Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **25:3** (2011), 163–173 DOI: 10.2298/FIL1103163P

ON AN INTEGRAL-TYPE OPERATOR FROM $\mathcal{Q}_K(p,q)$ SPACES TO α-BLOCH SPACES

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Abstract

Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be an analytic self-map of $\mathbb{D}$. We study the boundedness and compactness of the integral operator $C^n_{\varphi,g}$, which is defined by

$$(C^n_{\varphi,g}f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}),$$

from $\mathcal{Q}_K(p,q)$ and $\mathcal{Q}_{K,0}(p,q)$ spaces to α-Bloch spaces and little α-Bloch spaces.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let $\alpha > 0$. An $f \in H(\mathbb{D})$ is said to belong to the α-Bloch space, denoted by $\mathcal{B}^\alpha$, if

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty. \tag{1}$$

Under the above norm, $\mathcal{B}^\alpha$ is a Banach space. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the classical Bloch space. Let $\mathcal{B}^\alpha_0$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ for which $(1-|z|^2)^\alpha |f'(z)| \to 0$ as $|z| \to 1$. This space is called the little α-Bloch space.

Let $g(z,a)$ be the Green function with logarithmic singularity at a, i.e. $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$ (φ_a is a conformal automorphism defined by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in \mathbb{D}$). Let $p > 0$, $q > -2$, $K : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function. An $f \in H(\mathbb{D})$ is said to belong to $\mathcal{Q}_K(p,q)$ space if (see [9, 29])

$$\|f\| = \Big(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z)\Big)^{1/p} < \infty, \tag{2}$$

2010 *Mathematics Subject Classifications.* Primary 47B35, Secondary 30H05.
Key words and Phrases. $\mathcal{Q}_K(p,q)$ space, α-Bloch space, integral-type operator.
Received: March 07, 2011
Communicated by Dragana Cvetković Ilić
The author would like to thank the referee for his/her helpful comments.

where dA is the normalized Lebesgue area measure in $\mathbb{D}$. For $p \geq 1$, under the norm $\|f\|_{\mathcal{Q}_K(p,q)} = |f(0)| + \|f\|$, $\mathcal{Q}_K(p,q)$ is a Banach space. An $f \in H(\mathbb{D})$ is said to belong to $\mathcal{Q}_{K,0}(p,q)$ space if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) = 0. \tag{3}$$

Throughout the paper we assume that (see [29])

$$\int_0^1 (1-r^2)^q K(-\log r) r dr < \infty, \tag{4}$$

since otherwise $\mathcal{Q}_K(p,q)$ consists only of constant functions.

Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by $C_\varphi(f)(z) = f(\varphi(z)), f \in H(\mathbb{D})$. In [4], Li and Stević defined the generalized composition operator as follows

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

The generalized composition operator and its generalizations on various spaces were investigated in [4–7, 13, 14, 19, 21, 22, 24, 28, 30–33, 35, 36]. See, e.g., [1, 11] and the references therein for the study of the composition operator.

Let $g \in H(\mathbb{D})$, n be a nonnegative integer and φ be an analytic self-map of $\mathbb{D}$. In [38], the author defined a new integral-type operator as follows:

$$(C_{\varphi,g}^n f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).$$

$C_{\varphi,g}^1$ is the generalized composition operator C_φ^g. When $n = 0$, then $C_{\varphi,g}^0$ is the Volterra composition operator defined by Li in [3], extended by Stević in the n-dimensional case in [16] and subsequently studied in [15, 17, 18, 20, 23, 25–27].

Here we characterized the boundedness and compactness of the operator $C_{\varphi,g}^n$ from $\mathcal{Q}_K(p,q)$ and $\mathcal{Q}_{K,0}(p,q)$ to α-Bloch and little α-Bloch spaces.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

2 Main results and proofs

In this section we give our main results and proofs. For this purpose, we need some auxiliary results. The following lemma can be proved in a standard way (see, e.g., [10]).

