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Research Article

On an Integral-Type Operator from Zygmund-Type Spaces to Mixed-Norm Spaces on the Unit Ball

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The boundedness and compactness of an integral-type operator recently introduced by the author from Zygmund-type spaces to the mixed-norm space on the unit ball are characterized here.

1. Introduction

Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, $\partial\mathbb{B}$ its boundary, dV_N the normalized volume measure on $\mathbb{B}$, and $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$. Strictly positive, bounded, continuous functions on $\mathbb{B}$ are called *weights*.

For an $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\beta|\ge 0} a_\beta z^\beta$, let

$$\Re f(z) = \sum_{|\beta|\ge 0} |\beta| a_\beta z^\beta \tag{1.1}$$

be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \cdots + \beta_n$ and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$.

A positive, continuous function ν on the interval $[0,1)$ is called normal [1] if there are $\delta \in [0,1)$ and a and b, $0 < a < b$ such that

$$\begin{gathered}\frac{\nu(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1), \quad \lim_{r\to 1}\frac{\nu(r)}{(1-r)^a} = 0,\\ \frac{\nu(r)}{(1-r)^b} \text{ is increasing on } [\delta,1), \quad \lim_{r\to 1}\frac{\nu(r)}{(1-r)^b} = \infty.\end{gathered} \tag{1.2}$$

If we say that a function $\nu : \mathbb{B} \to [0, \infty)$ is normal, we also assume that it is radial, that is, $\nu(z) = \nu(|z|), z \in \mathbb{B}$.

Let μ be a weight. By $\mathcal{Z}_\mu(\mathbb{B}) = \mathcal{Z}_\mu$, we denote the class of all $f \in H(\mathbb{B})$ such that

$$z(f) := \sup_{z\in\mathbb{B}} \mu(z)\left|\Re^2 f(z)\right| < \infty, \tag{1.3}$$

and call it *the Zygmund-type class*. The quantity $z(f)$ is a seminorm. A norm on $\mathcal{Z}_\mu$ can be introduced by $\|f\|_{\mathcal{Z}} = |f(0)| + z(f)$. Zygmund-type class with this norm will be called the *Zygmund-type space*.

The *little Zygmund-type space* on $\mathbb{B}$, denoted by $\mathcal{Z}_{\mu,0}(\mathbb{B}) = \mathcal{Z}_{\mu,0}$, is the closed subspace of $\mathcal{Z}_\mu$ consisting of functions f satisfying the following condition

$$\lim_{|z|\to 1} \mu(z)\left|\Re^2 f(z)\right| = 0. \tag{1.4}$$

For $0 < p, q < \infty$, and ϕ normal, the mixed-norm space $H(p,q,\phi)(\mathbb{B}) = H(p,q,\phi)$ consists of all functions $f \in H(\mathbb{B})$ such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr\right)^{1/p} < \infty, \tag{1.5}$$

where

$$M_q(f,r) = \left(\int_{\partial\mathbb{B}} |f(r\zeta)|^q d\sigma(\zeta)\right)^{1/q}, \tag{1.6}$$

and $d\sigma$ is the normalized surface measure on $\partial\mathbb{B}$. For $p = q$, $\phi(r) = (1-r^2)^{(\alpha+1)/p}$, and $\alpha > -1$, the space is equivalent with the weighted Bergman space $A_\alpha^p(\mathbb{B})$.

In [2], the present author has introduced products of integral and composition operators on $H(\mathbb{B})$ as follows (see also [3–5]). Assume $g \in H(\mathbb{B})$, $g(0) = 0$, and φ is a holomorphic self-map of $\mathbb{B}$, then we define an operator on $H(\mathbb{B})$ by

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}),\ z \in \mathbb{B}. \tag{1.7}$$

The operator is an extension of the operator introduced in [6]. Here, we continue to study operator P_φ^g by characterizing the boundedness and compactness of the operator between Zygmund-type spaces and the mixed-norm space. For some results on related integral-type operators mostly in $\mathbb{C}^n$, see, for example, [3, 6–27] and the references therein.

