x)^{1/2}}{|x|^{3/2+p}} \int_{|x|^{1/2}s(x)^{1/2}/2}^{3|x|} \frac{t^2 dt}{t^{4-p+(2-p)\delta}} \int_{\zeta_0}^{\pi} \frac{\sin\zeta d\zeta}{(1-\cos\zeta)^{3/2+(2-p)\delta}} \\
&\leqq C_\delta \begin{cases} \dfrac{s(x)^{1/2}}{|x|^{3/2}\left(|x|^{1/2}s(x)^{1/2}\right)^{1+2\delta}} \left(\dfrac{|x|}{s(x)}\right)^{(1/2+2\delta)/3} & \text{for } p = 0 \\ \dfrac{s(x)^{1/2}}{|x|^{5/2}\left(|x|^{1/2}s(x)^{1/2}\right)^{\delta}} & \text{for } p = 1 \end{cases} \\
&\leqq \frac{C_\delta}{|x|^{3/2}s(x)^{1/2+\delta}}
\end{aligned}$$

because $1 \leqq s(x) \leqq 2|x|$. Also,

$$\begin{aligned}
M_2 &\leqq \frac{1}{|x|^{1+p}} \int_{|x|^{1/2}s(x)^{1/2}/2}^{3|x|} \frac{t^2 dt}{t^{4-p+(2-p)\delta}} \int_{\zeta_0}^{\pi} \frac{\sin\zeta d\zeta}{(1-\cos\zeta)^{1+(2-p)\delta}} \\
&\leqq C_\delta \begin{cases} \dfrac{1}{|x|\left(|x|^{1/2}s(x)^{1/2}\right)^{1+2\delta}} \left(\dfrac{|x|}{s(x)}\right)^{2\delta/3} & \text{for } p = 0 \\ \dfrac{1}{|x|^{2}\left(|x|^{1/2}s(x)^{1/2}\right)^{\delta}} & \text{for } p = 1 \end{cases} \\
&\leqq \frac{C_\delta}{|x|^{3/2}s(x)^{1/2+\delta}}
\end{aligned}$$

because $1 \leqq s(x) \leqq 2|x|$. Combining these estimations implies that

$$L_2 \leqq \frac{C_\delta}{|x|^{3/2}(1+s(x))^{1/2+\delta}} \quad \text{for } |x| \geqq 1 \text{ and } s(x) \geqq 1,$$

which together with (4.29), (4.27) and (4.22) implies (4.20). This completes the proof of the lemma.

5. On the existence of strong solutions to the non-stationary problem : Proofs of Theorems 1.4 and 1.5. Employing the argument due to Kato [29], we shall solve the integral equation (1.9). Recall that $T_{\mathbf{u}_\infty}(t)$ denotes the semigroup generated by the operator $\mathbb{O}(\mathbf{u}_\infty) = \mathbb{P}(-\Delta + (\mathbf{u}_\infty \cdot \nabla))$ with domain $\mathcal{D}_p = \mathbb{J}_p(\Omega) \cap \dot{\mathbb{W}}^1_p(\Omega) \cap \mathbb{W}^2_p(\Omega)$. In particular, $T_{\mathbf{0}}(t)$ is the semigroup generated by the Stokes operator $\mathbb{A} = \mathbb{O}(\mathbf{0}) = \mathbb{P}(-\Delta)$ when $\mathbf{u}_\infty = \mathbf{0}$. The L_p–L_q estimate of $T_{\mathbf{u}_\infty}(t)$ given in the following theorem plays an important role in our proof of Theorems 1.4 and 1.5.

THEOREM 5.1. Let $\sigma_0 > 0$ and assume that $|\mathbf{u}_\infty| \leqq \sigma_0$. (1) If $1 < p \leqq q < \infty$, then

$$\|T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_q \leqq C_{p,q,\sigma_0}\, t^{-\nu}\|\mathbf{a}\|_p, \quad \nu = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right), \quad \forall t > 0, {}^\forall\mathbf{a} \in \mathbb{J}_p(\Omega).$$

(2) If $1 < p < \infty$, then

$$\|T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_\infty \leqq C_{p,\sigma_0}\, t^{-3/2p}\, \|\mathbf{a}\|_p, \quad {}^\forall t \geqq 1,\ {}^\forall \mathbf{a} \in \mathbb{J}_p(\Omega).$$

(3) If $1 < p \leqq q \leqq 3$, then

$$\|\nabla T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_q \leqq C_{p,q,\sigma_0}\, t^{-(\nu+1/2)}\|\mathbf{a}\|_p, \quad {}^\forall t > 0,\ {}^\forall \mathbf{a} \in \mathbb{J}_p(\Omega).$$

(4) If $1 < p \leqq q < \infty$, then

$$\|T_{\mathbf{u}_\infty}(t)\mathbf{a}\|_{q,1} \leqq C_{p,q,\sigma_0}\, t^{-(\nu+1/2)}\, \|\mathbf{a}\|_p, \quad 0 < {}^\forall t \leqq 1,\ {}^\forall \mathbf{a} \in \mathbb{J}_p(\Omega).$$

REMARK 5.2. The assertions (1) and (3) were proved by Iwashita [28] when $\mathbf{u}_\infty = \mathbf{0}$ and by Kobayashi and Shibata [30] when $\mathbf{u}_\infty \neq \mathbf{0}$. The assertions (2) and (4) will be proved in the appendix below. When $\mathbf{u}_\infty = \mathbf{0}$ and $p = 6$, (2) was already proved by Chen [7]. (4) is well known as a property of the analytic semigroup, but the point is that the constant C_{p,q,σ_0} is independent of $\mathbf{u}_\infty$ provided that $|\mathbf{u}_\infty| \leqq \sigma_0$.

To handle with the linear perturbation term $\mathbb{P}[\mathcal{L}[\mathbf{w}]\mathbf{z}]$ in (1.9), we will use the following generalized Poincaré's inequality.

LEMMA 5.3. Let $0 \leqq \alpha < 1/3$ and $s_{\mathbf{u}_\infty}(x) = |x| - {}^{\mathrm{T}}x \cdot \mathbf{u}_\infty/|\mathbf{u}_\infty|$. Put $d_\alpha(x) = s_{\mathbf{u}_\infty}(x)^\alpha |x|^{1-\alpha}\log|x|$. Then, for any $R \geqq 3$ there exists a constant $C_{R,\alpha,\beta}$ independent of $\mathbf{u}_\infty$ such that

$$\int_{|x|\geqq R} \left|\frac{v(x)}{d_\alpha(x)}\right|^3 dx \leqq C_{R,\alpha,\beta}\left\{\int_{|x|\geqq R-1} |\nabla v(x)|^3\, dx + \int_{R-1\leqq|x|\leqq R} |v(x)|^3\right\} \tag{5.1}$$

for any $v \in \hat{W}^1_3(\mathbb{R}^3)$.

