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ON AN INVARIANT BILINEAR FORM ON THE SPACE OF AUTOMORPHIC FORMS VIA ASYMPTOTICS

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To my parents

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ABSTRACT

This thesis concerns the study of a new invariant bilinear form $\mathcal{B}$ on the space of automorphic forms of a split reductive group G over a global field. The form $\mathcal{B}$ is natural from the viewpoint of the geometric Langlands program.

First, we study a certain reductive monoid $\overline{M}$ associated to a parabolic subgroup P of G. The monoid $\overline{M}$ is used implicitly in the study of the geometry of Drinfeld's compactifications of the moduli stacks Bun_P and Bun_G. We show that $\overline{M}$ is a retract of the affine closure of the quasi-affine variety G/U, and we relate $\overline{M}$ to the Vinberg semigroup of G.

Second, we define the invariant bilinear form $\mathcal{B}$ over a function field using the asymptotics maps defined in Bezrukavnikov–Kazhdan [10] and Sakellaridis–Venkatesh [60] using the geometry of the wonderful compactification of G. We show that $\mathcal{B}$ is related to the miraculous duality functor studied by Drinfeld and Gaitsgory through the functions–sheaves dictionary. In the proof, we use the work of Schieder [62], which concerns the singularities of Drinfeld's compactification of Bun_G. We then give an alternate definition of $\mathcal{B}$, which extends to number fields, using the constant term operator and the inverse of the standard intertwining operator. The form $\mathcal{B}$ defines an invertible operator L from the space of compactly supported automorphic forms to a new space of "pseudo-compactly" supported automorphic forms. We give a formula for L^{-1} in terms of pseudo-Eisenstein series and constant term operators which suggests that L^{-1} is an analog of the Aubert–Zelevinsky involution.

Lastly, we study the Radon transform as an operator $R : \mathcal{C}_+ \to \mathcal{C}_-$ from the space of smooth K-finite functions on $F^n \setminus \{0\}$ with bounded support to the space of smooth K-finite functions on $F^n \setminus \{0\}$ supported away from a neighborhood of 0, where F is a (possibly Archimedean) local field. When $n = 2$, the Radon transform coincides with the standard intertwining operator. We prove that R is an isomorphism and provide explicit formulas for R^{-1}. These formulas in turn give a formula for $\mathcal{B}$ over a number field when $G = \mathrm{SL}(2)$.

CHAPTER 1

REDUCTIVE MONOID ASSOCIATED TO A PARABOLIC SUBGROUP

1.1 Introduction

1.1.1 Motivation

Let G be a connected reductive group over a perfect field k. Let U denote the unipotent radical of a parabolic subgroup P of G. Grosshans proved in [36] that the homogeneous space G/U is a quasi-affine variety and the algebra of regular functions $k[G/U]$ is finitely generated.

In [3], Arzhantsev and Timashev consider affine embeddings of G/U and give a detailed description of the *canonical embedding* $G/U \hookrightarrow \operatorname{Spec} k[G/U]$ under the assumption that the characteristic of k is 0. They establish a bijection between these affine embeddings and certain normal algebraic monoids with group of units equal to the Levi factor $M = P/U$. In particular, the canonical embedding corresponds to the monoid $\overline{M}$ defined as the closure of M in $\operatorname{Spec} k[G/U]$. This construction, which we first learned from [5], defines an affine algebraic monoid $\overline{M}$ in any characteristic. It is not *a priori* clear, however, whether the monoid $\overline{M}$ is normal in positive characteristic.

One of the goals of this chapter is to show that $\overline{M}$ is a normal algebraic monoid with group of units M in any characteristic, and to describe the combinatorial data it corresponds to under the classification of normal reductive monoids in [56, Theorem 5.4].

Let $\overline{G/U}$ denote the spectrum of $k[G/U]$. Then $\overline{G/U}$ is an affine variety of finite type, and it plays a prominent role in the definition of Drinfeld's compactification $\widetilde{\mathrm{Bun}}_P$ of the moduli stack of P-bundles over a smooth complete curve. Drinfeld's compactification is used to define the geometric Eisenstein series functors in [13]. As Baranovsky observes in [5, §6], the monoid $\overline{M}$ is used implicitly when studying the stratification of $\widetilde{\mathrm{Bun}}_P$. More specifically, the closed subscheme $\mathrm{Gr}_M^+ \subset \mathrm{Gr}_M$ of the affine Grassmannian (cf. [13, §6.2], [12, §1.6]) is just $(\overline{M}(O) \cap M(K))/M(O)$ inside $M(K)/M(O)$, where O is a complete discrete valuation ring with field of fractions K. The relative version of Gr_M^+ becomes $\mathcal{H}_M^+$, the positive part of the Hecke stack (cf. [12, §1.8]).

The stack $\mathcal{H}_M^+$ is therefore the global model for the formal arc space of the embedding $M \hookrightarrow \overline{M}$, as considered in [11, §2]. We hope that studying the properties of $\overline{M}$ will provide a better understanding of $\mathcal{H}_M^+$.

If P^- is a parabolic subgroup opposite to P, then G/U is closely related to the more symmetrically defined variety $\mathbb{X}_P = (G/U \times G/U^-)/(P \cap P^-)$, which is also quasi-affine. This variety $\mathbb{X}_P$ is called a *boundary degeneration* of G in [60] (when P is not a Borel subgroup, $\mathbb{X}_P$ is an *intermediate* degeneration), and it is a central object in the geometric proof of Bernstein's Second Adjointness Theorem in the theory of $\mathfrak{p}$-adic groups given in [10]. We note that this proof and the space $\mathbb{X}_P$ are closely related to the study of geometric constant term (and Eisenstein series) functors in [24].

The boundary degeneration $\mathbb{X}_P$ and its affine closure $\overline{\mathbb{X}}_P := \operatorname{Spec} k[\mathbb{X}_P]$ may be recovered from the *Vinberg semigroup* corresponding to G. The Vinberg semigroup $\overline{G_{\mathrm{enh}}}$ is used to define the Drinfeld-Lafforgue compactification $\overline{\mathrm{Bun}}_G$ (resp. the Drinfeld-Lafforgue-Vinberg compactification VinBun_G) of the moduli stack Bun_G in [62]. As one might expect, the positive part $\mathcal{H}_M^+$ of the Hecke stack appears in the stratification of $\overline{\mathrm{Bun}}_G$ (resp. VinBun_G), where P ranges over all conjugacy classes of parabolic subgroups, assuming that G is split. In this chapter we attempt to explain the relations between $\overline{M}$, $\overline{G/U}$, $\overline{\mathbb{X}}_P$, and $\overline{G_{\mathrm{enh}}}$ in hopes that it will elucidate the geometry underlying the aforementioned stratifications.

In [59], Sakellaridis fixes a strictly convex cone in the $\mathbb{Q}$-vector space spanned by the coweights of a split maximal torus T in G in order to "expand power series" on the boundary degeneration $\mathbb{X}_P$, under the assumption that the characteristic of k is 0. This cone is precisely the dual of what we call the *Renner cone* of $\overline{M}$. Thus the combinatorial description of $\overline{M}$ provides a first step towards generalizing the results of [59] to arbitrary characteristic.

The description of $\overline{M}$ is also of interest in the study of those local unramified automorphic L-functions associated to certain "basic functions" on $\overline{M}$ in the spirit of [11]. Such functions are considered in Chapter 2 in relation to the asymptotics map[1] and inversion of intertwining operators. The study of $\overline{M}$, and more generally of the intermediate boundary degenerations $\mathbb{X}_P$, is needed in Chapter 2 to generalize the results of [27], which treats the case when $G = \mathrm{SL}(2)$.

1.1.2 Contents

In §1.2, we recall the classification of normal reductive monoids proved by L. Renner. Given a reductive group and certain combinatorial data (what we call a *Renner cone*), we construct the associated normal algebraic monoid.

In §1.3, we define the normal reductive monoid $\overline{M}$ associated to a parabolic subgroup P of G. The group of units of $\overline{M}$ is the Levi factor M of P. We first give a combinatorial definition of $\overline{M}$ following Renner's classification. We then show in §1.3.2 that this monoid may be realized as a retract of $\overline{G/U}$, the spectrum of regular functions on the quasi-affine variety G/U. Lastly in §1.3.3 we describe $\overline{M}$ using the Tannakian formalism. This Tannakian description shows how $\overline{M}$ is used implicitly in [13], [12].

In §1.4, we first recall the definition of the boundary degeneration $\mathbb{X}_P$ associated to a pair of opposite parabolics. We show that $\overline{G/U}$ is a retract (and hence a closed subscheme) of $\overline{\mathbb{X}}_P := \operatorname{Spec} k[\mathbb{X}_P]$. Using the relation between the boundary degeneration and the Vinberg semigroup of G (i.e., the enveloping semigroup of G), we give another definition of the reductive monoid $\overline{M}$ using the existence of a certain idempotent in the Vinberg semigroup.

1. The asymptotics map, defined in [59, 60], coincides with the dual of the Bernstein map defined in [10].

1.1.3 Conventions

Let k be a perfect field of arbitrary characteristic. All schemes considered will be k-schemes. For a scheme S, let $k[S]$ denote the ring of regular functions $\Gamma(S, \mathcal{O}_S)$.

Fix an algebraic closure $\bar{k}$ of k, and let $\mathrm{Gal}(\bar{k}/k)$ denote its Galois group. For a k-scheme S, let $S_{\bar{k}}$ denote the base change $S \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}$, and let $\bar{k}[S] := \Gamma(S_{\bar{k}}, \mathcal{O}_{S_{\bar{k}}})$.

The group G. Let G be a connected reductive group over k. Let T denote its *abstract* Cartan[2] and W the corresponding Weyl group. We will denote by $\check{\Lambda}$ (resp. Λ) the weight (resp. coweight) lattice of $T_{\bar{k}}$, which is a $\mathrm{Gal}(\bar{k}/k)$-module.

The semigroup of dominant coweights (resp., weights) will be denoted by Λ_G^+ (resp., by $\check{\Lambda}_G^+$). The set of vertices of the Dynkin diagram of G will be denoted by Γ_G; for each $i \in \Gamma_G$ there corresponds a simple coroot α_i and a simple root $\check{\alpha}_i$. We denote the non-negative integral span of the set of positive coroots (resp. roots) by Λ_G^{pos} (resp. $\check{\Lambda}_G^{\mathrm{pos}}$). For $\lambda, \mu \in \Lambda$ we will write that $\lambda \geq \mu$ if $\lambda - \mu \in \Lambda_G^{\mathrm{pos}}$, and similarly for $\check{\Lambda}_G^{\mathrm{pos}}$. Let w_0 denote the longest element in the Weyl group of G.

Let P be a parabolic subgroup of G. Let U be its unipotent radical and $M := P/U$ the Levi factor. We use P to identify the abstract Cartan of M with T and let $W_M \subset W$ denote the corresponding Weyl group. There is a subdiagram $\Gamma_M \subset \Gamma_G$. We will denote by $\Lambda_M^{\mathrm{pos}} \subset \Lambda_G^{\mathrm{pos}}$, $\Lambda_M^+ \supset \Lambda_G^+$, $\geq_M$, $w_0^M \in W_M$, etc. the corresponding objects for M.

Let $\mathrm{Rep}(G)$ denote the abelian category of finite-dimensional G-modules. This category admits a forgetful functor to the abelian category of k-vector spaces. We define the functor

$$\mathrm{ind}_P^G : \mathrm{Rep}(P) \to \mathrm{Rep}(G)$$

as in [41, §I.3.3]. For a P-module $\overline{V}$, the induced module $\mathrm{ind}_P^G(\overline{V}) = (k[G] \otimes_k \overline{V})^P$ is finite-dimensional by properness of G/P. The functor ind_P^G is right adjoint to the restriction

2. When G is quasi-split, the abstract Cartan is defined as B/U_B for a Borel subgroup B. The definition is canonical and does not depend on the choice of Borel subgroup. When G is not quasi-split, the abstract Cartan is defined by Galois descent from the quasi-split case.

functor (cf. [41, Proposition I.3.4]). We also denote by ind_P^G the corresponding functor $\mathrm{Rep}(M) \to \mathrm{Rep}(G)$, where an M-module is considered as a P-module with trivial U-action. To a dominant weight $\check{\lambda} \in \check{\Lambda}_G^+$ one attaches the Weyl $G_{\bar{k}}$-module $\Delta(\check{\lambda})$, the dual Weyl module $\nabla(\check{\lambda})$, and the irreducible $G_{\bar{k}}$-module $L(\check{\lambda})$ of highest weight $\check{\lambda}$.

1.2 Recollections on normal reductive monoids

In this section we give a brief review of the classification of normal reductive monoids (i.e., normal, irreducible, affine algebraic monoids whose group of units is reductive), which is proved in [56, Theorem 5.4] by L. Renner. In [56], the base field is assumed to be algebraically closed, but the statements easily generalize to the case of a perfect base field by Galois descent.

To keep notation consistent with the rest of the chapter, we consider a connected reductive group M over k. Let T denote its *abstract* Cartan and W_M the corresponding Weyl group.

1.2.1 Renner cones

We denote by $\check{\Lambda}$ the weight lattice of $T_{\bar{k}}$ (i.e., the lattice of characters). Let $\check{\Lambda}^{\mathbb{Q}} := \check{\Lambda} \otimes \mathbb{Q}$, which is a $\mathbb{Q}$-vector space with a $\mathrm{Gal}(\bar{k}/k)$-action.

A *Renner cone* is a convex rational polyhedral cone in $\check{\Lambda}^{\mathbb{Q}}$ that is stable under the actions of W_M and $\mathrm{Gal}(\bar{k}/k)$. As the name suggests, the theorem of L. Renner shows that normal algebraic monoids with group of invertible elements M bijectively correspond to Renner cones generating $\check{\Lambda}^{\mathbb{Q}}$ as a vector space. The correspondence is as follows:

Let $\overline{M}$ be a reductive monoid with group of units M. Fix a Borel subgroup $B \subset M_{\bar{k}}$ and a Cartan subgroup (i.e., maximal torus) $T_{\mathrm{sub},\bar{k}} \subset B$, both defined over $\bar{k}$. This gives an identification of $T_{\mathrm{sub},\bar{k}}$ with the abstract Cartan $T_{\bar{k}}$. Consider the cone $\check{C} \subset \check{\Lambda}^{\mathbb{Q}}$ corresponding by [45] to the closure of $T_{\mathrm{sub},\bar{k}}$ in $\overline{M}_{\bar{k}}$. The pairs $(T_{\mathrm{sub},\bar{k}}, B)$ of a Cartan subgroup contained in a Borel subgroup are all conjugate by $M(\bar{k})$. Since M acts on $\overline{M}$ by conjugation, $\check{C}$ does not depend on the choice of $(T_{\mathrm{sub},\bar{k}}, B)$. The Weyl group of M acts on $T_{\mathrm{sub},\bar{k}}$ through the normalizer of $T_{\mathrm{sub},\bar{k}}$ in M, so $\check{C}$ is preserved by the action of W_M on $\check{\Lambda}^{\mathbb{Q}}$. The action of $\mathrm{Gal}(\bar{k}/k)$ on $G_{\bar{k}}$ induces an action on the set of pairs $(T_{\mathrm{sub},\bar{k}}, B)$. Since $\check{C}$ is canonically defined independently of the choice of $(T_{\mathrm{sub},\bar{k}}, B)$, the Galois action preserves $\check{C}$. Therefore $\check{C}$ is a Renner cone, and it is the Renner cone corresponding to $\overline{M}$.

Let $\check{C} \subset \check{\Lambda}^{\mathbb{Q}}$ be a Renner cone. We will construct the corresponding normal reductive monoid $\overline{M}$. Let us choose a Cartan subgroup (i.e., maximal torus) T_{sub} of M, defined over k. The construction of $\overline{M}$ will not depend on this choice.

1.2.2 The monoid $\overline{T_{\mathrm{sub}}}$

The characters $\check{\Lambda}$ form a basis of $\bar{k}[T]$. Let R' denote the subalgebra of $\bar{k}[T]$ spanned by the characters in $\check{C} \cap \check{\Lambda}$. The choice of a Borel $B \subset M_{\bar{k}}$ containing $T_{\mathrm{sub},\bar{k}}$ gives an isomorphism $T_{\mathrm{sub},\bar{k}} \cong T_{\bar{k}}$. All such Borel subgroups are conjugate by the normalizer of $T_{\mathrm{sub},\bar{k}}$ in $M_{\bar{k}}$. The subalgebra R' is preserved by the action of the Weyl group on T, so it defines a corresponding subalgebra $R \subset \bar{k}[T_{\mathrm{sub}}]$, which does not depend on the choice of a Borel subgroup.

Since $\check{C}$ is Galois stable, so is the subalgebra R. Set $\overline{T_{\mathrm{sub}}} := \mathrm{Spec}(R^{\mathrm{Gal}(\bar{k}/k)})$.

Lemma 1.2.1. *(i) $\overline{T_{\mathrm{sub}}}$ is a normal algebraic variety containing T_{sub} as a dense open subvariety.*

(ii) $\overline{T_{\mathrm{sub}}}$ has a (unique) monoidal structure extending the group structure on T_{sub}.

Proof. By Galois descent, it suffices to check the statements over $\bar{k}$, and we have $\bar{k}[\overline{T_{\mathrm{sub}}}] = R$. The submonoid $\check{C} \cap \check{\Lambda}$ is finitely generated and generates $\check{\Lambda}$ as a group. Moreover the submonoid is *saturated* (i.e., it is the intersection of a rational cone with the lattice). Statement (i) follows from [45, Ch. 1, Thm. 1].

To prove statement (ii), one must show that the map $k[T_{\mathrm{sub}}] \to k[T_{\mathrm{sub}}] \otimes k[T_{\mathrm{sub}}]$ sends the subalgebra R to $R \otimes R$. This is clear because $R \otimes \bar{k}$ has a basis consisting of characters of $T_{\mathrm{sub},\bar{k}}$. $\square$

1.2.3 The monoid $\overline{M}$

We will define a normal algebraic monoid $\overline{M}$ with group of units M such that the closure of T_{sub} in $\overline{M}$ equals $\overline{T_{\mathrm{sub}}}$. The monoid $\overline{M}$ will be the spectrum of a certain subalgebra A of the algebra of regular functions on M.

The algebra A. Let A denote the algebra of all $f \in k[M]$ such that for any $m_1, m_2 \in M(\bar{k})$ the function

$$t \mapsto f(m_1 t m_2)$$

belongs to the algebra R defined in §1.2.2. Since all Cartan subgroups of $M_{\bar{k}}$ are $M(\bar{k})$-conjugate, A does not depend on the choice of the subgroup $T_{\mathrm{sub}} \subset M$.

Proposition 1.2.2. *(i) A is a sub-bialgebra of the Hopf algebra $k[M]$.*

(ii) The map $M \to \operatorname{Spec} A$ is an open embedding.

(iii) A is an integrally closed domain.

(iv) The algebra A is finitely generated.

(v) The homomorphism $A \to k[\overline{T_{\mathrm{sub}}}]$ that takes a function to its restriction to T_{sub} is surjective.

Proof. All statements can be checked after base change to $\bar{k}$, so we will assume that k is algebraically closed.

Let A' denote the subalgebra of $k[M]$ generated by the matrix coefficients of a *finite* collection of Weyl[3] M-modules whose highest weights belong to $\check{C} \cap \check{\Lambda}^+_G$ and generate $\check{\Lambda}^+_G$ as a semigroup. The following properties of A' are easy to check:

(a) $A' \subset A$;

(a$'$) if k has characteristic 0 then $A' = A$;

(a$''$) the morphism $M \to \operatorname{Spec} A'$ is an open embedding;

(b) the composition $A' \hookrightarrow A \to R$ is surjective;

3. One can also take dual Weyl modules.

(c) the algebra A' is finitely generated.

The proof of statement (i) in the proposition is standard. Since $A' \subset A \subset k[M]$, property (a$''$) implies statement (ii). Statement (iii) follows from the normality of M and the normality part of Lemma 1.2.1(i). Statement (v) follows from (b).

Let A'' denote the integral closure of A' in the function field of M. Without any assumptions on the characteristic of k, we claim that $A = A''$. By (ii)-(iii), it suffices to check that A is contained in the localization O of A'' at any codimension 1 prime. Let K denote the field of fractions of O, which is also the field of rational functions on M. Then the normalization map $\operatorname{Spec} A'' \to \operatorname{Spec} A'$ induces a map $f' : \operatorname{Spec} O \to \operatorname{Spec} A'$ with $f'(\operatorname{Spec} K) \subset M$. We wish to lift f' to a morphism $f : \operatorname{Spec} O \to \operatorname{Spec} A$. Since

$$M(K) = M(O) \cdot T_{\mathrm{sub}}(K) \cdot M(O)$$

we can assume that $f'(\operatorname{Spec} K) \subset T_{\mathrm{sub}}(K)$. Then the existence of f follows from (b), which says that the closure of T_{sub} in $\operatorname{Spec} A$ maps isomorphically onto the closure of T_{sub} in $\operatorname{Spec} A'$. Therefore $A = A''$, and statement (iv) now follows. $\square$

The algebraic monoid $\overline{M}$. Now set $\overline{M} := \operatorname{Spec} A$.

By Proposition 1.2.2, $\overline{M}$ is a normal affine algebraic monoid equipped with an open embedding $M \hookrightarrow \overline{M}$ with dense image. By part (v) of the proposition, the closed embedding $T_{\mathrm{sub}} \hookrightarrow M$ extends to a closed embedding $\overline{T_{\mathrm{sub}}} \hookrightarrow \overline{M}$. By construction, the Renner cone corresponding to $\overline{M}$ is $\check{C}$.

Since $\overline{M}$ is an irreducible monoid and M is an open dense subgroup, M is necessarily the group of units of $\overline{M}$. The classification theorem of L. Renner ([56, Theorem 5.4]) says that every normal algebraic monoid with group of units M is isomorphic to a monoid $\overline{M}$ of the above form.

1.3 The monoid associated to a parabolic subgroup

Let P be a parabolic subgroup of G with Levi quotient $M := P/U$. We will define a canonical normal reductive monoid $\overline{M}$ with group of units M. This monoid appears implicitly in [13, 12], and it is explicitly considered in [3, §3.3] (in characteristic 0) and in [5, §6].

We identify the abstract Cartans of G and M as follows: for a Borel subgroup $B_M \subset M_{\bar{k}}$, the subgroup $B := B_M U \subset G_{\bar{k}}$ is a Borel subgroup, and $T_{\bar{k}} = B/U_B = B_M/U_{B_M}$.

1.3.1 The Renner cone of $\overline{M}$

We first give a combinatorial definition of $\overline{M}$ using Renner's classification, recalled in §1.2, by specifying the Renner cone $\check{C} \subset \check{\Lambda}^{\mathbb{Q}}$.

The submonoid Λ_U^{pos}. Let $\Lambda_U^{\mathrm{pos}} \subset \Lambda$ denote the non-negative integral span of the positive coroots of G that are not coroots of M. The submonoid Λ_U^{pos} is stable under the actions of W_M and $\mathrm{Gal}(\bar{k}/k)$ because M is defined over k.

Let $\check{G}$ (resp. $\check{M}$) denote the Langlands dual group of G (resp. M) over $\mathbb{C}$. Fix a maximal torus and a Borel subgroup containing it in the split group $\check{G}$. Then we may consider $\check{M}$ as a Levi subgroup of $\check{G}$. Let $\check{\mathfrak{u}}_P$ denote the nilpotent Lie algebra corresponding to the positive coroots of G that are not coroots of M. Then the symmetric algebra $\mathrm{Sym}(\check{\mathfrak{u}}_P)$ is a locally finite $\check{M}$-module by the adjoint action, and its set of weights equals Λ_U^{pos}.

Lemma 1.3.1. *Let $\lambda, \lambda' \in \Lambda_M^+$ with $\lambda \leq_M \lambda'$. If $\lambda' \in \Lambda_U^{\mathrm{pos}}$, then $\lambda \in \Lambda_U^{\mathrm{pos}}$.*

Proof. We have a decomposition of $\mathrm{Sym}(\check{\mathfrak{u}}_P)$ into irreducible highest weight $\check{M}$-modules $L_{\check{M}}(\gamma)$. Therefore λ' is a weight in $L_{\check{M}}(\gamma)$ for some $\gamma \in \Lambda_M^+$, and all the weights of $L_{\check{M}}(\gamma)$ lie in Λ_U^{pos}. Since $\lambda \in \Lambda_M^+$ and $\lambda \leq_M \lambda' \leq_M \gamma$, we deduce that λ is also a weight of $L_{\check{M}}(\gamma)$. Therefore $\lambda \in \Lambda_U^{\mathrm{pos}}$. □

Lemma 1.3.2. *The subset $\Lambda_U^{\mathrm{pos}} \subset \Lambda$ is equal to the intersection of $w(\Lambda_G^{\mathrm{pos}})$ for all $w \in W_M$. Consequently, $\Lambda_U^{\mathrm{pos}} \cap (-\Lambda_M^+) = \Lambda_G^{\mathrm{pos}} \cap (-\Lambda_M^+)$.*

Proof. Observe that Λ_U^{pos} is W_M-stable and hence contained in $w(\Lambda_G^{\mathrm{pos}})$ for all $w \in W_M$. To prove containment in the other direction, let $\lambda \in \bigcap_{w \in W_M} w(\Lambda_G^{\mathrm{pos}})$. Replacing λ by an element in the same W_M-orbit, we may assume that $\lambda \in -\Lambda_M^+$. By assumption $\lambda \in \Lambda_G^{\mathrm{pos}}$, so we can write $\lambda = \lambda_1 + \lambda_2$ where λ_1 is a linear combination of α_i for $i \in \Gamma_M$ and $\lambda_2 \in \Lambda_U^{\mathrm{pos}}$ is a linear combination of α_j for $j \in \Gamma_G \setminus \Gamma_M$. Note that $\lambda_2 \in \Lambda_U^{\mathrm{pos}} \cap (-\Lambda_M^+)$ and $\lambda \geq_M \lambda_2$. Then $w_0^M \lambda_2 \in \Lambda_U^{\mathrm{pos}} \cap \Lambda_M^+$ and $w_0^M \lambda \leq_M w_0^M \lambda_2$. Lemma 1.3.1 implies that $w_0^M \lambda \in \Lambda_U^{\mathrm{pos}}$, and hence $\lambda \in \Lambda_U^{\mathrm{pos}}$. One deduces the second statement of the lemma from the first because $\lambda \in -\Lambda_M^+$ satisfies $\lambda \leq_M w\lambda$ for all $w \in W_M$. $\square$

Remark 1.3.3. Lemma 1.3.2 implies that $\Lambda_U^{\mathrm{pos}} \cap \Lambda_M^+ = w_0^M(\Lambda_G^{\mathrm{pos}}) \cap \Lambda_M^+$. This submonoid of Λ_M^+ is denoted by $\Lambda_{M,G}^+$ in [13, §6.2.2, Proposition 6.2.3].

Lemma 1.3.4. *The submonoid $W_M \cdot \check{\Lambda}_G^+ \subset \check{\Lambda}$ is dual to Λ_U^{pos}, i.e.,*

$$W_M \cdot \check{\Lambda}_G^+ = \{\check{\lambda} \in \check{\Lambda} \mid \langle \check{\lambda}, \mu \rangle \geq 0 \text{ for all } \mu \in \Lambda_U^{\mathrm{pos}}\}. \tag{1.1}$$

Proof. Let $(\Lambda_U^{\mathrm{pos}})^\vee$ equal the r.h.s. of (1.1), which is evidently W_M-stable. If we consider an element in $(\Lambda_U^{\mathrm{pos}})^\vee \cap \check{\Lambda}_M^+$, then it pairs with positive coroots of M to non-negative integers since the element is M-dominant, and it pairs with all other positive coroots of G to non-negative integers by definition of the dual. Thus $(\Lambda_U^{\mathrm{pos}})^\vee \cap \check{\Lambda}_M^+ = \check{\Lambda}_G^+$, which implies that $(\Lambda_U^{\mathrm{pos}})^\vee$ is the union of $w(\check{\Lambda}_G^+)$ for all $w \in W_M$. $\square$

Corollary 1.3.5. *The submonoid $W_M \cdot \check{\Lambda}_G^+$ is saturated in $\check{\Lambda}$.*

The Renner cone $\check{C}$. Set $\check{C} \subset \check{\Lambda}^{\mathbb{Q}}$ to be the convex rational polyhedral cone generated by $W_M \cdot \check{\Lambda}_G^+$. Lemma 1.3.4 implies that $\check{C}$ is preserved by the action of $\mathrm{Gal}(\bar{k}/k)$, and Corollary 1.3.5 says that $\check{C} \cap \check{\Lambda} = W_M \cdot \check{\Lambda}_G^+$.

Definition of $\overline{M}$. Set $\overline{M}$ to be the normal reductive monoid with Renner cone $\check{C}$ constructed in Proposition 1.2.2. We will use this notation for the rest of the chapter.