In this paper, constants are denoted by C; they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Auxiliary Results

In this section, we quote several lemmas which are used in the proofs of the main results.

The first lemma was proved in [2].

Lemma 2.1. *Assume that φ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$, and $g(0) = 0$. Then, for every $f \in H(\mathbb{B})$ it holds*

$$\Re\left[P_{\varphi}^{g}(f)\right](z) = f(\varphi(z))g(z). \tag{2.1}$$

The next Schwartz-type characterization of compactness [28] is proved in a standard way (see, e.g., the proof of the corresponding lemma in [11]), hence we omit its proof.

Lemma 2.2. *Assume $p, q > 0$, φ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$, $g(0) = 0$, ϕ is normal, and μ is a weight. Then, the operator $P_{\varphi}^{g} : \mathcal{Z}_{\mu}$ (or $\mathcal{Z}_{\mu,0}$) $\rightarrow H(p,q,\phi)$ is compact if and only if for every bounded sequence $(f_k)_{k\in\mathbb{N}} \subset \mathcal{Z}_{\mu}$ (or $\mathcal{Z}_{\mu,0}$) converging to 0 uniformly on compacts of $\mathbb{B}$ we have $\lim_{k\to\infty}\|P_{\varphi}^{g} f_k\|_{H(p,q,\phi)} = 0$.*

The next lemma is folklore and can be found, for example, in [6] (one-dimensional case for standard power weights is due to Flett [29, Theorems 6 and 7]).

Lemma 2.3. *Assume that $0 < p, q < \infty$, ϕ is normal, and $m \in \mathbb{N}$. Then, the following asymptotic relationship holds for every $f \in H(\mathbb{B})$,*

$$\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr \asymp |f(0)|^p + \int_0^1 M_q^p(\Re^m f,r)(1-r)^{mp}\frac{\phi^p(r)}{1-r}dr. \tag{2.2}$$

Lemma 2.4. *Assume that μ is normal and $f \in \mathcal{Z}_{\mu}$. Then,*

$$|f(z)| \le C\|f\|_{\mathcal{Z}_{\mu}}\left(1 + \int_0^{|z|}\int_0^t \frac{ds}{\mu(s)}dt\right), \quad z \in \mathbb{B}. \tag{2.3}$$

Moreover, if

$$\int_0^1\int_0^t \frac{ds}{\mu(s)}dt < \infty, \tag{2.4}$$

then

$$|f(z)| \le C\|f\|_{\mathcal{Z}_{\mu}}, \tag{2.5}$$

for any $z \in \mathbb{B}$.

Proof. By Lemma 2.3.1 in [21] applied to $\Re f$ we have that

$$\left|\Re f(z)\right| \le C\|f\|_{\mathcal{Z}_\mu}\left(1+\int_0^{|z|}\frac{ds}{\mu(s)}\right). \tag{2.6}$$

Hence, for $|z| \ge 1/2$, we have that

$$\left|f(z)-f\left(\frac{z}{2}\right)\right| \le \int_{1/2}^{1}\left|\Re f(tz)\right|\frac{dt}{t} \le C\|f\|_{\mathcal{Z}_\mu}\int_{1/2}^{1}\left(1+\int_0^{t|z|}\frac{ds}{\mu(s)}\right)\frac{d(t|z|)}{|z|}, \tag{2.7}$$

so that

$$\left|f(z)\right| \le M_\infty\left(f,\frac{1}{2}\right)+C\|f\|_{\mathcal{Z}_\mu}\left(1+\int_0^{|z|}\int_0^{t}\frac{ds}{\mu(s)}dt\right), \tag{2.8}$$

where $M_\infty(f,1/2) = \max_{|z|\le 1/2}|f(z)|$.