Proof. First, we consider the case where $\alpha > 0$. Let $\epsilon > 0$ be a small number and $\rho(\theta)$ a function of $C^\infty(\mathbb{R})$ such that $\rho(\theta) = 1$ for $|\theta| < \epsilon$ and $\rho(\theta) = 0$ for $|\theta| \geqq 2\epsilon$. Let S be an orthogonal matrix such that $S\mathbf{u}_\infty = |\mathbf{u}_\infty|^{\mathrm{T}}(1,0,0)$ and put $y = Sx$. We shall use the polar coordinate (4.4) and the relation (4.5). If we put

$$I(r,\varphi) = \frac{1}{r(\log r)^3}\int_0^\pi \frac{\rho(\theta)|v(x)|^3\sin\theta}{(1-\cos\theta)^{3\alpha}}d\theta,$$

then we have

$$\int_{|x|\geqq R} \left|\frac{v(x)}{d_\alpha(x)}\right|^3 dx \leqq \int_R^\infty \int_0^{2\pi} I(r,\varphi)\, dr d\varphi + \frac{1}{(1-\cos\epsilon)^\alpha}\int_{|x|\geqq R}\left|\frac{v(x)}{|x|\,\log|x|}\right|^3 dx$$

because $1 - \cos\theta \leqq 1 - \cos\epsilon$ when $\epsilon \leqq \theta \leqq \pi$. First of all we shall estimate $I(r,\varphi)$.

Observe that

$$\begin{aligned}
&(1-3\alpha)r(\log r)^3 I(r,\varphi)\\
&= -\int_0^\pi \rho(\theta)(1-\cos\theta)^{1-3\alpha}\frac{\partial}{\partial\theta}|v(x)|^3 d\theta - \int_0^\pi \rho'(\theta)(1-\cos\theta)^{1-3\alpha}|v(x)|^3 d\theta\\
&\leqq 3r\int_0^\pi \rho(\theta)(1-\cos\theta)^{1-3\alpha}|v(x)|^2|\nabla v(x)|d\theta + \int_0^\pi |\rho'(\theta)|(1-\cos\theta)^{1-3\alpha}|v(x)|^3 d\theta\\
&\leqq 3r\left[\int_0^\pi \frac{\rho(\theta)(1-\cos\theta)^{3(1-3\alpha)/2}}{(\sin\theta)^{1/2}}|v(x)|^3 d\theta\right]^{3/2}\left[\int_0^\pi \rho(\theta)\sin\theta|\nabla v(x)|^3 d\theta\right]^{1/3}\\
&\qquad + \int_0^\pi |\rho'(\theta)|(1-\cos\theta)^{1-3\alpha}|v(x)|^3 d\theta.
\end{aligned}$$

Since

$$\frac{\rho(\theta)(1-\cos\theta)^{3(1-3\alpha)/2}}{(\sin\theta)^{1/2}} = \frac{\rho(\theta)\sin\theta}{(1-\cos\theta)^{3\alpha}}\left[\frac{(1-\cos\theta)^{1-\alpha}}{\sin\theta}\right]^{3/2} \leqq C_\epsilon\frac{\rho(\theta)\sin\theta}{(1-\cos\theta)^{3\alpha}}$$

as follows from the fact that $1-\alpha \geqq 1/2$, we have

$$\begin{aligned}
I(r,\varphi) \leqq C_{\epsilon,\alpha} I^{2/3}&\left(\frac{r^2}{(\log r)^3}\int_0^\pi \rho(\theta)\sin\theta|\nabla v(x)|^3 d\theta\right)^{1/3}\\
&+\frac{C_{\epsilon,\alpha}}{r(\log r)^3}\int_0^\pi |\rho'(\theta)|\sin\theta|v(x)|^3 d\theta
\end{aligned}$$

which implies that

$$\int\limits_{|x|\geqq R}\left|\frac{v(x)}{d_\alpha(x)}\right|^3 dx \leqq C_{\epsilon,\alpha}\left[\int\limits_{|x|\geqq R}\left(\frac{|v(x)|}{|x|\log|x|}\right)^3 dx + \frac{1}{(\log R)^3}\int\limits_{|x|\geqq R}|\nabla v(x)|^3 dx\right], \quad (5.2)$$

and hence the proof is reduced to the case where $\alpha = 0$, which is well known but for the completeness we shall give its proof. Let $\psi(r)$ be a function of $C^\infty(\mathbb{R})$ such that $\psi(r) \geqq 0$, $\psi(r) = 1$ for $r \geqq R$ and $\psi(r) = 0$ for $r \leqq R-1$. We use the polar coordinate (4.4) again, and then we have for any large $L > R$

$$\begin{aligned}
\int_R^L \frac{|v(y)|^3 dr}{r(\log r)^3} &\leqq \int_{R-1}^L \frac{\psi(r)|v(y)|^3 dr}{r(\log r)^3}\\
&= -\frac{1}{2}\frac{\psi(r)|v(y)|^3}{(\log r)^2}\Big|_{R-1}^L + \frac{1}{2}\int_{R-1}^L \frac{\frac{\partial}{\partial r}\left[\psi(r)|v(y)|^3\right]}{(\log r)^2}dr\\
&\leqq \frac{1}{2}\left\{\int_{R-1}^L \frac{\psi(r)|v(y)|^2|\nabla v(y)|}{(\log r)^2}dr + \int_{R-1}^L \frac{|\psi'(r)||v(y)|^3}{(\log r)^2}dr\right\}\\
&\leqq \frac{1}{2}\left(\int_{R-1}^L \frac{\psi(r)|v(y)|^3}{r(\log r)^3}dr\right)^{2/3}\left(\int_{R-1}^L \psi(r)|\nabla v(y)|^3 r^2 dr\right)^{1/3}\\
&\qquad + \frac{\max|\psi'(r)|}{(R-1)^2(\log(R-1))^2}\int_{R-1}^R |v(y)|^3 r^2 dr,
\end{aligned}$$

which implies that

$$\int_R^L \frac{|v(y)|^3}{r(\log r)^3} dr \leqq C_R \left\{ \int_{R-1}^L |\nabla v(y)|^3 r^2 dr + \int_{R-1}^R |v(y)|^3 r^2 dr \right\}. \tag{5.3}$$

Integrating (5.3) over S_1 and passing L to infinity, we have (5.1) for $\alpha = 0$, which together with (5.2) completes the proof of the lemma.

By (2.2) with $D = \Omega_{b_0+1}$ and Lemma 5.3, we have the following corollary.

COROLLARY 5.4. Let $0 \leqq \alpha < 1/3$ and let $d_\alpha(x)$ be the same function as in Lemma 5.3. Then, there exists a constant C_α such that

$$\|v/d_\alpha\|_3 \leqq C_\alpha \|\nabla v\|_3 \quad {}^\forall v \in \dot{W}_3^1(\Omega). \tag{5.4}$$

Below, $[\cdot]_{q,\rho,t}$ and $\mu(q)$ are the symbols defined in Theorem 1.4. Employing the argument due to Kato [29, p. 474] and using the fractional power of analytic semigroups and Theorem 5.1, we have the following lemma.