1.3.2 Relation to $\overline{G/U}$

In this subsection, we show (see Corollary 1.3.10) that $\overline{M}$ is isomorphic to the monoid constructed in [3, §3.3] and [5, §6]. First we recall some facts about the homogeneous space G/U.

A scheme S is *strongly quasi-affine* if the canonical map $S \to \operatorname{Spec} k[S]$ is an open embedding and $k[S]$ is a finitely generated k-algebra.

F. D. Grosshans proved that the quotient variety G/U is strongly quasi-affine in [36]. Recall that $\overline{G/U} = \operatorname{Spec} k[G/U]$, where $k[G/U]$ is the subalgebra of right U-invariant regular functions on G.

Weights of $k[G/U]$. The Levi factor $M := P/U$ acts on G/U from the right. Therefore we can consider $k[G/U]$ as an M-module and ask what is the set of weights[4] of this module with respect to the abstract Cartan of M.

Lemma 1.3.6. *The set of weights of the M-module $k[G/U]$ equals $W_M \cdot \check{\Lambda}_G^+ \subset \check{\Lambda}$.*

Proof. We may assume that k is algebraically closed. Choose a Borel subgroup B contained in P (so B/U is a Borel subgroup of M) and let $T_{\mathrm{sub}} \subset B$ be a maximal torus, which we identify with its image in M. The weights of $k[G/U]$ are the T_{sub}-eigenvalues with respect to right translations. Let $k[G/U]_{\check{\gamma}}$, $\check{\gamma} \in \check{\Lambda}$, denote a weight space.

Note that $k[G/U]_{\check{\gamma}}$ is a G-module by left translation. Let B^- denote the opposite Borel subgroup so that $B \cap B^- = T_{\mathrm{sub}}$. By unipotence of U_B^-, we deduce that $\check{\gamma}$ is a weight of $k[G/U]$ if and only if $k[G/U]_{\check{\gamma}}^{U_B^-} \neq 0$. Hence we are reduced to studying the weight spaces of $k[G/U]^{U_B^-}$. By considering the T-action by left translation, we have decompositions

$$k[U_B^- \backslash G] = \bigoplus_{\check{\lambda} \in \check{\Lambda}_G^+} \nabla(\check{\lambda}), \qquad k[U_B^- \backslash G]^U = \bigoplus_{\check{\lambda} \in \check{\Lambda}_G^+} \nabla(\check{\lambda})^U$$

4. Let V be an M-module over k. Choose a Borel subgroup $B_M \subset M_{\bar{k}}$ and a Cartan subgroup $T_{\mathrm{sub},\bar{k}} \subset B_M$, which is isomorphic to $T_{\bar{k}} = B_M/U_{B_M}$. We say that the set of weights of V is the set of $T_{\mathrm{sub},\bar{k}}$-eigenvalues of $V \otimes \bar{k}$. This set does not depend on the choice of $(T_{\mathrm{sub},\bar{k}}, B_M)$, so it can be considered as a subset of $\check{\Lambda}$, which is preserved by W_M and $\mathrm{Gal}(\bar{k}/k)$.

where U acts by right translation. Since $U_B^- P$ is dense in G, the restriction from G to P gives an injection

$$\nabla(\check{\lambda})^U \hookrightarrow \nabla_M(\check{\lambda}),$$

where $\nabla_M(\check{\lambda})$ is the dual Weyl M-module.

We now prove the 'only if' direction of the lemma. Suppose that $\check{\gamma}$ is a weight of $k[G/U]$. Then $\check{\gamma}$ must be a weight of $\nabla_M(\check{\lambda})$ for some $\check{\lambda} \in \check{\Lambda}_G^+$. There exists $w \in W_M$ such that $w(\check{\gamma}) \in \check{\Lambda}_M^+$. Since the set of weights of $\nabla_M(\check{\lambda})$ is W_M-stable, $w(\check{\gamma})$ is also a weight. Hence $w(\check{\gamma}) \leq_M \check{\lambda}$. Since $\langle \check{\alpha}_i, \alpha_j \rangle \leq 0$ for $i \in \Gamma_M, j \in \Gamma_G \setminus \Gamma_M$, we deduce that $w(\check{\gamma}) \in \check{\Lambda}_G^+$. This proves the 'only if' direction of the lemma.

Conversely, suppose $\check{\gamma}$ is a weight such that $w(\check{\gamma}) \in \check{\Lambda}_G^+$ for some $w \in W_M$. Then $\check{\lambda} := w(\check{\gamma})$ is the highest weight in $\nabla(\check{\lambda})^U$. Since the set of weights of an M-module is W_M-stable, we conclude that $\check{\gamma}$ is a weight of $k[G/U]$. $\square$

Corollary 1.3.7. *For any G-module V, the weights of the M-module V^U are a subset of $W_M \cdot \check{\Lambda}_G^+$.*

Proof. Any finite dimensional G-module V is a submodule of a direct sum of regular representations $k[G]$, so the weights of V^U are a subset of the weights of $k[G/U]$. $\square$

The closure of M in $\overline{G/U}$. The subgroup $P \subset G$ induces a closed embedding

$$M = P/U \hookrightarrow G/U, \tag{1.2}$$

i.e., we embed M in G/U by the right M-action on $1 \in G$. Then the closure of M in $\overline{G/U}$ has the structure of an irreducible algebraic monoid[5], and the right action of M on G/U extends to an action of this monoid on $\overline{G/U}$. We claim that the normalization of this monoid is isomorphic to the monoid $\overline{M}$.

Lemma 1.3.8. *The embedding* (1.2) *extends to a finite map $\overline{M} \to \overline{G/U}$.*

5. This monoid is denoted by M_+ in [5, §6].

Proof. Let T_{sub} be a Cartan subgroup of M and embed $T_{\text{sub}} \hookrightarrow G/U$ using (1.2). Let $\overline{T_{\text{sub}}}$ denote the closure of T_{sub} in $\overline{G/U}$. By the classification of normal reductive monoids in [56, Theorem 5.4], it suffices to show that the cone corresponding by [45] to $\overline{T_{\text{sub}}}$ is the Renner cone $\check{C}$ of $\overline{M}$.

By definition, $\overline{T_{\text{sub}}}$ is the spectrum of the image of the restriction map $k[G/U] \to k[T_{\text{sub}}]$. This map is equivariant with respect to right translations by T_{sub}, so $\bar{k}[\overline{T_{\text{sub}}}]$ decomposes into weight spaces. Let $\check{\gamma}$ be a weight of $\bar{k}[G/U]$. By left translation by G, one can find $f \in \bar{k}[G/U]_{\check{\gamma}}$ such that $f(1) = 1$. Therefore the weights of $k[\overline{T_{\text{sub}}}]$ coincide with the weights of $k[G/U]$, and the claim follows from Lemma 1.3.6. □

Fix a parabolic subgroup $P^- \subset G$ opposite to P. For the rest of this section we will identify M with the Levi subgroup $P \cap P^-$.

Theorem 1.3.9. *The composition*

$$\overline{M} \to \overline{G/U} \to \operatorname{Spec} k[G]^{U^- \times U}$$

is an isomorphism, where $U^- \times U$ acts on $k[G]$ by left and right translations, respectively.

Note that $\operatorname{Spec} k[G]^{U^- \times U}$ is the affine GIT quotient of $\overline{G/U}$ by the left action of $U^- \subset G$.

Corollary 1.3.10. *The (unique) map $\overline{M} \to \overline{G/U}$ extending the embedding* (1.2) *is a retract. In particular, it is a closed embedding.*

Proof. The fact that $\overline{M}$ is a retract of $\overline{G/U}$ follows immediately from the isomorphism in Theorem 1.3.9. To prove that it is a closed subscheme, it suffices to show that the algebra map $k[G/U] \to k[\overline{M}]$ is surjective. The theorem implies that the subalgebra $k[G]^{U^- \times U} \subset k[G/U]$ surjects onto $k[\overline{M}]$. □

For the purpose of proving Theorem 1.3.9, let $\tilde{M} = \operatorname{Spec} k[G]^{U^- \times U}$. The actions of M on G by left and right translations induce M-actions on $\tilde{M}$. We have a canonical $M \times M$ equivariant map $G \to \tilde{M}$.

Lemma 1.3.11. *The composition $M \to G \to \tilde{M}$ is an open embedding.*

Proof. We may check the assertion after base change to $\bar{k}$, so we assume k is algebraically closed. Choose Borel subgroups $B \subset P$ and $B^- \subset P^-$ such that $T_{\mathrm{sub}} := B \cap B^- \subset M$ is a maximal torus. Let

$$\tilde{T} = \operatorname{Spec} k[G]^{U_B^- \times U_B}.$$

Since $k[G]^{U_B^-}$ has a decomposition into dual Weyl G-modules, one deduces that $k[\tilde{T}]$ has a basis formed by $f_{\check{\lambda}}$ for $\check{\lambda} \in \check{\Lambda}_G^+$, where $f_{\check{\lambda}}(t) = \check{\lambda}(t)$, $t \in T_{\mathrm{sub}}$. From this explicit description, one sees that $\tilde{T}$ is a toric variety containing T_{sub} as a dense open subscheme.

Consider the composition $G \to \tilde{M} \to \tilde{T}$ and let $\overset{\circ}{G} \subset G$ denote the preimage of $T_{\mathrm{sub}} \subset \tilde{T}$. Then the preimage of T_{sub} in $\tilde{M}$, which we denote $\overset{\circ}{\tilde{M}}$, is equal to $\operatorname{Spec} k[\overset{\circ}{G}]^{U^- \times U}$. Observe that $B^-B = U_B^- \times T_{\mathrm{sub}} \times U_B$ is an open affine subset contained in $\overset{\circ}{G}$. Let us show that $\overset{\circ}{G} = B^-B$. By definition, $\overset{\circ}{G}$ consists of $g \in G$ such that $f_{\check{\lambda}}(g) \neq 0$ for all dominant weights $\check{\lambda}$. By the Bruhat decomposition, it suffices to show that if w belongs to the normalizer of T_{sub} but not to T_{sub} (i.e., w corresponds to a nontrivial element of W), then there exists $\check{\lambda}$ with $f_{\check{\lambda}}(w) = 0$. Indeed, for a dominant regular weight $\check{\lambda}$ we have $w\check{\lambda} \neq \check{\lambda}$. Thus the left and right T-actions on $w^{-1}f_{\check{\lambda}}$ do not have the same weight, which implies that $f_{\check{\lambda}}(w) = (w^{-1}f_{\check{\lambda}})(1) = 0$.

Let $B_M = B/U = B \cap M$ and $B_M^- = B^-/U^- = B^- \cap M$. From the equality $\overset{\circ}{G} = U_B^- \times T_{\mathrm{sub}} \times U_B$ we deduce that $\overset{\circ}{\tilde{M}} = U_{B_M}^- \times T_{\mathrm{sub}} \times U_{B_M}$ is an open dense subset of both M and $\tilde{M}$. Using left (or right) translations by M, we deduce that the whole group M is an open subset of $\tilde{M}$. $\square$

The field of rational functions on $\tilde{M}$ is contained in[6] the field of invariants $k(G)^{U^- \times U}$. Thus normality of G implies normality of $\tilde{M}$. Therefore Lemma 1.3.11 implies that $\tilde{M}$ is a normal reductive monoid with group of units M.

6. In fact one can show that the fraction field of $k[G]^{U^- \times U}$ is equal to $k(G)^{U^- \times U}$: given $f \in k(G)^{U^- \times U}$, consider the vector space of denominators $h \in k[G]$ such that $hf \in k[G]$. This is a $U \times U^-$-module, so there must exist an invariant element h. Then $hf \in k[G]^{U^- \times U}$.

Proof of Theorem 1.3.9. Let T_{sub} be a Cartan subgroup of $M \subset G$. Since $\tilde{M}$ is a normal reductive monoid with group of units M, it is determined by the closure of T_{sub} in $\tilde{M}$, which is the spectrum of the algebra

$$\tilde{R} := \operatorname{Im}(k[G]^{U^- \times U} \to k[T_{\text{sub}}]).$$

By unipotence of U^-, the algebra $\tilde{R}$ is the image of the restriction map $k[G/U] \to k[T_{\text{sub}}]$. Therefore $\operatorname{Spec} \tilde{R}$ is the closure of T_{sub} in $\overline{G/U}$. By the proof of Lemma 1.3.8, this is also the closure of T_{sub} in $\overline{M}$. Since $\tilde{M}$ and $\overline{M}$ are both normal algebraic monoids with unit group M, the classification of normal reductive monoids ([56, Theorem 5.4]) implies that the map $\overline{M} \to \tilde{M}$ is an isomorphism. $\square$

1.3.3 Tannakian description of $\overline{M}$

Let $\operatorname{Rep}(M)$ denote the monoidal category of finite-dimensional representations of M. Similarly, one has the monoidal category $\operatorname{Rep}(\overline{M})$. Since M is schematically dense in $\overline{M}$, the monoidal functor

$$\operatorname{Rep}(\overline{M}) \to \operatorname{Rep}(M)$$

corresponding to $M \hookrightarrow \overline{M}$ is fully faithful. So we can consider $\operatorname{Rep}(\overline{M})$ as a full subcategory of $\operatorname{Rep}(M)$.

The usual Tannakian formalism describes $\overline{M}$ in terms of $\operatorname{Rep}(\overline{M})$. Namely, for a test scheme S, an element of the monoid $\operatorname{Hom}(S, \overline{M})$ is a collection of assignments

$$\overline{V} \in \operatorname{Rep}(\overline{M}) \rightsquigarrow m_{\overline{V}} \in \operatorname{End}_{\mathcal{O}_S}(\overline{V} \otimes \mathcal{O}_S),$$

compatible with morphisms $\overline{V}_1 \to \overline{V}_2$ in $\operatorname{Rep}(\overline{M})$ and such that $m_{\overline{V}_1 \otimes \overline{V}_2} = m_{\overline{V}_1} \otimes m_{\overline{V}_2}$. The multiplication in $\operatorname{Hom}(S, \overline{M})$ corresponds to the multiplication in $\operatorname{End}_{\mathcal{O}_S}(\overline{V} \otimes \mathcal{O}_S)$.

Our goal is to prove Proposition 1.3.13 below, which describes the subcategory $\operatorname{Rep}(\overline{M})$.

Description of $\mathrm{Rep}(\overline{M})$**.** Fix a parabolic subgroup $P^- \subset G$ opposite to P, and identify the Levi subgroup $P \cap P^-$ with M.

For an M-module $\overline{V}$, we consider an element $f \in \mathrm{ind}_{P^-}^G(\overline{V})$ as a regular map $G \to \overline{V}$ (cf. [41, §I.3.3]) satisfying $f(gm\bar{u}) = m^{-1}f(g)$ for all $\bar{k}$-points $g \in G, m \in M, \bar{u} \in U^-$. Using this description, evaluation at 1 in G defines an M-morphism $\mathrm{ind}_{P^-}^G(\overline{V}) \to \overline{V}$.

Lemma 1.3.12. *Let* $\overline{V} \in \mathrm{Rep}(\overline{M})$*. Then evaluation at* 1 *induces an isomorphism*

$$\mathrm{ind}_{P^-}^G(\overline{V})^U \to \overline{V}. \tag{1.3}$$

Proof. Since UP^- is a dense open subset of G, the map (1.3) is injective. Let $v \in \overline{V}$. Then we can define a morphism $f : U \times M \times U^- \cong UP^- \to \overline{V}$ by $f(um\bar{u}) = m^{-1}v$ for $m \in M$, $u \in U$, $\bar{u} \in U^-$. For any $\xi \in \overline{V}^*$, the pairing $\langle \xi, f(um\bar{u})\rangle = \langle \xi, m^{-1}v\rangle$ extends to a regular function in $k[G]^{U \times U^-}$ by Theorem 1.3.9. Therefore f extends to a U-invariant function in $\mathrm{ind}_{P^-}^G(\overline{V})$, proving surjectivity of (1.3). □

Proposition 1.3.13. *Let* $\overline{V} \in \mathrm{Rep}(M)$*. Then the following are equivalent:*

(i) $\overline{V}$ *belongs to* $\mathrm{Rep}(\overline{M})$*.*

(ii) The weights of $\overline{V}$ *lie in* $W_M \cdot \check{\Lambda}_G^+ \subset \check{\Lambda}$*.*

(iii) There exists $V \in \mathrm{Rep}(G)$ *such that* $\overline{V} \cong V^U$*.*

Proof. The equivalence of (i) and (ii) follows from the definition of $\overline{M}$. Corollary 1.3.7 proves (iii) implies (ii). Lemma 1.3.12 shows that (i) implies (iii) by setting $V = \mathrm{ind}_{P^-}^G(\overline{V})$, which is a finite-dimensional G-module. □

Remark 1.3.14. Suppose that k is algebraically closed. One can deduce from Lemma 1.3.12 that $\nabla(\check{\lambda})^U$ is isomorphic to the dual Weyl M-module $\nabla_M(\check{\lambda})$. By [41, Remark II.2.11], the subspace $\nabla(\check{\lambda})^U$ also equals the sum of the weight spaces of $\nabla(\check{\lambda})$ with weights $\leq_M \check{\lambda}$. Dually, one sees that the sum of the weight spaces of $\Delta(\check{\lambda})$ with weights $\leq_M \check{\lambda}$ is isomorphic to $\Delta(\check{\lambda})_{U^-}$, which is in turn isomorphic to the Weyl M-module $\Delta_M(\check{\lambda})$.

Remark 1.3.15. Let O be a complete discrete valuation ring with field of fractions K and residue field k. By Proposition 1.3.13(iii) and the usual Tannakian formalism, one observes that the closed subscheme $\mathrm{Gr}_M^+ \subset \mathrm{Gr}_M = M(K)/M(O)$ defined in [13, §6.2], [12, §1.6] is equal to the subspace $(\overline{M}(O) \cap M(K))/M(O)$.

1.4 Relation to boundary degenerations

Let P and P^- be a pair of opposite parabolic subgroups in G. We identify the Levi subgroup $P \cap P^-$ with the Levi factor $M = P/U$. Let $\overline{M}$ be the normal reductive monoid with group of units M defined in §1.3.

In this section we will show that $\overline{G/U}$ embeds as a closed subscheme in the affine closure of the boundary degeneration defined in [10, 60, 59]. We will also describe the relation between the boundary degeneration and the Vinberg semigroup (i.e., enveloping semigroup) of G. This will give an alternate description of $\overline{M}$ as a subscheme of the Vinberg semigroup, using idempotents.

1.4.1 Boundary degenerations

Define the boundary degeneration

$$\mathbb{X}_P := (G \times G)/(P \underset{M}{\times} P^-) = (G/U \times G/U^-)/(P \cap P^-),$$

where $P \cap P^-$ acts diagonally on the right. It is known that $\mathbb{X}_P$ is quasi-affine (cf. [24, Proposition 2.4.4]), and $k[\mathbb{X}_P]$ is finitely generated by [36] and Hilbert's theorem on invariants. Therefore $\mathbb{X}_P$ is strongly quasi-affine.

Remark 1.4.1. The group $G \times G$ acts on $\mathbb{X}_P$ by left translations. Suppose that k is algebraically closed and choose a pair B, B^- of opposite Borel subgroups contained in P, P^-, respectively. Then the orbit of $B^- \times B$ acting on $(1,1) \in \mathbb{X}_P$ is a dense open subset. Therefore $\mathbb{X}_P$ is a spherical variety with respect to $G \times G$.

Let $\overline{\mathbb{X}}_P = \operatorname{Spec} k[\mathbb{X}_P]$. Since $\mathbb{X}_P$ is strongly quasi-affine, $\overline{\mathbb{X}}_P$ is affine of finite type and the canonical embedding $\mathbb{X}_P \hookrightarrow \overline{\mathbb{X}}_P$ is open.

Note that $\overline{\mathbb{X}}_P$ is the affine GIT quotient of $\overline{G/U} \times \overline{G/U^-}$ by the diagonal right M-action, but it is *not* the stack quotient (cf. [65, Tag 044Q] for the definition of quotient stacks).

Consider the map of strongly quasi-affine varieties

$$G/U \to \mathbb{X}_P : g \mapsto (g, 1). \tag{1.4}$$

The base change of (1.4) under the smooth cover $G \times G \to \mathbb{X}_P$ gives the natural closed embedding $G \times P^- \hookrightarrow G \times G$. Therefore (1.4) is also a closed embedding.

The composition $G/U \hookrightarrow \mathbb{X}_P \hookrightarrow \overline{\mathbb{X}}_P$ induces a map

$$\overline{G/U} \to \overline{\mathbb{X}}_P. \tag{1.5}$$

In characteristic 0, one easily deduces from [3, Proposition 5] that (1.5) is a closed embedding. In positive characteristic, this is not *a priori* clear, but the following theorem shows it is still true:

Theorem 1.4.2. *The map* (1.5) *is a closed embedding, and the composition*

$$\overline{G/U} \to \overline{\mathbb{X}}_P \to \operatorname{Spec} k[\mathbb{X}_P]^U$$

is an isomorphism, where $U \subset G$ acts on $\mathbb{X}_P$ by left translations in the second coordinate.

Proof. Observe that $k[\mathbb{X}_P]^U = (k[G/U] \otimes k[G]^{U \times U^-})^M$ where M acts diagonally by right translations. Using the inversion operator on G in the second coordinate, we get $k[\mathbb{X}_P]^U \cong (k[G/U] \otimes k[\overline{M}])^M$ where $k[\overline{M}] = k[G]^{U^- \times U}$ by Theorem 1.3.9 and M acts anti-diagonally on the right. Since M is dense in $\overline{M}$, the evaluation at $1 \in \overline{M}$ gives an injection

$$(k[G/U] \otimes k[\overline{M}])^M \hookrightarrow k[G/U].$$

On the other hand, $\overline{M}$ is the closure of M in $\overline{G/U}$ by Corollary 1.3.10. The right action of M on G/U therefore extends to a right action of $\overline{M}$ on $\overline{G/U}$, which corresponds to a comodule map $k[G/U] \to (k[G/U] \otimes k[\overline{M}])^M$. The composition

$$k[G/U] \to (k[G/U] \otimes k[\overline{M}])^M \hookrightarrow k[G/U]$$

is the identity, which proves that the composition $\overline{G/U} \to \operatorname{Spec} k[\mathbb{X}_P]^U$ is an isomorphism. It follows that the affine map (1.5) is a closed embedding. $\square$

Corollary 1.4.3. *Consider the embedding $M \hookrightarrow \mathbb{X}_P$ defined as the composition of the embeddings* (1.2) *and* (1.4)*. The closure of M in $\overline{\mathbb{X}}_P$ is isomorphic to $\overline{M}$. The composition*

$$\overline{M} \to \overline{\mathbb{X}}_P \to \operatorname{Spec} k[\mathbb{X}_P]^{U^- \times U}$$

is an isomorphism, where $U^- \times U \subset G \times G$ acts on $\mathbb{X}_P$ by left translations.

Proof. Combine Theorems 1.3.9 and 1.4.2. $\square$

1.4.2 Relation to Vinberg's semigroup

Recall that k is an arbitrary perfect field.

We first give a brief review of the standard material on the Vinberg semigroup, which is contained in [67, 58, 56].

Let $Z(G)$ denote the center of G. Consider the group

$$G_{\text{enh}} := (G \times T)/Z(G),$$

where $Z(G)$ maps to $G \times T$ anti-diagonally. Note that $Z(G_{\text{enh}}) = T$.

The Vinberg semigroup of G, denoted $\overline{G_{\text{enh}}}$, is a normal reductive k-monoid with group of units G_{enh}. The Renner cone of $\overline{G_{\text{enh}}}$ is by definition

$$\{(\check{\lambda}_1, \check{\lambda}_2) \in \check{\Lambda}^{\mathbb{Q}} \times \check{\Lambda}^{\mathbb{Q}} \mid \check{\lambda}_2 - w\check{\lambda}_1 \in \check{\Lambda}_G^{\text{pos},\mathbb{Q}} \text{ for all } w \in W\}, \tag{1.6}$$

where $\check{\Lambda}_G^{\text{pos},\mathbb{Q}}$ is the rational polyhedral cone generated by the positive roots of G. The Vinberg semigroup may be constructed from the Renner cone as described in §1.2.

The canonical homomorphism of algebraic groups $G_{\text{enh}} \to T_{\text{adj}} := T/Z(G)$ extends to a homomorphism of algebraic monoids

$$\bar{\pi} : \overline{G_{\text{enh}}} \to \overline{T_{\text{adj}}},$$

where $\overline{T_{\text{adj}}} := \mathfrak{t}_{\text{adj}}$ is the Cartan Lie algebra of the adjoint group. Let $\overset{\circ}{\overline{G_{\text{enh}}}}$ denote the non-degenerate locus of $\overline{G_{\text{enh}}}$. It is known that $\overset{\circ}{\overline{G_{\text{enh}}}}$ is smooth over $\overline{T_{\text{adj}}}$.

For a parabolic P with Levi factor M, let $\mathbf{c}_P \in \overline{T_{\text{adj}}}$ be the point defined by the condition that $\check{\alpha}_i(\mathbf{c}_P) = 1$ for simple roots $\check{\alpha}_i$, $i \in \Gamma_M$, and $\check{\alpha}_j(\mathbf{c}_P) = 0$ for all other simple roots. Note that $\mathbf{c}_P$ is an idempotent with respect to the monoid structure on $\overline{T_{\text{adj}}}$.

Let $(\overline{G_{\text{enh}}})_{\mathbf{c}_P}$ denote the fiber of $\bar{\pi}$ over $\mathbf{c}_P$. Note that by definition of $\mathbf{c}_P$, the center $Z(M)$ is the stabilizer of T acting on $\mathbf{c}_P$ in $\overline{T_{\text{adj}}}$.

Fix a pair of opposite parabolic subgroups P and P^-, and identify M with the Levi subgroup $P \cap P^-$. Since conjugation by M fixes $Z(M)$, the center of M can be embedded as a subgroup of the abstract Cartan T. Consider the anti-diagonal map

$$\mathfrak{s} : Z(M)/Z(G) \to (Z(M) \times T)/Z(G) \hookrightarrow (G \times T)/Z(G) = G_{\text{enh}}$$

defined by $\mathfrak{s}(t) = (t^{-1}, t)$. Observe that $Z(M)/Z(G) \subset T/Z(G) = T_{\text{adj}}$ coincides with the subtorus $\{t \in T_{\text{adj}} \mid \check{\alpha}_i(t) = 1,\, i \in \Gamma_M\}$. Let $\overline{Z(M)/Z(G)}$ denote the closure of $Z(M)/Z(G)$ in $\overline{T_{\text{adj}}}$.

Lemma 1.4.4. *(i) The map $\mathfrak{s}$ extends to a homomorphism*

$$\bar{\mathfrak{s}} : \overline{Z(M)/Z(G)} \to \overline{G_{\mathrm{enh}}}$$

of algebraic monoids.

(ii) The composition $\bar{\pi} \circ \bar{\mathfrak{s}}$ is the natural inclusion $\overline{Z(M)/Z(G)} \hookrightarrow \overline{T_{\mathrm{adj}}}$.

Proof. Since we know the composition $\pi \circ \mathfrak{s}$, it suffices to prove statement (i). We may assume that k is algebraically closed.

The weight lattice of $Z(M)/Z(G)$ is the free abelian group $\check{\Lambda}_{Z(M)/Z(G)}$ with basis consisting of the simple roots $\check{\alpha}_j$ for $j \in \Gamma_G \setminus \Gamma_M$. If $\check{\lambda} = \sum_{i \in \Gamma_G} n_i \check{\alpha}_i$ for $n_i \in \mathbb{Z}$, let $\mathrm{pr}(\check{\lambda}) := \sum_{j \notin \Gamma_M} n_j \check{\alpha}_j$. Let $\check{C}$ denote the Renner cone (1.6) of $\overline{G_{\mathrm{enh}}}$, and let $\check{C}_{\mathbb{Z}} := \check{C} \cap (\check{\Lambda} \times \check{\Lambda})$. Fix a Cartan subgroup $T_{\mathrm{sub}} \subset M$ and identify T_{sub} with T by choosing a Borel. The map $\mathfrak{s}$ lands in $(T_{\mathrm{sub}} \times T)/Z(G)$, so we have an induced map of weights (restricted to $\check{C}_{\mathbb{Z}}$):

$$\check{C}_{\mathbb{Z}} \to \check{\Lambda}_{Z(M)/Z(G)} : (\check{\lambda}_1, \check{\lambda}_2) \mapsto \mathrm{pr}(\check{\lambda}_2 - \check{\lambda}_1).$$

The image of this map is the non-negative span of the simple roots $\check{\alpha}_j, j \notin \Gamma_M$. Statement (i) follows. $\square$

Remark 1.4.5. If k is algebraically closed, then the map $\bar{\mathfrak{s}}$ we have constructed factors through the section $\overline{T_{\mathrm{adj}}} \to \overset{\circ}{\overline{G_{\mathrm{enh}}}}$ constructed in [24, Lemma D.5.2], which depends on a choice of Borel subgroup and maximal torus of G. In particular, $\bar{\mathfrak{s}}$ always lands in the non-degenerate locus of the Vinberg semigroup for arbitrary k.

