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Published on: 01 Jan 1980 - Journal of Mathematical Physics (American Institute of Physics)

Topics: Simple Lie group, Infinitesimal transformation, Representation theory, Ladder operator and Lie group

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Citation: Journal of Mathematical Physics **21**, 444 (1980); doi: 10.1063/1.524441
View online: http://dx.doi.org/10.1063/1.524441
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/21/3?ver=pdfcov
Published by the AIP Publishing

On an infinitesimal approach to semisimple Lie groups and raising and lowering operators of O(n) and U(n)

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(Received 7 February 1978; revised manuscript received 11 January 1979)

A purely algebraic approach to the evaluation of the fundamental Wigner coefficients and reduced matrix elements of O(n) and U(n) is given. The method employs the explicit use of projection operators which may be constructed using the polynomial identities satisfied by the infinitesimal generators of the group. As an application of this technique, a certain set of raising and lowering operators for O(n) and U(n) are constructed. They are simpler in appearance than those previously constructed since they may be written in a compact product form. They are, moreover, Hermitian conjugates of one another, and therefore are easily normalized.

1. INTRODUCTION

It has been shown[1-5] that the infinitesimal generators of semisimple Lie groups may be assembled into a matrix which satisfies a certain polynomial identity (herein called the characteristic identity) over the center of the universal enveloping algebra. On a representation admitting an infinitesimal character such an identity reduces to a polynomial identity over the underlying field (usually the real or complex field). In such a case the polynomial identity may be written in a convenient factored form.[2,3] In particular when acting on a finite dimensional irreducible representation of the group the polynomial identity reduces to the identities encountered recently by several authors for the various classical groups.[5-7]

The idea of assembling the infinitesimal generators of the classical groups into a matrix is not new and has proved in the past to be a useful technique.[8-10] By taking traces of powers of such a matrix one obtains a full set of invariants which serve to label the representations of the group completely, a fact which was recognized early.[4,10] The eigenvalues of such invariants (and invariants of a related nature) have been studied and computed by several authors.[11-13] Recently, however, a simple and systematic procedure for evaluating such invariants has been developed which employs the characteristic identities (see Ref. 11 and also compare with Ref. 13, Appendix B). In fact one may simply construct a full set of invariants for an arbitrary semisimple Lie group by the use of such methods. A general formula for the eigenvalues of these invariants, which applies to infinite as well as finite dimensional representations, has been developed in Ref. 3. This fact alone illustrates the potential importance of the characteristic identities in applications to group theory.

This paper is the first in a series which deals with the applications of the characteristic identities to the theory of groups. One of our principle aims is to show how one may evaluate the multiplicity free Wigner coefficients of a semisimple Lie group. It was noted early by Fano[14] that the characteristic identities (for the unitary groups of low order) were valuable for the explicit construction of projection operators. This idea was incorporated into subsequent work of Baird and Biedenharn[4] who noted that this algebraic technique would combine nicely with their evaluation of the fundamental Wigner coefficients for the general unitary groups. However, the idea was not considered further and it is our aim to pursue this matter in detail. The principle motivation for this paper, however, is that since the characteristic identities for arbitrary semisimple Lie groups are known the technique is generalizable to more general groups other than the relatively well known unitary groups.

The present paper deals solely with the orthogonal and unitary groups although possible extensions to more general groups are discussed in the concluding section. It is our principle aim to show how certain fundamental Wigner coefficients for O(n) and U(n) may be evaluated in a straightforward and simple manner by applying the use of projection operators which are constructed by means of the characteristic identities. At the same time we shall make an effort to relate our results to those obtained by Biedenharn, Louck, and Baird[15,16] who have given the evaluation of all multiplicity free Wigner coefficients for the unitary groups. Although our approach is intimately related to the approach employed by Baird and Biedenharn[16] there is one essential difference. The Wigner coefficients of the group are obtainable using only the properties of the projectors for which we have an explicit expression in terms of polynomials in the group generators. Calculations may then be carried out using only the Lie algebra commutation relations. Since the identities and associated projectors have been constructed explicitly for arbitrary semisimple Lie groups[17] this method is generalizable, in principle, to the general case.

As an application of this technique we shall construct a certain set of raising and lowering operators for U(n). These operators are different to those constructed by Nagel and Moshinsky[18] and may be written in a convenient product form. They are moreover Hermitian conjugates of one another which makes their normalization simple.

We shall also consider an extension of these results to the orthogonal subgroup of U(n). In particular the raising and lowering operators of the orthogonal groups are obtained which are different from those obtained by Wong[19] and Pang and Hecht.[20] Our approach to the orthogonal group in particular is considerably simpler than previous treatments and is no more difficult than the U(n) case. The raising and lowering operators for O(n) may also be written in a compact product form and are moreover Hermitian conjugates of one another.

2. WIGNER COEFFICIENTS OF U(n)

The generators a^i_j of the Lie group U(n) satisfy the commutation relations

$$[a^i_j, a^k_l] = \delta^k_j a^i_l - \delta^i_l a^k_j$$

and the hermiticity property

$$(a^i_j)^\dagger = a^j_i.$$

These generators may be assembled into a square matrix a whose (i, j) entry is the generator a^i_j. Polynomials in a may then be defined recursively by the formula

$$(a^m)^i_j = (a^{m-1})^i_k a^k_j = a^i_k (a^{m-1})^k_j.$$

By a simple induction argument one may show that if $p(x)$ is any polynomial then the entries of the matrix $p(a)$ satisfy the Hermiticity property

$$[p(a)^i_j]^\dagger = p(a)^j_i. \tag{1}$$

It has been shown[2,3] that the matrix a satisfies a polynomial identity over the center of the enveloping algebra which may be written in its factorized form as

$$\prod_{r=1}^{n} (a - \alpha_r) = 0, \tag{2}$$

where the α_r are invariants of the group whose eigenvalues on a representation of U(n) with representation label $(\lambda_1, \ldots, \lambda_n)$ are given by

$$\alpha_r = \lambda_r + n - r.$$

Associated with the matrix a is its adjoint $\bar{a}$ defined by $\bar{a}^i_j = -a^i_j$. As for the matrix a one may define polynomials in the matrix $\bar{a}$ recursively by the formula

$$(\bar{a}^m)^i_j = (\bar{a}^{m-1})^k_j \bar{a}^i_k = \bar{a}^k_j (\bar{a}^{m-1})^i_k.$$

The adjoint matrix $\bar{a}$ satisfies the polynomial identity

$$\prod_{r=1}^{n} (\bar{a} - \bar{\alpha}_r) = 0, \tag{3}$$

where the roots $\bar{\alpha}_r$ are related to the α_r by

$$\bar{\alpha}_r = n - 1 - \alpha_r.$$

By virtue of the identities (2) and (3) projection operators $P[r]$ and $\bar{P}[r]$ may be constructed by setting

$$P[r] = \prod_{l \neq r} \left(\frac{a - \alpha_l}{\alpha_r - \alpha_l} \right), \quad \bar{P}[r] = \prod_{l \neq r} \left(\frac{\bar{a} - \bar{\alpha}_l}{\bar{\alpha}_r - \bar{\alpha}_l} \right).$$

Such projection operators are useful since they may be used to define rather general functions of the matrix a by setting

$$p(a) = \sum_{r=1}^{n} p(\alpha_r) P[r], \quad p(\bar{a}) = \sum_{r=1}^{n} p(\bar{\alpha}_r) \bar{P}[r]. \tag{4}$$

Suppose now that π^* denotes the fundamental contragredient vector representation of U(n) and let V^* be the representation space of π^*. Then on V^* the generators a^i_j are represented by elementary matrices $\pi^*(a^i_j) = -E^j_i$, where E^j_i, has 1 in the (j,i) position and zeros elsewhere. It follows then that the matrix a may be written

$$a = \sum_{i,j=1}^{n} E^i_j a^i_j = -\sum_{i,j=1}^{n} \pi^*(a^j_i) a^i_j.$$

In passing it is interesting to note that in the case of SU(2) where π^* is the spin representation one obtians the matrix originally considered by Dirac in connection with wave equations for particles of higher spin. This same matrix also occurs in the quantum mechanics of a spinning electron in three dimensions (Landés interval rule) and appears again as one of the ladder operators considered by Louck[21] and others.