LEMMA 5.5. Let $\mathbf{c} \in \mathbb{J}_3(\Omega)$. Then,

$$t^{1/2}\nabla T_{\mathbf{u}_\infty}(t)\mathbf{c} \in \mathcal{B}([0,\infty);\mathbb{J}_3(\Omega)), \quad t^{\mu(q)}T_{\mathbf{u}_\infty}(t)\mathbf{c} \in \mathcal{B}([0,\infty);\mathbb{J}_q(\Omega)),$$
$$\lim_{t\to 0^+} \left(\|T_{\mathbf{u}_\infty}(t)\mathbf{c} - \mathbf{c}\|_3 + [T_{\mathbf{u}_\infty}(\cdot)\mathbf{c}]_{q,\mu(q),t} + [\nabla T_{\mathbf{u}_\infty}(\cdot)\mathbf{c}]_{3,1/2,t}\right) = 0.$$

Now, we shall give estimations of the right-hand side of (1.9). For notational simplicity, we introduce the following symbol:

$$[[v]]_{p,t} = [v]_{3,0,t} + [\nabla v]_{3,1/2,t} + [v]_{p,\mu(p),t} \quad 3 \leqq p < \infty. \tag{5.5}$$

LEMMA 5.6. Let $3 < p < \infty$ and $0 < \delta < \min(1/6, 4/p)$. Let $[\cdot]_{p,\rho,t}$, $|||\cdot|||_\delta$ and $[[\cdot]]_{p,t}$ be the same as in (1.2), (1.3) and (5.5), respectively. Put

$$L_{\mathbf{w}}(\mathbf{z})(t) = \int_0^t T_{\mathbf{u}_\infty}(t-s)\mathbb{P}[\mathcal{L}[\mathbf{w}]\mathbf{z}(s,\cdot)]ds,$$
$$N(\mathbf{z}_1,\mathbf{z}_2)(t) = \int_0^t T_{\mathbf{u}_\infty}(t-s)\mathbb{P}[(\mathbf{z}_1(s,\cdot)\cdot\nabla)\mathbf{z}_2(s,\cdot)]ds$$

where $\mathcal{L}[\mathbf{w}]\mathbf{z} = ((\mathbf{w}-\mathbf{u}_\infty)\cdot\nabla)\mathbf{z} + (\mathbf{z}\cdot\nabla)\mathbf{w}$ (cf. (1.7)). Then, we have the relations

$$[[L_{\mathbf{w}}(\mathbf{z})]]_{p,t} \leqq C_{p,\delta}\, |||\mathbf{w}-\mathbf{u}_\infty|||_\delta\, [\nabla \mathbf{z}]_{3,1/2,t} \quad {}^\forall t > 0, \tag{5.6}$$
$$[[N(\mathbf{z}_1,\mathbf{z}_2)]]_{p,t} \leqq C_p\, [\mathbf{z}_1]_{p,\mu(p)/2,t}\, [\nabla \mathbf{z}_2]_{3,1/2,t} \quad {}^\forall t > 0. \tag{5.7}$$

Proof. To prove (5.6), let us put $\alpha = \delta + 1/6$, $\gamma = 3\delta/4$ and $\epsilon = 1/(1+\gamma)$. Since $0 < \delta < 1/6$ and $p\delta < 4$, we have

$$0 < 3\delta\epsilon < 1, \ \ 0 < \alpha < 1/3, \ \ \gamma < 3/p, \ \ 0 < \epsilon < 1, \ \ (1+\delta)\epsilon > 1. \tag{5.8}$$

If we put

$$c_1 = \int_{\mathbb{R}^3} \left[(1+|x|)^{-1}\, s_{\mathbf{u}_\infty}(s)^{-\delta}\right]^{3\epsilon} dx,$$
$$c_2 = \int_{\mathbb{R}^3} \left[(1+|x|)^{-(1/2+\alpha)}\, s_{\mathbf{u}_\infty}(x)^{-(1/2+\delta-\alpha)} \log|x|\right]^{3\epsilon} dx$$

then by (4.4) and (4.5) we have

$$c_1 = 2\pi \int_0^\infty \frac{r^2\, dr}{((1+r)\, r^\delta)^{3\epsilon}} \int_0^\pi \frac{\sin\theta\, d\theta}{(1-\cos\theta)^{3\delta\epsilon}},$$
$$c_2 = 2\pi \int_0^\infty \frac{r^2 (\log r)^{3\epsilon}\, dr}{((1+r)^{1/2+\alpha}\, r^{1/3})^{3\epsilon}} \int_0^\pi \frac{\sin\theta d\theta}{(1-\cos\theta)^{\epsilon}}.$$

By (5.8), c_1 and c_2 are positive constants depending essentially only on δ. Therefore, by Hölder's inequality and Corollary 5.4 we have

$$\|((\mathbf{w}-\mathbf{u}_\infty)\cdot\nabla)\mathbf{z}(s,\cdot)\|_{3/(2+\gamma)} \leqq \|\mathbf{w}-\mathbf{u}_\infty\|_{3/(1+\gamma)}\|\nabla\mathbf{z}(s,\cdot)\|_3 \leqq c_1^{1/(3\epsilon)}\, |||\mathbf{w}-\mathbf{u}_\infty|||_\delta \|\nabla\mathbf{z}(s,\cdot)\|_3, \tag{5.9}$$

$$\|(\mathbf{z}(s,\cdot)\cdot\nabla)\mathbf{w}\|_{3/(2+\gamma)} \leqq \|\mathbf{z}(s,\cdot)/d_\alpha\|_3 \|d_\alpha\nabla\mathbf{w}\|_{3/(1+\gamma)} \leqq c_2^{1/(3\epsilon)}\, C_\alpha\, |||\mathbf{w}-\mathbf{u}_\infty|||_\delta \|\nabla\mathbf{z}(s,\cdot)\|_3. \tag{5.10}$$

Also, we have

$$\|((\mathbf{w}-\mathbf{u}_\infty)\cdot\nabla)\mathbf{z}(s,\cdot)\|_3 \leqq |||\mathbf{w}-\mathbf{u}_\infty|||_\delta \|\nabla\mathbf{z}(s,\cdot)\|_3, \tag{5.11}$$
$$\|(\mathbf{z}(s,\cdot)\cdot\nabla)\mathbf{w}\|_3 \leqq \|d_\alpha\nabla\mathbf{w}\|_\infty \|\mathbf{z}(s,\cdot)/d_\alpha\|_3 \leqq C_\delta |||\mathbf{w}-\mathbf{u}_\infty|||_\delta \|\nabla\mathbf{z}(s,\cdot)\|_3. \tag{5.12}$$

When $t \geqq 2$, by Theorem 5.1, Proposition 2.4, (5.9) to (5.12), we have

$$\begin{aligned} t^{1/2}\|\nabla L_{\mathbf{w}}(\mathbf{z})(t)\|_3 &\leqq C t^{1/2}\left\{\int_{t-1}^t (t-s)^{-1/2}\|\mathcal{L}[\mathbf{w}]\mathbf{z}(s,\cdot)\|_3 ds \right.\\ &\quad \left. + \int_0^{t-1} (t-s)^{-(3((2+\gamma)/3-1/3)/2+1/2)}\|\mathcal{L}[\mathbf{w}]\mathbf{z}(s,\cdot)\|_{3/(2+\gamma)} ds\right\} \\ &\leqq C_\delta t^{1/2} |||\mathbf{w}-\mathbf{u}_\infty|||_\delta [\nabla\mathbf{z}]_{3,1/2,t}\left\{\int_{t-1}^t (t-s)^{-1/2} ds (t-1)^{-1/2}\right.\\ &\quad \left. + \int_0^{t/2} s^{-1/2} ds (t/2)^{-(1+\gamma/2)} + \int_{t/2}^{t-1} (t-s)^{-(1+\gamma/2)} ds (t/2)^{-1/2}\right\} \\ &\leqq C_\delta |||\mathbf{w}-\mathbf{u}_\infty|||_\delta [\nabla\mathbf{z}]_{3,1/2,t}. \end{aligned}$$