The idempotent e_P. Observe that $\mathbf{c}_P$ lies in the submonoid $\overline{Z(M)/Z(G)} \subset \overline{T_{\mathrm{adj}}}$. Define the idempotent

$$e_P := \bar{\mathfrak{s}}(\mathbf{c}_P) \in \overset{\circ}{\overline{G_{\mathrm{enh}}}}(k),$$

which satisfies $\bar{\pi}(e_P) = \mathbf{c}_P$. In [24, Appendix C], it is shown (by passing to an algebraic closure $\bar{k}$) that

$$P = \{g \in G \mid g \cdot e_P = e_P \cdot g \cdot e_P\} \quad \text{and} \quad P^- = \{g \in G \mid e_P \cdot g = e_P \cdot g \cdot e_P\},$$

and the stabilizer of the $P \times P^-$ action on e_P equals $P \times_M P^-$. Note that if $g \in P \cap P^-$, then $g \cdot e_P = e_P \cdot g \cdot e_P = e_P \cdot g$. It follows that M is the centralizer of e_P in G.

Remark 1.4.6. It is known that $G \cdot e_P \cdot G$ is equal to the non-degenerate locus $(\overset{\circ}{\overline{G_{\text{enh}}}})_{\mathbf{c}_P}$ of the fiber (cf. [24, Corollary D.5.4]). One deduces from the above that the $G \times G$-action on e_P induces an isomorphism[7]

$$\mathbb{X}_P := (G \times G)/(P \underset{M}{\times} P^-) \cong (\overset{\circ}{\overline{G_{\text{enh}}}})_{\mathbf{c}_P}.$$

Remark 1.4.7. Suppose that k is algebraically closed. By a result of M. Putcha (cf. [56, Theorem 4.5]) for general reductive monoids, any idempotent in the non-degenerate locus of $\overline{G_{\text{enh}}}$ is $G(k)$-conjugate to e_P for some parabolic P. Moreover, the choice of P and P^- determines this idempotent in its conjugacy class.

Relating $\overline{M}$ to the Vinberg semigroup. Consider the map

$$G \to (\overline{G_{\text{enh}}})_{\mathbf{c}_P} : g \mapsto e_P \cdot g \cdot e_P. \tag{1.7}$$

Since $U \cdot e_P = e_P \cdot U^- = \{e_P\}$, this map is $U^- \times U$-invariant. By Theorem 1.3.9, we have an isomorphism $\overline{M} \cong \operatorname{Spec} k[G]^{U^- \times U}$. Since $(\overline{G_{\text{enh}}})_{\mathbf{c}_P}$ is affine, the map (1.7) must factor through a map

$$\overline{M} \to e_P \cdot (\overline{G_{\text{enh}}})_{\mathbf{c}_P} \cdot e_P. \tag{1.8}$$

Observe that $e_P \cdot (\overline{G_{\text{enh}}})_{\mathbf{c}_P} \cdot e_P$ is an irreducible algebraic monoid with identity e_P. The map (1.8) is an extension of the homomorphism of algebraic monoids $M \to e_P \cdot (\overline{G_{\text{enh}}})_{\mathbf{c}_P} \cdot e_P$

7. In fact, we learned from S. Schieder that this induces an isomorphism of affine varieties $\overline{\mathbb{X}}_P \cong (\overline{G_{\text{enh}}})_{\mathbf{c}_P}$. See Lemma A.3.2.

sending $m \mapsto m \cdot e_P = e_P \cdot m$. Therefore (1.8) must also be a homomorphism of algebraic monoids.

Theorem 1.4.8. *The homomorphism* (1.8) *is an isomorphism of algebraic monoids.*

By the definition of (1.7), we see that the image of (1.8) contains $e_P \cdot G \cdot e_P$. Since the latter map is a homomorphism of monoids, we deduce that the image contains $e_P \cdot G \cdot e_P \cdot G \cdot e_P$. By Remark 1.4.6, we have $G \cdot e_P \cdot G = \overset{\circ}{(\overline{G_{\text{enh}}})}_{\mathbf{c}_P}$ is dense in $(\overline{G_{\text{enh}}})_{\mathbf{c}_P}$. Multiplying on the left and right by e_P, we deduce that (1.8) has dense image. On the other hand, the restriction of (1.8) to M is injective. It follows that $M \cdot e_P$ is a dense sub*group* of $e_P \cdot (\overline{G_{\text{enh}}})_{\mathbf{c}_P} \cdot e_P$. Therefore $M \cdot e_P$ must be equal to the group of units of $e_P \cdot (\overline{G_{\text{enh}}})_{\mathbf{c}_P} \cdot e_P$.

We show that the monoid $e_P \cdot (\overline{G_{\text{enh}}})_{\mathbf{c}_P} \cdot e_P$ is normal and then use Renner's classification of normal monoids to prove the theorem.

Consider the larger algebraic monoid $e_P \cdot \overline{G_{\text{enh}}} \cdot e_P$ with unit e_P (where we do not restrict to a fiber). The action of $Z(G_{\text{enh}}) = T$ on $e_P \cdot \overline{G_{\text{enh}}} \cdot e_P$ induces an isomorphism

$$((e_P \cdot (\overline{G_{\text{enh}}})_{\mathbf{c}_P} \cdot e_P) \times T)/Z(M) \cong e_P \cdot \overline{G_{\text{enh}}} \cdot e_P, \tag{1.9}$$

so the two aforementioned monoids are closely related.

Since $e_P \cdot \overline{G_{\text{enh}}} \cdot e_P$ is the closed subscheme of the Vinberg semigroup fixed by left and right multiplications by e_P, it is a retract of $\overline{G_{\text{enh}}}$ in the category of schemes. The retraction is given by the formula $x \mapsto e_P \cdot x \cdot e_P$.

Lemma 1.4.9. *Let Y and S be integral affine schemes such that Y is a retract of S (i.e., there exist maps $Y \to S$ and $S \to Y$ such that their composition is the identity map on Y). If S is normal then so is Y.*

Proof. Since Y is a retract of S, we have an inclusion of algebras $k[Y] \hookrightarrow k[S]$. The algebra $k[S]$ is integrally closed, so if $\tilde{Y}$ denotes the normalization of Y in its field of fractions, then the previous inclusion factors as $k[Y] \hookrightarrow k[\tilde{Y}] \hookrightarrow k[S]$. On the other hand the map $Y \to S$

induces an algebra map $k[S] \to k[Y]$ which restricts to the identity on $k[Y]$. Localization implies that the composition $k[\tilde{Y}] \to k[Y]$ is injective and hence an isomorphism. $\square$

Corollary 1.4.10. *The algebraic monoid $e_P \cdot \overline{G_{\mathrm{enh}}} \cdot e_P$ is normal.*

Proof. The Vinberg semigroup is normal by definition, and we have observed that $e_P \cdot \overline{G_{\mathrm{enh}}} \cdot e_P$ is a retract of $\overline{G_{\mathrm{enh}}}$. $\square$

Corollary 1.4.11. *The algebraic monoid $e_P \cdot (\overline{G_{\mathrm{enh}}})_{\mathbf{c}_P} \cdot e_P$ is normal.*

Proof. We deduce from (1.9) that $e_P \cdot \overline{G_{\mathrm{enh}}} \cdot e_P$ is smooth locally isomorphic to $(e_P \cdot (\overline{G_{\mathrm{enh}}})_{\mathbf{c}_P} \cdot e_P) \times T$. It follows from Corollary 1.4.10 and ascending and descending properties of normality that $e_P \cdot (\overline{G_{\mathrm{enh}}})_{\mathbf{c}_P} \cdot e_P$ is normal. $\square$

Proof of Theorem 1.4.8. By Corollary 1.4.11 we know that $e_P \cdot (\overline{G_{\mathrm{enh}}})_{\mathbf{c}_P} \cdot e_P$ is a normal reductive monoid with group of units $M \cdot e_P$. Recall from §1.2 that normal reductive monoids are classified by their Renner cones. Since $\overline{M}$ is also a normal reductive monoid with group of units M, to prove the theorem it suffices to check that the Renner cones of $\overline{M}$ and $e_P \cdot (\overline{G_{\mathrm{enh}}})_{\mathbf{c}_P} \cdot e_P$ are equal. We may assume that k is algebraically closed.

Fix a Cartan subgroup $T_{\mathrm{sub}} \subset M \subset G$. Identify T_{sub} with the abstract Cartan T by choosing a Borel subgroup. Consider the embedding $T_{\mathrm{sub}} \hookrightarrow \overline{G_{\mathrm{enh}}}$ sending $t \mapsto t \cdot e_P$ and let $\overline{T_{\mathrm{sub}} \cdot e_P}$ denote the closure of the image. Set $T_{\mathrm{enh}} := (T_{\mathrm{sub}} \times T)/Z(G)$, which is a Cartan subgroup of G_{enh}, and let $\overline{T_{\mathrm{enh}}}$ denote its closure in $\overline{G_{\mathrm{enh}}}$. By definition, e_P lies in $\overline{T_{\mathrm{enh}}}$, so $T_{\mathrm{sub}} \hookrightarrow \overline{G_{\mathrm{enh}}}$ factors through the homomorphism of monoids

$$T_{\mathrm{sub}} \hookrightarrow \overline{T_{\mathrm{enh}}} : t \mapsto t \cdot e_P \tag{1.10}$$

Let $\check{C} \subset \check{\Lambda}^{\mathbb{Q}} \times \check{\Lambda}^{\mathbb{Q}}$ denote the Renner cone (1.6) of $\overline{G_{\mathrm{enh}}}$. Recall that the weights in $\check{C}_{\mathbb{Z}} := \check{C} \cap (\check{\Lambda} \times \check{\Lambda})$ form a basis of $k[\overline{T_{\mathrm{enh}}}]$. Let $(\check{\lambda}_1, \check{\lambda}_2) \in \check{C}_{\mathbb{Z}}$. Then $\check{\lambda}_2 - \check{\lambda}_1 \in \check{\Lambda}_G^{\mathrm{pos}}$, so it may be considered as a regular function on $\overline{T_{\mathrm{adj}}}$. Evaluating this function at $\mathbf{c}_P$ gives a number

$(\check{\lambda}_2 - \check{\lambda}_1)(\mathbf{c}_P)$, which is 1 if $\check{\lambda}_2 - \check{\lambda}_1 \in \check{\Lambda}_M^{\text{pos}}$ and 0 otherwise. By the definition of e_P, one sees that the homomorphism (1.10) corresponds to the map of weights

$$\check{C}_{\mathbb{Z}} \to \check{\Lambda} : (\check{\lambda}_1, \check{\lambda}_2) \mapsto (\check{\lambda}_2 - \check{\lambda}_1)(\mathbf{c}_P) \cdot \check{\lambda}_1. \tag{1.11}$$

The existence of the map (1.8) implies that the image of (1.11) must land in the Renner cone of $\overline{M}$, which is generated by the saturated submonoid $W_M \cdot \check{\Lambda}_G^+$. On the other hand, for $\check{\lambda} \in \check{\Lambda}_G^+$ and $w \in W_M$, one sees that $(w\check{\lambda}, \check{\lambda}) \mapsto w\check{\lambda}$. Thus the image of (1.11) equals $W_M \cdot \check{\Lambda}_G^+$.

Therefore the Renner cones of $\overline{M}$ and $e_P \cdot (\overline{G_{\text{enh}}})_{\mathbf{c}_P} \cdot e_P$ are equal, which proves the theorem. $\square$

CHAPTER 2

INVARIANT BILINEAR FORM DEFINED VIA ASYMPTOTICS

2.1 Introduction

2.1.1 The goal of this chapter

In [27], an invariant symmetric bilinear form $\mathcal{B}$ is defined on the space of automorphic forms for SL(2) over any global field. The goal of this chapter is to generalize the definition of $\mathcal{B}$ and the corresponding theory to any split reductive group G over a function field. The study of $\mathcal{B}$ is motivated by works [25, 30] on the geometric Langlands program. There is also a significant connection between $\mathcal{B}$ and the theory of Eisenstein series, as evidenced by [27].

Let G be a split reductive group over $\mathbb{F}_q$. Let X be a geometrically connected smooth projective curve over a finite field $\mathbb{F}_q$, and let F be the field of rational functions on X. Let $\mathbb{A}$ denote the adele ring of F.

For any place v of F, the completion of F with respect to v will be denoted F_v. Let $\mathfrak{o}_v$ denote the ring of integers of F_v, with residue field $\mathbb{F}_{q_v}$. We denote the standard maximal compact subgroup of $G(F_v)$ by K_v. Set $K := \prod_v K_v$; this is a maximal compact subgroup of $G(\mathbb{A})$.

We fix a field E of characteristic 0. Unless specified otherwise, all functions will take values in E.

Let $\mathcal{A}$ denote the space of K-finite C^∞ functions on $G(\mathbb{A})/G(F)$. Let $\mathcal{A}_c \subset \mathcal{A}$ denote the subspace of functions with compact support.

In this chapter we define and study a $G(\mathbb{A})$-invariant symmetric bilinear form $\mathcal{B}$ on $\mathcal{A}_c$. (The definition of $\mathcal{B}$ is given in §2.4.1.) Fix a Haar measure on $G(\mathbb{A})$. The form $\mathcal{B}$ is defined as an alternating sum of invariant bilinear forms $\mathcal{B}_P$ on $\mathcal{A}_c$, where the sum ranges over the conjugacy classes of parabolic subgroups of G. When $P = G$, the form $\mathcal{B}_G$ is the naive pairing

$$\mathcal{B}_{\mathrm{naive}}(f_1, f_2) = \int_{G(\mathbb{A})/G(F)} f_1(x) f_2(x) dx, \qquad f_1, f_2 \in \mathcal{A}_c.$$

The definition of $\mathcal{B}_P$ was suggested by Y. Sakellaridis in a private communication, and it uses the *local* asymptotics maps constructed in [10, 60] using the geometry of the De Concini–Procesi wonderful compactification of G. The asymptotics map is defined in the more general setting of harmonic analysis on spherical varieties in [60]. In [10], the asymptotics map is used to give a geometric proof of Bernstein's theorem on second adjointness. It is also shown ([10, Theorem 7.6]) that the asymptotics map is inverse to the standard (long) intertwining operator in the classical representation theory of $\mathfrak{p}$-adic groups. Using this relationship, one sees that the computation of the asymptotics of the characteristic function of K_v (cf. [59, §6]) goes back to the classical non-Archimedean Gindikin–Karpelevich formula due to [48, 50].

In order to study the form $\mathcal{B}$, we consider certain subspaces $\mathcal{C}_{P,\pm}$ of the space of smooth K-finite functions on $G(\mathbb{A})/M(F)U(\mathbb{A})$, where $P = MU$ is a standard parabolic subgroup with Levi subgroup M and unipotent radical U. The spaces $\mathcal{C}_{P,\pm}$ may be of independent interest as they are defined with respect to the same rational cones and support conditions as in the definition of Arthur's truncation operator (cf. the definition of $\widehat{\tau}_P$ in [2, §6]). In Proposition 2.5.5, we prove that the standard intertwining operator extends to an isomorphism $R_P : \mathcal{C}_{P^-,+} \to \mathcal{C}_{P,-}$.

Remark 2.1.1. We only consider the function field case in this chapter, but the reader may check that the definition of $\mathcal{B}$ on K-invariant automorphic forms extends to the number field case using the Archimedean Gindikin–Karpelevich formula. We hope to define $\mathcal{B}$ on the whole space $\mathcal{A}_c$ for an arbitrary global field F by better understanding the local Archimedean intertwining operator in the future.

2.1.2 Motivation from geometric Langlands

Let us explain the motivation for the existence of $\mathcal{B}$ from the geometric Langlands program. Here we assume that the field E equals $\overline{\mathbb{Q}}_\ell$ for a prime ℓ coprime to the characteristic of F.

A remarkable ℓ-adic complex on $\mathrm{Bun}_G \times \mathrm{Bun}_G$. Let Bun_G denote the stack of G-bundles on X. Let $\Delta : \mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$ be the diagonal morphism. We have the ℓ-adic complex $\Delta_*(\overline{\mathbb{Q}}_\ell)$ on $\mathrm{Bun}_G \times \mathrm{Bun}_G$.

This complex is the ℓ-adic analog of the complex of D-modules $\Delta_! \omega_{\mathrm{Bun}_G}$, which plays a crucial role in the theory of *miraculous duality* on Bun_G, which was developed in [25, §4.5] and [30]. Assume for the moment that X is over a ground field k of characteristic 0. Then miraculous duality gives an equivalence between the DG category of (complexes of) D-modules on Bun_G and its Lurie dual. Very roughly, the equivalence is defined as the functor

$$\mathrm{Ps\text{-}Id}_{\mathrm{Bun}_G,!} : \mathrm{D\text{-}mod}(\mathrm{Bun}_G)_{\mathrm{co}} \to \mathrm{D\text{-}mod}(\mathrm{Bun}_G)$$

given by the kernel $\Delta_! \omega_{\mathrm{Bun}_G}$ (whereas the identity functor is given by the kernel $\Delta_* \omega_{\mathrm{Bun}_G}$). The fact that this functor is an equivalence is a highly nontrivial theorem [30, Theorem 0.2.4].

The function b. Given G-bundles $\mathcal{F}^1_G, \mathcal{F}^2_G \in \mathrm{Bun}_G(\mathbb{F}_q)$, let $b(\mathcal{F}^1_G, \mathcal{F}^2_G)$ denote the trace of the geometric Frobenius acting on the $*$-stalk of the complex $\Delta_*(\overline{\mathbb{Q}}_\ell)$ over the point $(\mathcal{F}^1_G, \mathcal{F}^2_G) \in (\mathrm{Bun}_G \times \mathrm{Bun}_G)(\overline{\mathbb{F}}_q)$. Using results of [62], we deduce a formula for b in terms of the asymptotics maps (see Theorem A.3.12).

Relation between $\mathcal{B}$ and b. The quotient $K\backslash G(\mathbb{A})/G(F)$ identifies with $|\mathrm{Bun}_G(\mathbb{F}_q)|$, the set of isomorphism classes of G-bundles on X. So the function b can be considered as a function on $(G(\mathbb{A})/G(F)) \times (G(\mathbb{A})/G(F))$. The following theorem is one of our main results. The proof is given in §2.4.4.

Theorem 2.1.2. *Let $E = \overline{\mathbb{Q}}_\ell$ for ℓ coprime to the characteristic of F. Normalize the Haar* *measure on $G(\mathbb{A})$ so that K has measure 1. Then for any $f_1, f_2 \in \mathcal{A}_c^K$, one has*

$$\mathcal{B}(f_1, f_2) = \int_{(G\times G)(\mathbb{A})/(G\times G)(F)} b(g_1, g_2) f_1(g_1) f_2(g_2) dg_1 dg_2. \tag{2.1}$$

By non-degeneracy of the naive pairing $\mathcal{B}_{\text{naive}}$, defining the bilinear form $\mathcal{B}$ is equivalent to defining an operator $L : \mathcal{A}_c \to \mathcal{A}$ such that

$$\mathcal{B}(f_1, f_2) = \mathcal{B}_{\text{naive}}(Lf_1, f_2), \qquad f_1, f_2 \in \mathcal{A}_c.$$

Theorem 2.1.2 implies that the miraculous duality functor $\text{Ps-Id}_{\text{Bun}_G,!}$ is a D-module analog of the operator $q^{-\dim \text{Bun}_G} L$ via the functions–sheaves dictionary.

2.1.3 Analog of the Aubert–Zelevinsky involution

In §2.6.6, we define a subspace $\mathcal{A}_{ps\text{-}c} \subset \mathcal{A}$ of "pseudo-compactly" supported functions using the constant term operators and the spaces $\mathcal{C}_{P,+}$. We prove that the operator L above sends $\mathcal{A}_c$ to $\mathcal{A}_{ps\text{-}c}$, and the operator $L : \mathcal{A}_c \to \mathcal{A}_{ps\text{-}c}$ is an isomorphism (Theorem 2.6.12). The invertibility of L may be considered as a function-theoretic analog of the main result (Theorem 0.2.4) of [30].

Moreover, we give an explicit formula

$$L^{-1} f = \sum_P (-1)^{\dim Z(M)} (\text{Eis}_P \circ \text{CT}_P)(f), \qquad f \in \mathcal{A}_{ps\text{-}c},$$

for the inverse, where $\text{Eis}_P, \text{CT}_P$ denote respectively, the (pseudo-)Eisenstein operator and constant term operator. By considering $\text{Eis}_P, \text{CT}_P$ as global analogs of the parabolic induction functor and Jacquet functor, respectively, in the theory of smooth representations of a $\mathfrak{p}$-adic group, one can view the formula for L^{-1} as an analog of the formula for the Aubert–Zelevinsky involution on the Grothendieck group of smooth representations of finite length (see Remark 2.6.13). This involution was first defined and studied for $G = \text{GL}(n)$

by Zelevsinky [70] and later for general reductive groups by Aubert [4]. On Iwahori fixed vectors, it also corresponds to the Iwahori–Matsumoto involution (cf. [44]). There is an analogous involution for representations of a finite Chevalley group, often called the Alvis–Curtis involution, which was studied earlier in [1, 21].

The Aubert–Zelevinsky involution can be studied at the level of complexes. Such complexes were considered in [22] for representations of a finite Chevalley group. For every smooth representation M one can form a complex

$$0 \to M \to \bigoplus_{P} i_P^G r_P^G(M) \to \cdots \to i_B^G r_B^G(M) \to 0$$

where i_P^G, r_P^G denote, respectively, the parabolic induction and Jacquet functors, and the sum in the i-th term runs over standard parabolic subgroups of corank i in G. We call this complex the Deligne–Lusztig complex associated to M and denote it by $\mathrm{DL}(M)$. Aubert showed that for an irreducible module M, the complex $\mathrm{DL}(M)$ has cohomology in only one degree, which implies that the Aubert–Zelevsinky involution sends irreducible modules to irreducible modules (up to a sign). A new proof of this result was recently given in [9] using asymptotics maps and the geometry of the wonderful compactification of G.

2.1.4 Structure of the chapter

General remark. In the main body of the article we work with *classical* functions on $G(F_v)$ and $G(\mathbb{A})/G(F)$. These are, however, heavily motivated by geometric definitions and results appearing in the *geometric* Langlands program. We review the relevant geometry in Appendices A.1–A.3.

The main body of the chapter.

In Section 2.2, we study the asymptotics map and its relation to the intertwining operator over a local non-Archimedean field. In order to elucidate the support conditions of various functions, we give a combinatorial description of the bounded subsets of the boundary degenerations of G.

In Section 2.3, we compute the asymptotics of the characteristic function of K_v by reducing to the non-Archimedean Gindikin–Karpelevich formula using intertwining operators on K_v-invariants. To do so, we extend the classical Satake isomorphism to an isomorphism between certain completed Hecke algebras.

In Section 2.4, we define the bilinear form $\mathcal{B}$. After giving a geometric interpretation of the restriction of $\mathcal{B}$ to $\mathcal{A}_c^K$, we prove Theorem 2.1.2.

The definition of $\mathcal{B}$ we give differs slightly from the definition given in [27] for $G = \mathrm{SL}(2)$. In our definition, we use local asymptotics (which is essentially equivalent to local inverse intertwining operators) and then apply a local-to-global procedure.

In Section 2.5, we provide an alternate definition of $\mathcal{B}$, which directly generalizes the one in [27]. For a parabolic subgroup P with Levi factor M, we define subspaces $\mathcal{C}_{P,\pm}$ of the space of K-finite C^∞ functions on $G(\mathbb{A})/M(F)U(\mathbb{A})$. The definitions are such that the constant term operator CT_P (whose definition we recall) sends $\mathcal{A}_c$ to $\mathcal{C}_{P,-}$. The intertwining operator R_P (which is of local nature) is defined as a map $\mathcal{C}_{P^-,+} \to \mathcal{C}_{P,-}$, and we show that R_P is an isomorphism. Let $\langle\, ,\rangle$ denote the natural pairing between functions in $\mathcal{C}_{P^-}$ (when convergent). We prove the following in §2.5.6:

Theorem 2.1.3. *For any $f_1, f_2 \in \mathcal{A}_c$, one has*

$$\mathcal{B}(f_1, f_2) = \sum_P (-1)^{\dim Z(M)} \langle R_P^{-1} \mathrm{CT}_P(f_1), \mathrm{CT}_{P^-}(f_2)\rangle, \tag{2.2}$$

where the sum ranges over conjugacy classes of parabolic subgroups of G.

In Section 2.6, we use Theorem 2.1.3 to define the operator $L : \mathcal{A}_c \to \mathcal{A}$ and the subspace $\mathcal{A}_{ps\text{-}c} \subset \mathcal{A}$ of "pseudo-compactly" supported functions. We show that L sends $\mathcal{A}_c$ to $\mathcal{A}_{ps\text{-}c}$, and in Theorem 2.6.12 we prove that the operator $L : \mathcal{A}_c \to \mathcal{A}_{ps\text{-}c}$ is invertible. We give a formula (2.53) for L^{-1}, which is in fact simpler than the formula for L. This formula may be viewed as an analog of the definition of the Aubert–Zelevinsky involution.

Appendices A.1–A.3. In Appendix A.1, we consider the global model for the formal arc space of a group embedding into an algebraic monoid. This model was also used in [11, §2]. We realize the global model as a substack of a symmetrized version of the Hecke stack. We give a bound on the difference of the Harder–Narasimhan coweights of the two bundles corresponding to a point of the Hecke stack (Lemma A.1.7). In this article, we are primarily interested in the stack $\mathcal{H}_M^+$ attached to the monoid $\overline{M}$, defined as the closure of M in the affine closure of G/U, where P is a parabolic subgroup with Levi factor M. The stack $\mathcal{H}_M^+$ is a graded Ran version (in the sense of [29]) of the closed substack of the Hecke stack studied in [13] and [12, §1.8]. The Hecke stack is a twisted product of Bun_M and the Beilinson–Drinfeld (factorizable) affine Grassmannian, and it is more convenient to use the latter to talk about factorization properties. We briefly review the relevant notation and properties of the factorizable affine Grassmannian – we use a symmetrized version that does not explicitly mention the Ran space.

In Appendix A.2, we review the definition of the factorization algebras on the affine Grassmannian introduced in [14, 29] that act on geometric Eisenstein series. The main goal of this Appendix is to highlight the connection (via Grothendieck's functions–sheaves dictionary) between certain measures (related to unramified intertwining operators) appearing in the classical non-Archimedean Gindikin–Karpelevich formula and Gaitsgory's factorization algebras (see Proposition A.2.4, Lemma A.2.5). From this perspective, we point out how the main theorem of [12] may be interpreted as a categorical or geometric version of (Langlands' interpretation of) the Gindikin–Karpelevich formula.

In Appendix A.3, we study the compactification of the diagonal morphism of Bun_G using the results of [62]. The compactification $\overline{\mathrm{Bun}}_G$ we define is slightly different from the one found in the literature. We review the definition and relevant properties of the Drinfeld–Lafforgue–Vinberg degeneration of Bun_G. In particular we highlight the connection between the geometric Bernstein asymptotics studied in *loc. cit.* and Gaitsgory's factorization algebras to deduce that for an arbitrary parabolic subgroup, the geometric Bernstein asymptotics corresponds to the classical asymptotics of the characteristic function of K via the functions–sheaves dictionary.

2.1.5 Conventions

Throughout the chapter, G will be a connected split reductive group over $\mathbb{F}_q$. Fix a split torus $T \subset G$ and a Borel B containing T. Let W be the Weyl group of T. Let $\check{\Lambda}$ (resp. Λ) denote the weight (resp. coweight) lattice of T.

The monoid of dominant weights (resp., coweights) will be denoted by $\check{\Lambda}^+_G$ (resp., by Λ^+_G). The set of vertices of the Dynkin diagram of G will be denoted by Γ_G; for each $i \in \Gamma_G$ there corresponds a simple coroot α_i and a simple root $\check{\alpha}_i$. The set of coroots (resp. positive coroots) will be denoted by Φ_G (resp. Φ^+_G) and the positive span of Φ^+_G inside Λ by Λ^{pos}_G. Let $\check{\Delta}_G$ (resp. $\check{\Phi}^+_G$, $\check{\Phi}^-_G$, $\check{\Phi}_G$) denote the simple (resp. positive, negative, all) roots of G. By $2\check{\rho} \in \check{\Lambda}$ (resp. $2\rho \in \Lambda$) we will denote the sum of the positive roots (coroots) of G and by w_0 the longest element in the Weyl group of G. For $\lambda, \mu \in \Lambda$ we will write that $\lambda \geq \mu$ if $\lambda - \mu \in \Lambda^{\mathrm{pos}}_G$, and similarly for $\check{\Lambda}^{\mathrm{pos}}_G$.