Now let $V(\lambda)$ denote a finite dimensional irreducible module over U(n), with highest weight λ, and let π_λ be the representation afforded by $V(\lambda)$. On $V(\lambda)$ the entries of the matrix a are represented by endormorphisms $a^i_j \equiv \pi_\lambda(a^i_j)$. In such a case a may be written in the form

$$a = -\sum_{i,j=1}^{n} \pi^*(a^j_i) \pi_\lambda(a^i_j)$$

and hence a may be regarded as an operator on the product space $V^* \otimes V(\lambda)$.

Following Hannabuss[6] the matrix a may be written in invariant form, when acting on $V(\lambda)$, as

$$a = -\tfrac{1}{2}[\pi^* \otimes \pi_\lambda(I_2) - \pi^*(I_2) \otimes 1 - 1 \otimes \pi_\lambda(I_2)],$$

where I_2 is the universal Casimir element

$$I_2 = \sum_{i,j=1}^{n} a^i_j a^j_i.$$

To see this we note that a is an operator on the tensor product space $V^* \otimes V(\lambda)$. On V^* the generators are represented by elementary matrices $-E^j_i$ and on $V(\lambda)$ by $\pi_\lambda(a^i_j)$. On $V^* \otimes V(\lambda)$ the generators therefore are

$$\pi^* \otimes \pi_\lambda(a^i_j) = -E^j_i \otimes 1 + 1 \otimes \pi_\lambda(a^i_j),$$

so one may write

$$\pi^* \otimes \pi_\lambda(I_2) = \sum_{i,j=1}^{n} [-E^j_i \otimes 1 + 1 \otimes \pi_\lambda(a^i_j)] \times [-E^i_j \otimes 1 + 1 \otimes \pi_\lambda(a^j_i)].$$

Rearranging this expression we obtain

$$\sum_{i,j=1}^{n} E^j_i \pi_\lambda(a^j_i) = -\tfrac{1}{2}[\pi^* \otimes \pi_\lambda(I_2) - \pi^*(I_2) \otimes 1 - 1 \otimes \pi_\lambda(I_2)],$$

where the left-hand side is the matrix a of U(n) as required.

Decomposing the tensor product representation $V^* \otimes V(\lambda)$ we obtain the Clebsch–Gordon reduction

$$V^* \otimes V(\lambda) = \sum_{r=1}^{n} V(\lambda - \Delta_r),$$

where Δ_r is the weight with 1 in position r and zeros elsewhere. On each space $V(\lambda - \Delta_r)$ the operator a takes the constant value

$$-\tfrac{1}{2}[\pi_{\lambda-\Delta_r}(I_2) - \pi^*(I_2) - \pi_\lambda(I_2)] = \lambda_r + n - r,$$

which are the roots α_r of the characteristic identity. More generally if $p(x)$ is any polynomial then on the space $V(\lambda - \Delta_r)$, $p(a)$ takes the constant $p(\alpha_r)$. In particular the U(n) projector $P[r]$ takes the constant value 1 on the space $V(\lambda - \Delta_r)$ and zero on the remaining $V(\lambda - \Delta_k)$ $(k \neq r)$. Thus $P[r]$ projects $V^* \otimes V(\lambda)$ onto $V(\lambda - \Delta_r)$. We may express this in terms of Gel'fand patterns as

$$P[r]\left|\begin{matrix}\lambda-\Delta_s\\(\mu)\end{matrix}\right\rangle=\delta_{rs}\left|\begin{matrix}\lambda-\Delta_s\\(\mu)\end{matrix}\right\rangle, \tag{5}$$

where

$$\left|\begin{matrix}\lambda-\Delta_s\\(\mu)\end{matrix}\right\rangle$$

denotes a Gel'fand basis state for the space $V(\lambda-\Delta_s)$ and (μ) denotes a Gel'fand pattern for the subgroup $\mathrm{U}(n-1)$.

Now let

$$\left|\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle,\quad\left|\begin{matrix}\lambda\\(\nu')\end{matrix}\right\rangle$$

be two Gel'fand basis states in the space $V(\lambda)$. Then the matrix elements of the (i,j) entry of the projector $P[r]$ between these states are given by

$$\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix}\right|P[r]^i_j\left|\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle=\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix};\begin{matrix}\overline{1\dot{0}}\\i\end{matrix}\right|P[r]\left|\begin{matrix}\overline{1\dot{0}}\\j\end{matrix};\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle,$$

where

$$\left|\begin{matrix}\overline{1\dot{0}}\\i\end{matrix}\right\rangle$$

forms the usual basis for the contragredient vector representation and where

$$\left|\begin{matrix}\overline{1\dot{0}}\\j\end{matrix};\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle$$

denotes the product state

$$\left|\begin{matrix}\overline{1\dot{0}}\\j\end{matrix}\right\rangle\otimes\left|\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle.$$

Introducing a complete set of states for the product representation $V^*\otimes V(\lambda)$ these matrix elements may in turn be written

$$\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix}\right|P[r]^i_j\left|\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle=\sum_{\substack{(\mu),(\mu')\\s,t}}\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix};\begin{matrix}\overline{1\dot{0}}\\i\end{matrix}\middle|\begin{matrix}\lambda-\Delta_s\\(\mu')\end{matrix}\right\rangle$$
$$\times\left\langle\begin{matrix}\lambda-\Delta_s\\(\mu')\end{matrix}\right|P[r]\left|\begin{matrix}\lambda-\Delta_t\\(\mu)\end{matrix}\right\rangle$$
$$\times\left\langle\begin{matrix}\lambda-\Delta_t\\(\mu)\end{matrix}\middle|\begin{matrix}\overline{1\dot{0}}\\j\end{matrix};\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle,$$

where the sum on (μ) and (μ') is over all patterns in the spaces $V(\lambda-\Delta_t)$ and $V(\lambda-\Delta_s)$, respectively. Using property (5) of the projector $P[r]$ the right hand side equals

$$\sum_{(\mu)}\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix};\begin{matrix}\overline{1\dot{0}}\\i\end{matrix}\middle|\begin{matrix}\lambda-\Delta_r\\(\mu)\end{matrix}\right\rangle\left\langle\begin{matrix}\lambda-\Delta_r\\(\mu)\end{matrix}\middle|\begin{matrix}\overline{1\dot{0}}\\j\end{matrix};\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle \tag{6}$$

The numbers

$$\left\langle\begin{matrix}\lambda-\Delta_r\\(\mu)\end{matrix}\middle|\begin{matrix}\overline{1\dot{0}}\\j\end{matrix};\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle$$

are of fundamental importance since they are Wigner coefficients.

Note that Eq. (6) implies, in view of Eq. (4), that the matrix elements of the group generators are given by

$$\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix}\right|a^i_j\left|\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle=\sum_{r=1}^{n}(\lambda_r+n-r)\sum_{(\mu)}\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix};\begin{matrix}\overline{1\dot{0}}\\i\end{matrix}\middle|\begin{matrix}\lambda-\Delta_s\\(\mu)\end{matrix}\right\rangle.$$
$$\times\left\langle\begin{matrix}\lambda-\Delta_r\\(\mu)\end{matrix}\middle|\begin{matrix}\overline{1\dot{0}}\\j\end{matrix};\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle.$$

This equation is quite easily generalized to more general groups and indicates that in determining the matrix elements of the group generators only the associated Wigner coefficients are required.