If $|z| \le 1/2$, then by the mean value property of the function $f(z)-f(0)$ (see [30]), Jensen's inequality, and Parseval's formula, we obtain

$$\begin{aligned}\max_{|z|\le 1/2}\left|f(z)-f(0)\right|^2 &\le 4^n\int_{|z|\le 3/4}\left|f(w)-f(0)\right|^2 dV_N(w)\\ &\le 4^n\int_{|z|\le 3/4}\left|\Re f(w)\right|^2 dV_N(w)\\ &\le 3^n\max_{|z|\le 3/4}\left|\Re f(z)\right|^2.\end{aligned} \tag{2.9}$$

From (2.9) and (2.6), we obtain

$$\begin{aligned}M_\infty(f,1/2) &\le \left|f(0)\right|+\left(\sqrt{3}\right)^n\max_{|z|\le 3/4}\left|\Re f(z)\right|\\ &\le \left|f(0)\right|+\left(\sqrt{3}\right)^n C\|f\|_{\mathcal{Z}_\mu}\left(1+\int_0^{3/4}\frac{ds}{\mu(s)}\right)\\ &\le C\|f\|_{\mathcal{Z}_\mu}.\end{aligned} \tag{2.10}$$

From (2.8) and (2.10), (2.3) follows, from which by (2.4) the second statement follows. □

Lemma 2.5. *Assume μ is normal and* (2.4) *holds. Then, for every bounded sequence $(f_k)_{k\in\mathbb{N}} \subset \mathcal{Z}_\mu$ converging to 0 uniformly on compacts of $\mathbb{B}$, we have that*

$$\lim_{k\to\infty}\sup_{z\in\mathbb{B}}\left|f_k(z)\right| = 0. \tag{2.11}$$

Proof. From (2.4), we have that for every $\varepsilon > 0$, there is a $\delta \in (0, \min\{\varepsilon, 1/2\})$ such that

$$\int_{(1-\delta)|z|}^{|z|} \int_0^t \frac{ds}{\mu(s)} dt < \varepsilon, \tag{2.12}$$

for $|z| > 1 - \delta$.

Hence, from (2.12) it follows that for each $k \in \mathbb{N}$ and $|z| \geq 1 - \delta$

$$\begin{aligned} \left| f_k(z) - f_k((1-\delta)z) \right| &\leq \int_{1-\delta}^{1} \left| \Re f_k(tz) \right| \frac{dt}{t} \\ &\leq C \| f_k \|_{\mathcal{Z}_\mu} \int_{1-\delta}^{1} \left(1 + \int_0^{t|z|} \frac{ds}{\mu(s)} \right) dt \\ &\leq C \| f_k \|_{\mathcal{Z}_\mu} \left(\varepsilon + \int_{(1-\delta)|z|}^{|z|} \int_0^t \frac{ds}{\mu(s)} dt \right). \end{aligned} \tag{2.13}$$

From (2.12) and (2.13), we obtain

$$\left| f_k(z) \right| \leq \sup_{|w| \leq 1-\delta} \left| f_k(w) \right| + 2C\varepsilon \sup_{k \in \mathbb{N}} \| f_k \|_{\mathcal{Z}_\mu}. \tag{2.14}$$

Letting $k \to \infty$ in this inequality, using the assumption that f_k converges to 0 on the compact $|w| \leq 1 - \delta$, and using the fact that ε is an arbitrary positive number, the lemma follows. □

3. The Boundedness and Compactness of $P_\varphi^g : \mathcal{Z}_\mu$ (or $\mathcal{Z}_{\mu,0}$) $\to H(p,q,\phi)$

The boundedness and compactness of the operator $P_\varphi^g : \mathcal{Z}_\mu$ (or $\mathcal{Z}_{\mu,0}$) $\to H(p,q,\phi)$ are characterized in this section.