We will only consider parabolic subgroups that contain T. Let P be a standard[1] parabolic subgroup, i.e., P contains B. Then the Levi quotient can be canonically realized as a subgroup $M \subset P$. We have $P = MU$ where U is the unipotent radical of P. There is a unique parabolic P^- such that $P \cap P^- = M$. To M there corresponds a subdiagram $\Gamma_M \subset \Gamma_G$, coroots $\Phi_M \subset \Phi_G$, and positive coroots $\Phi^+_M \subset \Phi^+_G$. We will denote by $\Lambda^+_M \supset \Lambda^+_G$, $\Lambda^{\mathrm{pos}}_M \subset \Lambda^{\mathrm{pos}}_G$, $2\check{\rho}_M \in \check{\Lambda}$, $\geq_M$, etc. the corresponding objects for M.

Let $\mathrm{Rep}(G)$ denote the abelian category of finite-dimensional G-modules.

Given two G-spaces Y, Z such that the diagonal action of G on $Y \times Z$ is free, we let

1. Recall that in any conjugacy class of parabolic subgroups of G, there is exactly one standard parabolic subgroup.

$Y \times^G Z$ denote the quotient of $Y \times Z$ by the diagonal G-action.

For a scheme or stack $\mathcal{Y}$, we let $D(\mathcal{Y})$ denote the DG category of bounded constructible $\overline{\mathbb{Q}}_\ell$-sheaves on $\mathcal{Y}$. We will use 'sheaf' to mean a complex of sheaves. All functors between sheaves are derived functors. When $\mathcal{Y}$ is a stack over $\operatorname{Spec} \mathbb{F}_q$, we assume that ℓ is coprime to q. Choose a square root of q in $\overline{\mathbb{Q}}_\ell$ once and for all. The intersection cohomology sheaves are normalized so that they are pure of weight 0. In other words, for a smooth $\mathbb{F}_q$-stack $\mathcal{Y}$ of dimension n, $\mathrm{IC}_{\mathcal{Y}} \cong (\overline{\mathbb{Q}}_\ell(\frac{1}{2})[1])^{\otimes n}$.

2.2 Local intertwining operators and asymptotics

In this section, we work over a non-Archimedean local field F_v, and G is a connected split reductive group over F_v. The subscript v is only present to keep notation consistent throughout this article – the presence of a global field is not assumed, and the characteristic of F_v is arbitrary (and possibly zero).

Let $||_v$ denote the absolute value on F_v, let $\mathfrak{o}_v$ denote the ring of integers of F_v, and let q_v be the cardinality of the residue field. We will use G, P, $\mathbb{X}_P$, etc. to also denote the topological groups/spaces of F_v-points of the corresponding algebraic groups or varieties, e.g., $G = G(F_v)$, $P = P(F_v)$, $\mathbb{X}_P = \mathbb{X}_P(F_v)$. Let $K = K_v$ denote the standard maximal compact subgroup of G, and K_M denotes the standard maximal compact subgroup of M.

In §2.2.1–2.2.3, we define the space $\mathbb{X}_P$ and describe how to consider bounded subsets of G/U and $\mathbb{X}_P$ in terms of subsets of the lattice Λ. In §2.2.4–2.2.7, we review some definitions and results from [10] to introduce the local asymptotics map Asymp_P, which is "essentially the same" as the inverse of the standard intertwining operator. We observe that Asymp_P is determined by a generalized function ξ_P on $\mathbb{X}_P$. In §2.2.5–2.2.8, we give a formula for the inverse of the intertwining operator in terms of ξ_P.

2.2.1 *Bounded sets*

Let $\mathbb{X}$ be a quasi-affine variety over F_v (i.e., there exists a locally closed embedding of $\mathbb{X}$ into a finite dimensional affine space). We say that a subset $S \subset \mathbb{X}(F_v)$ is *bounded* if the following equivalent conditions are satisfied:

(i) for any regular function $f \in F_v[\mathbb{X}] := \Gamma(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$, the function $|f|_v$ is bounded on S,

(ii) for any locally closed embedding (in the sense of algebraic geometry) of $\mathbb{X}$ into an affine space, the image of S is bounded (with respect to the norm induced by the absolute value on F_v),

(iii) for any open embedding (in the sense of algebraic geometry) of X into an affine variety, the image of S is relatively compact (for the "usual" topology induced by the topology on F_v).

Recall that an F_v-scheme $\mathbb{X}$ is *strongly quasi-affine* if the canonical morphism

$$\mathbb{X} \to \operatorname{Spec} F_v[\mathbb{X}]$$

is an open embedding and $F_v[\mathbb{X}]$ is a finitely generated F_v-algebra. For a strongly quasi-affine variety $\mathbb{X}$, in condition (iii) it suffices to consider only the open embedding $\mathbb{X} \hookrightarrow \operatorname{Spec} F_v[\mathbb{X}]$.

2.2.2 *The strongly quasi-affine varieties G/U and $\mathbb{X}_P$*

Fix a standard parabolic subgroup $P \subset G$ with Levi subgroup M and unipotent radical U.

The quotient varieties G/U and G/U^- are strongly quasi-affine by [36]. Let $\overline{G/U} := \operatorname{Spec} F_v[G/U]$ and $\overline{G/U^-} := \operatorname{Spec} F_v[G/U^-]$ denote the affine closures.

We review the definition of the variety $\mathbb{X}_P$ introduced in [10, §2.2.1] below.

Define the *boundary degeneration*

$$\mathbb{X}_P := (G \times G)/(P \underset{M}{\times} P^-) = (G/U \times G/U^-)/M,$$

where M acts diagonally on the right. Recall (cf. [24, Proposition 2.4.4]) that $\mathbb{X}_P$ is a quasi-affine variety; let $\overline{\mathbb{X}}_P := \operatorname{Spec} F_v[\mathbb{X}_P]$ denote the affine closure. By [36], $F_v[G/U \times G/U^-]$ is finitely generated. Therefore Hilbert's theorem on invariants implies that $F_v[\mathbb{X}_P] = F_v[G/U \times G/U^-]^M$ is finitely generated (i.e., $\mathbb{X}_P$ is strongly quasi-affine). Thus a subset $S \subset \mathbb{X}_P$ is bounded if and only if $f(S) \subset F_v$ is bounded for every $f \in F_v[\mathbb{X}_P]$.

2.2.3 Combinatorial setup

We give a combinatorial description of bounded subsets of $\mathbb{X}_P$ in Proposition 2.2.2 below.

By the Cartan decomposition, $K_M\backslash M/K_M = (T/K_T)/W_M$. We have an isomorphism

$$\operatorname{ord}_T : T/K_T \to \Lambda$$

sending $\lambda(x) \mapsto \lambda \otimes (-\log_{q_v} |x|_v)$ where $\lambda \in \Lambda$, $x \in F_v^\times$. This induces an isomorphism

$$\operatorname{ord}_M : K_M\backslash M/K_M \to \Lambda_M^+. \tag{2.3}$$

By the Iwasawa decomposition, $G = K \cdot P = K \cdot P^-$. Therefore (2.3) induces the projections

$$\operatorname{ord}_M : G/U \to K\backslash(G/U)/K_M = K_M\backslash M/K_M = \Lambda_M^+ \tag{2.4}$$

and $\operatorname{ord}_M : G/U^- \to \Lambda_M^+$. We have a left $G \times G$-action on $\mathbb{X}_P$. Using (2.3) again, we also define the projection

$$\operatorname{ord}_M : \mathbb{X}_P \to (K \times K)\backslash\mathbb{X}_P = K_M\backslash M/K_M = \Lambda_M^+, \tag{2.5}$$

where the first equality sends $(m_1, m_2) \mapsto m_1^{-1} m_2$ when $m_1, m_2 \in M$.

Lemma 2.2.1. *Let $g_1 \in G/U$ and $g_2 \in G/U^-$. Consider the image of (g_1, g_2) in $\mathbb{X}_P$. Then*

$$w_0^M \mathrm{ord}_M(g_1, g_2) \leq_M \mathrm{ord}_M(g_2) - \mathrm{ord}_M(g_1) \leq_M \mathrm{ord}_M(g_1, g_2).$$

Proof. Let $\lambda_1 = \mathrm{ord}_M(g_1)$, $\lambda_2 = \mathrm{ord}_M(g_2)$, and $\theta = \mathrm{ord}_M(g_1, g_2)$. It follows from the definitions that $\theta(\varpi_v) \in K_M \lambda_1(\varpi_v)^{-1} K_M \lambda_2(\varpi_v) K_M$, where $\varpi_v \in \mathfrak{o}_v$ is a uniformizer. This is equivalent to $\lambda_2(\varpi_v) \in K_M \lambda_1(\varpi_v) K_M \theta(\varpi_v) K_M$. The usual properties of the (spherical) Hecke algebra imply that $\lambda_2 \leq_M \lambda_1 + \theta$. Similarly, we also have $\lambda_1(\varpi_v) \in K_M \lambda_2(\varpi_v) K_M \theta(\varpi_v)^{-1} K_M$, which implies that $\lambda_1 \leq_M \lambda_2 - w_0^M \theta$. □

Let $\Lambda^{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$. Let $\Lambda_G^{\mathrm{pos},\mathbb{Q}} \subset \Lambda^{\mathbb{Q}}$ denote the rational cone corresponding to Λ_G^{pos}. We define the rational ordering $\leq_G^{\mathbb{Q}}$ by $\mu \leq_G^{\mathbb{Q}} \lambda$ if and only if $\lambda - \mu \in \Lambda_G^{\mathrm{pos},\mathbb{Q}}$.

Let $\check{\Lambda}_G^{+,\mathbb{Q}} \subset \check{\Lambda}^{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \check{\Lambda}$ denote the rational cone corresponding to $\check{\Lambda}_G^+$.

We say that a subset $S \subset \Lambda^{\mathbb{Q}}$ is *bounded below* (with respect to $\leq_G^{\mathbb{Q}}$) if the following equivalent conditions are satisfied:

(i) For any $\check{\lambda} \in \check{\Lambda}_G^+$, the subset $\check{\lambda}(S) \subset \mathbb{Q}$ is bounded below.

(ii) There exists a subset $S_0 \subset \Lambda^{\mathbb{Q}}$ with compact closure in $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ such that $S \subset S_0 + \Lambda_G^{\mathrm{pos},\mathbb{Q}}$.

Define $S \subset \Lambda_G^{\mathbb{Q}}$ to be *bounded above* if $-S$ is bounded below.

Proposition 2.2.2. *A subset $S \subset \mathbb{X}_P$ is bounded if and only if $\mathrm{ord}_M(S) \subset \Lambda_M^+$ is bounded above.*

Proof. Consider the embedding $T \hookrightarrow \mathbb{X}_P : t \mapsto (t, 1)$ and let $\overline{T}$ denote the closure of T in $\overline{\mathbb{X}}_P$. Let $S_T \subset T$ denote the preimage of $(K \times K) \cdot S \subset \mathbb{X}_P$ under the previous embedding. Then S is bounded if and only if S_T is bounded in $\overline{T}$. Note that S_T is W_M-stable, and $-\mathrm{ord}_T(S_T) = W_M \cdot \mathrm{ord}_M(S)$. It is shown in Corollary 1.4.3 that $F_v[\overline{T}] \subset F_v[T]$ has a basis formed by the characters in $W_M \cdot \Lambda_G^+$. For a weight $\check{\lambda} \in \check{\Lambda}_G^+$ and $t \in T$, we have $-\log_{q_v} |\check{\lambda}(t)|_v = \langle \check{\lambda}, \mathrm{ord}_T(t) \rangle$. Therefore S_T is bounded in $\overline{T}$ if and only if $-\mathrm{ord}_T(S_T)$ is bounded above. Since $\mathrm{ord}_M(S) \subset \Lambda_M^+$, we conclude that $W_M \cdot \mathrm{ord}_M(S)$ is bounded above if and only if $\mathrm{ord}_M(S)$ is bounded above. $\square$

The rational cone $\Lambda_U^{\mathrm{pos},\mathbb{Q}}$. We introduce the rational cone $\Lambda_U^{\mathrm{pos},\mathbb{Q}}$, which is used throughout this chapter, and review some of its properties, which are proved in Chapter 1.

Let $\Lambda_U^{\mathrm{pos}} \subset \Lambda$ denote the non-negative integral span of the positive coroots of G that are not coroots of M. The submonoid Λ_U^{pos} is stable under the actions of W_M. Let $\Lambda_U^{\mathrm{pos},\mathbb{Q}}$ denote the corresponding rational cone.

Remark 2.2.3. Let X_P denote an $\mathfrak{o}_v$-model of $\mathbb{X}_P$, and set $\overline{\mathsf{X}}_\mathsf{P} := \mathrm{Spec}\,\Gamma(\mathsf{X}_\mathsf{P}, \mathcal{O}_{\mathsf{X}_\mathsf{P}})$. Then $\overline{\mathsf{X}}_\mathsf{P} \times_{\mathrm{Spec}\,\mathfrak{o}_v} \mathrm{Spec}\, F_v = \overline{\mathbb{X}}_P$, and $\overline{\mathsf{X}}_\mathsf{P}(\mathfrak{o}_v)$ is a $K \times K$-stable subset of $\overline{\mathbb{X}}_P(F_v)$. The proof of Proposition 2.2.2 shows that

$$\overline{\mathsf{X}}_\mathsf{P}(\mathfrak{o}_v) \cap \mathbb{X}_P(F_v) \subset \mathrm{ord}_M^{-1}((-\Lambda_U^{\mathrm{pos},\mathbb{Q}}) \cap \Lambda_M^+),$$

where $\Lambda_U^{\mathrm{pos},\mathbb{Q}}$ is the dual cone of $W_M \cdot \check{\Lambda}_G^{+,\mathbb{Q}}$ by Lemma 1.3.4.

We recall the definition of the Langlands retraction $\mathfrak{L} : \Lambda^{\mathbb{Q}} \to \Lambda_G^{+,\mathbb{Q}}$, which goes back to [49]. It is defined as follows: for $\lambda \in \Lambda^{\mathbb{Q}}$, let $\mathfrak{L}(\lambda)$ be the least element[2] in the set $\{\theta \in \Lambda_G^{+,\mathbb{Q}} \mid \lambda \leq_G^{\mathbb{Q}} \theta\}$ in the sense of the $\leq_G^{\mathbb{Q}}$ ordering. We refer the reader to [23] for further properties of the Langlands retraction.

Let $\Lambda_M^{+,\mathbb{Q}} \subset \Lambda^{\mathbb{Q}}$ denote the rational cone corresponding to Λ_M^+.

Lemma 2.2.4. *Let $\lambda \in \Lambda_M^{+,\mathbb{Q}}$. Then $\mathfrak{L}(\lambda) - \lambda \in \Lambda_U^{\mathrm{pos},\mathbb{Q}} \cap (-\Lambda_M^{+,\mathbb{Q}})$.*

Proof. Recall from [23, Proposition 2.1] that $\mathfrak{L}$ is piecewise linear, with linearity domains C_J indexed by subsets $J \subset \Gamma_G$: Let $V_J^\perp = \{\lambda \in \Lambda^{\mathbb{Q}} \mid \langle \check{\alpha}_j, \lambda \rangle = 0,\, j \in J\}$. Then C_J is the closed convex cone generated by $-\alpha_j, j \in J$ and $V_J^\perp \cap \Lambda_G^{+,\mathbb{Q}}$.

Suppose that $\lambda \in \Lambda_M^{+,\mathbb{Q}}$ lies in C_J. Then $\lambda - \mathfrak{L}(\lambda)$ belongs to the closed convex cone generated by $-\alpha_j$ for $j \in J$, and $\mathfrak{L}(\lambda) \in V_J^\perp$ by [23, Lemma 2.3]. Therefore $\langle \check{\alpha}_j, \lambda - \mathfrak{L}(\lambda) \rangle =$

2. The existence of the least element is not obvious; it was proved by R. P. Langlands in [49, §4].

$\langle \check{\alpha}_j, \lambda \rangle \geq 0$ for $j \in \Gamma_M \cap J$. Since $\langle \check{\alpha}_i, \alpha_j \rangle < 0$ for $i \in \Gamma_G - J$, $j \in J$, we also have $\langle \check{\alpha}_i, \lambda - \mathfrak{L}(\lambda) \rangle \geq 0$ for $i \in \Gamma_G - J$. Hence $\lambda - \mathfrak{L}(\lambda) \in (-\Lambda_G^{\mathrm{pos},\mathbb{Q}}) \cap \Lambda_M^{+,\mathbb{Q}}$. By Lemma 1.3.2, we have the equality $(-\Lambda_G^{\mathrm{pos},\mathbb{Q}}) \cap \Lambda_M^{+,\mathbb{Q}} = (-\Lambda_U^{\mathrm{pos},\mathbb{Q}}) \cap \Lambda_M^{+,\mathbb{Q}}$. □

Let $\Lambda^{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ and let $\Lambda_G^{+,\mathbb{R}}, \Lambda_G^{\mathrm{pos},\mathbb{R}}$ denote the real cones corresponding to $\Lambda_G^{+}, \Lambda_G^{\mathrm{pos}}$.

Corollary 2.2.5. *A subset $S \subset \Lambda_M^{+,\mathbb{Q}}$ is bounded above if and only if there exists a compact subset $S_0 \subset \Lambda_G^{+,\mathbb{R}}$ such that S is contained in the set $\{\theta - \mu \mid \theta \in S_0,\ \mu \in \Lambda_U^{\mathrm{pos},\mathbb{Q}}\}$.*

Proof. Suppose $S \subset \Lambda_M^{+,\mathbb{Q}}$ is bounded above. Then there exists a compact subset $S_1 \subset \Lambda^{\mathbb{R}}$ such that S is contained in $\{\theta - \mu \mid \theta \in S_1,\ \mu \in \Lambda_G^{\mathrm{pos},\mathbb{Q}}\}$. Let S_0 denote the closure of $\mathfrak{L}(S)$ in $\Lambda^{\mathbb{R}}$. Then S_0 is contained in $\Lambda_G^{+,\mathbb{R}} \cap \{\theta - \mu \mid \theta \in S_1,\ \mu \in \Lambda_G^{\mathrm{pos},\mathbb{R}}\}$, which is a compact set. Lemma 2.2.4 implies that S is contained in $\{\theta - \mu \mid \theta \in S_0,\ \mu \in \Lambda_U^{\mathrm{pos},\mathbb{Q}}\}$. The other direction is evident. □

The closed embedding $G \times P^- \hookrightarrow G \times G$ induces a closed embedding $G/U \hookrightarrow \mathbb{X}_P$ sending $g_1 \mapsto (g_1, 1)$. By Corollary 1.4.3, this embedding extends to a closed embedding of affine closures $\overline{G/U} \hookrightarrow \overline{\mathbb{X}}_P$. Similarly, the closed embedding $G/U^- \hookrightarrow \mathbb{X}_P : g_2 \mapsto (1, g_2)$ extends to a closed embedding $\overline{G/U^-} \hookrightarrow \overline{\mathbb{X}}_P$. Using these embeddings, we deduce the combinatorial description for bounded subsets of G/U and G/U^- from Proposition 2.2.2:

Proposition 2.2.6. *(i) A subset $S \subset G/U$ is bounded if and only if there exists a finite subset $S_0 \subset \Lambda$ such that* $\mathrm{ord}_M(S) \subset S_0 + \Lambda_U^{\mathrm{pos},\mathbb{Q}}$.

(ii) A subset $S \subset G/U^-$ is bounded if and only if there exists a finite subset $S_0 \subset \Lambda$ such that $\mathrm{ord}_M(S) \subset \{\theta - \mu \mid \theta \in S_0,\ \mu \in \Lambda_U^{\mathrm{pos},\mathbb{Q}}\}$.

Proof. Let $g_1 \in G/U$. Then $\mathrm{ord}_M(g_1) = -w_0^M \cdot \mathrm{ord}_M(g_1, 1)$. Using the closed embedding $\overline{G/U} \hookrightarrow \overline{\mathbb{X}}_P$, we deduce (i) from Proposition 2.2.2 and Corollary 2.2.5. For $g_2 \in G/U^-$, we have $\mathrm{ord}_M(g_2) = \mathrm{ord}_M(1, g_2)$, so we can similarly deduce (ii) using the closed embedding $\overline{G/U^-} \hookrightarrow \overline{\mathbb{X}}_P$. □

2.2.4 The space $\mathcal{C}_b(\mathbb{X}_P)$

We review the definitions of the space $\mathcal{C}_b(\mathbb{X}_P)$ from [10] in the context of our combinatorial setup.

Let $S^*(\mathbb{X}_P)$ denote the space of distributions on $\mathbb{X}_P$. Using our fixed choice of Haar measures, we identify distributions and generalized functions on $\mathbb{X}_P$. Given a generalized function $\xi \in S^*(\mathbb{X}_P)$, one can define a map $T_\xi : C_c^\infty(G/U) \to C^\infty(G/U^-)$ by the formula

$$T_\xi(\varphi)(g_2) = \int_{G/U} \varphi(g_1)\xi(g_1, g_2)dg_1, \qquad \varphi \in C_c^\infty(G/U),\ g_2 \in G/U^-. \tag{2.6}$$

Let $\mathcal{C}(\mathbb{X}_P)$ denote the space of $K \times K$-finite C^∞ functions on $\mathbb{X}_P$.

Let $\mathcal{C}_b(\mathbb{X}_P) \subset \mathcal{C}(\mathbb{X}_P)$ denote the subspace of functions with bounded support. Proposition 2.2.2 implies that $\mathcal{C}_b(\mathbb{X}_P)$ is the set of functions $\xi \in \mathcal{C}(\mathbb{X}_P)$ such that $\mathrm{ord}_M(\mathrm{supp}\,\xi)$ is bounded above.

We say that a generalized function $\xi \in S^*(\mathbb{X}_P)$ has *essentially bounded support* if the convolution of ξ with any element of $C_c^\infty(G) \otimes C_c^\infty(G)$ has bounded support. Let $S_b^*(\mathbb{X}_P)^G$ denote the space of generalized function with essentially bounded support that are invariant under the diagonal G-action on $\mathbb{X}_P$.

2.2.5 The spaces $\mathcal{C}_{P,\pm}$

Let $\Lambda_{G,P}^{\mathbb{Q}} = \Lambda_{M/[M,M]}^{\mathbb{Q}}$. This vector space is the quotient of $\Lambda^{\mathbb{Q}}$ by the subspace spanned by the coroots of M. For $\lambda \in \Lambda^{\mathbb{Q}}$, let $[\lambda]_P$ denote the projection of λ to $\Lambda_{G,P}^{\mathbb{Q}}$. We define the map

$$\deg_P : G/U \to \Lambda_{G,P}^{\mathbb{Q}}$$

by $\deg_P(x) = [\mathrm{ord}_M(x)]_P$.

Let $\Lambda_{G,P}^{\mathrm{pos},\mathbb{Q}}$ denote the image of $\Lambda_G^{\mathrm{pos},\mathbb{Q}}$ (equivalently $\Lambda_U^{\mathrm{pos},\mathbb{Q}}$) under the projection $\Lambda^{\mathbb{Q}} \to \Lambda_{G,P}^{\mathbb{Q}}$.

Let $\mathcal{C}_P$ denote the space of K-finite C^∞ functions on G/U. Let $\mathcal{C}_{P,c} \subset \mathcal{C}_P$ stand for the subspace of compactly supported functions.

Let $\mathcal{C}_{P,+} \subset \mathcal{C}_P$ denote the set of all functions $\varphi \in \mathcal{C}_P$ such that $\deg_P(\operatorname{supp}\varphi)$ is contained in $S_0 + \Lambda^{\mathrm{pos},\mathbb{Q}}_{G,P}$ for some finite subset $S_0 \subset \Lambda^{\mathbb{Q}}_{G,P}$. Similarly, let $\mathcal{C}_{P,-} \subset \mathcal{C}_P$ denote the set of all $\varphi \in \mathcal{C}_P$ such that $-\deg_P(\operatorname{supp}\varphi)$ is contained in $S_0 + \Lambda^{\mathrm{pos},\mathbb{Q}}_{G,P}$ for some finite set S_0.

One similarly defines the spaces $\mathcal{C}_{P^-,\pm} \subset \mathcal{C}_{P^-}$. We emphasize that $\mathcal{C}_{P^-,+}$ is defined with respect to the cone $-\Lambda^{\mathrm{pos},\mathbb{Q}}_{G,P}$. So $\mathcal{C}_{P^-,\pm}$ is the set of all $\varphi \in \mathcal{C}_{P^-}$ such that $\mp\deg_{P^-}(\operatorname{supp}\varphi)$ is contained in $S_0 + \Lambda^{\mathrm{pos},\mathbb{Q}}_{G,P}$ for some finite set S_0.

Lemma 2.2.7. *Let $\xi \in S^*_b(\mathbb{X}_P)^G$ be a generalized function with essentially bounded support. Then formula* (2.6) *defines a map $T_\xi : \mathcal{C}_{P,-} \to \mathcal{C}_{P^-,+}$.*

Proof. Let $\varphi \in \mathcal{C}_{P,-}$. Since φ is K-finite, there exists a compact open subgroup $K' \subset K$ such that φ is K'-invariant. Let $\delta_{K'} \in C^\infty_c(G)$ equal $\frac{1}{\mathrm{mes}(K')}$ times the characteristic function of K'. Then

$$\xi' := (\delta_{K'} \otimes 1) * \xi = (1 \otimes \delta_{K'}) * \xi \in C^\infty_b(\mathbb{X}_P)$$

is a smooth function with bounded support, and it suffices to show that $T_{\xi'}(\varphi) = T_\xi(\varphi)$ is well-defined and belongs to $\mathcal{C}_{P^-,+}$.

Fix $g_2 \in G/U^-$. For any $g_1 \in G/U$, Lemma 2.2.1 gives the inequality $\mathrm{ord}_M(g_1) \leq_M \mathrm{ord}_M(g_2) - w_0^M \mathrm{ord}_M(g_1, g_2)$. Then Corollary 2.2.5 implies that there is a finite subset $S_0 \subset \Lambda$ such that if $\xi'(g_1, g_2) \neq 0$, then $\mathrm{ord}_M(g_1) \subset S_0 + w_0^M \Lambda^{\mathrm{pos},\mathbb{Q}}_G$. From this combinatorial description, we deduce that the function sending $g_1 \in G/U$ to $\varphi(g_1)\xi'(g_1, g_2)$ is compactly supported. Therefore $T_{\xi'}(\varphi)$ is well-defined.

Moreover, if $T_{\xi'}(\varphi)(g_2) \neq 0$, then there must exist $g_1 \in G/U$ such that $g_1 \in \operatorname{supp}\varphi$ and $(g_1, g_2) \in \operatorname{supp}\xi' \subset \mathbb{X}_P$. Observe that $\deg_{P^-}(g_2) = \deg_P(g_1) + [\mathrm{ord}_M(g_1, g_2)]_P$ in $\Lambda^{\mathbb{Q}}_{G,P}$. Since ξ' has bounded support, $[\mathrm{ord}_M(\operatorname{supp}\xi')]_P$ is contained in $-\Lambda^{\mathrm{pos},\mathbb{Q}}_{G,P} + S_1$ for a finite set S_1. By definition of $\mathcal{C}_{P,-}$, we deduce that $\deg_{P^-}(g_2)$ must lie in $-\Lambda^{\mathrm{pos},\mathbb{Q}}_{G,P} + S_2$ for some finite set S_2. Thus $T_{\xi'}(\varphi) \in \mathcal{C}_{P^-,+}$. $\square$

2.2.6 Intertwining operator

Define the intertwining operator $R_P : C_c^\infty(G/U^-) \to C^\infty(G/U)$ by the formula

$$R_P(\varphi)(g) = \int_U \varphi(gu)du, \qquad g \in G. \tag{2.7}$$

Let $\mathbb{X}_P^-$ denote the space $(G/U^- \times G/U)/M$. Any generalized function $\eta \in S^*(\mathbb{X}_P^-)$ defines a map $T_\eta : C_c^\infty(G/U^-) \to C^\infty(G/U)$ as in formula (2.6). Let $\eta_P \in S^*(\mathbb{X}_P^-)$ denote the generalized function such that $R_P = T_{\eta_P}$, i.e.,

$$\int_U \varphi(g_2u)du = \int_{G/U^-} \varphi(g_1)\eta_P(g_1, g_2)dg_1, \qquad \varphi \in C_c^\infty(G/U^-),\ g_2 \in G. \tag{2.8}$$

Define the projection

$$\mathrm{ord}_M : \mathbb{X}_P^- \to (K \times K)\backslash \mathbb{X}_P^- = \Lambda_M^+$$

by sending $(m_1, m_2) \mapsto \mathrm{ord}_M(m_1^{-1}m_2)$ for $m_1, m_2 \in M$.