In certain special cases the sum (6) reduces to a single term enabling an evaluation of certain Wigner coefficients by an independent evaluation of the left hand side. The case of primary interest to us is the matrix element of the (r,r) entry of the projector $P[r]$ between the maximal state $|\lambda\rangle$ of $V(\lambda)$. In this case one obtains

$$\langle\lambda|P[r]^r_r|\lambda\rangle=\sum_{(\mu)}\left|\left\langle\lambda;\begin{matrix}\overline{1\dot{0}}\\r\end{matrix}\middle|\begin{matrix}\lambda-\Delta_r\\(\mu)\end{matrix}\right\rangle\right|^2.$$

However, the state

$$\left|\lambda;\begin{matrix}\overline{1\dot{0}}\\r\end{matrix}\right\rangle$$

has weight $\lambda-\Delta_r$ which is the highest weight occurring in $V(\lambda-\Delta_r)$. Hence, the term

$$\left\langle\lambda;\begin{matrix}\overline{1\dot{0}}\\r\end{matrix}\middle|\begin{matrix}\lambda-\Delta_r\\(\mu)\end{matrix}\right\rangle$$

vanishes unless

$$\left|\begin{matrix}\lambda-\Delta_r\\(\mu)\end{matrix}\right\rangle$$

coincides with the unique maximal state $|\lambda-\Delta_r\rangle$ of $V(\lambda-\Delta_r)$. One therefore obtains

$$\langle\lambda|P[r]^r_r|\lambda\rangle=\left|\left\langle\lambda;\begin{matrix}\overline{1\dot{0}}\\r\end{matrix}\middle|\lambda-\Delta_r\right\rangle\right|^2. \tag{7}$$

Another case of interest, which we shall not treat here, is the case where $i=j=n$ in Eq. (6). In this case one obtains the important result

$$\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix}\right|P[r]^n_n\left|\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle=\delta_{(\nu')(\nu)}\left|\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix};\begin{matrix}\overline{1\dot{0}}\\n\end{matrix}\middle|\begin{matrix}\lambda-\Delta_r\\(\nu)\end{matrix}\right\rangle\right|^2.$$

The matrix elements of $P[r]^n_n$ are quite easily evaluated as demonstrated in Ref. 22. This then enables a complete determination of all fundamental Wigner coefficients and ultimately the matrix elements of the group generators. We shall illustrate this procedure in a forthcoming publication. It is also interesting to note that the technique is capable of generalization to infinite dimensions enabling a treatment of the noncompact groups.

The matrix element (7) is quite easily determined as demonstrated in Ref. 22, according to which we may write

$$\left|\left\langle\lambda;\begin{matrix}1\dot{0}\\r\end{matrix}\middle|\lambda-\Delta_r\right\rangle\right|^2=\prod_{l>r}\left(\frac{\lambda_r-\lambda_l+l-r-1}{\lambda_r-\lambda_l+l-r}\right). \tag{8}$$

Now if $|\lambda\rangle$ is the maximal state in $V(\lambda)$ then one obtains

$$P[r]^i_j|\lambda\rangle=0,\quad\text{for } j>r \text{ and } i=1,\ldots,n. \tag{9}$$

To see this suppose

$$\left|\begin{matrix}\lambda\\(\nu')\end{matrix}\right\rangle$$

is an arbitrary state in $V(\lambda)$. Application of Eq. (6) immediately gives

$$\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix}\right|P[r]^i_j\left|\lambda\right\rangle=\sum_{(\mu)}\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix};\begin{matrix}\overline{1\dot{0}}\\i\end{matrix}\middle|\begin{matrix}\lambda-\Delta_r\\(\mu)\end{matrix}\right\rangle\left\langle\begin{matrix}\lambda-\Delta_r\\(\mu)\end{matrix}\middle|\begin{matrix}\overline{1\dot{0}}\\j\end{matrix};\lambda\right\rangle. \tag{10}$$

However, if $j > r$ then the state

$$\left| \begin{matrix} \dot{10} \\ j \end{matrix} ;\lambda \right\rangle$$

has weight $\lambda - \Delta_j$, which is greater than the highest weight occurring in $V(\lambda - \Delta_r)$. It follows immediately that the sum (10) vanishes. Since

$$\left| \begin{matrix} \lambda \\ (\nu) \end{matrix} \right\rangle$$

was chosen arbitrarily one obtains Eq. (9).

By applying similar considerations to the matrix $\bar{a}$ one obtains the equation

$$\left\langle \begin{matrix} \lambda \\ (\nu') \end{matrix} \right| \bar{P}[r]^i_j \left| \begin{matrix} \lambda \\ (\nu) \end{matrix} \right\rangle = \sum_{(\mu)} \left\langle \begin{matrix} \lambda \\ (\nu') \end{matrix} ; \begin{matrix} \dot{10} \\ j \end{matrix} \right| \begin{matrix} \lambda + \Delta_r \\ (\mu) \end{matrix} \right\rangle \left\langle \begin{matrix} \lambda + \Delta_r \\ (\mu) \end{matrix} \right| \begin{matrix} \dot{10} \\ j \end{matrix} ; \begin{matrix} \lambda \\ (\nu) \end{matrix} \right\rangle$$

where

$$\left| \begin{matrix} \dot{10} \\ i \end{matrix} \right\rangle$$

forms the usual basis for the fundamental vector representation. One may then deduce the results

$$\langle \lambda | \bar{P}[r]^r_r | \lambda \rangle = \left| \left\langle \lambda ; \begin{matrix} \dot{10} \\ r \end{matrix} \right| \lambda + \Delta_r \right\rangle \right|^2, \tag{11}$$

$$\bar{P}[r]^i_j | \lambda \rangle = 0, \quad \text{for } i < r, \quad j = 1,\ldots,n.$$

According to Ref. 22 the Wigner coefficient (11) may be evaluated using the formula

$$\left| \left\langle \lambda ; \begin{matrix} \dot{10} \\ r \end{matrix} \right| \lambda + \Delta_r \right\rangle \right|^2 = \prod_{l<r} \left(\frac{\lambda_r - \lambda_l + l - r + 1}{\lambda_r - \lambda_l + l - r} \right). \tag{12}$$

3. REDUCED MATRIX ELEMENTS OF U(*n*)

Recall that a U(n) vector operator ψ is defined as a collection of components $\psi^i (i = 1,\ldots n)$ which satisfy

$$[a^i_j, \psi^k] = \delta^k_j \psi^i. \tag{13}$$

By taking the Hermitian conjugate of this relation one obtains the transformation law of contragredient vector operators

$$[a^i_j, \psi^\dagger_k] = -\delta^i_k \psi^\dagger_j. \tag{14}$$

It is known[5,16] that a vector operator ψ may be resolved into a sum of shift vectors $\psi[r]$:

$$\psi = \sum_{r=1}^{n} \psi[r],$$

where $\psi[r]$ increases the eigenvalue of the representation label λ_r by one unit leaving the other λ_k unchanged;

$$\lambda_k \psi[r] = \psi[r](\lambda_k + \delta_{kr}).$$

Similarly, a contragredient vector operator $\psi^\dagger$ may be resolved into shift components $\psi^\dagger[r] = (\psi[r])^\dagger$ which decrease the representation labels by one unit:

$$\lambda_k \psi^\dagger[r] = \psi^\dagger[r](\lambda_k - \delta_{kr}).$$

Such shift vectors may be constructed by application of the projectors $P[r]$ and $\bar{P}[r]$ as follows:

$$\psi[r] = P[r]\psi = \psi \bar{P}[r], \quad \psi^\dagger[r] = \psi^\dagger P[r] = \bar{P}[r]\psi^\dagger. \tag{15}$$

One may show (see Ref. 22 for details) that the following relations hold:

$$\psi[r]\psi^\dagger[r] = \bar{m}_r P[r], \quad \psi^\dagger[r]\psi[r] = m_r \bar{P}[r]. \tag{16}$$

The invariants m_r and $\bar{m}_r$ are of particular interest since their eigenvalues on finite-dimensional irreducible representations are the squares of the reduced matrix elements of ψ and $\psi^\dagger$, respectively. Equation (16) may then be regarded as an operator generalization of the Wigner–Eckart theorem.

The operator $\tilde{\psi}[r] = (\bar{m}_r)^{-1/2}\psi[r]$ is therefore a fundamental Wigner operator which have been treated in detail by Biedenharn, Giovannini, Louck, and Baird.[4,15,23] Equation (16) may then be rewritten in the form

$$\tilde{\psi}[r]\tilde{\psi}^\dagger[r] = P[r],$$

an observation previously made by Louck and Biedenharn.[13] The important thing from our point of view is that we have an explicit expression for $P[r]$ as a polynomial in the matrix a. Equation (16) was arrived at using only purely algebraic techniques.