Lemma 1. *Let $\alpha, p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0, \infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$ and n is a nonnegative integer. Then $C_{\varphi,g}^n : \mathcal{Q}_K(p,q)$(or $\mathcal{Q}_{K,0}(p,q)$) $\to \mathcal{B}^\alpha$ is compact if and only if $C_{\varphi,g}^n$:*

$\mathcal{Q}_K(p,q)$(or $\mathcal{Q}_{K,0}(p,q)$) $\to \mathcal{B}^\alpha$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{Q}_K(p,q)$(or $\mathcal{Q}_{K,0}(p,q)$) which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|C^n_{\varphi,g}f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k\to\infty$.

The following lemma is essentially proved in [8], hence we omit its proof.

Lemma 2. *A closed set K in $\mathcal{B}^\alpha_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1^-}\sup_{f\in K}(1-|z|^2)^\alpha|f'(z)|=0.$$

Lemma 3. [29] *Let $p>0$, $q>-2$ and K is a nonnegative nondecreasing function on $[0,\infty)$. For $f\in\mathcal{Q}_K(p,q)$, we have $f\in\mathcal{B}^{\frac{q+2}{p}}$ and*

$$\|f\|_{\mathcal{B}^{\frac{q+2}{p}}}\le\|f\|_{\mathcal{Q}_K(p,q)}. \tag{5}$$

Lemma 4. [12] *Let $f\in\mathcal{B}^\alpha, 0<\alpha<\infty$. Then*

$$|f(z)|\le\begin{cases} C\|f\|_{\mathcal{B}^\alpha} & ,\quad 0<\alpha<1;\\ C\|f\|_{\mathcal{B}^\alpha}\ln\frac{e}{1-|z|} & ,\quad \alpha=1;\\ C\frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}} & ,\quad \alpha>1.\end{cases}$$

Now we are in a position to state and prove the main results of this paper.

Theorem 1. *Let $\alpha,p>0$, $q>-2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that*

$$\int_0^1 K(-\log r)(1-r)^{\min\{-1,q\}}\Big(\log\frac{1}{1-r}\Big)^{\chi_{-1}(q)}rdr<\infty, \tag{6}$$

where $\chi_O(x)$ denote the characteristic function of the set O. Assume that φ is an analytic self-map of $\mathbb{D}$ and $n\in\mathbb{N}$. Then the following statements are equivalent.

(i) $C^n_{\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{B}^\alpha$ is bounded;

(ii) $C^n_{\varphi,g}:\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}^\alpha$ is bounded;

(iii)

$$M_1:=\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}<\infty. \tag{7}$$

Proof. $(iii)\Rightarrow(i)$. Suppose that (7) holds. First it is easy to see that $(C^n_{\varphi,g}f)(0)=0$ and $(C^n_{\varphi,g}f)'(z)=f^{(n)}(\varphi(z))g(z)$ for every $f\in H(\mathbb{D})$. For any

$z \in \mathbb{D}$ and $f \in \mathcal{Q}_K(p,q)$, by Lemma 3 we have

$$\begin{aligned}(1-|z|^2)^\alpha |(C^n_{\varphi,g}f)'(z)| &= (1-|z|^2)^\alpha |f^{(n)}(\varphi(z))g(z)| \\ &\leq \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} \|f\|_{\mathcal{B}^{\frac{q+2}{p}}} \\ &\leq \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} \|f\|_{\mathcal{Q}_K(p,q)}, \qquad (8)\end{aligned}$$

where we have used the following well-known characterization for α-Bloch functions (see, e.g., [34])

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| \asymp |f'(0)| + \cdots + |f^{(n-1)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{n+\alpha-1}|f^{(n)}(z)|.$$

Taking the supremum in (8) for $z \in \mathbb{D}$, then employing (7) we obtain that $C^n_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is bounded.

$(i) \Rightarrow (ii)$. It is clear.