Lemma 2.2.8. *The subset* $\mathrm{ord}_M(\mathrm{supp}\,\eta_P)$ *is contained in* $(-\Lambda_U^{\mathrm{pos},\mathbb{Q}}) \cap \Lambda_M^+$. *In particular,* $\mathrm{ord}_M(\mathrm{supp}\,\eta_P)$ *is bounded above.*

Proof. Suppose that $(k_1a, k_2) \in \mathrm{supp}\,\eta_P$ for $k_1, k_2 \in K$ and $a \in T$ with $\mathrm{ord}_T(a) \in \Lambda_M^+$. Then by definition of η_P, there exists $u \in U$ such that $u \in kaU^-$ for $k = k_2^{-1}k_1$.

Fix a dominant weight $\check{\lambda} \in \check{\Lambda}_G^+$. Let $\Delta(\check{\lambda})$ denote the Weyl G-module with highest weight $\check{\lambda}$. This F_v-vector space is the extension of scalars of a free $\mathfrak{o}_v$-module, and the latter determines a K-invariant norm $||_v$ on $\Delta(\check{\lambda})$. We give $\Delta(\check{\lambda})^*$ the dual norm.

Let $\phi \in \Delta(\check{\lambda})^*$ be a norm 1 weight vector of weight $-w_0^M\check{\lambda}$. Since $-w_0^M\check{\lambda} \geq_M -\check{\lambda}$ and all weights of $\Delta(\check{\lambda})^*$ are $\geq_G -\check{\lambda}$, we observe that ϕ is U^--invariant. By orthogonality of weight spaces, there exists a weight vector $\xi \in \Delta(\check{\lambda})$ of weight $w_0^M\check{\lambda}$ and norm 1 such that

$\langle \phi, \xi \rangle = 1$. Note that ξ is automatically U-invariant. We have the inequality

$$0 = \log_{q_v} |\langle u \cdot \phi, \xi \rangle|_v = \langle w_0^M \check{\lambda}, \mathrm{ord}_T(a) \rangle + \log_{q_v} |\langle k \cdot \phi, \xi \rangle|_v \leq \langle w_0^M \check{\lambda}, \mathrm{ord}_T(a) \rangle.$$

Since $w\check{\lambda} \geq_M w_0^M \check{\lambda}$ for any $w \in W_M$ and $\mathrm{ord}_T(a) \in \Lambda_M^+$, we conclude that $\mathrm{ord}_T(a) \in w\Lambda_G^{\mathrm{pos},\mathbb{Q}}$ for all $w \in W_M$. Lemma 1.3.2 implies that $\mathrm{ord}_T(a) \in \Lambda_U^{\mathrm{pos},\mathbb{Q}}$. Observe that $\mathrm{ord}_M(k_1 a, k_2) = -w_0^M \mathrm{ord}_T(a)$, so we are done. $\square$

The following result is [10, Corollary 7.4, Proposition 7.5(a)]. We give a proof using our combinatorial description of bounded subsets of $\mathbb{X}_P$.

Proposition 2.2.9. *Formula* (2.7) *defines an operator* $R_P : \mathcal{C}_{P^-,+} \to \mathcal{C}_{P,-}$.

Proof. The proof is exactly the same as the proof of Lemma 2.2.7. We repeat the argument for completeness: Let $\varphi \in \mathcal{C}_{P^-,+}$. Fix $g_2 \in G/U$. For any $g_1 \in G/U^-$, Lemma 2.2.1 gives the inequality $\mathrm{ord}_M(g_1) \leq_M \mathrm{ord}_M(g_2) - w_0^M \mathrm{ord}_M(g_1, g_2)$. Since $\mathrm{ord}_M(\mathrm{supp}\, \eta_P) \subset -\Lambda_U^{\mathrm{pos},\mathbb{Q}}$, the inequality implies that if $\xi'(g_1, g_2) \neq 0$, then $\mathrm{ord}_M(g_1) = \mathrm{ord}_M(g_2) + \mu$ for some $\mu \in w_0^M \Lambda_G^{\mathrm{pos},\mathbb{Q}}$. From this combinatorial description, we deduce that the function sending $g_1 \in G/U^-$ to $\varphi(g_1)\eta_P(g_1, g_2)$ is compactly supported. Therefore (2.8) is well-defined.

Moreover, if $R_P(\varphi)(g_2) \neq 0$, then there must exist $g_1 \in G/U^-$ such that $g_1 \in \mathrm{supp}\, \varphi$ and $(g_1, g_2) \in \mathrm{supp}\, \eta_P \subset \mathbb{X}_P^-$. Observe that $\deg_P(g_2) = \deg_{P^-}(g_1) + [\mathrm{ord}_M(g_1, g_2)]_P$ in $\Lambda_{G,P}^{\mathbb{Q}}$. Lemma 2.2.8 shows that $[\mathrm{ord}_M(\mathrm{supp}\, \eta_P)]_P$ is contained in $-\Lambda_{G,P}^{\mathrm{pos},\mathbb{Q}}$. By definition of $\mathcal{C}_{P^-,+}$, we deduce that $\deg_P(g_2)$ must lie in $-\Lambda_{G,P}^{\mathrm{pos},\mathbb{Q}} + S_0$ for some finite set S_0. Thus $R_P(\varphi) \in \mathcal{C}_{P,-}$. $\square$

2.2.7 Local asymptotics map

The work [10] gives a geometric proof of the second adjointness between parabolic induction and restriction (Jacquet) functors by defining the Bernstein map $\mathsf{B} : C_c^\infty(\mathbb{X}_P) \to C_c^\infty(G)$. If F_v has characteristic 0, this map is the "asymptotics" map constructed in [60] in the more general setting of spherical varieties. The dual of B gives a map

$$\mathrm{Asymp}_P : S^*(G) \to S^*(\mathbb{X}_P),$$

where $S^*(G), S^*(\mathbb{X}_P)$ are the spaces of distributions on $G, \mathbb{X}_P$, respectively.

Fix the Haar measures on G, M, U so that $K, K_M, K \cap U$ have measure 1. Using these measures, we identify distributions and generalized functions on $G, \mathbb{X}_P$.

The Bernstein map B is $G \times G$-equivariant, and hence so is Asymp_P. Therefore Asymp_P preserves $K \times K$-finiteness, and [10, Proposition 7.1] shows that it restricts to a map

$$\mathrm{Asymp}_P : C_c^\infty(G) \to \mathcal{C}_b(\mathbb{X}_P).$$

Let $\delta_g \in S^*(G)$ denote the delta (generalized) function at $g \in G$. Set

$$\xi_P := \mathrm{Asymp}_P(\delta_1) \in S^*(\mathbb{X}_P), \tag{2.9}$$

which we consider as a generalized function on $\mathbb{X}_P$. Let $f_1, f_2 \in C_c^\infty(G)$ and set $f_2^\vee(g) = f_2(g^{-1})$. Then $G \times G$-equivariance of $\mathrm{Asymp}_{P,v}$ implies that

$$(f_1, f_2) * \xi_P = \mathrm{Asymp}_P(f_1 * f_2^\vee), \tag{2.10}$$

where $*$ denotes convolution with respect to the $G \times G$-action on $\mathbb{X}_P$ (resp. the usual convolution on G). In particular, ξ_P has *essentially bounded support* in the sense that the convolution of ξ_P with any element of $C_c^\infty(G)$ has bounded support.

Note that ξ_P depends on the choice of Haar measure on G.

We have the following relationship between the asymptotics map and the intertwining operator:

Theorem 2.2.10 ([10, Theorem 7.6]). *Let $\varphi \in C_c^\infty(G/U)$. We have an equality*

$$\varphi = (R_P \circ T_{\xi_P})(\varphi).$$

In particular, the integral defining $(R_P \circ T_{\xi_P})(\varphi)$ converges.

The theorem is stated for the case $P = B$ in [10], but one can check that the proof generalizes to the case of an arbitrary parabolic subgroup.

2.2.8 Invertibility of R_P

We use Theorem 2.2.10 to show that R_P is an isomorphism in Proposition 2.2.13 below.

Let $\widetilde{\mathcal{C}}_{P,-}$ denote the smooth part of the linear dual representation $\mathcal{C}^*_{P,-}$. Let

$$R_P^* : \widetilde{\mathcal{C}}_{P,-} \to \widetilde{\mathcal{C}}_{P^-,+}$$

denote the linear dual of the operator R_P. Using the measure on G/U determined by the fixed Haar measures on G and U, we identify $\widetilde{\mathcal{C}}_{P,-}$ with a subspace of $\mathcal{C}_P$ containing $\mathcal{C}_{P,c}$. Similarly, we consider $\widetilde{\mathcal{C}}_{P^-,+} \subset \mathcal{C}_{P^-}$.

Let $R_{P^-} : \mathcal{C}_{P,+} \to \mathcal{C}_{P^-,-}$ denote the intertwining operator with respect to the opposite parabolic P^- (i.e., we integrate over U^- in formula (2.7)).

Lemma 2.2.11. *We have an equality $R_P^* = R_{P^-} : \mathcal{C}_{P,c} \to \mathcal{C}_{P^-}$.*

Proof. Let $\tilde{\varphi}, \varphi \in \mathcal{C}_{P,c}$. Then

$$\begin{aligned}\langle R_P^*(\tilde{\varphi}), \varphi \rangle = \int_{G/U} \tilde{\varphi}(g) \int_U \varphi(gu) du dg = \int_G \tilde{\varphi}(g)\varphi(g) dg \\ = \int_{G/U^-} \int_{U^-} \tilde{\varphi}(g\bar{u})\varphi(g) d\bar{u} dg = \langle R_{P^-}(\tilde{\varphi}), \varphi \rangle,\end{aligned}$$

where all the integrals are finite. This proves the lemma. $\square$

Let $\xi_P \in S^*(\mathbb{X}_P)$ be the generalized function defined in (2.9). Note that $\xi_P \in S_b^*(\mathbb{X}_P)^G$, so Lemma 2.2.7 defines a map $T_{\xi_P} : \mathcal{C}_{P,-} \to \mathcal{C}_{P^-,+}$. Theorem 2.2.10 has the following reformulation:

Lemma 2.2.12. *We have the equality* $R_P \circ T_{\xi_P} = \mathrm{id}$ *on* $\mathcal{C}_{P,-}$.

Proof. Let $\varphi \in \mathcal{C}_{P,-}$. Since φ is K-finite, there exists a compact open subgroup $K' \subset K$ such that φ is K'-invariant. The proof of Lemma 2.2.7 shows that

$$\mathrm{ord}_M(\mathrm{supp}\, T_{\xi_P}(\varphi)) \subset S := \{\lambda + \theta - \mu \mid \lambda \in \mathrm{ord}_M(\mathrm{supp}\, \varphi),\, \theta \in S_0,\, \mu \in \Lambda_G^{\mathrm{pos},\mathbb{Q}}\},$$

where $S_0 \subset \Lambda$ is a finite subset depending only on K' and not φ. The proof of Proposition 2.2.9 shows that $\mathrm{ord}_M(\mathrm{supp}\, R_P(T_{\xi_P}(\varphi)))$ is also contained in S. Therefore we deduce that it suffices to prove that $\varphi = R_P(T_{\xi_P}(\varphi))$ for compactly supported $\varphi \in \mathcal{C}_{P,c}$, which is Theorem 2.2.10. □

Proposition 2.2.13. *The map* $R_P : \mathcal{C}_{P^-,+} \to \mathcal{C}_{P,-}$ *is an isomorphism. The inverse is given by the formula*

$$R_P^{-1}(\varphi)(g_2) = \int_{G/U} \varphi(g_1)\xi_P(g_1, g_2)dg_1, \qquad \varphi \in \mathcal{C}_{P,-},\, g_2 \in G/U^- \tag{2.11}$$

Remark 2.2.14. Our proof of Proposition 2.2.13 is different from the one in [10, Proposition 7.5(b)]. This proof was suggested by V. Drinfeld.

We give a separate, self-contained proof of invertibility of $R_P^K : \mathcal{C}_{P^-,+}^K \to \mathcal{C}_{P,-}^K$ with explicit formulas in Corollary 2.3.7.

Proof. Lemma 2.2.12 implies that R_P has a right inverse given by (2.11). It remains to show that R_P has a left inverse. Apply Lemma 2.2.12 to the *opposite parabolic* P^- to get a map $T_{\xi_{P^-}} : \mathcal{C}_{P^-,-} \to \mathcal{C}_{P,+}$ such that $R_{P^-} \circ T_{\xi_{P^-}} = \mathrm{id}$ on $\mathcal{C}_{P^-,-}$. Taking the dual operators gives an equality

$$T_{\xi_{P^-}}^* \circ R_{P^-}^* = \mathrm{id} \tag{2.12}$$

on $\mathcal{C}_{P^-,c} \subset \widetilde{\mathcal{C}}_{P^-,-}$. Let $\varphi \in \mathcal{C}_{P^-,c}$. Lemma 2.2.11 (applied to P^-) implies that $R^*_{P^-}(\varphi) = R_P(\varphi)$. Define $\tilde{\xi} \in S^*_b(\mathbb{X}_P)^G$ by

$$\tilde{\xi}(g_1, g_2) = \xi_{P^-}(g_2, g_1).$$

It follows formally from the definition of $T_{\xi_{P^-}}$ that $T^*_{\xi_{P^-}}(R_P(\varphi)) = T_{\tilde{\xi}}(R_P(\varphi))$, where the map $T_{\tilde{\xi}} : \mathcal{C}_{P,-} \to \mathcal{C}_{P^-,+}$ is defined by Lemma 2.2.7. Therefore (2.12) implies that $T_{\tilde{\xi}} \circ R_P = \mathrm{id}$ on $\mathcal{C}_{P^-,c}$. Then the same argument as in the proof of Lemma 2.2.12 shows that $T_{\tilde{\xi}} \circ R_P = \mathrm{id}$ on $\mathcal{C}_{P^-,+}$, so we conclude that R_P has a left inverse. $\square$

2.3 Formulas on K-invariants

Let F_v be an arbitrary non-Archimedean local field. We use the same notation and conventions as in §2.2 (e.g., $G = G(F_v)$, $K = K_v$, etc.).

Restricting to K-invariants, we see that the intertwining operator is essentially convolution with a measure μ_M on M. We compute the Satake transform of μ_M using the non-Archimedean Gindikin–Karpelevich formula. We give a formula for $\mathrm{Asymp}_P(\delta_K)$, where δ_K is the characteristic function of K, in terms of the convolution inverse of μ_M.

Fix the Haar measures on G, M, T, U, U^- so that K, K_M, K_T, $K \cap U$, $K \cap U^-$ all have measure 1.

2.3.1 Intertwining operator on K-invariants

Let $K \subset G$ act on G/U, G/U^- on the left. Recall that $K\backslash G/U = K\backslash G/U^- = K_M\backslash M$.

The measure μ_M. Let $\bar{\mu}$ denote the direct image of the Haar measure on U under the map

$$U \hookrightarrow G \to K\backslash G/U^- = K_M\backslash M, \tag{2.13}$$

where $G = K \cdot M \cdot U^-$ by the Iwasawa decomposition. In other words, $\bar{\mu}(\Omega)$ is the measure of $U \cap (K \cdot \Omega \cdot U^-) \subset U$, where $\Omega \subset K_M \backslash M$. Since (2.13) is equivariant with respect to the action of K_M by conjugation, $\bar{\mu}$ is right K_M-invariant. Define μ_M to be the K_M-bi-invariant measure on M whose pushforward to $K_M \backslash M$ equals $\bar{\mu}$.

A formula in terms of convolution. Let R_P^K denote the restriction of the intertwining operator (2.7) to $\mathcal{C}^K_{P^-,+} \to \mathcal{C}^K_{P,-}$. Then we have the formula

$$R_P^K(\varphi)(m) = \delta_P(m)^{-1} \int_U \varphi(um)du = \delta_P(m)^{-1} \int_M \varphi(m_1 m)\mu_M(m_1) \tag{2.14}$$

where $\varphi \in \mathcal{C}^K_{P^-,+}$, $m \in M$.

Comparing with (2.8), we see that

$$((\delta_K \otimes 1) * \eta_P)(m, 1) = \mu_M(m), \qquad m \in M, \tag{2.15}$$

where we consider μ_M as a K_M-bi-invariant function using the fixed Haar measure on M.

2.3.2 Satake isomorphism

We extend the classical Satake isomorphism to an isomorphism between certain larger algebras (defined below) that are convenient for our purposes.

The completed Hecke algebra. Let H_M^+ denote the space of K_M-bi-invariant measures on M (with values in E) whose support is contained in $\mathrm{ord}_M^{-1}(\Lambda_U^{\mathrm{pos},\mathbb{Q}} \cap \Lambda_M^+)$, where $\Lambda_U^{\mathrm{pos},\mathbb{Q}}$ is the rational cone defined in §2.2.3.

Remark 2.3.1. Lemma 2.2.8 and (2.15) imply that μ_M belongs to H_M^+.

Lemma 2.3.2. *(i) Suppose that Σ is a submonoid of Λ_M^+ such that if $\lambda \in \Lambda_M^+$ and there exists $\nu \in \Sigma$ such that $\lambda \leq_M \nu$, then $\lambda \in \Sigma$. Then the vector space of compactly supported K_M-bi-invariant measures on M whose support is contained in* $\mathrm{ord}_M^{-1}(\Sigma)$ *is closed under the convolution product, so the vector space becomes an algebra.*

(ii) If, in addition, Σ generates a strongly convex cone and the intersection of Σ with

any shift of Λ_M^{pos} *is finite, then convolution extends, by continuity, to the space of all* K_M*-bi-invariant measures on* M *whose support is contained in* $\mathrm{ord}_M^{-1}(\Sigma)$*. Then this space is also an algebra.*

Proof. The lemma follows from the usual properties of the Hecke algebra. $\square$

Lemma 1.3.1 implies that $\Sigma = \Lambda_U^{\mathrm{pos},\mathbb{Q}} \cap \Lambda_M^+$ satisfies the condition in Lemma 2.3.2(i), and one observes from the definition that Λ_U^+ also satisfies condition (ii). Therefore H_M^+ is an algebra with respect to convolution.

Let H_T^+ denote the space of K_T-bi-invariant measures on T whose support is contained in $\mathrm{ord}_T^{-1}(\Lambda_U^{\mathrm{pos},\mathbb{Q}} \cap \Lambda)$. Lemma 2.3.2(ii) implies that H_T^+ is an algebra with respect to convolution. Observe that the Weyl group of M acts on H_T^+.

Using our fixed Haar measures, we identify locally constant functions on M (resp. T) with locally constant measures on M (resp. T). We also fix the Haar measure on $U_B^- \cap M$ such that $U_B^- \cap K_M$ has measure 1.

Lemma 2.3.3. *The usual Satake transform extends to an isomorphism* $\mathrm{CT} : H_M^+ \to (H_T^+)^{W_M}$ *given by the formula*

$$\mathrm{CT}(h)(t) = \delta_{B\cap M}(t)^{-1/2} \int_{U_B^-\cap M} h(t\bar{n})d\bar{n}, \tag{2.16}$$

where $h \in H_M^+$ *is considered as a function on* M*.*

Here CT stands for 'constant term'. Since the image of the Satake transform is W_M-invariant, (2.16) does not depend on the choice of Borel subgroup of M.

Proof. Let $1_{\mathrm{ord}_M^{-1}(\lambda)}$ denote the characteristic function of $\mathrm{ord}_M^{-1}(\lambda) \subset M$ for $\lambda \in \Lambda_M^+$. It is known (cf. [17, §4.2]) that $\mathrm{CT}(1_{\mathrm{ord}_M^{-1}(\lambda)})$ does not vanish on $\mathrm{ord}_T^{-1}(\lambda')$, $\lambda' \in \Lambda_M^+$, only if $\lambda' \leq_M \lambda$. Thus we deduce that CT is well-defined and an isomorphism from the usual Satake isomorphism and the fact that $\Lambda_U^{\mathrm{pos},\mathbb{Q}} \cap \Lambda_M^+$ satisfies Lemma 2.3.2. $\square$

Remark 2.3.4. Note that the algebra H_T^+ is isomorphic to the completion of the semigroup algebra of $\Lambda_U^{\mathrm{pos},\mathbb{Q}} \cap \Lambda$ at the augmentation ideal. In particular, it is a local ring, and $\hat{h} \in H_T^+$ is a unit if and only if $\hat{h}(1) \neq 0$. We deduce that H_M^+ and $(H_T^+)^{W_M}$ are also complete local rings.

2.3.3 Gindikin–Karpelevich formula

In this subsection we rewrite the non-Archimedean Gindikin–Karpelevich formula[3] as a formula for $\mathrm{CT}(\mu_M) \in (H_T^+)^{W_M}$.

Recall that we defined $\delta_P(m) = |\det \mathrm{Ad}_{\mathrm{Lie}(U)}(m)|$ for $m \in M$. Let $2\check{\rho}_P := 2\check{\rho} - 2\check{\rho}_M$ be the sum of the positive roots in G that are not roots of M. For $m \in M$, we have $\delta_P(m) = q_v^{-\langle 2\check{\rho}_P, \mathrm{ord}_M(m)\rangle}$.

For $\lambda \in \Lambda$, let $1_{\mathrm{ord}_T^{-1}(\lambda)}$ denote the characteristic function of $\mathrm{ord}_T^{-1}(\lambda) \subset T$. Set

$$e^\lambda = q_v^{\langle \check{\rho}_P, \lambda\rangle} \cdot 1_{\mathrm{ord}_T^{-1}(\lambda)} \in H_T^+.$$

Proposition 2.3.5. *We have*

$$\mathrm{CT}(\mu_M) = \prod_{\alpha \in \Phi_G^+ - \Phi_M} \frac{1 - q_v^{-1} e^\alpha}{1 - e^\alpha}, \tag{2.17}$$

where the r.h.s. is considered as an element of $(H_T^+)^{W_M}$ by Remark 2.3.4.

Proof. For the purpose of this proof, we may assume $E = \mathbb{Q}$ (since μ_M takes values in $\mathbb{Q}$). Let $\check{\lambda} \in \check{\Lambda} \otimes \mathbb{C}$ satisfy $\mathrm{Re}\langle \check{\lambda}, \alpha\rangle > 0$ for every positive coroot α of G. Let $\chi_{\check{\lambda}}$ be the unramified character $T \to \mathbb{C}^\times$ sending $t \mapsto q_v^{-\langle \check{\lambda}, \mathrm{ord}_T(t)\rangle}$. Define the function $\phi_{K,\check{\lambda}}$ on G by

$$\phi_{K,\check{\lambda}}(k \cdot t \cdot \bar{n}) = \chi_{\check{\lambda}}(t) \delta_B^{1/2}(t), \quad k \in K,\, t \in T,\, \bar{n} \in U(B^-)$$

3. The Gindikin–Karpelevich formula for non-Archimedean local fields is due to Langlands [48] and MacDonald [50] independently, with a generalization by Casselman [18].

where $G = K \cdot B^-$ by the Iwasawa decomposition. The Gindikin–Karpelevich formula for non-Archimedean local fields [48, p. 18] implies that

$$\int_M \phi_{K,\check{\lambda}}(m)\mu_{M,v}(m) = \int_U \phi_{K,\check{\lambda}}(u)du = \prod_{\alpha \in \Phi^+ - \Phi_M} \frac{1 - q_v^{-1-\langle \check{\lambda}, \alpha \rangle}}{1 - q_v^{-\langle \check{\lambda}, \alpha \rangle}}$$

where the l.h.s. converges absolutely. Integrating over $M = K_M \cdot (B^- \cap M)$ using the Iwasawa decomposition (cf. [17, Equations (5), (10)]), the l.h.s. equals $\int_T \chi_{\check{\lambda}}(t)\delta_P^{1/2}(t)\,\mathrm{CT}(\mu_{M,v})(t)dt$. Note that for $\nu \in \Lambda$, we have $\int_T \chi_{\check{\lambda}}(t)\delta_P^{1/2}(t)e^{\nu}(t)dt = q_v^{-\langle \check{\lambda}, \nu \rangle}$. Therefore equation (2.17) holds after integrating against $\chi_{\check{\lambda}}$ for any $\check{\lambda} \in \check{\Lambda} \otimes \mathbb{C}$ satisfying $\mathrm{Re}\langle \check{\lambda}, \alpha \rangle > 0$ for all $\alpha \in \Phi_G^+$. This implies the equality (2.17) of elements in $(H_T^+)^{W_M}$. □

Corollary 2.3.6. *The measure μ_M is invertible in H_M^+.*

Proof. The Gindikin–Karpelevich formula (2.17) implies that $\mathrm{CT}(\mu_M)(1) = 1$, so $\mathrm{CT}(\mu_M)$ is invertible by Remark 2.3.4. The extended Satake isomorphism (2.16) then implies that μ_M is invertible. □

Let $\nu_M \in H_M^+$ denote the convolution inverse of μ_M. We consider it as a K_M-bi-invariant measure on M.

Corollary 2.3.7. *The operator $R_P^K : \mathcal{C}_{P^-,+}^K \to \mathcal{C}_{P,-}^K$ is an isomorphism. The inverse is given by the formula*

$$(R_P^K)^{-1}(\varphi)(m) = \int_M \delta_P(m_1 m)\varphi(m_1 m)\nu_M(m_1), \tag{2.18}$$

where $\varphi \in \mathcal{C}_{P,-}^K$, $m \in M$.

Proof. Define $\xi \in \mathcal{C}_b(\mathbb{X}_P)^{K \times K}$ by $\xi(m, 1) = \nu_M(m)$ for $m \in M$. The fact that ν_M belongs to H_M^+ implies that ξ indeed has bounded support. Then the r.h.s. of (2.18) equals $T_\xi(\varphi)$, where $T_\xi : \mathcal{C}_{P,-} \to \mathcal{C}_{P^-,+}$ is defined in Lemma 2.2.7. In particular, the r.h.s. of (2.18) is well-defined. Note that by the Iwasawa decomposition, $\mathcal{C}_{P^-,+}^K$ and $\mathcal{C}_{P,-}^K$ identify with the same space of K_M-invariant functions on M. Equation (2.14) expresses R_P^K in terms of the convolution action of μ_M on $\mathcal{C}^K_{P^-,+} = \mathcal{C}^K_{P,-}$. This action is compatible with the convolution product of H_M^+. Thus we deduce from invertibility of μ_M that R_P^K is an isomorphism with inverse given by (2.18). $\square$

2.3.4 Langlands' reformulation

We will reformulate the Gindikin–Karpelevich formula in terms of Langlands' reinterpretation of the classical Satake isomorphism.

Let $\check{G}$ (resp. $\check{M}$, $\check{T}$) denote the Langlands dual group of G (resp. M, T) over E. Let $\check{\mathfrak{u}}_P$ be the Lie algebra corresponding to the roots $\Phi_G^+ - \Phi_M$ in $\check{G}$, so $\check{\mathfrak{u}}_P$ is a $\check{M}$-module by the adjoint action $\mathrm{Ad}_{\check{\mathfrak{u}}_P}$.

The map $1_{\mathrm{ord}_T^{-1}(\lambda)} \mapsto \lambda$ defines an isomorphism $C_c^\infty(T)^{K_T} \cong E[\check{T}]$, which is compatible with the W_M-action. Recall that $E[\check{T}]^{W_M} = E[\check{M}]^{\check{M}}$, where $\check{M}$ acts on itself by conjugation. Let $\mathbf{K}(\mathrm{Rep}(\check{M}))$ denote the Grothendieck group of the abelian category of finite dimensional $\check{M}$-modules. Give $\mathbf{K}(\mathrm{Rep}(\check{M}))$ the tensor product multiplication. Then we have an algebra isomorphism by taking characters:

$$\mathbf{K}(\mathrm{Rep}(\check{M})) \underset{\mathbb{Z}}{\otimes} E \xrightarrow{\mathrm{Ch}} E[\check{M}]^{\check{M}} : [V] \mapsto \mathrm{tr}(\sigma, V), \qquad \sigma \in \check{M}.$$

Let $\mathrm{Rep}^+(\check{M})$ denote the subcategory of $\check{M}$-modules with weights contained in $\Lambda_U^{\mathrm{pos},\mathbb{Q}}$. Since $2\check{\rho}_P \in \check{\Lambda}$ is perpendicular to all coroots of M, we may consider it as a central cocharacter of $\check{M}$. We have a non-negative grading of the Grothendieck group $\mathbf{K}(\mathrm{Rep}^+(\check{M}))$ by the eigenvalues of $2\check{\rho}_P$. Let $\mathbf{K}^+(\mathrm{Rep}(\check{M}))$ be the completion of $\mathbf{K}(\mathrm{Rep}^+(\check{M}))$ with respect to the augmentation ideal of this grading. Then one sees that $\mathrm{Ch}^{-1} \circ \mathrm{CT}$ extends to an algebra isomorphism

$$\mathcal{S} : H_M^+ \to \mathbf{K}^+(\mathrm{Rep}(\check{M})) \hat{\otimes} E,$$

where $\hat{\otimes}$ is the completed tensor product.