Also, when $t \geqq 2$, by Theorem 5.1, Proposition 2.4, (5.9) and (5.10) we have

$$\begin{aligned} &\|L_{\mathbf{w}}(\mathbf{z})(t)\|_3 + t^{\mu(p)} \quad \|L_{\mathbf{w}}(\mathbf{z})(t)\|_p \\ &\quad \leqq C_\delta |||\mathbf{w}-\mathbf{u}_\infty|||_\delta [\nabla\mathbf{z}]_{3,1/2,t} \times \\ &\quad \left\{\int_0^t (t-s)^{-3((2+\gamma)/3-1/3)/2} s^{-1/2} ds + t^{\mu(p)} \int_0^t (t-s)^{-3((2+\gamma)/3-1/p)/2} s^{-1/2} ds\right\} \\ &= C_\delta |||\mathbf{w}-\mathbf{u}_\infty|||_\delta [\nabla\mathbf{z}]_{3,1/2,t}\, (B(1/2-\gamma/2, 1/2) + B(3/2p-\gamma/2, 1/2)) \end{aligned}$$

where $B(\alpha,\beta)$ denotes the beta function. When $0<t\leqq 2$, by Theorem 5.1, Proposition 2.4, (5.11) and (5.12),

$$
\begin{aligned}
&\|L_{\mathbf{w}}(\mathbf{z})(t)\|_3 + t^{\mu(p)}\|L_{\mathbf{w}}(\mathbf{z})(t)\|_p + t^{1/2}\|L_{\mathbf{w}}(\mathbf{z})(t)\|_3 \\
&\quad\leqq C_\delta |||\mathbf{w}-\mathbf{u}_\infty|||_\delta [\nabla \mathbf{z}]_{3,1/2,t}\times \\
&\quad\left\{\int_0^t s^{-1/2}ds + t^{\mu(p)}\int_0^t (t-s)^{-3(1/3-1/p)/2}s^{-1/2}ds + t^{1/2}\int_0^t (t-s)^{-1/2}s^{-1/2}ds\right\} \\
&\quad= C_\delta |||\mathbf{w}-\mathbf{u}_\infty|||_\delta [\nabla \mathbf{z}]_{3,1/2,t} t^{1/2}(2+B(1/2+3/2p,1/2)+B(1/2,1/2))
\end{aligned}
$$

Combining these estimations implies (5.6).

(2) Define ℓ by the relation: $1/\ell = 1/3+1/p$. By Proposition 2.4 and Hölder's inequality,

$$
\begin{aligned}
\|\mathbb{P}[\mathbf{z}_1(s,\cdot)\cdot\nabla)\mathbf{z}_2(s,\cdot)]\|_\ell &\leqq C_q\|\mathbf{z}_1(s,\cdot)\|_p\|\nabla\mathbf{z}_2(s,\cdot)\|_3 \\
&\leqq C_p s^{-(1-3/2p)}[\mathbf{z}_1]_{p,\mu(p),s}[\nabla\mathbf{z}_2]_{3,1/2,s}
\end{aligned}
\tag{5.13}
$$

and hence by Theorem 5.1

$$
\begin{aligned}
&\|N(\mathbf{z}_1,\mathbf{z}_2)(t)\|_3 + t^{\mu(p)}\|N(\mathbf{z}_1,\mathbf{z}_2)(t)\|_p + t^{1/2}\|N(\mathbf{z}_1,\mathbf{z}_2)(t)\|_3 \\
&\leqq C_{\delta,p}[\mathbf{z}_1]_{p,\mu(p),t}[\nabla\mathbf{z}_2]_{3,1/2,t}\left\{\int_0^t (t-s)^{-3(1/\ell-1/3)/2}s^{-(\mu(p)+1/2)}ds\right. \\
&\qquad + t^{\mu(p)}\int_0^t (t-s)^{-3(1/\ell-1/p)/2}s^{-(\mu(p)+1/2)}ds \\
&\qquad \left. + t^{1/2}\int_0^t (t-s)^{-3(1/\ell-1/3)/2+1/2)}s^{-(\mu(p)+1/2)}ds\right\} \\
&\leqq C_{\delta,p}[\mathbf{z}_1]_{p,\mu(p),t}[\nabla\mathbf{z}_2]_{3,1/2,t}\times \\
&\qquad (B(1-3/2p,3/2p)+B(1/2,3/2p)+B(1/2-3/2p,3/2p)),
\end{aligned}
$$

which implies (5.7). This completes the proof of the lemma.

Under these preparations, by the contraction mapping principle we shall solve (1.9). Below, p, β and δ are constants given in Theorem 1.4, that is, $p>3$, $0<\delta<\beta<1-\delta$ and $0<\delta<\min(1/6,4/p)$ and all the constants will depend on p, b and δ, but for simplicity we will omit to write this dependence. By Theorem 5.1 there exists a $\sigma>0$ such that if

$$
0<|\mathbf{u}_\infty|\leqq\epsilon\leqq\min(\sigma,1) \tag{A.3}
$$

then (SP) admits solutions $\mathbf{w}$ and $\mathfrak{p}$ satisfying the estimate :

$$
\|\mathbf{w}-\mathbf{u}_\infty\|_{p,2} + |||\mathbf{w}-\mathbf{u}_\infty|||_\delta + \|\mathfrak{p}\|_{p,1}\leqq|\mathbf{u}_\infty|^\beta \tag{5.14}
$$

which in particular implies that

$$
\|\mathbf{w}-\mathbf{u}_\infty\|_3 + \|\nabla\mathbf{w}\|_{3/2}\leqq C_\delta|\mathbf{u}_\infty|^\beta, \tag{5.15}
$$

where

$$C_\delta = \int_{\mathbb{R}^3} (1+|x|)^{-3} s_{\mathbf{u}_\infty}(x)^{-\delta}\,dx + \int_{\mathbb{R}^3} (1+|x|)^{-9/4} s_{\mathbf{u}_\infty}(x)^{-(3/4+\delta)}\,dx$$
$$= 2\pi \int_0^\infty \frac{r^2\,dr}{1+r)^3 r^\delta} \int_0^\pi \frac{\sin\theta\,d\theta}{(1-\cos\theta)^\delta} + 2\pi \int_0^\infty \frac{r^2\,dr}{(1+r)^{9/4} r^{3/4+\delta}} \int_0^\pi \frac{\sin\theta\,d\theta}{(1-\cos\theta)^{3/4+\delta}}.$$

Note that C_δ is independent of $\mathbf{u}_\infty$. According to (1.9), we put $\mathbf{u}_0(t) = T_{\mathbf{u}_\infty}(t)\mathbf{b}$ and $Q(\mathbf{z})(t) = \mathbf{u}_0(t) - L_{\mathbf{w}}(\mathbf{z})(t) - N(\mathbf{z},\mathbf{z})(t)$, where we have used the symbols defined in Lemma 5.6. To solve (1.9) by the contraction mapping principle, we introduce the invariant space $\mathcal{I}$ as follows:

$$\mathcal{I} = \{\mathbf{z} \in C^0((0,\infty); \mathbb{J}_3(\Omega) \cap \mathbb{L}_p(\Omega) \cap \mathbb{W}^1_3(\Omega)) \mid$$
$$[[\mathbf{z}]]_{p,t} \leqq \sqrt{\epsilon} \qquad {}^\forall t > 0, \tag{5.16}$$
$$\lim_{t\to 0+} (\,\|\,\mathbf{z}(t,\cdot) - \mathbf{b}\,\|_3 + [\nabla \mathbf{z}\,]_{3,1/2,t} + [\mathbf{z}\,]_{p,\mu(p),t}) = 0\,\}. \tag{5.17}$$