$(ii) \Rightarrow (iii)$. Suppose that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded, i.e. there exists a constant C such that $\|C^n_{\varphi,g}f\|_{\mathcal{B}^\alpha} \leq C\|f\|_{\mathcal{Q}_K(p,q)}$ for all $f \in \mathcal{Q}_{K,0}(p,q)$. Taking the function $f(z) \equiv z^n$, which belongs to $\mathcal{Q}_{K,0}(p,q)$, we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |g(z)| < \infty. \qquad (9)$$

For $w \in \mathbb{D}$, let $f_w(z) = \frac{1-|w|^2}{(1-z\overline{w})^{\frac{q+2}{p}}}$. Using the condition (6), we see that $f_w \in \mathcal{Q}_{K,0}(p,q)$, for each $w \in \mathbb{D}$ (see [2]), moreover there is a positive constant C such that $\sup_{w\in\mathbb{D}} \|f_w\|_{\mathcal{Q}_K(p,q)} \leq C$ and

$$|f_w^{(n)}(w)| = \prod_{j=0}^{n-1}\Big(\frac{q+2}{p}+j\Big)\frac{|w|^n}{(1-|w|^2)^{\frac{q+2-p}{p}+n}}.$$

Hence,

$$\begin{aligned}\infty &> C\|C^n_{\varphi,g}\|_{\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}^\alpha} \geq \|C^n_{\varphi,g}f_{\varphi(\lambda)}\|_{\mathcal{B}^\alpha} \\ &\geq \prod_{j=0}^{n-1}\Big(\frac{q+2}{p}+j\Big)\frac{(1-|\lambda|^2)^\alpha |g(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^{\frac{q+2-p}{p}+n}} \qquad (10)\end{aligned}$$

for each $\lambda \in \mathbb{D}$.

From (10), we have

$$\begin{aligned}\sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{(1-|\lambda|^2)^\alpha |g(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{q+2-p}{p}+n}} &\leq 2^n \sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{(1-|\lambda|^2)^\alpha |g(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^{\frac{q+2-p}{p}+n}} \\ &\leq C\|C^n_{\varphi,g}\|_{\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}^\alpha} < \infty. \qquad (11)\end{aligned}$$

Inequality (9) gives

$$\sup_{|\varphi(\lambda)|\le\frac{1}{2}}\frac{(1-|\lambda|^2)^\alpha|g(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\frac{q+2-p}{p}+n}} \le \frac{4^{\frac{q+2-p}{p}+n}}{3^{\frac{q+2-p}{p}+n}}\sup_{|\varphi(\lambda)|\le\frac{1}{2}}(1-|\lambda|^2)^\alpha|g(\lambda)|<\infty, \quad (12)$$

where we used the assumption $(q+2-p)/p+n>0$. Therefore, (7) follows from (11) and (12). This completes the proof of Theorem 1. □

Theorem 2. *Let $\alpha, p>0$, $q>-2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$ and $n\in\mathbb{N}$. Then the following statements are equivalent.*

(i) $C^n_{\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{B}^\alpha$ is compact;

(ii) $C^n_{\varphi,g}:\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}^\alpha$ is compact;

(iii) $C^n_{\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{B}^\alpha$ is bounded and

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}=0. \quad (13)$$

Proof. *(iii)* ⇒ *(i)*. Suppose that $C^n_{\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{B}^\alpha$ is bounded and (13) holds. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{Q}_K(p,q)$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{Q}_K(p,q)}\le C$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. By the assumption, for any $\varepsilon>0$, there exists a $\delta\in(0,1)$ such that

$$\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}<\varepsilon \quad (14)$$

when $\delta<|\varphi(z)|<1$. Since $C^n_{\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{B}^\alpha$ is bounded, then from the proof of Theorem 1 we have

$$M_2:=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g(z)|<\infty. \quad (15)$$