Note that by taking the trace of Eq. (16) one obtains the relations

$$\bar{m}_r = \frac{\psi[i]^i \psi^\dagger[r]_i}{t_r(P[r])}, \quad m_r = \frac{\psi^\dagger[r]_i \psi[r]^i}{t_r(\bar{P}[r])}, \tag{17}$$

which enables a systematic method for determining the reduced matrix elements. (The traces of the projectors $P[r]$ and $\bar{P}[r]$ have been evaluated in Refs. 11 and 22.)

We now note that if $|\lambda\rangle$ is a maximal weight state of weight λ then one obtains, in view of Eqs. (9) and (11),

$$\psi^\dagger[r]_i |\lambda\rangle = \psi^\dagger[r]_j P[r]^j_i |\lambda\rangle = 0, \quad \text{for } i > r,$$

$$\psi[r]^i |\lambda\rangle = \psi^j \bar{P}[r]^i_j |\lambda\rangle = 0, \quad \text{for } i < r.$$

From this it follows that $\psi[r]^r|\lambda\rangle$ and $\psi^\dagger[r]_r|\lambda\rangle$ are maximal weight states of U(n) of weight $(\lambda + \Delta_r)$ and $(\lambda - \Delta_r)$, respectively. One may secure the normalization of these states from the equations

$$\psi[r]^r \psi^\dagger[r]_r = \bar{m}_r P[r]^r_r, \quad \psi^\dagger[r]_r \psi[r]^r = m_r \bar{P}[r]^r_r \tag{18}$$

which may be evaluated using Eqs. (8) and (12).

Suppose now we look at the subgroup embedding $U(n) \subset U(n+1)$. The generators of $U(n+1)$ may also be assembled into a matrix $\hat{a}$ as we did for U(n). This matrix satisfies an $(n+1)$ degree polynomial identity analogous to Eq. (2). We denote the $U(n+1)$ characteristic identity by

$$\prod_{r=1}^{n+1} (\hat{a} - \beta_r) = 0,$$

where the β_r are invariants of the group which take constant values $\beta_r = \lambda_r + n + 1 - r$ on a representation of $U(n+1)$ with highest weight $(\lambda_1,\ldots,\lambda_{n+1})$.

In our previous notation let ψ denote the U(n) vector operator with components $\psi^i = a^i_{n+1}$ and let $\psi^\dagger_i = a^{n+1}_i$ $(i = 1,\ldots n)$ denote its contragredient. In this case the reduced matrix elements of ψ and $\psi^\dagger$ may be evaluated as a function of the β_k and α_r by applying formula (17) (see Ref. 22 for details). One obtains the formulas

$$m_r = (-1)^n \prod_{p=1}^{n+1} (\beta_p - \alpha_r - 1) \prod_{\substack{l=1 \\ \neq r}}^{n} (\alpha_r - \alpha_l + 1)^{-1}, \tag{19}$$

$$\bar{m}_r = (-1)^n \prod_{p=1}^{n+1} (\beta_p - \alpha_r) \prod_{\substack{l=1\\ \neq r}}^{n} (\alpha_r - \alpha_l - 1)^{-1}.$$

The raising and lowering operators $\psi[r]^r$ and $\psi^\dagger[r]_r$ will shift between maximal weight states of U(n) in a given irreducible representation of U($n+1$). The normalization of these operators may be obtained directly from Eq. (18) since the quantities $m_r, \bar{m}_r, P[r]^r_r$, and $\bar{P}[r]^r_r$ may all be evaluated using Eqs. (8), (12), and (19).

4. RAISING AND LOWERING OPERATORS OF U(n)

The U(n) generators a^i_j, where i and j are restricted to values $1,\dots,m$ (for some positive integer m less than n), form the generators of the unitary subgroup U(m) of U(n). We see therefore that U(n) admits the canonical chain of subgroups

$$\mathrm{U}(n) \supset \mathrm{U}(n-1) \supset \cdots \supset \mathrm{U}(1). \tag{20}$$

The irreducible representations of the groups U(m), $1 \leqslant m \leqslant n$, may be characterized by partitions $(\lambda_{1m}, \lambda_{2m}, \dots, \lambda_{mm})$ where the λ_{im} are integers satisfying

$$\lambda_{1m} \geqslant \lambda_{2m} \geqslant \cdots \geqslant \lambda_{mm} \geqslant 0.$$

The partitions of two groups U($m+1$) and U(m) in the chain (20) are related by the inequalities

$$\lambda_{1m+1} \geqslant \lambda_{1m} \geqslant \lambda_{2m+1} \geqslant \lambda_{2m} \geqslant \cdots \geqslant \lambda_{mm} \geqslant \lambda_{m+1\,m+1}.$$

The set of partitions for the chain (20) is most conveniently arranged into a Gel'fand pattern which has been described by Gel'fand and Zetlin[24] and appears in the paper by Nagel and Moshinsky.[18]

If U($m+1$) is a subgroup in the chain (20) we shall denote, for convenience, a maximal weight vector of U(m) [i.e., a semimaximal state of U($m+1$)] simply by the pattern

$$\left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle .$$

This pattern denotes a maximal weight state of U(m) of weight (λ_{im}) contained in an irreducible representation of U($m+1$) with highest weight (λ_{im+1}).

When acting on the above state the shift components $\psi_m[r]$ of the U(m) vector operator

$$\psi^i_m = a^i_{m+1}$$

are given by [see Eq. (15)]

$$\psi_m[r]^i = a^j_{m+1} \prod_{\substack{l=1\\ \neq r}}^{m} \left[\frac{\bar{a} + \lambda_{lm} - l + 1}{\lambda_{lm} - \lambda_{rm} + r - l} \right]^i_j,$$

where $\bar{a}$ is the U(m) adjoint matrix. Similarly, the shift components of the U(m) contragredient vector operator $\psi^\dagger_m$ are given by

$$\psi^\dagger_m[r]_i = a^{m+1}_j \prod_{\substack{l=1\\ \neq r}}^{m} \left[\frac{a - \lambda_{lm} - m + l}{\lambda_{rm} - \lambda_{lm} + l - r} \right]^j_i.$$

For convenience we denote the rth component of $\psi_m[r]$ by ψ^r_m, viz., $\psi^r_m = \psi_m[r]^r$. In view of our previous remarks we have

$$\psi^r_m \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle \propto \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} + \delta_{ir} \end{matrix} \right\rangle, \psi^{\dagger r}_m \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle \propto \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} - \delta_{ir} \end{matrix} \right\rangle.$$

We therfore have our required raising and lowering operators and it just remains to obtain the normalization. Using the hermiticity relation

$$\left(\psi^{\dagger r}_m \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle \right)^\dagger = \left\langle \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right| \psi^r_m$$

the normalization constants $N^r_m(\lambda_{im+1}, \lambda_{im})$ of the lowering operators $\psi^{\dagger r}_m$ are given by

$$N^r_m = \left\langle \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right| \psi^r_m \psi^{\dagger r}_m \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle^{1/2},$$

which, using Eq. (18), may in turn be written

$$N^r_m = \left\langle \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right| \bar{m}_r P[r]^r_r \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle^{1/2},$$

where $P[r]$ is the U(m) projector

$$P[r] = \prod_{\substack{l=1\\ \neq r}}^{m} \left(\frac{a - \lambda_{lm} - m + l}{\lambda_{rm} - \lambda_{lm} + l - r} \right).$$

Substituting formulas (8) and (19) into the above expression, noting that the operators β_k take constant values $\lambda_{km+1} + m + 1 - k$, while the α_r take constant values $\lambda_{rm} + n - r$, one obtains the formula

$$N^r_m = \Big[(-1)^m \prod_{p=1}^{m+1} (\lambda_{pm+1} - \lambda_{rm} + r - p + 1) \times \prod_{l<r} (\lambda_{rm} - \lambda_{lm} + l - r - 1)^{-1} \prod_{l>r} (\lambda_{rm} - \lambda_{lm} + l - r)^{-1} \Big]^{1/2} \tag{21}$$