Let q denote any number $\geqq 3$, below. By (1) and (3) of Theorem 5.1, we have

$$[[\mathbf{u}_0]]_{p,t} \leqq M_1 \|\mathbf{b}\,\|_3, \quad [\mathbf{u}_0]_{q,\mu(q),t} \leqq C_q \|\mathbf{b}\,\|_3 \tag{5.18}$$

for any $t > 0$ where M_1 is a constant depending only on p, essentially. If $\|\mathbf{b}\|_3 \leqq \epsilon$, then we choose $\epsilon > 0$ so small that

$$M_1 \sqrt{\epsilon} \leqq 1, \tag{A.4}$$

and hence by (5.18) and Lemma 5.5 we see that $\mathbf{u}_0 \in \mathcal{I}$. In particular, $\mathcal{I}$ is not empty. By Lemma 5.6, (5.18), (5.16) and (5.14),

$$[[Q(\mathbf{z})]]_{p,t} \leqq M_1\,\epsilon + C_{p,\delta}\,|\mathbf{u}_\infty|^\beta \sqrt{\epsilon} + C_p\,\epsilon \qquad {}^\forall t > 0$$

provided that $\mathbf{z} \in \mathcal{I}$ and $\|\mathbf{b}\|_3 \leqq \epsilon$. Since $|\mathbf{u}_\infty| \leqq \epsilon$, if we choose $\epsilon > 0$ so small that

$$M_1 \sqrt{\epsilon} + C_{p,\delta}\,\epsilon^\beta + C_p \sqrt{\epsilon} \leqq 1, \tag{A.5}$$

we have $[[Q(\mathbf{z})]]_{p,t} \leqq \sqrt{\epsilon}$, ${}^\forall t > 0$, which together with Lemma 5.5 implies that $Q(\mathbf{z}) \in \mathcal{I}$ for any $\mathbf{z} \in \mathcal{I}$. Since $Q(\mathbf{z}_1)(t) - Q(\mathbf{z}_2)(t) = -\{L_{\mathbf{w}}(\mathbf{z}_1 - \mathbf{z}_2) + N(\mathbf{z}_1 - \mathbf{z}_2, \mathbf{z}_1) + N(\mathbf{z}_2, \mathbf{z}_1 - \mathbf{z}_2)\}$, by Lemma 5.6, (5.14) and (5.16) we have

$$[[Q(\mathbf{z}_1) - Q(\mathbf{z}_2)]]_{p,t} \leqq M_2(\epsilon^\beta + 2\sqrt{\epsilon})[[\mathbf{z}_1 - \mathbf{z}_2]]_{p,t}$$

for some M_2 independent of $\mathbf{u}_\infty$ provided that $\mathbf{z}_1$, $\mathbf{z}_2 \in \mathcal{I}$ and $|\mathbf{u}_\infty| \leqq \epsilon$. If we choose $\epsilon > 0$ so small that

$$M_2(\epsilon^\beta + 2\sqrt{\epsilon}) \leqq 1/2, \tag{A.6}$$

we see that Q is a contraction, and hence Q has a unique fixed point $\mathbf{z} \in \mathcal{I}$, from which Theorem 1.4 follows except for (1.14).

Now, we shall show (1.14) for any q. Since we have already proved (1.14) for $q = 3$ and $q = p$, by the interpolation we see that (1.14) holds for $3 \leqq q \leqq p$. Therefore, we may assume that $p < q < \infty$. Let $\gamma > 0$ be the same as in (5.8) and put $r = 3/(2+\gamma)$. Note that $1 < r < 3/2$. By (5.9) to (5.12) and (5.14) we know that

$$\|\mathbb{P}[\mathcal{L}[\mathbf{w}]\mathbf{z}(s,\cdot)]\|_n \leqq C_n\,|\mathbf{u}_\infty|^\beta\, s^{-1/2}\,[\nabla \mathbf{z}]_{3,1/2,s} \qquad r \leqq {}^\forall n \leqq 3. \tag{5.19}$$

When $3 < q < \infty$, by (5.19) with $n = 3/2$ and (1) of Theorem 5.1 we have

$$\begin{aligned} \|L_{\mathbf{w}}(\mathbf{z})(t)\|_q &\leqq C_q \int_0^t (t-s)^{-3(2/3-1/q)/2}\, s^{-1/2}\, ds\, |\mathbf{u}_\infty|^\beta\, [\nabla \mathbf{z}]_{3,1/2,t} \\ &\leqq C_q\, B(3/2q, 1/2)\, |\mathbf{u}_\infty|^\beta\, t^{-\mu(q)}\, \sqrt{\epsilon} \end{aligned} \tag{5.20}$$

where we have used (5.16). To estimate $N(\mathbf{z},\mathbf{z})$, in view of (5.13) let ℓ be a number such that $1/\ell = 1/3 + 1/p$. Since $\|(\mathbf{z}(s,\cdot)\cdot\nabla)\mathbf{z}(s,\cdot)\|_\ell \leqq s^{-(1-3/2p)}\,\epsilon$ as follows from (5.13) and (5.16), by (1) of Theorem 5.1 we have

$$\begin{aligned} \|N(\mathbf{z},\mathbf{z})(t)\|_q &\leqq C_q \int_0^t (t-s)^{-3(1/\ell-1/q)/2}\, s^{-(1-3/2p)}\, ds\, \epsilon \\ &\leqq C_q\, B(3(1/q-1/p)/2 + 1/2, 3/2p)\, \epsilon\, t^{-\mu(q)} \end{aligned} \tag{5.21}$$

for any $t > 0$. Combining (5.18), (5.20) and (5.21), we have (1.14) for $p < q < \infty$. Finally, we shall show (1.14) for $q = \infty$. Since we do not know the L_∞–L_p estimate of $T_{\mathbf{u}_\infty}(t)$ for small $t > 0$, we have to use the Sobolev's imbedding theorem and (4) of Theorem 5.1 to estimate $T_{\mathbf{u}_\infty}(t)$ for small $t > 0$, so that let m be a fixed number such that $3 < m < p$. We shall always use the relation $\|v\|_\infty \leqq C_m\|v\|_{m,1}$ in our treatment for small $t > 0$, below. Keeping this in mind, by (2) and (4) of Theorem 5.1 and (5.19) with $n = 3$ and $n = r$ we have

$$\|L_{\mathbf{w}}(\mathbf{z})(t)\|_\infty \leqq C_m|\mathbf{u}_\infty|^\beta\,\sqrt{\epsilon}\,(\chi(t)\,t^{-\mu(m)} + (1-\chi(t))t^{-1/2}) \quad {}^\forall t > 0. \tag{5.22}$$

Here and hereafter, we put $\chi(t) = 1$ for $t \leqq 1$ and $\chi(t) = 0$ for $t \geqq 1$. In fact, (5.22) follows from the relations

$$\begin{aligned} &\int_0^t (t-s)^{-(3(1/3-1/m)/2+1/2)}\, s^{-1/2}\, ds = B(3/2m, 1/2)\, t^{-\mu(m)} && {}^\forall t > 0; \\ &\int_0^{t-1} (t-s)^{-3/(2r)}\, s^{-1/2}\, ds \leqq C_r t^{-1/2} && {}^\forall t \geqq 1, \end{aligned}$$

where we have used the fact that $3/(2r) > 1$. By (5.13), (5.16) and (2) and (4) of Theorem 5.1 we have also

$$\|N(\mathbf{z},\mathbf{z})(t)\|_\infty \leqq C\,\epsilon\left(\chi(t)\,t^{-(1-3/(2m))} + (1-\chi(t))\,t^{-1/2}\right) \quad {}^\forall t > 0. \tag{5.23}$$