Let $\Omega=\{z\in\mathbb{D}:|\varphi(z)|\le\delta\}$. Then, we have

$$\begin{aligned}&\|C^n_{\varphi,g}f_k\|_{\mathcal{B}^\alpha}=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(C^n_{\varphi,g}f_k)'(z)|\\ \le\;&\sup_{\Omega}(1-|z|^2)^\alpha|g(z)||f_k^{(n)}(\varphi(z))|+\sup_{\mathbb{D}\backslash\Omega}(1-|z|^2)^\alpha|g(z)||f_k^{(n)}(\varphi(z))|\\ \le\;&\sup_{\Omega}(1-|z|^2)^\alpha|g(z)||f_k^{(n)}(\varphi(z))|+C\sup_{\mathbb{D}\backslash\Omega}\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}\|f_k\|_{\mathcal{Q}_K(p,q)}\\ \le\;&M_2\sup_{|w|\le\delta}|f_k^{(n)}(w)|+C\varepsilon\|f_k\|_{\mathcal{Q}_K(p,q)}. \quad (16)\end{aligned}$$

¿From Cauchy's estimate and the assumption that $f_k\to0$ as $k\to\infty$ on compact subsets of $\mathbb{D}$, we see that $f_k^{(n)}\to0$ as $k\to\infty$ on compact subsets of $\mathbb{D}$. Letting

$k \to \infty$ in (16) and using the fact that ε is an arbitrary positive number, we obtain $\lim_{k\to\infty} \|C^n_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} = 0$. Applying Lemma 1, the result follows.

$(i) \Rightarrow (ii)$. This implication is obvious.

$(ii) \Rightarrow (iii)$. Suppose that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is compact. Then it is clear that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded and from Theorem 1 we see that $C^n_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is bounded. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist then condition (13) is vacuously satisfied). Let $f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^{\frac{q+2}{p}}}$. Then, $f_k \in \mathcal{Q}_{K,0}(p,q)$, $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{Q}_K(p,q)} < \infty$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is compact, by Lemma 1 we have

$$\lim_{k\to\infty} \|C^n_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} = 0. \tag{17}$$

On the other hand, from (10) we have

$$\|C^n_{\varphi,g} f_k\|_{\mathcal{B}^\alpha} \geq \prod_{j=0}^{n-1} \Big(\frac{q+2}{p} + j\Big) \frac{(1-|z_k|^2)^\alpha |g(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\frac{2+q-p}{p}+n}}$$

which together with (17) implies that

$$\lim_{|\varphi(z_k)|\to 1} \frac{(1-|z_k|^2)^\alpha |g(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\frac{2+q-p}{p}+n}} = \lim_{k\to\infty} \frac{(1-|z_k|^2)^\alpha |g(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\frac{2+q-p}{p}+n}} = 0, \tag{18}$$

from which (13) easily follows. $\square$

Theorem 3. *Let $\alpha, p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$ and $n \in \mathbb{N}$. Then $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is bounded if and only if $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded and*

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |g(z)| = 0. \tag{19}$$

Proof. Suppose that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is bounded. It is obvious that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded. Taking the function $f(z) = z^n$, and employing the boundedness of $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ we see that (19) holds.

Conversely, assume that $C^n_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded and (19) holds. Then, for each polynomial $p(z)$, we have that

$$(1-|z|^2)^\alpha |(C^n_{\varphi,g} p)'(z)| \leq (1-|z|^2)^\alpha |g(z)| \|p^{(n)}\|_\infty,$$

from which it follows that $C^n_{\varphi,g} p \in \mathcal{B}^\alpha_0$. Since the set of all polynomials is dense in $\mathcal{Q}_{K,0}(p,q)$ (see [2]), we have that for every $f \in \mathcal{Q}_{K,0}(p,q)$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|f - p_k\|_{\mathcal{Q}_K(p,q)} \to 0$, as $k \to \infty$. Hence

$$\|C^n_{\varphi,g} f - C^n_{\varphi,g} p_k\|_{\mathcal{B}^\alpha} \leq \|C^n_{\varphi,g}\|_{\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}^\alpha} \|f - p_k\|_{\mathcal{Q}_K(p,q)} \to 0$$

as $k\to\infty$. Since $\mathcal{B}_0^\alpha$ is closed subset of $\mathcal{B}^\alpha$, we obtain $C^n_{\varphi,g}(\mathcal{Q}_{K,0}(p,q))\subset\mathcal{B}_0^\alpha$. Therefore $C^n_{\varphi,g}:\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}_0^\alpha$ is bounded. □