Our normalized lowering operators therefore are

$$(N^r_m)^{-1} \psi^{\dagger r}_m.$$

Proceeding in a similar fashion the normalization constants $\bar{N}^r_m$ of the raising operators ψ^r_m are given by

$$\bar{N}^r_m = \left\langle \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right| m_r \bar{P}[r]^r_r \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle^{1/2},$$

which may be evaluated using Eqs. (12) and (19). One thereby obtains

$$\bar{N}^r_m = \Big[(-1)^m \prod_{p=1}^{m+1} (\lambda_{pm+1} - \lambda_{rm} + r - p) \times \prod_{l>r} (\lambda_{rm} - \lambda_{lm} + l - r + 1)^{-1} \prod_{l<r} (\lambda_{rm} - \lambda_{lm} + l - r)^{-1} \Big]^{1/2}.$$

We may now write down an arbitrary Gel'fand basis state in terms of lowering operators acting on the maximal weight state $|\mathrm{max}\rangle$ of U(n). We have

$$\left| \begin{matrix} \lambda_{1n} & \lambda_{2n} \cdots\cdots \lambda_{nn} \\ \lambda_{1n-1} & \lambda_{2n-1} \cdots \lambda_{n-1\,n-1} \\ \vdots & \vdots \\ \lambda_{12} & \lambda_{22} \\ \lambda_{11} & \end{matrix} \right\rangle = N[\lambda]^{-1} \prod_{m=1}^{n-1} \prod_{r=1}^{m} (\psi^{\dagger r}_m)^{\lambda_{rm+1} - \lambda_{rm}} |\mathrm{max}\rangle. \tag{22}$$

The normalization constant $N[\lambda]$ appearing in this expression is easily computed by repeated application of Eq. (21). We readily obtain

$$N[\lambda]=\Bigg(\prod_{m=1}^{n-1}\prod_{r=1}^{m}\prod_{l<r}\frac{(\lambda_{rm+1}-\lambda_{lm}+l-r-1)!}{(\lambda_{rm}-\lambda_{lm}+l-r-1)!}\times\prod_{l>r}\frac{(\lambda_{rm}-\lambda_{lm+1}+l-r)!}{(\lambda_{rm+1}-\lambda_{lm+1}+l-r)!}\times\prod_{p=1}^{m+1}\frac{(\lambda_{pm+1}-\lambda_{rm}+r-p+1)!}{(\lambda_{pm+1}-\lambda_{rm+1}+r-p+1)!}\Bigg)^{1/2}.$$

It should be noted that the products of lowering operators appearing in Eq. (22) are ordered in such a way that the lowering operators for the group U(m) appear on the right of the lowering operators for the group U($m-1$). The lowering operators $\psi_m^{\dagger r}$ and $\psi_m^{\dagger k}$ for U(m) are ordered so that $\psi_m^{\dagger r}$ appears on the right of $\psi_m^{\dagger k}$ when $r<k$. However, changing the order of two lowering operators for U(m) will only result in a change of normalization constant $N[\lambda]$.

5. EXTENSION TO O(n)

Without loss of generality we may take the generators of the orthogonal subgroup of U(n) to be

$$\alpha^i_j=a^i_j-a^{\ j}_i, \tag{23}$$

where the a^i_j are the generators of U(n). This corresponds to the choice of O(n) metric $g_{ij}=\delta_{ij}$. The generators (23) satisfy the commutation relations

$$[\alpha^{\ i}_j,\alpha^k_l]=\delta^k_j\,\alpha^i_l-\delta^i_l\,\alpha^k_j-\delta^k_i\alpha^{\ j}_l+\delta^{\ j}_l\alpha^k_i \tag{24}$$

and the hermiticity requirement

$$(\alpha^i_j)^\dagger=\alpha^{\ j}_i.$$

The representations of O(n) may be labeled by the maximum eigenvalues of the operators

$$-i\alpha^{2r-1}_{2r},\quad r=1,\dots,h,$$

where

$$h=\left[\frac{n}{2}\right]=\begin{cases}\frac{1}{2}n & n\ \text{even},\\ \frac{1}{2}(n-1) & n\ \text{odd}.\end{cases}$$

As for U(n) one may consider the O(n) matrix α, whose (i,j) entry is the generator α^i_j, and its adjoint $\bar{\alpha}$ with entries $\bar{\alpha}^{\ i}_j=-\alpha^i_j$. The matrices α and $\bar{\alpha}$ satisfy polynomial identities of the form

$$\prod_{r=1}^{n}(\alpha-\alpha_r)=0,\quad \prod_{r=1}^{n}(\bar{\alpha}-\bar{\alpha}_r)=0,$$

where the α_r are invariants of the group whose eigenvalues on an irreducible representation with highest weight $(\lambda_1,\dots,\lambda_h)$ are given by

$$\alpha_r=\lambda_r+n-1-r,\quad \alpha_{n+1-r}=r-1-\lambda_r,\quad r=1,\dots,h,$$

with

$$\alpha_{h+1}=h,\quad \text{for}\quad n=2h+1.$$

Following the notation of Green[5] we may define labels λ_r for $r>h$ by setting

$$\lambda_{n+1-r}=1-\lambda_r,\quad r=1,\dots h, \tag{25}$$

with

$$\lambda_{h+1}=1,\quad \text{for}\quad n=2h+1.$$

The roots $\bar{\alpha}_r$ appearing in the adjoint identity are related to the roots α_r by $\bar{\alpha}_r=\alpha_{n+1-r}$.

Unlike the U(n) case the O(n) generators defined by Eq. (23) are not in Cartan form. However, it is easily checked,[14] that the matrix a defined by

$$a^i_j=(M^{-1})^i_p\alpha^p_qM^q_j, \tag{26}$$

where M is the numerical unitary matrix with entries

$$M^{2j-1}_j=\frac{1}{\sqrt{2}}=M^{2j-1}_{n+1-j},$$

$$M^{2j}_j=-\frac{i}{\sqrt{2}}=-M^{2j}_{n+1-j},\quad j=1,\dots,h,$$

all other entries being zero except when $n=2h+1$, where we have an additional nonzero entry $M^n_{h+1}=1$, which has entries consisting of the O(n) generators in their root space forms.

These generators satisfy the commutation relations

$$[a^i_j,a^k_l]=\delta^k_ja^i_l-\delta^i_la^k_j-\delta^k_{n+1-i}a^{n+1-j}_l+\delta^{n+1-j}_la^k_{n+1-i}. \tag{27}$$

The diagonal entries of the matrix a are given by

$$a^r_r=-a^{n+1-r}_{n+1-r}=-i\alpha^{2r-1}_{2r},\quad r=1,\dots h,$$

with

$$a^{h+1}_{h+1}=0,\quad \text{for}\ n=2h+1.$$

In view of the commutation relations (27) the entries a^i_j are in Cartan form with the positive roots above the diagonal and negative roots below the diagonal of the matrix a in analogy with U(n).

More generally, if $p(x)$ is any polynomial, then the matrices $p(a)$ and $p(\alpha)$ are related by

$$p(a)=M^{-1}p(\alpha)M. \tag{28}$$

Similarly, we have

$$p(\bar{a})=(\bar{M})^{-1}p(\bar{\alpha})\bar{M},$$

where

$$\bar{M}^p_q=M^p_q,\quad (\bar{M})_q^{-1p}=(M^{-1})^p_q.$$

From this we see that the matrices a and α satisfy the same characteristic identity.

As for U(n) one may construct a set of projectors

$$P[r]=\prod_{l\neq r}\left(\frac{a-\alpha_l}{\alpha_r-\alpha_l}\right),\quad \bar{P}[r]=\prod_{l\neq r}\left(\frac{\bar{a}-\bar{\alpha}_l}{\alpha_r-\bar{\alpha}_l}\right),$$

from which one may define arbitrary functions of a and $\bar{a}$ as in Eq. (4).