In fact, since $\|(\mathbf{z}(s,\cdot)\cdot\nabla)\mathbf{z}(s,\cdot)\|_\ell \leqq C\,\epsilon\, s^{-(1-3/(2p))}$, (5.23) follows from the relations

$$\begin{aligned} &\int_0^t (t-s)^{-(3(1/\ell-1/m)/2+1/2)}\, s^{-(1-3/(2p))}\, ds \\ &\qquad = B(3(1/m-1/p)/2, 3/(2p))\, t^{-(1-3/(2m))} \qquad {}^\forall t > 0; \\ &\int_0^{t-1} (t-s)^{-3/(2\ell)}\, s^{-(1-3/(2p))}\, ds \leqq B(1/2-3/(2p), 3/(2p))\, t^{-1/2} \qquad {}^\forall t \geqq 1, \end{aligned}$$

where we have used the fact that $m < p$ to obtain the fact that $3(1/m - 1/p)/2 > 0$ in the beta function. Since

$$\|\mathbf{u}_0(t,\cdot)\|_\infty \leqq C\left(\chi(t)t^{-(1-3/(2m))} + (1-\chi(t))\,t^{-1/2}\right)\|\mathbf{b}\|_3 \quad {}^\forall t > 0$$

as follows immediately from Theorem 5.1 and Sobolev's imbedding theorem, we have (1.14) for $q = \infty$. This completes the proof of Theorem 1.4.

A proof of Theorem 1.5. Let $\mathbf{w}_{\mathbf{u}_\infty}$ be a solution of (SP) in the case that $\mathbf{f}(x) = \mathbf{g}(x) = \mathbf{0}$, and let $\mathbf{z}_{\mathbf{u}_\infty}$ be a solution of the integral equation (1.9) in the case that $\mathbf{w} = \mathbf{w}_{\mathbf{u}_\infty}$ and that $\mathbf{b}$ is replaced by $\mathbf{b} + \mathbf{u}_\infty - \mathbf{w}_{\mathbf{u}_\infty}$. If we put $\mathbf{v}_{\mathbf{u}_\infty} = \mathbf{w}_{\mathbf{u}_\infty} + \mathbf{z}_{\mathbf{u}_\infty}$, then by Theorem 1.4 and Iwashita's result [28 ,Theorem 1.4] we see easily that the statement of Theorem 1.5, except for (1.15), is valid. Therefore, we shall show (1.15) only, below. The argument below is almost the same as in the proof of Theorem 1.4. For notational simplicity, we write $\mathbf{z} = \mathbf{z}_{\mathbf{u}_\infty}$ and $\mathbf{w} = \mathbf{w}_{\mathbf{u}_\infty}$ for $\mathbf{u}_\infty \neq \mathbf{0}$. Since $\mathbf{w}_\mathbf{0} = \mathbf{0}$, if we put $\mathbf{v} = \mathbf{z} - \mathbf{z}_\mathbf{0}$, then by (1.9) we have

$$\begin{aligned}\mathbf{v}(t) = T_{\mathbf{u}_\infty}(t)(\mathbf{u}_\infty - \mathbf{w}) + (T_{\mathbf{u}_\infty}(t) - T_\mathbf{0}(t))\mathbf{b}\\ - \mathbb{L}_\mathbf{w}[\mathbf{z}](t) - N[\mathbf{v},\mathbf{z}](t) - N[\mathbf{z}_\mathbf{0},\mathbf{v}](t) - I(t)\end{aligned} \tag{5.24}$$

where

$$I(t) = \int_0^t (T_{\mathbf{u}_\infty}(t-s) - T_\mathbf{0}(t))\,\mathbf{b}\,ds = -\int_0^t T_{\mathbf{u}_\infty}(t-s)\,\mathbb{P}[(\mathbf{u}_\infty\cdot\nabla)T_\mathbf{0}(s)\mathbf{b}]\,ds. \tag{5.25}$$

By Theorem 1.1 we know that $|||\mathbf{u}_\infty - \mathbf{w}|||_\delta \leqq |\mathbf{u}_\infty|^\beta$, which implies that $\|\mathbf{u}_\infty - \mathbf{w}\|_3 \leqq C|\mathbf{u}_\infty|^\beta$. By Theorem 5.1, we have

$$\begin{aligned}\|T_{\mathbf{u}_\infty}(t)(\mathbf{u}_\infty - \mathbf{w})\|_q &\leqq C_q\,t^{-\mu(q)}|\mathbf{u}_\infty|^\beta \qquad 3 \leqq {}^\forall q < \infty,\\ \|\nabla T_{\mathbf{u}_\infty}(t)(\mathbf{u}_\infty - \mathbf{w})\|_3 &\leqq C\,t^{-1/2}\,|\mathbf{u}_\infty|^\beta,\\ \|T_{\mathbf{u}_\infty}(t)(\mathbf{u}_\infty - \mathbf{w})\|_\infty &\leqq C_m\,\omega(t)\,|\mathbf{u}_\infty|^\beta\end{aligned} \tag{5.26}$$

for any $t > 0$. Here and hereafter, we put

$$\omega(t) = \chi(t)t^{-(1-3/(2m))} + (1-\chi(t))t^{-(1/2-3/(2m))}.$$

Applying Theorem 5.1 to (5.25), we have

$$\begin{aligned}\|(T_{\mathbf{u}_\infty}(t) - T_\mathbf{0}(t))\mathbf{b}\|_q &\leqq C_q\,|\mathbf{u}_\infty|\|\mathbf{b}\|_3 t^{3/(2q)} \qquad 3 \leqq {}^\forall q < \infty,\\ \|\nabla(T_{\mathbf{u}_\infty}(t) - T_\mathbf{0}(t))\mathbf{b}\|_3 &\leqq C\,|\mathbf{u}_\infty|\|\mathbf{b}\|_3,\\ \|(T_{\mathbf{u}_\infty}(t) - T_\mathbf{0}(t))\mathbf{b}\|_\infty &\leqq C_m\,|\mathbf{u}_\infty|\|\mathbf{b}\|_3\omega(t)\end{aligned} \tag{5.27}$$

for any $t > 0$. By (5.20), Lemma 5.6 and (5.22), we have

$$\begin{aligned}\|L_\mathbf{w}[\mathbf{z}](t)\|_q &\leqq C_q\,t^{-(\mu(q)}|\mathbf{u}_\infty|^\beta\sqrt{\epsilon} \qquad 3 \leqq {}^\forall q < \infty,\\ \|\nabla L_\mathbf{w}[\mathbf{z}](t)\|_3 &\leqq C\,t^{-1/2}\,|\mathbf{u}_\infty|^\beta\sqrt{\epsilon},\\ \|L_\mathbf{w}[\mathbf{z}](t)\|_\infty &\leqq C_m\,\omega(t)\,|\mathbf{u}_\infty|^\beta\sqrt{\epsilon}\end{aligned} \tag{5.28}$$

for any $t > 0$. Here and hereafter, we use (1.10) and (1.14) to estimate $\mathbf{z} = \mathbf{z}_{\mathbf{u}_\infty}$ and $\mathbf{z_0}$. For simplicity, we put

$$\{\mathbf{v}\}_{q,\rho_1,\rho_2,t} = \sup_{0<s\leqq t} \chi(s)\, s^{\rho_1} \|\mathbf{v}(s,\cdot)\|_q + \sup_{0<s\leqq t} (1-\chi(s))\, s^{\rho_2} \|\mathbf{v}(s,\cdot)\|_q.$$