Let $V \in \operatorname{Rep}(\check{M})$. Consider the expression $\sum_n t^n[\operatorname{Sym}^n V]$, which is a formal series in $\mathbf{K}(\operatorname{Rep}(\check{M}))[\![t]\!]$. Here t is a formal parameter (that is unrelated to the torus). It is well-known that the inverse of this series equals

$$\Lambda(t, V) := \sum_n (-t)^n [\wedge^n V].$$

If we consider coefficients in $E[t]$ rather than E, we have $\operatorname{tr}(\sigma, \Lambda(t, V)) = \det(\operatorname{Id} - \sigma \cdot t, V)$ for $\sigma \in \check{M}(E)$.

Suppose that the weights of V are contained in $\Lambda_U^{\mathrm{pos},\mathbb{Q}} - \{0\}$. Let $\tau \in E^\times$. Then the series

$$\mathrm{S}(\tau, V) := \sum_n \tau^n [\operatorname{Sym}^n V]$$

is a well-defined element of the completed Grothendieck group $\mathbf{K}^+(\operatorname{Rep}(\check{M}))\hat{\otimes} E$, and it is the inverse of $\Lambda(\tau, V) \in \mathbf{K}^+(\operatorname{Rep}(\check{M}))\hat{\otimes} E$.

The central cocharacter $2\check{\rho}_P$ defines a non-negative $\check{M}$-module grading of $\check{\mathfrak{u}}_P$ by the eigenspace decomposition. Let $\operatorname{gr}^i(\check{\mathfrak{u}}_P)$ denote the eigenspace of $\operatorname{Ad}_{\check{\mathfrak{u}}_P}(2\check{\rho}_P)$ with weight $2a_i$, where a_i is a positive integer. Then in the above language, equation (2.17) and its multiplicative inverse have the reformulations

$$\mathcal{S}(\mu_{M,v}) = \prod_i \frac{\Lambda(q_v^{-1+a_i}, \operatorname{gr}^i(\check{\mathfrak{u}}_P))}{\Lambda(q_v^{a_i}, \operatorname{gr}^i(\check{\mathfrak{u}}_P))}, \qquad \mathcal{S}(\nu_{M,v}) = \prod_i \frac{\Lambda(q_v^{a_i}, \operatorname{gr}^i(\check{\mathfrak{u}}_P))}{\Lambda(q_v^{-1+a_i}, \operatorname{gr}^i(\check{\mathfrak{u}}_P))}. \tag{2.19}$$

The formula for $\mathcal{S}(\mu_{M,v})$ essentially appears in [48, p. 33].

Using the equality $\Lambda(q_v^{-1+a_i}, \operatorname{gr}^i(\check{\mathfrak{u}}_P))^{-1} = \mathrm{S}(q_v^{-1+a_i}, \operatorname{gr}^i(\check{\mathfrak{u}}_P))$, we have the expansion

$$\frac{\Lambda(q_v^{a_i}, \operatorname{gr}^i(\check{\mathfrak{u}}_P))}{\Lambda(q_v^{-1+a_i}, \operatorname{gr}^i(\check{\mathfrak{u}}_P))} = \left(\sum_n (-1)^n [\wedge^n \operatorname{gr}^i(\check{\mathfrak{u}}_P)] \cdot q_v^{a_i n}\right)\left(\sum_n [\operatorname{Sym}^n \operatorname{gr}^i(\check{\mathfrak{u}}_P)] \cdot q_v^{-n+a_i n}\right) \tag{2.20}$$

in $\mathbf{K}^+(\operatorname{Rep}(\check{M}))\hat{\otimes} E$.

2.3.5 Asymptotics on K-invariants

Let $\delta_K \in C_c^\infty(G)$ denote the characteristic function of K. Note that $\mathrm{Asymp}_P(\delta_K) = (\delta_K \otimes 1) * \xi_P = (1 \otimes \delta_K) * \xi_P$ is $K \times K$-invariant. Using (2.18) and (2.11), we deduce the formula

$$\mathrm{Asymp}_P(\delta_K)(m, 1) = \nu_M(m). \tag{2.21}$$

When $P = B$ is a Borel subgroup and F_v has characteristic 0, (2.21) is proved in [59, Theorem 6.8] in the more general setting of spherical varieties.

Remark 2.3.8. Note that $\nu_M(1) = \mathrm{CT}(\nu_M)(1) = 1$ by the explicit formula (2.17). Thus (2.21) implies that $\mathrm{Asymp}_P(\delta_K)$ takes constant value 1 on the $K \times K$ orbit of $(1, 1) \in \mathbb{X}_P$.

In the notation of Remark 2.2.3 we also see that $\mathrm{Asymp}_P(\delta_K)$ has support contained in $\overline{\mathsf{X}}_{\mathsf{P}}(\mathfrak{o}_v)$ since $\nu_M \in H_M^+$.

2.4 The bilinear form $\mathcal{B}$

We work over the function field F with adele ring $\mathbb{A}$. Let X be the corresponding geometrically connected smooth projective curve over $\mathbb{F}_q$. In this section, we define the bilinear form $\mathcal{B}$ and prove Theorem 2.1.2.

In our notation, we will add a subscript v when referring to the objects or spaces defined in §2.2 over F_v (e.g., $\mathcal{C}_P$ becomes $\mathcal{C}_{P,v}$, Asymp_P becomes $\mathrm{Asymp}_{P,v}$).

2.4.1 Definition of $\mathcal{B}$

Fix a Haar measure on $G(\mathbb{A})$. For $f_1, f_2 \in \mathcal{A}_c$, set

$$\mathcal{B}(f_1, f_2) := \sum_P (-1)^{\dim Z(M)} \cdot \mathcal{B}_P(f_1, f_2) \tag{2.22}$$

where the sum ranges over standard parabolic subgroups $P \subset G$ with Levi subgroup M, and $\mathcal{B}_P$ is a $G(\mathbb{A})$-invariant bilinear form defined in §2.4.2 below. The form $\mathcal{B}$ is $G(\mathbb{A})$-invariant since each $\mathcal{B}_P$ is. It will also be evident that $\mathcal{B}$ is symmetric. Let us note that $\mathcal{B}_P$ and $\mathcal{B}$ slightly depend on the choice of a Haar measure on $G(\mathbb{A})$.

2.4.2 Definition of $\mathcal{B}_P$

Fix a standard parabolic subgroup P. Define the boundary degeneration

$$\mathbb{X}_P = (G \times G)/(P \underset{M}{\times} P^-)$$

as in §2.2.2, where $\mathbb{X}_P$ is now a strongly quasi-affine variety over $\mathbb{F}_q$. Let $\overline{\mathbb{X}}_P$ denote the affine closure.

The topological space $\mathbb{X}_P(\mathbb{A})$ is isomorphic to the restricted product of $\mathbb{X}_P(F_v)$ with respect to the compact subspaces $\mathbb{X}_P(\mathfrak{o}_v)$. The topological space $\overline{\mathbb{X}}_P(\mathbb{A})$ is isomorphic to the restricted product of $\overline{\mathbb{X}}_P(F_v)$ with respect to $\overline{\mathbb{X}}_P(\mathfrak{o}_v)$. Recall that the topology on $\mathbb{X}_P(\mathbb{A})$ is *not* the subspace topology induced from $\overline{\mathbb{X}}_P(\mathbb{A})$.

We say that a function on $\mathbb{X}_P(\mathbb{A})$ has bounded support if the support is relatively compact in $\overline{\mathbb{X}}_P(\mathbb{A})$. Let $\mathcal{C}_b(\mathbb{X}_P(\mathbb{A}))$ denote the space of $K \times K$-finite C^∞ functions on $\mathbb{X}_P(\mathbb{A})$ with bounded support.

Note that the action of $P^- \times P$ on $1 \in G$ and $(1,1) \in \mathbb{X}_P$ have the same stabilizer equal to the diagonal embedding of M. Fix the measure on $\mathbb{X}_P(\mathbb{A})$ to be the unique $G(\mathbb{A}) \times G(\mathbb{A})$-invariant measure such that on the $P^-(\mathbb{A}) \times P(\mathbb{A})$-orbit of $(1,1)$, it coincides with the restriction of the chosen Haar measure on $G(\mathbb{A})$ to $P^-(\mathbb{A}) \cdot P(\mathbb{A})$.

Define $\mathrm{Asymp}_P : C_c^\infty(G(\mathbb{A})) \to \mathcal{C}_b(\mathbb{X}_P(\mathbb{A}))$ by

$$\mathrm{Asymp}_P(\bigotimes_v f_v) = \bigotimes_v \mathrm{Asymp}_{P,v}(f_v) \tag{2.23}$$

where $f_v \in C_c^\infty(G(F_v))$ and $f_v = \delta_{K_v}$ is the characteristic function of K_v for almost all v. Observe that Asymp_P is well-defined since $\mathrm{Asymp}_{P,v}(\delta_{K_v})$ equals 1 on $\mathbb{X}_P(\mathfrak{o}_v)$ by Remark 2.3.8. The product $\otimes_v \mathrm{Asymp}_{P,v}(f_v)$ has bounded support in $\mathbb{X}_P(\mathbb{A})$ because the support of $\mathrm{Asymp}_{P,v}(\delta_{K_v})$ is contained in $\overline{\mathbb{X}}_P(\mathfrak{o}_v)$ by Remark 2.3.8.

Define the generalized function $\xi_P \in S^*(\mathbb{X}_P(\mathbb{A}))$ by

$$\xi_P = \bigotimes_v \xi_{P,v} \tag{2.24}$$

where $\xi_{P,v} \in S^*(\mathbb{X}_P(F_v))$ is defined by (2.9). Equation (2.24) is well-defined because any element of $C_c^\infty(\mathbb{X}_P(\mathbb{A}))$ is K_v-invariant for almost all v, and $\delta_{K_v} * \xi_{P,v} = \mathrm{Asymp}_{P,v}(\delta_{K_v})$ equals 1 on $\mathbb{X}_P(\mathfrak{o}_v)$ by Remark 2.3.8. From (2.10) we also deduce that ξ_P has essentially bounded support, i.e., for any $\tilde{f} \in C_c^\infty(G(\mathbb{A}))$, the convolution $(\tilde{f} \otimes 1) * \xi_P = (1 \otimes \tilde{f}^\vee) * \xi_P = \mathrm{Asymp}_P(\tilde{f})$ has bounded support, where $\tilde{f}^\vee(g) := \tilde{f}(g^{-1})$.

Define a bilinear form $\tilde{\mathcal{B}}_P : C_c^\infty(G(\mathbb{A})) \otimes C_c^\infty(G(\mathbb{A})) \to E$ by the formula

$$\tilde{\mathcal{B}}_P(\tilde{f}_1, \tilde{f}_2) := \sum_{x \in \mathbb{X}_P(F)} \mathrm{Asymp}_P(\tilde{f}_1^\vee * \tilde{f}_2)(x), \qquad \tilde{f}_1, \tilde{f}_2 \in C_c^\infty(G(\mathbb{A})), \tag{2.25}$$

where $\tilde{f}_1^\vee(g) := \tilde{f}_1(g^{-1})$, and $*$ denotes convolution over $G(\mathbb{A})$. The sum is finite because $\mathrm{Asymp}_P(\tilde{f}_1^\vee * \tilde{f}_2)$ has bounded support, and the intersection of the discrete subset $\mathbb{X}_P(F) \subset \overline{\mathbb{X}}_P(\mathbb{A})$ with a bounded subset of $\mathbb{X}_P(\mathbb{A})$ is finite.

Using (2.10), one can also write (2.25) as

$$\tilde{\mathcal{B}}_P(\tilde{f}_1, \tilde{f}_2) = \sum_{x \in \mathbb{X}_P(F)} \int_{(G \times G)(\mathbb{A})} \tilde{f}_1(g_1)\tilde{f}_2(g_2)\xi_P((g_1, g_2)x)dg_1dg_2. \tag{2.26}$$

For $g \in G(\mathbb{A})$, let δ_g denote the delta (generalized) function at g. Observe that

$$\tilde{\mathcal{B}}_P(\delta_g * \tilde{f}_1, \delta_g * \tilde{f}_2) = \tilde{\mathcal{B}}_P(\tilde{f}_1, \tilde{f}_2), \qquad g \in G(\mathbb{A}). \tag{2.27}$$

By $(G \times G)(\mathbb{A})$-equivariance of Asymp_P, we have

$$\tilde{\mathcal{B}}_P(\tilde{f}_1 * \delta_{g_1}, \tilde{f}_2 * \delta_{g_2}) = \tilde{\mathcal{B}}_P(\tilde{f}_1, \tilde{f}_2), \qquad g_1, g_2 \in G(F). \tag{2.28}$$

We define the bilinear form $\mathcal{B}_P : \mathcal{A}_c \otimes \mathcal{A}_c \to E$ as follows. For $f_1, f_2 \in \mathcal{A}_c$, there exist $\tilde{f}_1, \tilde{f}_2 \in C_c^\infty(G(\mathbb{A}))$ whose direct images are f_1, f_2. Set

$$\mathcal{B}_P(f_1, f_2) = \tilde{\mathcal{B}}_P(\tilde{f}_1, \tilde{f}_2), \tag{2.29}$$

which does not depend on the choices of $\tilde{f}_1, \tilde{f}_2$ by (2.28). The form $\mathcal{B}_P$ is $G(\mathbb{A})$-invariant by (2.27). Formula (2.29) was suggested by Y. Sakellaridis in a private communication.

2.4.3 Restriction of $\mathcal{B}_P$ to $\mathcal{A}_c^K$

Fix the Haar measure on $G(\mathbb{A})$ so that K has measure 1. Let $\tilde{f}_1, \tilde{f}_2 \in C_c^\infty(G(\mathbb{A}))^K$ be left K-invariant functions. Let $\delta_K = \otimes \delta_{K_v}$ denote the characteristic function of K on $G(\mathbb{A})$. Note that averaging $\delta_K * \delta_1 = \delta_1 * \delta_K = \delta_K$. Let $f_1, f_2 \in \mathcal{A}_c^K$ denote the direct images of $\tilde{f}_1, \tilde{f}_2$. We deduce from (2.26) that

$$\mathcal{B}_P(f_1, f_2) = \int_{(G\times G)(\mathbb{A})/(G\times G)(F)} f_1(g_1) f_2(g_2) b_P(g_1, g_2) dg_1 dg_2, \tag{2.30}$$

where $b_P(g_1, g_2) = \sum_{\mathbb{X}_P(F)} \mathrm{Asymp}_P(\delta_K)((g_1, g_2)x)$.

Observe that b_P is obtained from $\mathrm{Asymp}_P(\delta_K) \in \mathcal{C}_b(\mathbb{X}_P(\mathbb{A}))^{K\times K}$ by pull-push along the diagram

$$(G\times G)(\mathbb{A})/(G\times G)(F) \leftarrow (G\times G)(\mathbb{A}) \overset{(G\times G)(F)}{\times} \mathbb{X}_P(F) \to \mathbb{X}_P(\mathbb{A}). \tag{2.31}$$

2.4.4 Geometric interpretation

As explained in [32, §1.2.3, Remark 1.2.17], we can identify[4] the double cosets $K\backslash G(\mathbb{A})/G(F)$ with $|\mathrm{Bun}_G(\mathbb{F}_q)|$, the isomorphism classes of G-bundles on X. Let us give a geometric

4. The identification relies on the assumptions that any G-bundle $\mathcal{F}_G$ on X is trivial when restricted to $\mathrm{Spec}\, F$ and $\mathrm{Spec}\, \mathfrak{o}_v$ for each place v. We know the restriction of $\mathcal{F}_G$ to $\mathrm{Spec}\, \mathfrak{o}_v$ is trivial by smoothness of G and Lang's theorem (any G-bundle over a finite field is trivial). The generic triviality of $\mathcal{F}_G|_{\mathrm{Spec}\, F}$ follows from the Hasse principle for split reductive groups over a function field, which is proved by [37].

interpretation of b_P as a function on $(\mathrm{Bun}_G \times \mathrm{Bun}_G)(\mathbb{F}_q)$.

Let $\mathcal{F}_G^1, \mathcal{F}_G^2 \in \mathrm{Bun}_G(\mathbb{F}_q)$ be G-bundles. Fixing trivializations of $\mathcal{F}_G^i \times_X \mathrm{Spec}(\mathfrak{o}_v)$ for all places v, we get lifts of $\mathcal{F}_G^i \in K\backslash G(\mathbb{A})/G(F)$ to $g_i \in G(\mathbb{A})/G(F)$ for $i = 1, 2$. The pre-image of (g_1, g_2) in $(G \times G)(\mathbb{A}) \times^{(G \times G)(F)} \mathbb{X}_P(F)$ under the left arrow of (2.31) is in bijection with the set of rational sections of the morphism

$$(\mathbb{X}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2} := (\mathcal{F}_G^1 \underset{X}{\times} \mathcal{F}_G^2) \overset{G \times G}{\times} \mathbb{X}_P \to X.$$

Given a rational section $\beta \in (\mathbb{X}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2}(F)$, we can restrict to $(\mathbb{X}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2}(F_v)$ for any place v, which is isomorphic to $\mathbb{X}_P(F_v)$ by the trivializations of $\mathcal{F}_G^i \times_X \mathrm{Spec}(\mathfrak{o}_v)$, $i = 1, 2$. This describes the right arrow in (2.31).

Let $(\overline{\mathbb{X}}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2} := \overline{\mathbb{X}}_P \overset{G \times G}{\times} (\mathcal{F}_G^1 \underset{X}{\times} \mathcal{F}_G^2)$. We have an isomorphism $(\overline{\mathbb{X}}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2}(\mathfrak{o}_v) \cong \overline{\mathbb{X}}_P(\mathfrak{o}_v)$ compatible with the aforementioned identification $(\mathbb{X}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2}(F_v) \cong \mathbb{X}_P(F_v)$.

Remark 2.3.8 implies that the support of $\mathrm{Asymp}_P(\delta_K)$ is contained in $\mathbb{X}_P(\mathbb{A}) \cap \overline{\mathbb{X}}_P(\mathfrak{o}_{\mathbb{A}})$. Therefore $\mathrm{Asymp}_P(\delta_K)$ does not vanish at the image of β in $\mathbb{X}_P(\mathbb{A})$ only if β extends to a *regular* section $X \to (\overline{\mathbb{X}}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2}$. Such an extension is unique since $\overline{\mathbb{X}}_P$ is separated. Thus

$$b_P(\mathcal{F}_G^1, \mathcal{F}_G^2) = \sum_{\beta} \prod_{v} \mathrm{Asymp}_{P,v}(\delta_{K_v})(\beta_v) \tag{2.32}$$

where the sum is over sections $\beta : X \to (\overline{\mathbb{X}}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2}$ that generically land in the non-degenerate locus $(\mathbb{X}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2}$, and $\beta_v \in \mathbb{X}_P(F_v)$ is the image of β under the right arrow in (2.31). The $K_v \times K_v$-orbit of β_v does not depend on the choice of trivializations.

Note that $\mathrm{Asymp}_{P,v}(\delta_{K_v})(\beta_v) = 1$ if $\beta_v \in \mathbb{X}_P(\mathfrak{o}_v)$. Thus the product is only over those places v that β sends to the degenerate locus $(\overline{\mathbb{X}}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2} - (\mathbb{X}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2}$.

Remark 2.4.1. The product $\prod_v \mathrm{Asymp}_{P,v}(\delta_{K_v})(\beta_v)$ is a K-invariant function on $\mathbb{X}_P(\mathbb{A}) \cap \overline{\mathbb{X}}_P(\mathfrak{o}_{\mathbb{A}})$. Its value does not depend on the choice of trivializations of $\mathcal{F}_G^i \times_X \mathrm{Spec}(\mathfrak{o}_v)$, so we may also consider it as a function $\mathrm{Asymp}_P(\delta_K)(\beta)$ of β.

Remark 2.4.2. Theorem A.3.16 interprets $\mathrm{Asymp}_P(\delta_K)(\beta)$ as the trace of the geometric Frobenius acting on the $*$-stalks of an ℓ-adic sheaf.

Proof of Theorem 2.1.2. Let $\mathcal{F}^1_G, \mathcal{F}^2_G \in \mathrm{Bun}_G(\mathbb{F}_q)$. Using the geometric interpretation (2.32) and Theorem A.3.12, we get the equality

$$b(\mathcal{F}^1_G, \mathcal{F}^2_G) = \sum_P (-1)^{\dim Z(M)} b_P(\mathcal{F}^1_G, \mathcal{F}^2_G),$$

where the sum ranges over standard parabolic subgroups. The theorem now follows from the definition of $\mathcal{B}$ and the formula (2.30). $\square$

2.5 Global intertwining operators

Let P denote a standard parabolic subgroup. We define the subspaces $\mathcal{C}_{P,\pm}$ of functions on $G(\mathbb{A})/M(F)U(\mathbb{A})$ and recall the definition of the constant term operator. We show that the product of the local intertwining operators induces an operator $R_P : \mathcal{C}_{P^-,+} \to \mathcal{C}_{P,-}$, and we prove that R_P is invertible (Proposition 2.5.5). We prove Theorem 2.1.3 at the end of the section.

We continue to add a subscript v to the notation of §2.2 when appropriate.

2.5.1 The spaces $\mathcal{C}_P$, $\mathcal{C}_{P,\pm}$

Let $\mathcal{C}_P$ denote the space of K-finite C^∞ functions on $G(\mathbb{A})/M(F)U(\mathbb{A})$. Let $\mathcal{C}_{P,c} \subset \mathcal{C}_P$ stand for the subspace of compactly supported functions.

As in §2.4.4, the quotient $K_M\backslash M(\mathbb{A})/M(F)$ identifies with $|\mathrm{Bun}_M(\mathbb{F}_q)|$, the set of isomorphism classes of M-bundles on X. Recall that this identification uses the fact that any M-bundle on X is generically trivial. Since we have an exact sequence

$$0 = H^1(\mathrm{Spec}\, F, U) \to H^1(\mathrm{Spec}\, F, P) \to H^1(\mathrm{Spec}\, F, M),$$

we deduce that any P-bundle on X is also generically trivial. This allows us to make the identification $K\backslash G(\mathbb{A})/P(F) = |\mathrm{Bun}_P(\mathbb{F}_q)|$ by the decomposition $G(\mathbb{A}) = K \cdot P(\mathbb{A})$. This space projects to $K\backslash G(\mathbb{A})/M(F)U(\mathbb{A}) = |\mathrm{Bun}_M(\mathbb{F}_q)|$.

Let $\Lambda_{G,P} = \pi_1(M)$ denote the quotient of Λ by the subgroup generated by the coroots of M. It is well-known that there is a bijection $\deg_M : \pi_0(\mathrm{Bun}_M) \simeq \pi_1(M)$. Note that $\Lambda^{\mathbb{Q}}_{G,P} := \Lambda_{G,P} \otimes \mathbb{Q} = \Lambda^{\mathbb{Q}}_{M/[M,M]} = \Lambda^{\mathbb{Q}}_{Z_0(M)}$. We call the composition

$$\mathrm{Bun}_M \to \pi_1(M) \to \Lambda^{\mathbb{Q}}_{G,P}$$

the *slope* map. We define the map

$$\deg^{\mathbb{Q}}_P : G(\mathbb{A})/U(\mathbb{A}) \to \Lambda^{\mathbb{Q}}_{G,P}$$

by setting $\deg^{\mathbb{Q}}_P(g)$ equal to the slope of the M-bundle corresponding to $g \in G(\mathbb{A})$. Equivalently, if $g = (g_v)$, $g_v \in G(F_v)$, then $\deg^{\mathbb{Q}}_P(g) = \sum_v \deg_{P,v}(g_v)$, where $\deg_{P,v}$ is as defined in §2.2.5.

Let $\Lambda^{\mathrm{pos},\mathbb{Q}}_{G,P}$ denote the image of $\Lambda^{\mathrm{pos},\mathbb{Q}}_G$ under the projection $\Lambda^{\mathbb{Q}} \to \Lambda^{\mathbb{Q}}_{G,P}$. We define the global spaces $\mathcal{C}_{P,\pm}$ analogously to the definitions of the local spaces $\mathcal{C}_{P,\pm,v}$ in §2.2.5:

Let $\mathcal{C}_{P,+} \subset \mathcal{C}_P$ be the set of all functions $\varphi \in \mathcal{C}_P$ such that $\deg^{\mathbb{Q}}_P(\operatorname{supp}\varphi)$ is contained in $S_0 + \Lambda^{\mathrm{pos},\mathbb{Q}}_{G,P}$ for some finite subset $S_0 \subset \Lambda^{\mathbb{Q}}_{G,P}$. Similarly, let $\mathcal{C}_{P,-} \subset \mathcal{C}_P$ denote the set of all $\varphi \in \mathcal{C}_P$ such that $-\deg^{\mathbb{Q}}_P(\operatorname{supp}\varphi)$ is contained in $S_0 + \Lambda^{\mathrm{pos},\mathbb{Q}}_{G,P}$ for some finite set S_0.

One similarly defines the spaces $\mathcal{C}_{P^-,\pm} \subset \mathcal{C}_{P^-}$. We emphasize that $\mathcal{C}_{P^-,+}$ is defined with respect to the cone $-\Lambda^{\mathrm{pos},\mathbb{Q}}_{G,P}$. So $\mathcal{C}_{P^-,\pm}$ is the space of all $\varphi \in \mathcal{C}_{P^-}$ such that $\mp\deg^{\mathbb{Q}}_{P^-}(\operatorname{supp}\varphi)$ is contained in $S_0 + \Lambda^{\mathrm{pos},\mathbb{Q}}_{G,P}$ for some finite set S_0.

Remark 2.5.1. In the case $P = G$, we have $\Lambda^{\mathrm{pos}}_{G,G} = 0$. So we observe that $\mathcal{C}_{G,+} = \mathcal{C}_{G,-} \subset \mathcal{A}$ is the set of functions $f \in \mathcal{A}$ such that $\deg^{\mathbb{Q}}_G(\operatorname{supp} f)$ is finite.

2.5.2 The Harder–Narasimhan–Shatz stratification

Before discussing the constant term operator, we need to recall some reduction theory, which we state in terms of the Harder–Narasimhan–Shatz stratification of Bun_M. This stratification of Bun_M was defined in [38, 63, 64] in the case $M = \mathrm{GL}(n)$. For any reductive M it was defined in [52, 53, 54] and [7, 6]. We also refer the reader to [61].

Let $\Lambda_M^{+,\mathbb{Q}}$ denote the rational cone corresponding to the monoid Λ_M^+. For $\lambda \in \Lambda_M^{+,\mathbb{Q}}$, we follow the notation of [25, Theorem 7.4.3] and let $\mathrm{Bun}_M^{(\lambda)} \subset \mathrm{Bun}_M$ denote[5] the quasi-compact locally closed reduced substack of M-bundles with Harder–Narasimhan coweight λ. We have a map $\mathrm{HN} : |\mathrm{Bun}_M(\mathbb{F}_q)| \to \Lambda_M^{+,\mathbb{Q}}$, which sends an M-bundle to its unique Harder–Narasimhan coweight. We will also use HN to denote the composition

$$\mathrm{HN} : G(\mathbb{A})/M(F)U(\mathbb{A}) \to |\mathrm{Bun}_M(\mathbb{F}_q)| \to \Lambda_M^{+,\mathbb{Q}}.$$

The map HN will be our global analog of the map $\mathrm{ord}_{M,v}$ defined in (2.4).

For $\lambda \in \Lambda^{\mathbb{Q}}$, let $[\lambda]_P$ denote the projection of λ to $\Lambda_{G,P}^{\mathbb{Q}}$. Then for $x \in G(\mathbb{A})/M(F)U(\mathbb{A})$, we have $[\mathrm{HN}(x)]_P = \deg_P^{\mathbb{Q}}(x)$.

Remark 2.5.2. There exists an integer N such that the image of HN lies in $\frac{1}{N}\Lambda_M^+$.

2.5.3 The constant term operator

We will always fix the Haar measure on $U(\mathbb{A})$ so that $U(\mathbb{A})/U(F)$ has measure 1.

In §2.1.1, we defined the spaces $\mathcal{A}$ and $\mathcal{A}_c \subset \mathcal{A}$. The *constant term* operator $\mathrm{CT}_P : \mathcal{A} \to \mathcal{C}_P$ is defined by the formula

$$\mathrm{CT}_P(f)(g) = \int_{U(\mathbb{A})/U(F)} f(gu)du, \qquad f \in \mathcal{A},\ g \in G(\mathbb{A}). \tag{2.33}$$

5. In *loc. cit.* the stratification is defined over an algebraically closed field. To define the stratification over $\mathbb{F}_q$, we first base change to $\overline{\mathbb{F}}_q$ and then note that the Harder–Narasimhan strata are defined over $\mathbb{F}_q$ by Galois invariance.

In other words, CT_P is the pull-push along the diagram

$$G(\mathbb{A})/G(F) \leftarrow G(\mathbb{A})/P(F) \rightarrow G(\mathbb{A})/M(F)U(\mathbb{A}). \tag{2.34}$$

Recall from §2.2.3 what it means for a subset of $\Lambda^{\mathbb{Q}}$ to be bounded above (resp. below) with respect to the partial (rational) ordering $\leq_G^{\mathbb{Q}}$.