Now let π_λ denote a finite dimensional irreducible represenation of O(n) with highest weight λ, and let $V(\lambda)$ be the representation space of π_λ. As before the matrix elements of entries of the projectors $P[r]$ and $\bar{P}[r]$ are bilinear combinations of Wigner coefficients

$$\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix}\right|P[r]^i_j\left|\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle=\sum_{(\mu)}\left\langle\begin{matrix}\lambda\\(\nu')\end{matrix};\begin{matrix}\overline{10}\\i\end{matrix}\right|\left.\begin{matrix}\lambda-\Delta_r\\(\mu)\end{matrix}\right\rangle\left\langle\begin{matrix}\lambda-\Delta_r\\(\mu)\end{matrix}\right|\left.\begin{matrix}\overline{10}\\j\end{matrix};\begin{matrix}\lambda\\(\nu)\end{matrix}\right\rangle,$$

$$r=1,\dots,n \tag{29}$$

$$\left\langle {\lambda \atop (\nu')} \right| \bar{P}[r]^i_j \left| {\lambda \atop (\nu)} \right\rangle = \sum_{(\mu)} \left\langle \lambda; {\dot{10} \atop (\nu')\; j} \middle| {\lambda - \Delta_r \atop (\mu)} \right\rangle \left\langle {\lambda - \Delta_r \atop (\mu)} \middle| {\dot{10} \atop i}; {\lambda \atop (\nu)} \right\rangle,$$

where

$$\left| {\dot{10} \atop i} \right\rangle$$

constitutes a basis state of the fundamental vector representation of weight Δ_i where we define labels Δ_i for $i > h$ by $\Delta_i = -\Delta_{n+1-i}$. From Eq. (29) one may deduce the relations

$$\langle \lambda | P[r]^r_r | \lambda \rangle = \left| \left\langle \lambda; {\overline{\dot{10}} \atop r} \middle| \lambda - \Delta_r \right\rangle \right|^2,$$

$$\langle \lambda | \bar{P}[r]^r_r | \lambda \rangle = \left| \left\langle \lambda; {\dot{10} \atop r} \middle| \lambda + \Delta_r \right\rangle \right|^2,$$

which may be evaluated using the formulas (see Ref. 22)

$$\left| \left\langle \lambda; {\overline{\dot{10}} \atop r} \middle| \lambda - \Delta_r \right\rangle \right|^2 = \prod_{l>r} \left(\frac{\lambda_r - \lambda_l - 1 + \delta_{l,h+1} - \delta_{l,n+1-r} + l - r}{\lambda_r - \lambda_l + l - r} \right), \quad n = 2h+1,$$

$$= \prod_{l>r} \left(\frac{\lambda_r - \lambda_l - 1 + \delta_{l,n+1-r} + l - r}{\lambda_r - \lambda_l + l - r} \right), \quad n = 2h, \tag{30}$$

$$\left| \left\langle \lambda; {\dot{10} \atop r} \middle| \lambda + \Delta_r \right\rangle \right|^2 = \prod_{l<r} \left(\frac{\lambda_r - \lambda_l + l - r + 1 - 2\delta_{l,h+1} + \delta_{l,n+1-r}}{\lambda_r - \lambda_l + l - r} \right), \quad n = 2h+1,$$

$$= \prod_{l<r} \left(\frac{\lambda_r - \lambda_l + 1 + l - r - \delta_{l,n+1-r}}{\lambda_r - \lambda_l + l - r} \right), \quad n = 2h, \tag{31}$$

where we define labels λ_r for $r > h$ in accordance with Eq. (25).

Using Eq. (29) one may also deduce the relations

$$P[r]^i_j |\lambda\rangle = 0, \quad \text{for } j > r, \quad i = 1,...,n,$$

$$\bar{P}[r]^i_j |\lambda\rangle = 0, \quad \text{for } i < r, \quad j = 1,...,n. \tag{32}$$

6. REDUCED MATRIX ELEMENTS OF O(*n*)

With respect to the generators (23) we define an O(n) vector operator as an operator with n components $\tilde{\psi}^i$ which satisfy

$$[\alpha^i_j, \tilde{\psi}^k] = \delta^k_j \tilde{\psi}^i - \delta^k_i \tilde{\psi}^j. \tag{33}$$

By taking the Hermitian conjuate of this relation one obtains the transformation law of contragredient vector operators

$$[\alpha^i_j, \tilde{\psi}^\dagger_k] = -\delta^i_k \tilde{\psi}^\dagger_j + \delta^j_k \tilde{\psi}^\dagger_i. \tag{34}$$

By applying the change of basis matrix M the vector operator $\tilde{\psi}$ gets transformed to

$$\psi^i = (M^{-1})^i_j \tilde{\psi}^j.$$

The vector operator ψ therefore has components

$$\psi^j = \tilde{\psi}^{2j-1} + i\tilde{\psi}^{2j},$$

$$\psi^{n+1-j} = \tilde{\psi}^{2j-1} - i\tilde{\psi}^{2j}, \quad j = 1,...,h,$$

with

$$\psi^{h+1} = \tilde{\psi}^n, \quad \text{for } n = 2h+1,$$

which transform according to

$$[a^i_j, \psi^k] = \delta^k_j \psi^i - \delta^k_{n+1-i} \psi^{n+1-j}. \tag{35}$$

Similarly, the contragredient vector operator $\tilde{\psi}^\dagger$ gets transformed into a contragredient vector operator $\psi^\dagger$ with components

$$\psi^\dagger_i = \tilde{\psi}^\dagger_j M^j_i,$$

which transforms according to

$$[a^i_j, \psi^\dagger_k] = -\delta^i_k \psi^\dagger_j + \delta^{n+1-j}_k \psi^\dagger_{n+1-i}. \tag{36}$$

From now on we refer to a vector (contragredient vector) operator as an operator with n components transforming according to Eq. (35) [Eq. (36)].

The O(n) vector operator ψ may be resolved into a sum of shift vectors

$$\psi^\bullet = \sum_{r=1}^{n} \psi[r]$$

which alter the representation labels according to

$$\lambda_k \psi[r] = \psi[r](\lambda_k + \delta_{kr}),$$

$$\lambda_k \psi[n+1-r] = \psi[n+1-r](\lambda_k - \delta_{kr}), \quad r = 1,...,h,$$

with

$$\lambda_k \psi[h+1] = \psi[h+1]\lambda_k, \quad \text{for } n = 2h+1.$$

The shift components $\psi^\dagger[r]$ of $\psi^\dagger$ therefore alter the representation labels according to

$$\lambda_k \psi^\dagger[r] = \psi^\dagger[r](\lambda_k - \delta_{kr}),$$

$$\lambda_k \psi^\dagger[n+1-r] = \psi^\dagger[n+1-r](\lambda_k + \delta_{kr}), \quad r = 1,...,h,$$

with

$$\lambda_k \psi^\dagger[h+1] = \psi^\dagger[h+1]\lambda_k, \quad \text{for } n = 2h+1.$$

These shifts components may be constructed by application of the projectors $P[r]$ and $\bar{P}[r]$ as in Eq. (15).

One also obtains the relations [cf. Eq. (16)]

$$\psi[r]\psi^\dagger[r] = \bar{m}_r P[r], \quad \psi^\dagger[r]\psi[r] = m_r \bar{P}[r],$$

where $\bar{m}_r = m_{n+1-r}$ are the squares of the reduced matrix elements of the vector operator ψ. These reduced matrix elements may be evaluated using Eq. (17).

The generators α^i_j of O($n+1$) may also be assembled into a matrix $\hat{\alpha}$ as for O(n). The matrix $\hat{\alpha}$ satisfies a polynomial identity

$$\prod_{r=1} (\hat{\alpha} - \beta_r) = 0,$$

where the β_r take constant values on a finite dimensional irreducible representation of O($n+1$) with highest weight λ given by

$$\beta_r = \lambda_r + n - r, \quad r = 1,...,n+1,$$

where we define labels λ_r for $r > [(n+1)/2]$ as in Eq. (25).