Theorem 3.1. *Assume that $p, q > 0$, φ is a holomorphic self-map of $\mathbb{B}$, $g \in H(\mathbb{B})$, $g(0) = 0$, ϕ and μ are normal, and μ satisfies condition* (2.4). *Let*

$$G(z) = \int_0^1 g(tz) \frac{dt}{t}. \tag{3.1}$$

Then, the following statements are equivalent:

(a) $P_\varphi^g : \mathcal{Z}_{\mu,0} \to H(p,q,\phi)$ *is bounded;*
(b) $P_\varphi^g : \mathcal{Z}_{\mu} \to H(p,q,\phi)$ *is bounded;*
(c) $P_\varphi^g : \mathcal{Z}_{\mu,0} \to H(p,q,\phi)$ *is compact;*
(d) $P_\varphi^g : \mathcal{Z}_{\mu} \to H(p,q,\phi)$ *is compact;*
(e) $G \in H(p,q,\phi)$.

Moreover, if $P_\varphi^g : \mathcal{Z}_\mu \to H(p,q,\phi)$ is bounded, then the following asymptotic relations hold:

$$\left\| P_\varphi^g \right\|_{\mathcal{Z}_\mu \to H(p,q,\phi)} \asymp \left\| P_\varphi^g \right\|_{\mathcal{Z}_{\mu,0} \to H(p,q,\phi)} \asymp \| G \|_{H(p,q,\phi)}. \tag{3.2}$$

Proof. The implications (d)$\Rightarrow$(b), (b)$\Rightarrow$(a), (d)$\Rightarrow$(c), and (c)$\Rightarrow$(a) are obvious.

(a)$\Rightarrow$(e) Since $P_\varphi^g : \mathcal{Z}_{\mu,0} \to H(p,q,\phi)$ is bounded and $f(z) \equiv 1 \in \mathcal{Z}_{\mu,0}$, by Lemma 2.1 we have that $G(z) = P_\varphi^g(1)(z) \in H(p,q,\phi)$. Moreover,

$$\|G\|_{H(p,q,\phi)} = \left\|P_\varphi^g(1)\right\|_{H(p,q,\phi)} \le \left\|P_\varphi^g\right\|_{\mathcal{Z}_{\mu,0}\to H(p,q,\phi)}. \tag{3.3}$$

(e)$\Rightarrow$(d) Assume that $(f_k)_{k\in\mathbb{N}} \subset \mathcal{Z}_\mu$ is a bounded sequence converging to 0 uniformly on compacts of $\mathbb{B}$. Then, by Lemmas 2.1, 2.3, and 2.5, we have

$$\begin{aligned}\left\|P_\varphi^g f_k\right\|_{H(p,q,\phi)} &\asymp \left|P_\varphi^g f_k(0)\right| + \left(\int_0^1 M_q^p(g f_k\circ\varphi, r)\frac{\phi^p(r)}{(1-r)^{1-p}}dr\right)^{1/p}\\ &\le C\|G\|_{H(p,q,\phi)}\sup_{z\in\mathbb{B}}\left|f_k(z)\right| \longrightarrow 0, \quad \text{as } k\longrightarrow\infty,\end{aligned} \tag{3.4}$$

which along with Lemma 2.2 implies the compactness of $P_\varphi^g : \mathcal{Z}_\mu \to H(p,q,\phi)$.

From (2.4) and by Lemmas 2.3 and 2.4, we have

$$\begin{aligned}\left\|P_\varphi^g f\right\|_{H(p,q,\phi)} &\le C\left(\int_0^1 M_q^p(g f\circ\varphi, r)\frac{\phi^p(r)}{(1-r)^{1-p}}dr\right)^{1/p}\\ &\le C\|f\|_{\mathcal{Z}_\mu}\left(\int_0^1 M_q^p(g, r)\frac{\phi^p(r)}{(1-r)^{1-p}}dr\right)^{1/p}\\ &\le C\|f\|_{\mathcal{Z}_\mu}\|G\|_{H(p,q,\phi)}.\end{aligned} \tag{3.5}$$

This, together with (3.3) and the inequality

$$\left\|P_\varphi^g\right\|_{\mathcal{Z}_{\mu,0}\to H(p,q,\phi)} \le \left\|P_\varphi^g\right\|_{\mathcal{Z}_\mu\to H(p,q,\phi)}, \tag{3.6}$$

implies the asymptotic relations in (3.2), as desired. □

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