Lemma 2.5.3. *Let* $f \in \mathcal{A}_c$*. Then* $\mathrm{HN}(\operatorname{supp} \mathrm{CT}_P(f)) \subset \Lambda_M^{+,\mathbb{Q}}$ *is bounded above. Consequently,* $\mathrm{CT}_P : \mathcal{A} \to \mathcal{C}_P$ *sends* $\mathcal{A}_c$ *to* $\mathcal{C}_{P,-}$*.*

Proof. If we pass to K-orbits in the diagram (2.34), then we get the $\mathbb{F}_q$-points of the diagram of stacks

$$\mathrm{Bun}_G \leftarrow \mathrm{Bun}_P \rightarrow \mathrm{Bun}_M\,. \tag{2.35}$$

For $\theta \in \Lambda_G^{+,\mathbb{Q}}$, let $\mathrm{Bun}_G^{(\leq\theta)} \subset \mathrm{Bun}_G$ denote the open substack of G-bundles having Harder-Narasimhan coweight $\leq_G^{\mathbb{Q}} \theta$. Let $f \in \mathcal{A}_c$. Then the K-orbits of its support are contained in

$$\bigcup_{\theta \in S} \mathrm{Bun}_G^{(\leq\theta)}(\mathbb{F}_q)$$

for a finite subset $S \subset \Lambda_G^{+,\mathbb{Q}}$. It follows from the definition of Harder-Narasimhan coweight that the image of

$$\mathrm{Bun}_P^{(\lambda)} := \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_M^{(\lambda)}, \qquad \lambda \in \Lambda_M^{+,\mathbb{Q}}$$

intersects $\mathrm{Bun}_G^{(\leq\theta)}$ only if $\lambda \leq_G^{\mathbb{Q}} \theta$ (cf. [25, Theorem 7.4.3(3)]). Now by pull-push along the diagram (2.35), we conclude that $\mathrm{HN}(\operatorname{supp} \mathrm{CT}_P(f))$ is contained in the set of $\lambda \in \Lambda_M^{+,\mathbb{Q}}$ such that $\lambda \leq_G^{\mathbb{Q}} \theta$ for some $\theta \in S$. Therefore $\mathrm{HN}(\operatorname{supp} \mathrm{CT}_P(f))$ is bounded above.

Since $[\mathrm{HN}(x)]_P = \deg_P^{\mathbb{Q}}(x)$, we deduce that $\deg_P^{\mathbb{Q}}(\operatorname{supp} \mathrm{CT}_P(f)) \subset \{[\theta]_P - \mu \mid \theta \in S, \mu \in \Lambda_{G,P}^{\mathrm{pos},\mathbb{Q}}\}$. By definition, this means that $\mathrm{CT}_P(f) \in \mathcal{C}_{P,-}$. $\square$

2.5.4 The operator $R_P : \mathcal{C}_{P^-,+} \to \mathcal{C}_{P,-}$

Let Z denote the space of pairs (g_1, g_2), where $g_1 \in G(\mathbb{A})/P^-(F)$, $g_2 \in G(\mathbb{A})/P(F)$ have equal image in $G(\mathbb{A})/(P^- \cdot P)(F)$. We have projections from Z to $G(\mathbb{A})/P^-(F)$ and $G(\mathbb{A})/P(F)$.

Define $R_P : \mathcal{C}_{P^-,+} \to \mathcal{C}_P$ to be the pull-push along the diagram

$$G(\mathbb{A})/M(F)U^-(\mathbb{A}) \leftarrow G(\mathbb{A})/P^-(F) \leftarrow Z \to G(\mathbb{A})/P(F) \to G(\mathbb{A})/M(F)U(\mathbb{A}).$$

Equivalently, R_P is given by the explicit formula

$$R_P(\varphi)(g) = \int_{U(\mathbb{A})} \varphi(gu)du, \qquad \varphi \in \mathcal{C}_{P^-,+},\ g \in G(\mathbb{A}). \tag{2.36}$$

It is evident from the definition that R_P is $G(\mathbb{A})$-equivariant.

Proposition 2.5.4. *The operator $R_P : \mathcal{C}_{P^-,+} \to \mathcal{C}_P$ is well-defined, and the image of R_P is contained in $\mathcal{C}_{P,-}$. More specifically, for $\varphi \in \mathcal{C}_{P^-,+}$ we have*

$$\mathrm{HN}(\operatorname{supp} R_P(\varphi)) \subset \{\lambda - \mu \mid \lambda \in \mathrm{HN}(\operatorname{supp}\varphi),\ \mu \in \Lambda_G^{\mathrm{pos},\mathbb{Q}}\}.$$

Proof. Let $(g_1, g_2) \in Z$. The quotient $(K \times K)\backslash Z$ identifies with the set of isomorphism classes of the $\mathbb{F}_q$-points of the stack $\mathrm{Maps}^\circ(X, P^-\backslash G/P)$ of maps generically landing in $P^-\backslash(P^- \cdot P)/P$. By [12, Proposition 3.2], the stack $\mathrm{Maps}^\circ(X, P^-\backslash G/P)$ is isomorphic to the relative version of the open Zastava space $\overset{\circ}{\mathcal{Z}}_{\mathrm{Bun}_M}$. In particular, there is a map $\overset{\circ}{\mathcal{Z}}_{\mathrm{Bun}_M} \to \mathcal{H}_M^+$, where $\mathcal{H}_M^+ := \mathrm{Maps}^\circ(X, M\backslash\overline{M}/M)$ is the Hecke substack introduced in §A.1.5. This map is induced from the contraction $G \to \overline{M}$. The image of g_1 in $K\backslash G(\mathbb{A})/M(F)U^-(\mathbb{A}) = |\mathrm{Bun}_M(\mathbb{F}_q)|$ defines an M-bundle $\mathcal{F}_M^1$. Similarly, the image of g_2 in $K\backslash G(\mathbb{A})/M(F)U(\mathbb{A}) = |\mathrm{Bun}_M(\mathbb{F}_q)|$ defines $\mathcal{F}_M^2$. Then $(g_1, g_2) \in Z$ maps to a point $(\mathcal{F}_M^1, \mathcal{F}_M^2, \beta_M) \in \mathcal{H}_M^+(\mathbb{F}_q)$, where β_M is an $\overline{M}$-morphism $\mathcal{F}_M^2 \to \mathcal{F}_M^1$ in the language of §A.1.1.

Let $\lambda_1, \lambda_2 \in \Lambda_M^{+,\mathbb{Q}}$ be the Harder-Narasimhan coweights of $\mathcal{F}_M^1, \mathcal{F}_M^2$ respectively. Then Remark A.1.9 implies that $\lambda_1 - \lambda_2 \in w_0^M \Lambda_G^{\mathrm{pos},\mathbb{Q}}$. The definition of $\mathcal{C}_{P^-,+}$ implies that the set of $\lambda_1 \in \Lambda_M^{+,\mathbb{Q}}$ for which $\varphi(g_1) \neq 0$ satisfy $-[\lambda_1]_P \in S_0 + \Lambda_{G,P}^{\mathrm{pos},\mathbb{Q}}$ for a finite set S_0. We deduce that the intersection of $\mathrm{HN}(\operatorname{supp}\varphi)$ with $\lambda_2 + w_0^M \Lambda_G^{\mathrm{pos},\mathbb{Q}}$ is finite. Thus $R_P(\varphi)(g_2)$ is an integral over the K-orbits of $G(\mathbb{A})/M(F)U^-(\mathbb{A})$ corresponding to the union of $\mathrm{Bun}_M^{(\lambda_1)}(\mathbb{F}_q)$ ranging over a finite set of λ_1, i.e., $R_P(\varphi)$ is well-defined.

Remark A.1.9 also gives the inequality $\lambda_2 \leq_G^{\mathbb{Q}} \lambda_1$, which proves the second statement of the proposition. It immediately follows that $R_P(\varphi) \in \mathcal{C}_{P,-}$. □

2.5.5 Invertibility of R_P

Below we will prove Proposition 2.5.5, which says that the operator $R_P : \mathcal{C}_{P^-,+} \to \mathcal{C}_{P,-}$ is invertible. We deduce the proposition from the local results of §2.2.

Fix the Haar measure on $G(\mathbb{A})$. Recall that we defined a generalized function ξ_P on $\mathbb{X}_P(\mathbb{A})$ by (2.24), which slightly depends on the choice of measure on $G(\mathbb{A})$. The Haar measures on $G(\mathbb{A})$ and $U(\mathbb{A})$ induce a $G(\mathbb{A})$-invariant measure on $G(\mathbb{A})/U(\mathbb{A})$.

Proposition 2.5.5. *The map $R_P : \mathcal{C}_{P^-,+} \to \mathcal{C}_{P,-}$ is an isomorphism. The inverse is given by the formula*

$$R_P^{-1}(\varphi)(g_2) = \int_{G(\mathbb{A})/U(\mathbb{A})} \varphi(g_1)\xi_P(g_1, g_2)dg_1, \qquad \varphi \in \mathcal{C}_{P,-},\ g_2 \in G(\mathbb{A}). \tag{2.37}$$

Proof. Let us first show that the right hand side of (2.37) is well-defined for any $\varphi \in \mathcal{C}_{P,-}$ and $g_2 \in G(\mathbb{A})$. Since φ is K-finite, there exists a compact open subgroup $K' = \prod K'_v \subset K$ such that φ is K'-invariant. Let $\delta_{K'} \in C_c^\infty(G(\mathbb{A}))$ equal $\frac{1}{\mathrm{mes}(K')}$ times the characteristic function of K'. Recall that $(\delta_{K'} \otimes 1) * \xi_P = \mathrm{Asymp}_P(\delta_{K'}) \in \mathcal{C}_b(\mathbb{X}_P(\mathbb{A}))$, where Asymp_P is defined in §2.4.2. Thus the r.h.s. of (2.37) equals

$$\int_{G(\mathbb{A})/U(\mathbb{A})} \varphi(g_1)\,\mathrm{Asymp}_P(\delta_{K'})(g_1, g_2)dg_1.$$

Let $\mathrm{ord}_{M,v} : \mathbb{X}_P(F_v) \to \Lambda_M^+$ be the map (2.5). Proposition 2.2.2 implies that for every v there exists a finite subset $S_v \subset \Lambda$ such that

$$\mathrm{ord}_{M,v}(\mathrm{supp}(\mathrm{Asymp}_{P,v}(\delta_{K'_v}))) \subset \{\theta - \mu \mid \theta \in S_v,\, \mu \in \Lambda_U^{\mathrm{pos},\mathbb{Q}}\},$$

and we can take $S_v = \{0\}$ for almost all v by Remark 2.3.8. For $g_1 \in G(\mathbb{A})/U(\mathbb{A})$, consider the image of (g_1, g_2) in $\mathbb{X}_P(\mathbb{A})$. Let $\mathcal{F}_M^1 \in \mathrm{Bun}_M^{(\lambda_1)}(\mathbb{F}_q)$ (resp. $\mathcal{F}_M^2 \in \mathrm{Bun}_M^{(\lambda_2)}(\mathbb{F}_q)$) be the image of g_1 in $K\backslash G(\mathbb{A})/M(F)U(\mathbb{A})$ (resp. g_2 in $K\backslash G(\mathbb{A})/M(F)U^-(\mathbb{A})$). Lemma A.1.7 implies that

$$-\sum_v \log_q(q_v) \cdot \mathrm{ord}_{M,v}(g_1, g_2) \leq_M^{\mathbb{Q}} \lambda_1 - \lambda_2 \leq_M^{\mathbb{Q}} -w_0^M \sum_v \log_q(q_v) \cdot \mathrm{ord}_{M,v}(g_1, g_2). \quad (2.38)$$

Let $S = \{\sum_v \log_q(q_v) \cdot \lambda_v \mid \lambda_v \in S_v\}$. Suppose $\mathrm{Asymp}_P(\delta_{K'})(g_1, g_2) \neq 0$. Then we have

$$\lambda_1 \in \{\lambda_2 - w_0^M \theta + \mu \mid \theta \in S, \mu \in w_0^M \Lambda_G^{\mathrm{pos},\mathbb{Q}}\} \quad (2.39)$$

$$\lambda_2 \in \{\lambda_1 + \theta - \mu \mid \theta \in S,\, \mu \in \Lambda_G^{\mathrm{pos},\mathbb{Q}}\} \quad (2.40)$$

and we emphasize that S is a finite set depending only on the stabilizer in K of φ. From (2.39) and the definition of $\varphi \in \mathcal{C}_{P,-}$, we conclude that the r.h.s. of (2.37) is a finite integral, which we temporarily denote $T(\varphi)(g_2)$. From (2.40) we deduce that $T(\varphi)$ defines an element in $\mathcal{C}_{P^-,+}$.

It remains to show that $T : \mathcal{C}_{P,-} \to \mathcal{C}_{P^-,+}$ is inverse to R_P. Let $\varphi \in \mathcal{C}_{P,-}$. Then (2.40) implies that $\mathrm{HN}(\mathrm{supp}\, T(\varphi)) \subset \{\lambda + \theta - \mu \mid \lambda \in \mathrm{HN}(\mathrm{supp}\, \varphi),\, \theta \in S,\, \mu \in \Lambda_G^{\mathrm{pos},\mathbb{Q}}\}$. Proposition 2.5.4 implies in turn that

$$\mathrm{HN}(\mathrm{supp}\, R_P(T\varphi)) \subset \{\lambda + \theta - \mu \mid \lambda \in \mathrm{HN}(\mathrm{supp}\, \varphi),\, \theta \in S,\, \mu \in \Lambda_G^{\mathrm{pos},\mathbb{Q}}\}.$$

Therefore we deduce that to show $R_P \circ T = \mathrm{id}$, it suffices to check the equality for $\varphi \in \mathcal{C}_{P,c}$.

Any such φ is the pushforward of an element in $C_c^\infty(G(\mathbb{A})/U(\mathbb{A}))$, which is isomorphic to the restricted tensor product of $\mathcal{C}_{P,c,v}$ over all places v. Since $R_P \circ T$ is defined as a product of local integrals, $(R_P \circ T)(\varphi) = \varphi$ follows from Proposition 2.2.13.

Similarly, it suffices to check that $T \circ R_P = \mathrm{id}$ on $\varphi \in \mathcal{C}_{P^-,c}$. This again follows from the corresponding local statement Proposition 2.2.13. □

2.5.6 A formula for $\mathcal{B}_P$ in terms of R_P

We give a formula for the bilinear form $\mathcal{B}_P$ defined in §2.4.2 in terms of the global intertwining operator R_P. This formula is the analog of [27, Definition 3.1.1] for a general reductive group G.

Fix some Haar measure on $G(\mathbb{A})$, and fix the Haar measure on $U^-(\mathbb{A})$ such that the measure of $U^-(\mathbb{A})/U^-(F)$ equals 1. Then we get an invariant measure on $G(\mathbb{A})/M(F)U^-(\mathbb{A})$ and therefore a pairing between $\mathcal{C}_{P^-,c}$ and $\mathcal{C}_{P^-}$ defined by

$$\langle \varphi_1, \varphi_2 \rangle := \int_{G(\mathbb{A})/M(F)U^-(\mathbb{A})} \varphi_1(x)\varphi_2(x)dx. \tag{2.41}$$

Lemma 2.5.3 implies that this pairing is well-defined when $\varphi_1 \in \mathcal{C}_{P^-,+}$ and $\varphi_2 \in \mathrm{CT}_{P^-}(\mathcal{A}_c)$.

Proposition 2.5.6. *For any $f_1, f_2 \in \mathcal{A}_c$, one has*

$$\mathcal{B}_P(f_1, f_2) = \langle R_P^{-1}\,\mathrm{CT}_P(f_1), \mathrm{CT}_{P^-}(f_2)\rangle. \tag{2.42}$$

Proof. Choose $\tilde{f}_1, \tilde{f}_2 \in C_c^\infty(G(\mathbb{A}))$ that pushforward to f_1, f_2. Then

$$\mathrm{CT}_P(f_1)(g) = \sum_{\gamma \in G(F)/U(F)} \int_{U(\mathbb{A})} \tilde{f}_1(gu\gamma^{-1})du, \qquad g \in G(\mathbb{A}).$$

The formula (2.37) directly gives

$$(R_P^{-1} \circ \mathrm{CT}_P)(f_1)(g_2) = \sum_{\gamma \in G(F)/U(F)} \int_{G(\mathbb{A})} \tilde{f}_1(g_1\gamma^{-1})\xi_P(g_1, g_2)dg_1.$$

Let $\tilde{f}_1^\vee(g) := \tilde{f}_1(g^{-1})$. By left $G(\mathbb{A})$-equivariance of Asymp_P, the right hand side of (2.42) equals

$$\int_{G(\mathbb{A})/P^-(F)} \sum_{\gamma \in G(F)/U(F)} \mathrm{Asymp}_P(\tilde{f}_1^\vee)(\gamma, g) f_2(g)dg$$
$$= \int_{G(\mathbb{A})/G(F)} \sum_{x \in \mathbb{X}_P(F)} \mathrm{Asymp}_P(\tilde{f}_1^\vee)((1,g)x) f_2(g)dg.$$

By right $G(\mathbb{A})$-equivariance of Asymp_P and (2.25), we conclude that the right hand side equals $\mathcal{B}_P(f_1, f_2)$. □

Proof of Theorem 2.1.3. The theorem follows immediately from (2.22) and Proposition 2.5.6. □

2.6 The operator L and its inverse

We first recall basic facts from the theory of Eisenstein series. Then we define the operator $L : \mathcal{A}_c \to \mathcal{A}$ in terms of the Eisenstein operator, the inverse of the standard intertwining operator, and the constant term operator. Motivated by a characterization of $\mathcal{A}_c$ due to Harder, we define the subspace $\mathcal{A}_{ps\text{-}c} \subset \mathcal{A}$ of "pseudo-compactly" supported functions in §2.6.6. We check that L sends $\mathcal{A}_c$ to this new subspace $\mathcal{A}_{ps\text{-}c}$. Lastly, we prove that $L : \mathcal{A}_c \to \mathcal{A}_{ps\text{-}c}$ is invertible in Theorem 2.6.12 and give the formula for its inverse.

We will continue to use the notation from §2.5. In this section, we will assume that the field of coefficients E equals $\mathbb{C}$.

2.6.1 *Constant term revisited*

Recall that in Lemma 2.5.3, we showed that the constant term operator $\mathrm{CT}_P : \mathcal{A} \to \mathcal{C}_P$ sends $\mathcal{A}_c$ to $\mathcal{C}_{P,-}$ for all parabolic subgroups P. The proof of [37, Theorem 1.2.1] shows that the converse is also true:

Lemma 2.6.1. *For $f \in \mathcal{A}$, we have f is compactly supported if and only if $\mathrm{CT}_P(f)$ lies in $\mathcal{C}_{P,-}$ for all standard parabolics P.*

Proof. The "only if" direction is proven by Lemma 2.5.3.

Note that if $f \in \mathcal{C}_{G,-} \subset \mathcal{A}$, then $\deg_G^{\mathbb{Q}}(\operatorname{supp} f)$ is finite. Therefore to prove the "if" direction, we may assume that $f \in \mathcal{A}$ and there is a fixed $\theta \in \Lambda^{\mathbb{Q}}$ such that

$$\deg_P^{\mathbb{Q}}(\operatorname{supp} \mathrm{CT}_P(f)) \subset -\Lambda_{G,P}^{\mathrm{pos},\mathbb{Q}} + [\theta]_P$$

for all standard parabolics P. By reduction theory, there exists a number c such that any $x \in G(\mathbb{A})/G(F)$ has a representative $g \in G(\mathbb{A})$ with $\langle \check{\alpha}, \deg_B^{\mathbb{Q}}(g) \rangle > c$ for all simple roots $\check{\alpha}$ of G. Suppose $f(g) \neq 0$ for $g \in G(\mathbb{A})$ with $\lambda = \deg_B^{\mathbb{Q}}(g) \in \Lambda^{\mathbb{Q}}$ as above. By [51, Lemma I.2.7] (cf. [37, Lemma 1.2.2]), there exists c' such that if $\langle \check{\alpha}, \lambda \rangle > c'$ for all simple roots $\check{\alpha}$ which are not simple roots of M for some standard parabolic P with Levi M, then $\mathrm{CT}_P(f)(g) = f(g)$. Let M be the Levi such that the simple roots of M are precisely the simple roots $\check{\alpha}$ of G such that $\langle \check{\alpha}, \lambda \rangle \le c'$. Then $\mathrm{CT}_P(f)(g) = f(g) \neq 0$ implies that $\deg_P^{\mathbb{Q}}(g) = [\lambda]_P$ lies in $-\Lambda_{G,P}^{\mathrm{pos},\mathbb{Q}} + [\theta]_P$. On the other hand the choice of M implies that λ belongs in a translate of $-\Lambda_M^{+,\mathbb{Q}} \subset -\Lambda_M^{\mathrm{pos},\mathbb{Q}} + \Lambda_{Z(M)}^{\mathbb{Q}}$. We deduce that the set of all possible λ is bounded above. We also have $\langle \check{\alpha}, \lambda \rangle > c$ for all simple roots $\check{\alpha}$, so there are in fact only finitely many possibilities for λ. We conclude that f is compactly supported. $\square$

For $P = MU$ a standard parabolic, we also use the notation $\mathrm{CT}_M^G := \mathrm{CT}_P$ below.

Let $P_1 \subset P$ be standard parabolic subgroups with Levi subgroups $M_1 \subset M$. Then the constant term operator $\mathrm{CT}_{M_1}^M$ can be considered as an operator $\mathrm{CT}_{M_1}^M : \mathcal{C}_P \to \mathcal{C}_{P_1}$. We say that $\varphi \in \mathcal{C}_P$ is M-cuspidal if $\mathrm{CT}^M_{M_1}(\varphi) = 0$ for all standard Levi subgroups $M_1 \subset M$.

2.6.2 Eisenstein operator

Let $P = MU$ be a parabolic subgroup of G. We define the Eisenstein operator[6] $\mathrm{Eis}_P : \mathcal{C}_{P,c} \to \mathcal{A}_c$ to be the pull-push along the diagram

$$G(\mathbb{A})/M(F)U(P)(\mathbb{A}) \leftarrow G(\mathbb{A})/P(F) \to G(\mathbb{A})/G(F), \tag{2.43}$$

where the left arrow is proper. Explicitly,

$$\mathrm{Eis}_P(\varphi)(g) := \sum_{\gamma \in G(F)/P(F)} \varphi(g\gamma), \qquad \varphi \in \mathcal{C}_{P,c},\ g \in G(\mathbb{A}).$$

We also use the notation $\mathrm{Eis}^G_M := \mathrm{Eis}_P$ for P standard.

It is well known that

$$\langle \mathrm{CT}_P(f_1), \varphi_2 \rangle = \mathcal{B}_{\mathrm{naive}}(f_1, \mathrm{Eis}_P(\varphi_2)) \tag{2.44}$$

for $f_1 \in \mathcal{A}$, $\varphi_2 \in \mathcal{C}_{P,c}$. By this adjunction, we see that it is actually possible to define $\mathrm{Eis}_P(\varphi_2)$ for any φ_2 such that $\langle \varphi_1, \varphi_2 \rangle$ is finite for all $\varphi_1 \in \mathrm{CT}_P(\mathcal{A}_c)$. Lemma 2.5.3 implies that all $\varphi_2 \in \mathcal{C}_{P,+}$ satisfy this condition. Thus Eis_P extends to an operator

$$\mathrm{Eis}_P : \mathcal{C}_{P,+} \to \mathcal{A}.$$

2.6.3 Intertwining operators revisited

In this section we recall some facts about the standard intertwining operators corresponding to elements of the Weyl group.

6. The authors of [51] call it “pseudo-Eisenstein”.

Let $P = MU$ and $P' = M'U'$ be *standard* parabolic subgroups such that $M' = wMw^{-1}$ for some $w \in W$. Then the intertwining operator[7] R_w is an operator $\mathcal{C}_{P,c} \to \mathcal{C}_{P'}$ defined by the explicit formula

$$(R_w\varphi)(g) = \int_{U'(\mathbb{A})/(U'(\mathbb{A})\cap wU(\mathbb{A})w^{-1})} \varphi(guw)du. \tag{2.45}$$

Proposition 2.6.4 below implies that if $\varphi \in \mathcal{C}_{P,c}$, then $R_w\varphi \in \mathcal{C}_{P',-}$. The same proposition also implies that $R_w\varphi \in \mathcal{C}_{P'}$ converges absolutely for any $\varphi \in \mathcal{C}_{P,+}$.

We will use the extra notation $R^G_{M,w} = R_w$ when necessary for clarity (note that the standard parabolic P' is determined by its Levi wMw^{-1}).

When $P = MU$ and $P' = M'U'$ are two (not necessarily standard) parabolic subgroups containing T and $M' = wMw^{-1}$ for $w \in W$, we will use the notation

$$R_{P':P,w} : \mathcal{C}_{P,c} \to \mathcal{C}_{P',-}$$

for the intertwining operator, which is defined by the same formula (2.45). We also use $R_{P':P}$ to denote $R_{P':P,1}$. So the operator R_P considered in §2.5.4 is now denoted by $R_{P:P^-}$.

Let $V \subset \mathcal{C}_P$ denote the subspace of functions $\varphi \in \mathcal{C}_P$ such that $R_{P':P,w}(\varphi)$ converges absolutely. Let $V' \subset V$ denote the subspace of $\varphi \in V$ such that $R_{P':P,w}(\varphi)$ is compactly supported.

Proposition 2.6.2. *The map $R_{P':P,w} : V' \to \mathcal{C}_{P',c}$ is an isomorphism. The inverse map $R^{-1}_{P':P,w} : \mathcal{C}_{P',c} \to \mathcal{C}_P$ is an integral operator.*

Proof. It suffices to consider the case $w = 1$. Let $\varphi \in V'$. Then $\varphi' := R_{P':P}\varphi \in \mathcal{C}_{P',c}$ and $R_{P^-:P'}(\varphi') \in \mathcal{C}_{P^-,-}$ must equal $R_{P^-:P}\varphi$. The operator $R_{P^-:P} : \mathcal{C}_{P,+} \to \mathcal{C}_{P^-,-}$ is invertible by Proposition 2.5.5, so we have $(R^{-1}_{P^-:P} \circ R_{P^-:P'})(R_{P':P}\varphi) = \varphi$, which proves injectivity.

7. In [51, II.1.6], R_w is denoted $M(w, \pi)$ for a cuspidal representation π. But we prefer to avoid multiple uses of the letter M.

Now take $\varphi' \in \mathcal{C}_{P',c}$. Since $\mathcal{C}_{P',c} \subset \mathcal{C}_{P',-}$, we have $R^{-1}_{P':P'-}(\varphi') \in \mathcal{C}_{P'-,+}$. Then $\varphi := (R_{P:P'-} \circ R^{-1}_{P':P'-})(\varphi') \in \mathcal{C}_P$ is well-defined. Moreover, $R_{P':P}(\varphi) = (R_{P':P'-} \circ R^{-1}_{P':P'-})(\varphi') = \varphi'$. Hence $\varphi \in V'$, and we have shown surjectivity of $R_{P':P}$. □

Remark 2.6.3. The operators $R_{P':P,w}$ and $R^{-1}_{P':P,w}$ are defined on larger spaces of functions, but we will not consider the corresponding support conditions in this article.

2.6.4 The composition $\mathrm{CT}_{P'} \circ \mathrm{Eis}_P$

Let $P = MU$ and $P' = M'U'$ be standard parabolic subgroups of G. Let $\check{\Phi}^+_M$ denote the set of positive roots of M.

Let $\varphi \in \mathcal{C}_{P,c}$ be M-cuspidal. Then [51, Proposition II.1.7] gives the formula

$$(\mathrm{CT}_{P'} \circ \mathrm{Eis}_P)(\varphi) = \sum_{w \in W(M,M')} (\mathrm{Eis}^{M'}_{wMw^{-1}} \circ R^G_{M,w})(\varphi). \tag{2.46}$$

where $W(M, M') := \{w \in W \mid w^{-1}\check{\alpha} > 0, \forall \check{\alpha} \in \check{\Phi}^+_{M'}$ and wMw^{-1} is a standard Levi of $M'\}$.