In the special case where ψ is the vector operator $\psi^i = (M^{-1})^i_j \alpha^j_{n+1}$ (where we sum on j from 1 to n), with adjoint $\psi^\dagger_i = \alpha^{n+1}_j M^j_i$, the reduced matrix elements m_r and

$\bar{m}_r$ may be evaluated as a function of the β_k and α_r using Eq. (17). We readily obtain (see Ref. 22)

$$\bar{m}_r = m_{n+1-r} = (-1)^n \prod_{p=1}^{n+1} (\beta_p - \alpha_r) \times \prod_{l \neq r} (\alpha_r - \alpha_l - 1 - \delta_{l,n+1-r})^{-1}, \quad n = 2h,$$

$$\bar{m}_r = m_{n+1-r} = (-1)^n \prod_{p=1}^{n+1} (\beta_p - \alpha_r) \times \prod_{l \neq r} (\alpha_r - \alpha_l - l + \delta_{l,h+1} - \delta_{l,n+1-r})^{-1}, \quad n = 2h+1. \quad (37)$$

7. RAISING AND LOWERING OPERATORS OF $O(n)$

The orthogonal group admits the canonical chain of subgroups

$$O(n) \supset O(n-1) \supset \cdots \supset O(2). \quad (38)$$

The irreducible representations of the groups $O(m)$ $(m = 2,...,n)$ are characteristized by partitions $(\lambda_{1m}, \lambda_{2m}, ..., \lambda_{hm})$ $(h = [m/2])$ which satisfy the inequalities

$$\lambda_{1,m} \geqslant \lambda_{2m} \geqslant \cdots \geqslant \lambda_{h-1,m} \geqslant |\lambda_{h,m}|, \quad m = 2h,$$
$$\lambda_{1,m} \geqslant \lambda_{2,m} \geqslant \cdots \geqslant \lambda_{h-1,m} \geqslant \lambda_{h,m} \geqslant 0., \quad m = 2h+1,$$

where the labels are simultaneously all integers or all half odd integers.

The partitions of two groups $O(m+1)$ and $O(m)$ in the canonical chain (38) are related by inequalities

$$\lambda_{1,2p+1} \geqslant \lambda_{1,2p} \geqslant \lambda_{2,2p+1} \geqslant \lambda_{2,2p} \geqslant \cdots \geqslant \lambda_{p,2p+1} \geqslant \lambda_{p,2p} \geqslant -\lambda_{p,2p+1},$$
$$\lambda_{1,2p} \geqslant \lambda_{1,2p-1} \geqslant \lambda_{2,2p} \geqslant \lambda_{2,2p-1} \geqslant \cdots \geqslant \lambda_{p-1,2p-1} \geqslant |\lambda_{p2p}|.$$

The set of partitions for the chain (38) may be rearranged into a Gel'fand pattern which has been described by Gel'fand and Zetlin[24] and appears in the paper by Pang and Hecht.[20]

Following our $U(n)$ notation we shall denote a maximal weight state of $O(m)$ with representation label (λ_{im}) contained in a representation of $O(m+1)$ with representation label (λ_{im+1}) by

$$\left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle.$$

Let us also denote the components $\psi_m[r]^r$ of the $O(m)$ vector $\psi_m^i = (M^{-1})^i_j \alpha^j_{m+1}$ simply by ψ_m^r.

In order to incorporate all possible shifts we need only consider the operators ψ_m^r and their adjoints $\psi_m^{\dagger r}$ for values of r in the range $r = 1,...,h$. Of course, in the case when $m = 2h+1$ is odd one may also consider the zero shift operator ψ_m^0 and its adjoint $\psi_m^{\dagger 0}$ defined by $\psi_m^0 = \psi_m[h+1]^{h+1}$. However, we do not require the zero shift operator for our purposes.

We then have

$$\psi_m^r \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle \propto \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} + \delta_{ir} \end{matrix} \right\rangle,$$

$$\psi_m^{\dagger r} \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle \propto \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} - \delta_{ir} \end{matrix} \right\rangle, \quad r = 1,...,h.$$

The normalization constants N_m^r of the lowering operators $\psi_m^{\dagger r}$ and the normalization constants $\bar{N}_m^r$ of the raising operators ψ_m^r are given by

$$N_m^r = \left\langle \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right| \bar{m}_r P[r]^r_r \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle^{1/2}, \quad (39)$$

$$\bar{N}_m^r = \left\langle \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right| m_r \bar{P}[r]^r_r \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle^{1/2}. \quad (40)$$

Due to differences in the normalization associated with $O(m)$ for m odd and even we shall now consider each case separately.

$O(m = 2h+1)$

In this case we have, in accordance with Eqs. (30) and (37),

$$\bar{m}_r \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle = (-1)^m \prod_{p=1}^{m+1} (\lambda_{pm+1} - \lambda_{rm} + r - p + 1) \times \prod_{l \neq r} (\lambda_{rm} - \lambda_{lm} + l - r - 1 + \delta_{l,h+1} - \delta_{l,n+1-r})^{-1} \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle,$$

$$P[r]^r_r \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle = \prod_{l>r} \left(\frac{\lambda_{rm} - \lambda_{lm} - 1 + \delta_{l,h+1} - \delta_{l,n+1-r} + l - r}{\lambda_{rm} - \lambda_{lm} + l - r} \right) \times \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle,$$

which gives

$$N_m^r = \left[(-1)^m \prod_{p=1}^{m+1} (\lambda_{pm+1} - \lambda_{rm} + r - p + 1) \times \prod_{l>r} (\lambda_{rm} - \lambda_{lm} + l - r)^{-1} \prod_{l<r} (\lambda_{rm} - \lambda_{lm} + l - r - 1)^{-1} \right]^{1/2}.$$

We also have, in accordance with Eqs. (31) and (37)

$$m_r \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle = (-1)^m \prod_{p=1}^{m+1} (\lambda_{pm+1} - \lambda_{rm} + r - p) \times \prod_{l \neq r} (\lambda_{rm} - \lambda_{lm} + l - r + 1 - 2\delta_{l,h+1} + \delta_{l,n+1-r})^{-1} \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle,$$

$$\bar{P}[r]^r_r \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle = \prod_{l<r} \left(\frac{\lambda_{rm} - \lambda_{lm} + l - r + 1 - 2\delta_{l,h+1} + \delta_{l,n+1-r}}{\lambda_{rm} - \lambda_{lm} + l - r} \right) \times \left| \begin{matrix} \lambda_{im+1} \\ \lambda_{im} \end{matrix} \right\rangle,$$

where we have made use of the identities

$$\bar{\beta}_p - \bar{\alpha}_r = \alpha_r - \beta_p + 1,$$
$$\bar{\alpha}_r - \bar{\alpha}_l = \alpha_l - \alpha_r + \delta_{l,h+1}, \quad \text{for } r \neq h+1.$$

Substituting these expressions into Eq. (40) one obtains

$$\bar{N}^r_m = \Big[(-1)^m \prod_{p=1}^{m+1} (\lambda_{pm+1} - \lambda_{rm} + r - p) \times \prod_{l<r} (\lambda_{rm} - \lambda_{lm} + l - r)^{-1} \prod_{l>r} (\lambda_{rm} - \lambda_{lm} + l - r + 1 - 2\delta_{l,h+1} + \delta_{l,n+1-r})^{-1}\Big]^{1/2}.$$

O(m = 2h)

As for the case m odd we may substitute the formulas (30), (31), and (37) into Eqs. (39) and (40) to give

$$N^r_m = \Big[(-1)^m \prod_{p=1}^{m+1} (\lambda_{pm+1} - \lambda_{rm} + r - p + 1 - \delta_{p,h+1}) \times \prod_{l>r} (\lambda_{rm} - \lambda_{lm} + l - r)^{-1} \prod_{l<r} (\lambda_{rm} - \lambda_{lm} + l - r - 1)^{-1}\Big]^{1/2},$$

$$\bar{N}^r_m = \Big[(-1)^m \prod_{p=1}^{m+1} (\lambda_{pm+1} - \lambda_{rm} + r - p - \delta_{p,h+1}) \prod_{l<r} (\lambda_{rm} - \lambda_{lm} + l - r)^{-1} \prod_{l>r} (\lambda_{rm} - \lambda_{lm} + l - r + 1 + \delta_{l,n+1-r})^{-1}\Big]^{1/2},$$

where we have made use of the identities

$$\bar{\beta}_p - \bar{\alpha}_r = \alpha_r - \beta_p + 1 + \delta_{p,h+1}$$

and

$$\bar{\alpha}_r - \bar{\alpha}_l = \alpha_l - \alpha_r.$$

Our normalized lowering and raising operators are therefore given by $(N^r_m)^{-1}\psi^{\dagger r}_m$ and $(\bar{N}^r_m)^{-1}\psi^r_m$, respectively, for values of r in the range $r = 1,\dots,h$.