Theorem 4. *Let $\alpha,p>0$, $q>-2$ and K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$ and $n\in\mathbb{N}$. Then the following statements are equivalent.*

(i) $C^n_{\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{B}_0^\alpha$ is compact;

(ii) $C^n_{\varphi,g}:\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}_0^\alpha$ is compact;

(iii)

$$\lim_{|z|\to1}\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}=0. \tag{20}$$

Proof. $(iii)\Rightarrow(i)$. Assume that (20) holds. Let $f\in\mathcal{Q}_K(p,q)$. By the proof of Theorem 1 we have

$$(1-|z|^2)^\alpha|(C^n_{\varphi,g}f)'(z)|\le C\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}\|f\|_{\mathcal{Q}_K(p,q)}. \tag{21}$$

Taking the supremum in (21) over all $f\in\mathcal{Q}_K(p,q)$ such that $\|f\|_{\mathcal{Q}_K(p,q)}\le1$, then letting $|z|\to1$, we get

$$\lim_{|z|\to1}\sup_{\|f\|_{\mathcal{Q}_K(p,q)}\le1}(1-|z|^2)^\alpha|(C^n_{\varphi,g}f)'(z)|=0.$$

¿From which by Lemma 2 we see that $C^n_{\varphi,g}:\mathcal{Q}_K(p,q)\to\mathcal{B}_0^\alpha$ is compact.

$(i)\Rightarrow(ii)$. This implication is obvious.

$(ii)\Rightarrow(iii)$. Suppose that $C^n_{\varphi,g}:\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}_0^\alpha$ is compact. Then $C^n_{\varphi,g}:\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}_0^\alpha$ is bounded and by Theorem 3 we get

$$\lim_{|z|\to1}(1-|z|^2)^\alpha|g(z)|=0. \tag{22}$$

If $\|\varphi\|_\infty<1$, from (22), we obtain that

$$\lim_{|z|\to1}\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}\le\frac{1}{(1-\|\varphi\|_\infty^2)^{\frac{2+q-p}{p}+n}}\lim_{|z|\to1}(1-|z|^2)^\alpha|g(z)|=0,$$

from which the result follows in this case.

Assume that $\|\varphi\|_\infty=1$. Let $(\varphi(z_k))_{k\in\mathbb{N}}$ be a sequence such that $\lim_{k\to\infty}|\varphi(z_k)|=1$. From the compactness of $C^n_{\varphi,g}:\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}_0^\alpha$ we see that the operator $C^n_{\varphi,g}:\mathcal{Q}_{K,0}(p,q)\to\mathcal{B}^\alpha$ is compact. ¿From Theorem 2 we get

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}=0 \tag{23}$$

From (23), we have that for every $\varepsilon>0$, there exists an $r\in(0,1)$ such that

$$\frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}}<\varepsilon \tag{24}$$

when $r < |\varphi(z)| < 1$. From (22), there exists a $\sigma \in (0,1)$ such that

$$(1-|z|^2)^\alpha |g(z)| \le \varepsilon(1-r^2)^{\frac{2+q-p}{p}+n} \tag{25}$$

when $\sigma < |z| < 1$.

Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have

$$\frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} < \varepsilon. \tag{26}$$

On the other hand, if $\sigma < |z| < 1$ and $|\varphi(z)| \le r$, we obtain

$$\frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}+n}} < \frac{1}{(1-r^2)^{\frac{2+q-p}{p}+n}}(1-|z|^2)^\alpha |g(z)| < \varepsilon. \tag{27}$$

¿From (26) and (27) we get (20), as desired. The proof is completed. □

Next, we consider the case $n = 0$.

