# ON AN INEQUALITY OF LYAPUNOV

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Nehari [1] claims that if an interval $[a, b]$ contains $n$ zeros of a nontrivial solution of

$$y^{(n)} + p_n y^{(n-1)} + \cdots + p_1 y = 0 \tag{1}$$

then

$$\sum_{k=1}^{n} 2^k (b-a)^{n-k} \int_a^b |p_k| \, dt \geqq 2^{n+1}. \tag{2}$$

According to Fink and St. Mary [2] the proof of (2) given in [1] is incorrect, and therefore the inequality is, as yet, undecided. For $n=2$ and $p_2 \equiv 0$, (2) is known to be correct and also the best possible. This case was first proved by Lyapunov and is generally referred to as Lyapunov's theorem. A recent proof of this result may be found in Hochstadt [3]. In [2] a similar technique is used to prove (2) for $n=2$, where $p_2$ is merely integrable on $[a, b]$. In fact, one can extract from [2] the inequality

$$\left[(b-a)\int_a^b |p_1| \, dt\right]^{1/2} + \frac{1}{2}\int_a^b |p_2| \, dt \geqq 2. \tag{3}$$

The purpose of this note is to provide a generalization of (3) to certain $n$th order differential equations.

THEOREM. *Consider the differential equation*

$$y^{(n)} - p y^{(n-1)} - q y = 0, \qquad n \geqq 2, \tag{4}$$

*where $p$ and $q$ are integrable on $[a, b]$. Suppose that a nontrivial solution of* (4) *has at least $n$ zeros on $[a, b]$. Then*

$$\left[(b-a)^{n-1}\int_a^b |q| \, dt\right]^{1/n} + \frac{1}{n}\int_a^b |p| \, dt \geqq 2. \tag{5}$$

Clearly for $n=2$, (5) reduces to (3). In order to prove (5) we reduce (4) to a system, by letting

$$x_i = y^{(i-1)}, \qquad i = 1, 2, \cdots, n,$$

so that

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$$x_i' = x_{i+1}, \qquad i = 1, 2, \cdots, n-1, \tag{6}$$
$$x_n' = px_n + qx_1.$$

Since $y$ vanishes $n$ times on $[a, b]$, each $x_i$ vanishes at least once on that interval. We can, therefore, split $[a, b]$ into two subintervals $[a, c]$ and $[c, b]$ where $a<c<b$ such that on each, each $x_i$ vanishes at least once.

First we shall consider the interval $[a, c]$, and let $\bar{x}_i$ denote the maximum of $|x_i|$ on that interval. Using (6) and the fact that each $x_i$ vanishes at some point on $[a, c]$ we have

$$\bar{x}_i \leqq \bar{x}_{i+1}(c-a), \qquad i = 1, 2, \cdots, n-1, \tag{7}$$

$$|x_n| \leqq \bar{x}_1 \int_a^c |q|\,dt + \int_t^c |p|\,|x_n|\,dt, \tag{8}$$

where $x_n(c)=0$. From (7) we see that

$$\bar{x}_1 \leqq \bar{x}_n(c-a)^{n-1}$$

and combined with (8) we finally have

$$|x_n| \leqq \bar{x}_n(c-a)^{n-1} \int_a^c |q|\,dt + \int_t^c |p|\,|x_n|\,dt. \tag{9}$$

From (9) by means of Gronwall's inequality [4, p. 24] we find that

$$\bar{x}_n \leqq \bar{x}_n(c-a)^{n-1} \int_a^c |q|\,dt \exp \int_a^c |p|\,dt$$

and finally

$$\int_a^c |q|\,dt \geqq \frac{\exp \int_a^c |p|\,dt}{(c-a)^{n-1}}. \tag{10}$$

Similarly

$$\int_c^b |q|\,dt \geqq \frac{\exp \int_c^b |p|\,dt}{(b-c)^{n-1}}. \tag{11}$$

We can combine (10) and (11) and use the inequality

$$A^n/a^{n-1} + B^n/b^{n-1} \geqq (A+B)^n/(a+b)^{n-1}$$

to obtain

$$(b-a)^{n-1}\int_a^b |q|\,dt \geqq \left[\exp\left(-\frac{1}{n}\int_a^c |p|\,dt\right) + \exp\left(-\frac{1}{n}\int_c^b |p|\,dt\right)\right]^n. \tag{12}$$

(12) is an interesting inequality in its own right and for $n=2$ is also stated in [2]. In order to derive the simpler and desired inequality (5) we use the fact that

$$\exp(-x) \geqq 1-x$$

in (12) and extract the $n$th root of both sides. Then

$$\left[(b-a)^{n-1}\int_a^b |q|\,dt\right]^{1/n} \geqq 2-\frac{1}{n}\left(\int_a^c |p|\,dt+\int_c^b |p|\,dt\right) = 2-\frac{1}{n}\int_a^b |p|\,dt,$$

which is equivalent to (5).

## Bibliography

1. Z. Nehari, *On an inequality of Lyapunov*, Studies in Mathematical Analysis and Related Topics, Stanford Univ. Press, Stanford, Calif., 1962, pp. 256–261.

2. A. M. Fink and D. F. St. Mary, *On an inequality of Nehari*, Proc. Amer. Math. Soc. **21** (1969), 640–642.

3. H. Hochstadt, *A new proof of a stability estimate of Lyapunov*, Proc. Amer. Math. Soc. **14** (1963), 525–526.

4. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

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