8. GENERALIZATIONS (see Refs. 3 and 17)

Let L be a semisimple Lie algebra with Cartan subalgebra H and let Φ be the set of roots of L relative to H. Let Δ denote a base of L and let Φ^+ denote the corresponding set of positive roots. Finally let Λ denote the set of integral weights and $\Lambda^+ \subset \Lambda$ the set of dominant integral weights.

Fix a basis $\{x_1,\dots,x_l\}$ ($l = \dim L$) of L and let $\{x^1,\dots,x^l\}$ be the corresponding dual basis with respect to the Killing form on L. Let $V(\lambda)$, $\lambda \in \Lambda^+$, be a finite dimensional irreducible module over L with highest weight λ and let π_λ be the representation afforded by $V(\lambda)$. One may then consider the operator

$$A = -\frac{1}{2}\sum_{r=1}^{l} [\pi_\lambda(x^r) \otimes x_r + \pi_\lambda(x_r) \otimes x^r],$$

which may be regarded as a $d \times d$ matrix [$d = \dim V(\lambda)$] with entries from L:

$$A_{ij} = -\frac{1}{2}\sum_{r=1}^{l} [\pi_\lambda(x^r)_{ij} x_r + \pi_\lambda(x_r)_{ij} x^r],$$

where $\pi_\lambda(x)$ is the matrix [with respect to some suitably chosen basis in $V(\lambda)$] representing the generator under the representation π_λ.

The matrix A is clearly a generalization of the matrices considered for U(n) and O(n). Upon setting π_λ to be the contragredient vector representation of U(n) and choosing the basis $a^i_j (i,j = 1,\dots,n)$ one obtains the matrix with entries (a^i_j).

If $V(\mu)$, $\mu \in \Lambda^+$, is a finite dimensional representation of L, with highest weight μ, then acting on $V(\mu)$ the matrix A may be written

$$A = -\frac{1}{2}\sum_{r=1}^{l} [\pi_\lambda(x^r) \otimes \pi_\mu(x_r) + \pi_\lambda(x_r) \otimes \pi_\mu(x^r)].$$

A may clearly be regarded as an operator on the product space $V(\lambda) \otimes V(\mu)$. If $\lambda_1,\dots,\lambda_k$ are the distinct weights occurring in $V(\lambda)$ then the matrix A, when acting on $V(\mu)$, satisfies the polynomial identity[3]

$$\prod_{i=1}^{k} [A - \tfrac{1}{2}(\lambda,\lambda + 2\delta) - \tfrac{1}{2}(\lambda_i, 2(\mu + \delta) + \lambda_i)] = 0, \tag{41}$$

where (,) is the inner product induced on the weights by the Killing form and δ is the half sum of the positive roots (see Humphreys[25]).

One may write the Clebsch–Gordon reduction of the product space $V(\lambda) \otimes V(\mu)$ in the symbolic form[3,25]

$$V(\lambda) \otimes V(\mu) = \bigoplus_{i=1}^{k} n(i) V(\mu + \lambda_i),$$

where $V(\mu + \lambda_i)$ is a finite dimensional irreducible representation, which admits the infinitesimal character $\chi_{\mu+\lambda_i}$ (in the notation of Humphreys[25]), which is unique if it exists. The multiplicities $n(i)$ may be obtained using Klimyk's formula[26] (see also Refs. 3).

Using the identity (41) one may construct the projection operators

$$P[i] = \prod_{\substack{j=1 \\ \neq i}}^{k} \left\{ \frac{A - \frac{1}{2}(\lambda,\lambda + 2\delta) - \frac{1}{2}(\lambda_j, 2(\mu + \delta) + \lambda_j)}{\frac{1}{2}[(\lambda_i, 2(\mu + \delta) + \lambda_i) - \frac{1}{2}(\lambda_j, 2(\mu + \delta) + \lambda_j)]} \right\}$$

$$i = 1,\dots,k,$$

which project the space $V(\lambda) \otimes V(\mu)$ onto the subspace $V(\mu + \lambda_i)$. The matrix elements of the entries of the projector between basis states in the space $V(\mu)$ are therefore bilinear combinations of Clebsch–Gordan coefficients of the form

$$\left\langle \begin{matrix} \mu + \lambda_i \\ (\nu) \end{matrix} \middle| \begin{matrix} \lambda \\ (\rho) \end{matrix} ; \begin{matrix} \mu \\ (\tau) \end{matrix} \right\rangle, \tag{42}$$

where

$$\left| \begin{matrix} \lambda \\ (\rho) \end{matrix} ; \begin{matrix} \mu \\ (\tau) \end{matrix} \right\rangle,$$

denotes the product state

$$\left| \begin{matrix} \lambda \\ (\rho) \end{matrix} \right\rangle \otimes \left| \begin{matrix} \mu \\ (\tau) \end{matrix} \right\rangle$$

and where

$$\left| \begin{matrix} \lambda \\ (\rho) \end{matrix} \right\rangle$$

refers to a state in the representation $V(\lambda)$ where (ρ) denotes a set of labels used to distinguish the basis states. This then opens up the possibility of determining the Wigner coefficients (42) by exploiting the properties of the projectors $P[r]$. The nicest case occurs when the weights in $V(\lambda)$ all occur with multiplicity one. Then the tensor product is multiplicity free, which is precisely the case considered in our treatment of O(n) and U(n).

In particular, one may consider more general Wigner coefficients for the unitary group by choosing our reference representation π_λ to be one of the tensor representations.

Returning to the general case, on unitary representations of the group, the generators behave under Hermitian conjugation like $(x^i)^\dagger = x_i$. In particular, if x_α is an element of the root space L_α, corresponding to the root $\alpha\in\Phi$, then one may deduce $(x_\alpha)^\dagger \in L_{-\alpha}$.

A tensor operator with highest weight $\lambda\in\Lambda^+$ is defined to be a collection of operators T_i, $i = 1,\ldots,d$, which transform under commutation with elements of L like the representation π_λ:

$$[x,T_i] = \pi_\lambda(x)^j_{\ i} T_j.$$

One may project out the shift components $T[i]$ of the tensor T by applying the projector $P[i]$[15]:

$$T[i] = TP[i].$$

One may deduce the relation

$$T[i]T^\dagger[i] = M(i)P[i], \tag{43}$$

where $T^\dagger[i] = (T[i])^\dagger$ are the shift components of the contragredient tensor $T^\dagger$. The quantity $M(i)$ appearing in Eq. (43) is the reduced matrix element which may be evaluated using

$$M(i) = \frac{\Sigma_{j=1}^d T[i]_j T^\dagger[i]^j}{t_r(P[i])},$$

where $t_r(P[i])$ (the trace of the projector $P[i]$) may be evaluated from the formula[3]

$$t_r(P[i]) = c(i) \prod_{\alpha\in\Phi^+} \frac{(\mu+\delta+\lambda_i,\alpha)}{(\mu+\delta,\alpha)},$$

where $c(i)$ is the multiplicity of the weight λ_i in $V(\lambda)$.

Equation (43) is clearly a generalization of Eq. (16) and may be used in a similar way (at least for unit multiplicities). One may choose a basis for $V(\lambda)$ to be a weight basis (i.e., a basis of simultaneous eigenvectors of the Cartan subalgebra). Suppose that the basis is arranged so that the ith basis vector has weight λ_i. Then as before one may show that the operator $T[i]_i$ takes a maximal weight vector of L of weight μ into a maximal weight vector of weight $\mu+\lambda_i$. By this means we may construct generalization raising and lowering operators for more general groups which may be normalized using the equation

$$T[i]^i T^\dagger[i]_i = M(i)(P[i])^i_i.$$

In particular, such considerations are important in the labeling problems where a Lie algebra is embedded in a larger Lie algebra K which is separated from K by an irreducible tensor operator

$$K = L \oplus_\perp T$$

where

$$[L,T]\subseteq T, \quad [T,T]\subseteq L.$$

ACKNOWLEDGMENTS

The author would like to thank Professor H.S. Green who provided the foundations on which much of our work has been based. He would also like to thank Professor C.A. Hurst for his encouragement and for proof reading the manuscript. Finally, the author acknowledges the financial support of a Commonwealth Postgraduate Research Award.

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