Theorem 5. *Let $\alpha, p > 0$, $q > -2$ such that $q + 2 \ge p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is bounded;

(ii) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded;

(iii)

$$\begin{cases} \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |g(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \infty & , \quad q+2 = p; \\ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}}} < \infty & , \quad q+2 > p. \end{cases}$$

Proof. *(ii)* ⇒ *(iii)*. Assume that $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded. For $w \in \mathbb{D}$, let

$$f_w(z) = \begin{cases} \ln \frac{e}{1-z\overline{w}} & , \quad q+2 = p; \\ \frac{1-|w|^2}{(1-z\overline{w})^{\frac{q+2}{p}}} & , \quad q+2 > p. \end{cases}$$

Then $f_w \in \mathcal{Q}_{K,0}(p,q)$ (see [2]). The other proof is similar to the proof of Theorem 1 and hence we omit it.

(i) ⇒ *(ii)* is obvious.

(iii) ⇒ *(i)*. Using Lemma 4, similar to the proof of Theorem 1, the implication follows. We omit the details of the proofs.

Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Taking the test function

$$f_{\varphi(z_k)}(z) = \begin{cases} \ln \frac{e}{1-z\overline{\varphi(z_k)}} & , \quad q+2 = p; \\ \frac{1-|\varphi(z_k)|^2}{(1-z\overline{\varphi(z_k)})^{\frac{q+2}{p}}} & , \quad q+2 > p, \end{cases}$$

similar to the proof of Theorem 2, we obtain the following result.

Theorem 6. *Let $\alpha, p > 0$, $q > -2$ such that $q+2 \geq p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*
(i) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is compact;
(ii) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is compact;
(iii) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is bounded and

$$\begin{cases} \lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha |g(z)| \ln \frac{e}{1-|\varphi(z)|^2} = 0 & , \quad q+2 = p; \\ \lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}}} = 0 & , \quad q+2 > p. \end{cases}$$

Similar to the proofs of Theorems 3 and 4, we obtain Theorems 7 and 8 respectively. We omit the proofs.

Theorem 7. *Let $\alpha, p > 0$, $q > -2$ such that $q+2 \geq p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is bounded if and only if $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded and*

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |g(z)| = 0.$$

Theorem 8. *Let $\alpha, p > 0$, $q > -2$ such that $q+2 \geq p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*
(i) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha_0$ is compact;
(ii) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is compact;
(iii)

$$\begin{cases} \lim_{|z|\to 1}(1-|z|^2)^\alpha |g(z)| \ln \frac{e}{1-|\varphi(z)|^2} = 0 & , \quad q+2 = p; \\ \lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q-p}{p}}} = 0 & , \quad q+2 > p. \end{cases}$$

The proof of the following two theorems are similar to the proofs of Theorems 12-14 of [37]. We omit the details.

Theorem 9. *Let $\alpha, p > 0$, $q > -2$ such that $q+2 < p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*
(i) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is bounded;
(ii) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is bounded;
(iii) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha$ is compact;
(iv) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha$ is compact;

(v)

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |g(z)| < \infty.$$

Theorem 10. *Let $\alpha, p > 0$, $q > -2$ such that $q+2 < p$. Let K be a nonnegative nondecreasing function on $[0,\infty)$ such that (6) holds. Assume that φ is an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha_0$ is bounded;

(ii) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is bounded;

(iii) $C^0_{\varphi,g} : \mathcal{Q}_K(p,q) \to \mathcal{B}^\alpha_0$ is compact;

(iv) $C^0_{\varphi,g} : \mathcal{Q}_{K,0}(p,q) \to \mathcal{B}^\alpha_0$ is compact;

(v)

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |g(z)| = 0.$$

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