By using this notation, we put

$$\{\{\mathbf{v}\}\}_{p,t} = \{\mathbf{v}\}_{3,1/2,0,t} + \{\mathbf{v}\}_{p,\mu(p),-3/(2p),t} + \{\mathbf{v}\}_{3,0,-1/2,t}.$$

Then we have the relations

$$\begin{aligned} \{N[\mathbf{v},\mathbf{z}]\}_{q,\mu(q),-3/(2q),t} + \{N[\mathbf{z},\mathbf{v}_0]\}_{q,\mu(q),-3/(2q),t} &\leqq C_{p,q}\sqrt{\epsilon}\, \{\{\mathbf{v}\}\}_{p,t} \\ \{\nabla N[\mathbf{v},\mathbf{z}]\}_{3,1/2,0,t} + \{\nabla N[\mathbf{z_0},\mathbf{v}]\}_{3,1/2,0,t} &\leqq C_{p,q}\sqrt{\epsilon}\, \{\{\mathbf{v}\}\}_{p,t}, \\ \{N[\mathbf{v},\mathbf{z}]\}_{\infty,1-3/(2m),0,t} + \{N[\mathbf{z_0},\mathbf{v}]\}_{\infty,1-3/(2m),0,t} &\leqq C_m\sqrt{\epsilon}\, \{\{\mathbf{v}\}\}_{p,t} \end{aligned} \tag{5.29}$$

for any $t > 0$, where $N[\cdot,\cdot](t)$ is the same as in Lemma 5.6. To obtain (5.29), we have used the relation

$$\begin{aligned} &\|(\mathbf{v}(s,\cdot)\cdot\nabla)\mathbf{z}(s,\cdot)\|_\ell + \|(\mathbf{z_0}(s,\cdot)\cdot\nabla)\mathbf{v}(s,\cdot)\|_\ell \\ &\quad\leqq \{\{\mathbf{v}\}\}_{p,t}([\nabla\mathbf{z}]_{3,1/2,s} + [\mathbf{z}_0]_{p,\mu(p),s}) \left(\chi(s)\, s^{-(1-3/(2p))} + (1-\chi(s))\, s^{-(1/2-3/(2p))}\right) \end{aligned}$$

and the fact that $[\nabla\mathbf{z}\,]_{3,1/2,s} \leqq \sqrt{\epsilon}$ and $[\mathbf{z}_0]_{p,\mu(p),s} \leqq \sqrt{\epsilon}$. Finally, applying Theorem 5.1 to (5.28) we have

$$\begin{aligned} \|I(t)\|_q &\leqq C_q|\mathbf{u}_\infty|\,\epsilon\, t^{3/(2q)} \qquad 3 \leqq {}^\forall q < \infty, \\ \|\nabla I(t)\|_3 &\leqq C|\mathbf{u}_\infty|\,\epsilon, \\ \|I(t))\|_\infty &\leqq C|\mathbf{u}_\infty|\,\epsilon\,\omega(t) \end{aligned} \tag{5.30}$$

for any $t > 0$. In fact, to show (5.30) we use the relation

$$\|(\mathbf{z_0}(s,\cdot)\cdot\nabla)\mathbf{z_0}(s,\cdot)\|_\ell \leqq s^{-(1-3/(2p))}\,\epsilon,$$

which follows from (5.13) and the facts that $[\nabla\mathbf{z_0}]_{3,1/2,t} \leqq \sqrt{\epsilon}$ and $[\mathbf{z_0}]_{p,\mu(p),t} \leqq \sqrt{\epsilon}$ for any $t > 0$. Then, the first inequality in (5.30) follows from the relation

$$\int_0^t \left(\int_0^{t-s} (t-s-r)^{-3(1/\ell-1/q)/2}\, r^{-1/2}\, dr\right) s^{-(1-3/(2p))}\, ds \leqq C_{p,q}\, t^{3/(2q)} \quad t > 0.$$

The second inequality in (5.30) follows from the relation

$$\int_0^t \left(\int_0^{t-s} (t-s-r)^{-3((1/\ell-1/q)/2+1/2)}\, r^{-1/2}\, dr\right) s^{-(1-3/(2p))}\, ds \leqq C_{p,q} \quad t > 0.$$

The third inequality in (5.30) follows from the relations

$$\int_0^t \left(\int_0^{t-s} (t-s-r)^{-3((1/\ell-1/m)/2+1/2)} r^{-1/2}\, dr \right) s^{-(1-3/(2p))}\, ds \leqq C_{p,m}\, t^{-(1/2-3/(2m))} \quad t>0$$

$$\int_0^t \left(\int_0^{t-s} (t-s-r)^{-3/\ell} r^{-1/2}\, dr \right) s^{-(1-3/(2p))}\, ds \leqq C_p \qquad t \geqq 1.$$

Combining (5.26)–(5.30), we have

$$\{\{\mathbf{v}\}\}_{p,t} \leqq C\left\{ |\mathbf{u}_\infty|^\beta + |\mathbf{u}_\infty| \|\mathbf{b}\|_3 + |\mathbf{u}_\infty|\epsilon + \sqrt{\epsilon}\{\{\mathbf{v}\}\}_{p,t} \right\}$$

and hence choosing $\epsilon > 0$ so small that $C\sqrt{\epsilon} \leqq 1/2$, we have

$$\{\{\mathbf{v}\}\}_{p,t} \leqq C|\mathbf{u}_\infty|^\beta \tag{5.31}$$

because $|\mathbf{u}_\infty| \leqq 1$ and $\|\mathbf{b}\|_3 \leqq \epsilon \leqq 1$. Inserting (5.31) into (5.29), by (5.26) to (5.30) we have (1.15) which completes the proof of Theorem 1.5.

Appendix. L_∞–L_p decay estimate of $T_{\mathbf{u}_\infty}(t)$. In this appendix we shall show (2) and (4) in Theorem 5.1. By Kobayashi and Shibata [30, (4.26)], we know that

$$|\lambda| \|(\mathbb{O}(\mathbf{u}_\infty) + \lambda \mathbb{I})^{-1}\mathbf{f}\|_p + \|(\mathbb{O}(\mathbf{u}_\infty) + \lambda \mathbb{I})^{-1}\mathbf{f}\|_{p,2} \leqq C_{p,\sigma_0}\|\mathbf{f}\|_p \quad {}^\forall \mathbf{f} \in \mathbb{J}_p(\Omega)$$

provided that $|\mathbf{u}_\infty| \leqq \sigma_0$, $|\lambda| \geqq R_0$ and $|\arg \lambda| < \pi - \delta_0$ for some $R_0 > 0$ and $0 < \delta_0 < \pi/2$. Therefore, employing the argument in Pazy [39, Theorem 6.13] we have (4) of Theorem 5.1. In order to prove (2) of Theorem 5.1, we put $\mathbf{u}(t,\cdot) = T_{\mathbf{u}_\infty}(t+1)\mathbf{a}$. Then, by Kobayashi and Shibata [30, (6.18) and (6.27)] when $\mathbf{u}_\infty \neq \mathbf{0}$ and by Iwashita [28, Lemmas 5.3 and 5.4] when $\mathbf{u}_\infty = \mathbf{0}$, we know that

$$\|\mathbf{u}(t,\cdot)\|_{p,2m,\Omega_b} + \|\partial_t \mathbf{u}(t,\cdot)\|_{p,2m,\Omega_b} + \|\mathfrak{p}(t,\cdot)\|_{p,2m,\Omega_b} \leqq C_{p,m,b,\sigma_0}(1+t)^{-3/(2p)}\|\mathbf{a}\|_p \tag{Ap.1}$$

for any $t \geqq 0$ and integer $m \geqq 0$ where $\mathfrak{p}$ is the pressure associated with $\mathbf{u}$, that is, $\mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u} + \nabla\mathfrak{p} = \mathbf{0}$, and b is a fixed constant $> b_0 + 3$. By Sobolev's imbedding theorem and (Ap.1), we have

$$\|\mathbf{u}(t,\cdot)\|_{\infty,\Omega_b} \leqq C_{p,b,\sigma_0}(1+t)^{-3/(2p)}\, \|\mathbf{a}\|_p. \tag{Ap.2}$$

Therefore, our task is to estimate $\mathbf{u}(t,x)$ for $|x| \geqq b$.

Let $\psi \in C^\infty(\mathbb{R}^3)$ be such that $\psi(x) = 0$ for $|x| \leqq b-2$ and $\psi(x) = 1$ for $|x| \geqq b-1$ and put

$$\begin{aligned}
\mathbf{z}(t,\cdot) &= \psi\, \mathbf{u}(t,\cdot) - \mathbb{B}[(\nabla\psi)\cdot \mathbf{u}(t,\cdot)], \\
\mathbf{e} &= \psi\, T_{\mathbf{u}_\infty}(1)\mathbf{a} - \mathbb{B}[(\nabla\psi)\cdot T_{\mathbf{u}_\infty}(1)\mathbf{a}], \\
\mathbf{h}(t,\cdot) &= -\{(\nabla\psi)\mathfrak{p}(t,\cdot) + 2(\nabla\psi) : \nabla\mathbf{u}(t,\cdot) + (\Delta\psi)\mathbf{u}(t,\cdot) - ((\mathbf{u}_\infty\cdot\nabla)\psi)\mathbf{u}(t,\cdot) \\
&\quad + (\partial_t - \Delta + (\mathbf{u}_\infty\cdot\nabla))\mathbb{B}[(\nabla\psi)\cdot\mathbf{u}(t,\cdot)]\}.
\end{aligned}$$

By Proposition 2.3 and (Ap.1), we have

$$\begin{cases} \mathbf{z}_t - \Delta\,\mathbf{z} + (\mathbf{u}_\infty \cdot \nabla)\,\mathbf{z} + \nabla(\psi\mathfrak{p}) = \mathbf{h}, \ \ \nabla \cdot \mathbf{z} = 0 & \text{in } (0,\infty)\times\mathbb{R}^3, \\ \mathbf{z}(0,x) = \mathbf{e}(x) & \text{in } \mathbb{R}^3, \end{cases} \tag{Ap.3}$$

$$\begin{cases} |\mathbf{h}(t,\cdot)|_{p,2m-1} \leqq C_{p,m,b,\sigma_0}(1+t)^{-3/(2p)}\|\mathbf{a}\|_p & \forall m \geqq 1, \\ \|\mathbf{e}|_{p,2m} \leqq C_{p,m,\sigma_0}\|\mathbf{a}\|_p & \forall m \geqq 0, \end{cases} \tag{Ap.4}$$

$$\begin{cases} \mathbf{z}(t,x) = \mathbf{u}(t,x) \qquad \text{for } |x| \geqq b-1, \\ \operatorname{supp}\mathbf{h}(t,\cdot) \subset D_{b-1}. \end{cases} \tag{Ap.5}$$

Let $S_{\mathbf{u}_\infty}(t)$ denote the semigroup generated by $\mathbb{O}(\mathbf{u}_\infty)$ on $\mathbb{J}_p(\mathbb{R}^3)$, that is,

$$S_{\mathbf{u}_\infty}(t)\mathbf{f} = \left(\frac{1}{4\pi\,t}\right)^{3/2} \int_{\mathbb{R}^3} e^{-|x-t\mathbf{u}_\infty - y|^2/(4t)}\,\mathbb{P}_0\,\mathbf{f}(y)\,dy,$$

where we have put $\mathbb{P}_0 = \mathbb{P}_{\mathbb{R}^3}$ for notational simplicity. By Young's inequality and the $L_p(\mathbb{R}^3)$ boundedness of Riesz's transform, we see easily that

$$|\partial_x^\alpha\, S_{\mathbf{u}_\infty}(t)\,\mathbf{f}|_q \leqq C_{p,q,\sigma_0} t^{-(\nu+|\alpha|/2)}|\mathbf{f}|_p \qquad \forall\alpha \tag{Ap.6}$$

where $1 < p \leqq q \leqq \infty$ and $\nu = 3(1/p - 1/q)/2$. Since $\nabla \cdot \mathbf{e} = 0$, applying Duhamel's principle to (Ap.3), we have

$$\mathbf{z}(t,\cdot) = S_{\mathbf{u}_\infty}(t)\mathbf{e} + \mathbf{z}_1(t,\cdot), \quad \mathbf{z}_1(t,\cdot) = \int_0^t S_{\mathbf{u}_\infty}(t-s)\mathbf{h}(s,\cdot)\,ds.$$

By (Ap.6) and (Ap.4) we have

$$|S_{\mathbf{u}_\infty}(t)\mathbf{e}|_\infty \leqq C_{p,\sigma_0}\, t^{-3/(2p)}\|\mathbf{a}\|_p \quad \forall t > 0. \tag{Ap.7}$$

When $t \geqq 1$, we observe that

$$|\mathbf{z}_1(t,\cdot)|_\infty \leqq \int_{t-1}^t (t-s)^{-3/(2q)}|\mathbf{h}(s,\cdot)|_q\,ds + \int_0^{t-1} (t-s)^{-3/(2r)}|\mathbf{h}(s,\cdot)|_r\,ds \tag{Ap.8}$$

where q and r are suitable numbers such that $q > 3/2$ and $1 < r < 3/2$. By Sobolev's imbedding theorem and the fact that supp $\mathbf{h}(t,x)$ is compact (cf. (Ap.5)), by (Ap.4) we have

$$|\mathbf{h}(s,\cdot)|_q, \ \ |\mathbf{h}(s,\cdot)|_r \leqq C(1+s)^{-3/(2p)}\|\mathbf{a}\|_p. \tag{Ap.9}$$

Applying (Ap.9) to (Ap.8) we see easily that

$$|\mathbf{z}_1(t,\cdot)|_\infty \leqq C(1+t)^{-3/(2p)}\|\mathbf{a}\|_p \quad t \geqq 1. \tag{Ap.10}$$

When $0 \leqq t \leqq 1$, we take $q > \max(3,p)$, and then by Sobolev's imbedding theorem, (Ap.6) and (Ap.4) we have

$$\begin{aligned} |\mathbf{z}_1(t,\cdot)|_\infty &\leqq C_p|\mathbf{z}_1(t,\cdot)|_{q,1} \\ &\leqq C_q \int_0^t (t-s)^{-1/2}|\mathbf{h}(s,\cdot)|_q\,ds \\ &\leqq C_{p,q} \int_0^t (t-s)^{-1/2}(1+s)^{-3/(2p)}\,ds\,\|\mathbf{a}\|_p \\ &\leqq C_{p,q}\sqrt{t}\|\mathbf{a}\|_p \leqq C_{p,q}\|\mathbf{a}\|_p. \end{aligned} \tag{Ap.11}$$

Since $\mathbf{u}(t,x) = \mathbf{z}(t,x)$ for $|x| \geqq b$, combining (Ap.2), (Ap.7), (Ap.10), and (Ap.11) implies (2) of Theorem 5.1, which completes the proof.

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