Proposition 2.6.4. *Let $\varphi \in \mathcal{C}_{P,+}$ be arbitrary. Then one has*

$$(\mathrm{CT}_{P'} \circ \mathrm{Eis}_P)(\varphi) = \sum_{w \in W^\bullet_{M,M'}} (\mathrm{Eis}^{M'}_{wM_1w^{-1}} \circ R^G_{M_1,w} \circ \mathrm{CT}^M_{M_1})(\varphi) \tag{2.47}$$

where $W^\bullet_{M,M'} := \{w \in W \mid \forall \check{\alpha} \in \check{\Phi}^+_M,\ w\check{\alpha} > 0,\ \forall \check{\alpha} \in \check{\Phi}^+_{M'},\ w^{-1}\check{\alpha} > 0\}$ *and* $M_1 := M \cap w^{-1}M'w$*, and every term on the right hand side converges absolutely.*

Proof. See the proof of [51, Proposition II.1.7]. □

2.6.5 The operator $L : \mathcal{A}_c \to \mathcal{A}$

One has the operators

$$\mathcal{A}_c \xrightarrow{\mathrm{CT}_P} \mathcal{C}_{P,-} \xrightarrow{R^{-1}_{P:P^-}} \mathcal{C}_{P^-,+} \xrightarrow{\mathrm{Eis}_{P^-}} \mathcal{A}.$$

Thus we deduce that

$$\mathcal{B}(f_1, f_2) = \mathcal{B}_{\text{naive}}(Lf_1, f_2) \tag{2.48}$$

where the operator $L : \mathcal{A}_c \to \mathcal{A}$ is defined by

$$L := \sum_P (-1)^{\dim Z(M)} \operatorname{Eis}_{P^-} \circ R^{-1}_{P:P^-} \circ \operatorname{CT}_P$$

and the sum ranges over the standard parabolic subgroups. Unlike the form $\mathcal{B}$, the operator L does not depend on the choice of Haar measure on $G(\mathbb{A})$.

Observe that for $f \in \mathcal{A}_c$ cuspidal, we have $Lf = (-1)^{\dim Z(G)} f$.

Remark 2.6.5. Theorem 2.1.2 implies that the miraculous duality functor $\text{Ps-Id}_{\text{Bun}_G,!}$ defined in [25, §4.4.8] is the D-module analog of the operator $q^{-\dim \text{Bun}_G} \cdot L^K : \mathcal{A}_c^K \to \mathcal{A}^K$ on K-invariants via the functions–sheaves dictionary (cf. [27, §A.8.4]).

The following proposition shows the interplay between the operator L and the Eisenstein operators.

Proposition 2.6.6. *Let P be a standard parabolic subgroup. Let $\varphi \in \mathcal{C}_{P,c}$ be M-cuspidal. Then*

$$(L \circ \operatorname{Eis}_P)(\varphi) = (-1)^{\dim Z(M)} (\operatorname{Eis}_{P^-} \circ R^{-1}_{P:P^-})(\varphi).$$

Proof. Let P' be another standard parabolic subgroup. By formula (2.46),

$$(\operatorname{Eis}_{P'^-} \circ R^{-1}_{P':P'^-} \circ \operatorname{CT}_{P'} \circ \operatorname{Eis}_P)(\varphi) = \sum_{w \in W(M,M')} (\operatorname{Eis}_{P'^-} \circ R^{-1}_{P':P'^-} \circ \operatorname{Eis}^{M'}_{wMw^{-1}} \circ R^G_{M,w})(\varphi) \tag{2.49}$$

and each term on the r.h.s. converges absolutely. Let $P'_1 \subset M'$ be the standard parabolic subgroup with Levi wMw^{-1}, so $\operatorname{Eis}^{M'}_{wMw^{-1}} = \operatorname{Eis}_{P'_1}$. One can check that

$$R^{-1}_{P':P'^-} \circ \operatorname{Eis}_{P'_1} = \operatorname{Eis}_{P'_1} \circ R^{-1}_{P'_1U':P'_1U'^-}$$

when either side converges absolutely (recall that $R^{-1}_{P':P'-}$ is defined as a product of local integrals by (2.37)). Let $P'_2 = w_0^{M'} P_1'^{-} w_0^{M'} \subset M'$. Let $Q = P'_2 U'$ be the standard parabolic subgroup of G with Levi $w'Mw'^{-1}$, $w' = w_0^{M'} w$. Then $\mathrm{Eis}_{P'_1} \circ R^{-1}_{P'_1U':P'_1U'-} = \mathrm{Eis}_{P_2'^{-}} \circ R^{-1}_{P_2'^{-}U':Q^-} \circ w_0^{M'}$, where $w_0^{M'}$ denotes right translation by a representative of $w_0^{M'}$ in $M'(F)$.

From the definition (2.45), we have

$$(w_0^{M'} \circ R^G_{M,w})(\varphi)(g) = \int_{(U^-_{P'_2}U')(\mathbb{A}) \cap w'U^-(\mathbb{A})w'^{-1}} \varphi(guw')du, \qquad g \in G(\mathbb{A}).$$

If $\check{\alpha}$ is a positive root of G not in M, then $w'\check{\alpha} = w_0^{M'} w\check{\alpha}$ must be negative if it is a root of M', by definition of $W(M, M')$. Thus $U^-_{P'_2}U' \cap w'U^- w'^{-1} = U' \cap w'U^-w'^{-1}$ and $U_{P'_2}U' \cap w'Uw'^{-1} = U' \cap w'Uw'^{-1}$. We deduce that $R_{P_2'^{-}U':Q^-} = w_0^{M'} \circ R^G_{M,w} \circ R_{P:Q^-,w'^{-1}}$ and therefore

$$(R^{-1}_{P'_1U':P'_1U'-} \circ w_0^{M'} \circ R^G_{M,w})(\varphi) = R^{-1}_{P:Q^-,w'^{-1}}(\varphi)$$

for $\varphi \in \mathcal{C}_{P,c}$. Next, observe that $\mathrm{Eis}_{P'-} \circ \mathrm{Eis}_{P_2'^{-}} = \mathrm{Eis}_{Q^-}$. Thus the r.h.s. of (2.49) equals

$$\sum_{w \in W(M,M')} (\mathrm{Eis}_{Q^-} \circ R^{-1}_{P:Q^-,w'^{-1}})(\varphi).$$

Note that Q depends only on $w'Mw'^{-1}$ and not on M'. Summing over all standard parabolics P', we get the formula

$$(L \circ \mathrm{Eis}_P)(\varphi) = \sum_{w' \in W} \left(\sum_{M' \left| \substack{M' \cap B^- \subset w'Bw'^{-1} \\ w'Mw'^{-1} \subset M'} \right.} (-1)^{\dim Z(M')} \right) (\mathrm{Eis}_{Q^-} \circ R^{-1}_{P:Q^-,w'^{-1}})(\varphi). \tag{2.50}$$

Fixing $w' \in W$, let us classify all M' such that $M' \cap B^- \subset w'Bw'^{-1}$ and $w'Mw'^{-1} \subset M'$. The two conditions above are equivalent to requiring $\check{\Delta}_{M'} \subset w'\check{\Phi}^-_G$ and $\check{\Delta}_G \cap w'\check{\Phi}_M \subset \check{\Delta}_{M'}$.

Thus the inner sum in (2.50) equals

$$(1+(-1))^{|\check{\Delta}_G \cap w'(\check{\Phi}_G^- - \check{\Phi}_M)|},$$

which vanishes unless $w'(\check{\Phi}_G^- - \check{\Phi}_M)$ contains no simple roots. If $w' \notin W_M$, then $B \not\subset w'Pw'^{-1}$, so $w'(\check{\Phi}_G^+ \cup \check{\Phi}_M)$ does not contain all the simple roots, i.e., $w'(\check{\Phi}_G^- - \check{\Phi}_M)$ contains a simple root. Therefore in order for the sum to not vanish, w' must be in W_M and $M \cap B^- \subset w'Bw'^{-1}$. Hence $w' = w_0^M$, and we have $M' = M$, $Q = P$. Since $R_{P:P^-, w_0^M} = R_{P:P^-}$ by $M(F)$-invariance, we get $(L \circ \mathrm{Eis}_P)(\varphi) = (-1)^{\dim Z(M)}(\mathrm{Eis}_{P^-} \circ R_{P:P^-}^{-1})(\varphi)$. $\square$

For a standard parabolic P, define the "second Eisenstein" operator $\mathrm{Eis}'_P : \mathcal{C}_{P,-} \to \mathcal{A}$ by

$$\mathrm{Eis}'_P := \mathrm{Eis}_{P^-} \circ R_{P:P^-}^{-1}.$$

Let $\mathcal{A}^M$ denote the space of smooth K-finite functions on $M(\mathbb{A})/M(F)$. Let $L^M : \mathcal{A}_c^M \to \mathcal{A}^M$ denote the operator L with respect to the reductive group M. Applying $\mathrm{ind}_{P(\mathbb{A})}^{G(\mathbb{A})}$, we also let L^M denote the induced operator $\mathcal{C}_{P,c} \to \mathcal{C}_P$. Recall that L^M is $(-1)^{\dim Z(M)}$ times the identity on M-cuspidal functions in $\mathcal{C}_{P,c}$.

Corollary 2.6.7. *One has the equality*

$$L \circ \mathrm{Eis}_P = \mathrm{Eis}'_P \circ L^M : \mathcal{C}_{P,c} \to \mathcal{A}. \tag{2.51}$$

The operator L is self-adjoint with respect to $\mathcal{B}_{\text{naive}}$, so (2.51) gives

$$\mathrm{CT}_P \circ L = L^M \circ \mathrm{CT}'_P : \mathcal{A}_c \to \mathcal{C}_P, \tag{2.52}$$

where $\mathrm{CT}'_P : \mathcal{A}_c \to \mathcal{C}_{P,+}$ is defined by $\mathrm{CT}'_P = R_{P^-:P}^{-1} \circ \mathrm{CT}_{P^-}$.

Remark 2.6.8. As explained in [27, §A.11.7], equation (2.51) is an analog of the "strange" functional equation for geometric Eisenstein series stated in [30, Theorem 4.1.2].

Remark 2.6.9. Observe that for any $f \in \mathcal{A}_c$, we have $\deg_G^{\mathbb{Q}}(\operatorname{supp} f) = \deg_G^{\mathbb{Q}}(\operatorname{supp} Lf) \subset \Lambda_{G,G}^{\mathbb{Q}}$ from the definitions (i.e., L does not change the connected component of the image of the support in Bun_G). As a consequence, for any $\varphi \in \mathcal{C}_P$ on which $L^M\varphi$ converges, we have $\deg_P^{\mathbb{Q}}(\operatorname{supp} \varphi) = \deg_P^{\mathbb{Q}}(\operatorname{supp} L^M\varphi)$.

2.6.6 The space $\mathcal{A}_{ps\text{-}c}$ of "pseudo-compactly" supported functions

We are inspired by Lemma 2.6.1 to make the following definition.

Definition 2.6.10. Let $\mathcal{A}_{ps\text{-}c}$ be the space of all functions $f \in \mathcal{A}$ such that $\mathrm{CT}_P(f) \in \mathcal{C}_{P,+}$ for all standard parabolic subgroups $P \subset G$.

Proposition 2.6.11. *For any $f \in \mathcal{A}_c$, one has $Lf \in \mathcal{A}_{ps\text{-}c}$.*

Proof. Let $P = MU$ be a standard parabolic subgroup of G. By (2.52), we have $\mathrm{CT}_P(Lf) = L^M(\mathrm{CT}'_P(f))$ and $\mathrm{CT}'_P(f) \in \mathcal{C}_{P,+}$. Remark 2.6.9 implies that $L^M(\mathrm{CT}'_P(f)) \in \mathcal{C}_{P,+}$ as well. Hence $\mathrm{CT}_P(Lf) \in \mathcal{C}_{P,+}$ and $Lf \in \mathcal{A}_{ps\text{-}c}$. □

Theorem 2.6.12. *The operator $L : \mathcal{A}_c \to \mathcal{A}_{ps\text{-}c}$ is invertible. For $f \in \mathcal{A}_{ps\text{-}c}$ one has*

$$L^{-1}f = \sum_P (-1)^{\dim Z(M)} (\mathrm{Eis}_P \circ \mathrm{CT}_P)(f). \tag{2.53}$$

where the sum ranges over standard parabolic subgroups.

Proof. For $f \in \mathcal{A}_{ps\text{-}c}$, set $L'f$ equal to the r.h.s. of (2.53). First we need to check that $L'f \in \mathcal{A}_c$. Let P' be another standard parabolic. Then by (2.47),

$$(\mathrm{CT}_{P'} \circ \mathrm{Eis}_P \circ \mathrm{CT}_P)(f) = \sum_{w \in W^{\bullet}_{M,M'}} (\mathrm{Eis}^{M'}_{wM_1w^{-1}} \circ R^G_{M_1,w} \circ \mathrm{CT}^G_{M_1})(f)$$

where $W^{\bullet}_{M,M'} = \{w \in W \mid w\check{\alpha} > 0, \forall \check{\alpha} \in \check{\Phi}^+_M,\ w^{-1}\check{\alpha} > 0, \forall \check{\alpha} \in \check{\Phi}^+_{M'}\}$ is a set of representatives for $W_{M'}\backslash W/W_M$, and $M_1 = M \cap w^{-1}M'w$. Summing over P, we have

$$\mathrm{CT}_{P'}(L'f) = \sum_{w\in W} \sum_{M_1 | wM_1w^{-1}\subset M'} \left(\sum_{M \,\middle|\, \substack{M_1 = M\cap w^{-1}M'w \\ w\in W^{\bullet}_{M,M'}}} (-1)^{\dim Z(M)} \right) (\mathrm{Eis}^{M'}_{wM_1w^{-1}} \circ R^{G}_{M_1,w} \circ \mathrm{CT}^{G}_{M_1})(f).$$

Note that $\check{\Delta}_{M_1}$ is any subset of $w^{-1}\check{\Phi}^{+}_{M'} \cap \check{\Delta}_G$, while $\check{\Delta}_M - \check{\Delta}_{M_1}$ is any subset of $w^{-1}(\check{\Phi}^{+}_{G} - \check{\Phi}^{+}_{M'}) \cap \check{\Delta}_G$. Thus the inner sum over M vanishes unless $w^{-1}(\check{\Phi}^{+}_{G} - \check{\Phi}^{+}_{M'})$ contains no simple roots. This only occurs if $w = w_0^{M'}w_0$ and $M_1 = M$. By considering $M'_1 = w_0^{M'}w_0 M_1 w_0 w_0^{M'}$ instead of M_1, we see that

$$\mathrm{CT}_{P'}(L'f) = \sum_{M'_1\subset M'} (-1)^{\dim Z(M'_1)} (\mathrm{Eis}^{M'}_{M'_1} \circ R_{P'_1:(M'\cap P'_1)U'^{-}} \circ \mathrm{CT}_{(M'\cap P'_1)U'^{-}})(f) \qquad (2.54)$$

where the sum is over all Levi subgroups of M', and P'_1 is the standard parabolic subgroup of G with Levi M'_1. Let $Q = (M'\cap P'_1)U'^{-}$. Then $w_0 w_0^{M'} Q w_0^{M'} w_0$ is a standard parabolic subgroup. Since $f \in \mathcal{A}_{ps\text{-}c}$, we have $\mathrm{CT}_Q(f) \in \mathcal{C}_{Q,+}$. Observe that $\mathrm{Eis}^{M'}_{M'_1} \circ R_{P'_1:Q} = R_{P':P'^{-}} \circ \mathrm{Eis}^{M'}_{M'_1}$ when either side converges absolutely. Also note that $\deg^{\mathbb{Q}}_{P'}(\operatorname{supp} \mathrm{Eis}^{M'}_{M'_1}(\mathrm{CT}_Q f)) = \deg^{\mathbb{Q}}_{P'}(\operatorname{supp}(\mathrm{CT}_Q f))$. From this we deduce that $(\mathrm{Eis}^{M'}_{M'_1} \circ \mathrm{CT}_Q)(f) \in \mathcal{C}_{P'^{-},+}$. Therefore

$$(R_{P':P'^{-}} \circ \mathrm{Eis}^{M'}_{M'_1} \circ \mathrm{CT}_Q)(f) \in \mathcal{C}_{P',-}.$$

Consequently, $\mathrm{CT}_{P'}(L'f) \in \mathcal{C}_{P',-}$ and L' defines an operator $\mathcal{A}_{ps\text{-}c} \to \mathcal{A}_c$.

Now we check that L' is inverse to L. For $f \in \mathcal{A}_{ps\text{-}c}$, it follows from (2.54) and the ensuing discussion that

$$LL'f = \sum_{P'} \sum_{P'_1 \subset P'} (-1)^{\dim Z(M') - \dim Z(M'_1)} (\mathrm{Eis}_{P'-} \circ R^{-1}_{P':P'-} \circ R_{P':P'-} \circ \mathrm{Eis}^{M'}_{M'_1} \circ \mathrm{CT}_{(M' \cap P'_1)U'-})(f).$$

Observe that $\mathrm{Eis}_{P'-} \circ \mathrm{Eis}^{M'}_{M'_1} = \mathrm{Eis}_{(M' \cap P'_1)U'-}$. Set $M'_2 := w_0^{M'} M'_1 w_0^{M'}$ and let P'_2 be the corresponding standard parabolic subgroup. Then $\mathrm{Eis}_{(M' \cap P'_1)U'-} \circ \mathrm{CT}_{(M' \cap P'_1)U'-} = \mathrm{Eis}_{P'^-_2} \circ \mathrm{CT}_{P'^-_2}$. Hence

$$LL'f = \sum_{P'_2} \left(\sum_{P'_2 \subset P'} (-1)^{\dim Z(M') - \dim Z(M'_2)} \right) (\mathrm{Eis}_{P'^-_2} \circ \mathrm{CT}_{P'^-_2})(f).$$

The inner sum vanishes unless $P'_2 = G$. Therefore $LL'f = f$.

For $f \in \mathcal{A}_c$, we apply (2.52) to get

$$L'Lf = \sum_{P} \sum_{M_1 \subset M} (-1)^{\dim Z(M) - \dim Z(M_1)} (\mathrm{Eis}_{(M \cap P_1^-)U} \circ \mathrm{CT}'^{M}_{M \cap P_1^-} \circ \mathrm{CT}'_P)(f),$$

where $P = MU$ ranges over the standard parabolic subgroups, M_1 ranges over Levi subgroups of M, and $P_1 \subset G$ is the standard parabolic subgroup with Levi M_1. Observe that $\mathrm{CT}'^{M}_{M \cap P_1^-} \circ \mathrm{CT}'_P = \mathrm{CT}'_{(M \cap P_1^-)U}$. Conjugating by w_0^M, one sees that

$$\mathrm{Eis}_{(M \cap P_1^-)U} \circ \mathrm{CT}'_{(M \cap P_1^-)U} = \mathrm{Eis}_{P_2} \circ \mathrm{CT}'_{P_2},$$

where P_2 is the standard parabolic subgroup with Levi $M_2 := w_0^M M_1 w_0^M$. Then

$$L'Lf = \sum_{P_2} \left(\sum_{P_2 \subset P} (-1)^{\dim Z(M) - \dim Z(M_2)} \right) (\mathrm{Eis}_{P_2} \circ \mathrm{CT}'_{P_2})(f),$$

and the inner sum vanishes unless $P_2 = G$. Therefore $L'Lf = f$, and we have proved that

L' is the inverse of L. $\square$

Remark 2.6.13. We observe that formula (2.53) for L^{-1} may be thought of as an analog of the Aubert–Zelevinsky involution for smooth representatives of a $\mathfrak{p}$-adic group: Let F_v denote a non-Archimedean local field. For every smooth $G(F_v)$-module M one can form a complex

$$0 \to M \to \bigoplus_P i_P^G r_P^G(M) \to \cdots \to i_B^G r_B^G(M) \to 0$$

where i_P^G, r_P^G denote, respectively, the parabolic induction and Jacquet functors, and the sum in the i-th term runs over standard parabolic subgroups of corank i in G. We call this complex the Deligne–Lusztig complex associated to M and denote it by $\mathrm{DL}(M)$. Analogous complexes were considered in [22] for representations of a finite Chevalley group. In the Grothendieck group, we have

$$[\mathrm{DL}(M)] = \sum_P (-1)^{\dim Z(G) - \dim Z(M)} [i_P^G r_P^G(M)]$$

where the sum ranges over standard parabolic subgroups. In fact, this defines an involution of the Grothendieck group, which is often called the Aubert–Zelevinsky involution. If one considers $\mathrm{Eis}_P, \mathrm{CT}_P$ as global analogs of i_P^G, r_P^G, respectively, then formula (2.53) suggests that L^{-1} is a global analog of the Aubert–Zelevinsky involution (although L^{-1} is no longer an involution).

Remark 2.6.14. The main result of [30] (namely, Theorem 0.1.6) says that the stack Bun_G is miraculous, i.e., the functor $\text{Ps-Id}_{\mathrm{Bun}_G,!} : \text{D-mod}(\mathrm{Bun}_G)_{\mathrm{co}} \to \text{D-mod}(\mathrm{Bun}_G)$ is an equivalence. This equivalence is a D-module analog of the part of Theorem 2.6.12 that says that the operator $L^K : \mathcal{A}_c^K \to \mathcal{A}^K$ induces an isomorphism $\mathcal{A}_c^K \to \mathcal{A}_{ps\text{-}c}^K$.

The formula (2.53) for L^{-1} may be useful in describing the functor inverse to $\text{Ps-Id}_{\mathrm{Bun}_G,!}$ (we expect that one can mimic the construction of the Deligne–Lusztig complex using the functors $\mathrm{Eis}_P^{\mathrm{enh}}, \mathrm{CT}_P^{\mathrm{enh}}$ defined in [31, §6]). We refer the reader to [27, Conjecture C.2.1] for an explicit conjecture in the case $G = \mathrm{SL}(2)$.

CHAPTER 3

RADON INVERSION FORMULAS OVER LOCAL FIELDS

3.1 Introduction

3.1.1 Some notation

Let F be a local field (i.e., F is either non-Archimedean or $\mathbb{R}$ or $\mathbb{C}$). Let G denote the topological group $\mathrm{GL}_n(F)$ for an integer $n \geq 2$.

Let K be the standard maximal compact subgroup of G (i.e., if F is non-Archimedean then $K = \mathrm{GL}_n(\mathcal{O})$, where $\mathcal{O} \subset F$ is the ring of integers, if $F = \mathbb{R}$ then $K = O(n)$, and if $F = \mathbb{C}$ then $K = U(n)$).

We fix a field E of characteristic 0; if F is Archimedean we assume that E equals $\mathbb{C}$. Unless otherwise specified, all functions will take values in E.

Let $\mathcal{C}$ denote the space of K-finite C^∞ functions on $F^n \setminus \{0\}$. In §3.2.2 we define the subspace $\mathcal{C}_+ \subset \mathcal{C}$ consisting of functions with bounded support and the subspace $\mathcal{C}_- \subset \mathcal{C}$ consisting of functions supported away from a neighborhood of 0.

3.1.2 Subject of this chapter

In this chapter we consider the Radon transform as an operator

$$R : \mathcal{C}_+ \to \mathcal{C}_-.$$

When F is non-Archimedean, R is known to be an isomorphism [10]. An explicit formula for the inverse was, however, not present in the literature. There is a 'classical' inversion formula due to Černov [20] on the space of Schwartz functions, but its relation to R^{-1} is not obvious. We formulate and prove a simple formula for R^{-1} in the non-Archimedean case and relate it to Černov's formula.

In the Archimedean case, the invertibility of R was *a priori* unclear due to the nonstandard nature of the function spaces $\mathcal{C}_{\pm}$. We prove that R is indeed an isomorphism when F is Archimedean and provide formulas for R^{-1} (here K-finiteness of $\mathcal{C}$ plays a crucial role).

3.1.3 Motivation

Our interest in the operator R originates from the classical theory of automorphic forms. Let G denote the algebraic group SL_2 and U (resp. U^-) the subgroup of strictly upper (resp. lower) triangular matrices and T the maximal torus of diagonal matrices. Let B denote the subgroup of upper triangular matrices. Then $G(F)/U(F) = F^2 \backslash \{0\}$ and $G(F)/U^-(F) = F^2 \setminus \{0\}$. The operator R coincides with the (local) intertwining operator $R_B : \mathcal{C}_{B^-,+} \to \mathcal{C}_{B,-}$ defined in §2.2.6. The spaces $\mathcal{C}_{\pm}$ also coincide with $\mathcal{C}_{B,\pm}$ as defined in §2.2.5

While we work only with the local field F, one also has the global analog of R defined in Section 2.5. The global intertwining operator plays an important role in the theory of Eisenstein series and their constant terms [16, §3.7]. The constant terms of automorphic forms reside in the space $\mathcal{C}_-(G(\mathbb{A})/T(F)U(\mathbb{A}))$, which makes it a natural space to study in this setting. The results of this chapter are used to prove invertibility of the global intertwining operator in [27]. This in turn gives explicit formulas for the bilinear form $\mathcal{B}$ defined in Chapter 2 in the case $G = \mathrm{SL}(2)$ over any *global* field. These results may also be reinterpreted as explicit formulas for the distributions ξ_P defined in §2.2.7. In particular, they give definitions of ξ_P over an Archimedean local field, while ξ_P was *a priori* defined only over non-Archimedean local fields.

In the situation where F is a non-Archimedean local field, the operator R^{-1} is essentially the same as the 'Bernstein map' introduced in [10, Definition 5.3]; the precise relation between the two is explained in [10, Theorem 7.5]. The Bernstein map is also studied in [60] (there it is called the asymptotic map) in the more general context of spherical varieties.

In the real case, the Radon transform has been studied extensively by analysts ([33], [39], [40]) over slightly different function spaces.

3.1.4 Structure of the chapter

In §3.2 we define the subspaces $\mathcal{C}_\pm \subset \mathcal{C}$ and recall the definition of the Radon transform over a general local field F.

In §3.3, we consider the case when F is non-Archimedean. We prove that R is invertible and give a formula for R^{-1} in Theorem 3.3.6. This is done by relating the Radon transform to the Fourier transform (§3.3.3-3.3.5). We deduce the previously known Radon inversion formula of Černov [20] from Theorem 3.3.6 in §3.3.6.

We consider the real case in §3.4. The formula for R^{-1} is given on each K-isotypic component of $\mathcal{C}_-$ in Theorem 3.4.2 in terms of convolution with a distribution on $\mathbb{R}_{>0}$. The Mellin transform of this distribution is computed in Theorem 3.4.4. The proof of the theorems is in §3.4.6. The invertibility of R heavily relies on the K-finiteness assumption in the definition of $\mathcal{C}$. In §3.4.7, we prove (Corollary 3.4.8) that the analog of R is not surjective when K-finiteness is dropped from the definitions.

In §3.5, the complex case is developed in the same way as the real case. The inversion formula is given in Theorem 3.5.2 and the reformulation using the Mellin transform is Theorem 3.5.4.

3.2 Recollections on the Radon transform

3.2.1 The norm on F^n

Let $|\cdot|$ denote the normalized absolute value on F when F is non-Archimedean and the usual absolute value[1] when F is Archimedean. For $a \in F^\times$, set $v(a) := -\log|a|$. If F is non-Archimedean log stands for $\log_q$, where q is the order of the residue field of F. If F is Archimedean, log is understood as the natural logarithm.

We define a norm $\|\cdot\|$ on F^n as follows. If F is non-Archimedean, then $\|\cdot\|$ is the norm

1. If $F = \mathbb{R}$, then the normalized absolute value coincides with the usual absolute value. If $F = \mathbb{C}$, then the normalized absolute value is the square of the usual absolute value.

induced by the standard lattice $\mathcal{O}^n$ (i.e., $\|x\|$ is the maximum of the absolute values of the coordinates of $x \in F^n$). If F is Archimedean, then $\|\cdot\|$ is induced by the standard Euclidean/Hermitian inner product (i.e., the square root of the sum of the absolute values squared).

For $x \in F^n \setminus \{0\}$, set $v(x) := -\log\|x\|$.

3.2.2 The spaces $\mathcal{C}, \mathcal{C}_c, \mathcal{C}_\pm$

Let $\mathcal{C}$ denote the space of K-finite C^∞ functions on $F^n \setminus \{0\}$ (recall that if F is non-Archimedean, C^∞ means locally constant). Let $\mathcal{C}_c \subset \mathcal{C}$ be the subspace of compactly supported functions on $F^n \setminus \{0\}$.

Given a real number N, let $\mathcal{C}_{\leq N} \subset \mathcal{C}$ denote the set of all functions $\varphi \in \mathcal{C}$ such that $\varphi(\xi) \neq 0$ only if $v(\xi) \leq N$. Similarly, we have $\mathcal{C}_{\geq N}, \mathcal{C}_{>N}$, and so on. Let $\mathcal{C}_-$ denote the union of the subspaces $\mathcal{C}_{\leq N}$ for all N. Let $\mathcal{C}_+$ denote the union of the subspaces $\mathcal{C}_{\geq N}$ for all N. Clearly $\mathcal{C}_- \cap \mathcal{C}_+ = \mathcal{C}_c$ and $\mathcal{C}_- + \mathcal{C}_+ = \mathcal{C}$.

3.2.3 Radon transform

Equip F with the following Haar measure: if F is non-Archimedean we require that $\mathrm{mes}(\mathcal{O}) = 1$; if F is Archimedean we use the usual Lebesgue measure. Let the measure on F^n be the product of the measures on n copies of F. Fix the Haar measure on $F^\times$ to be $d^\times t := \frac{dt}{|t|}$.

Let $f \in \mathcal{C}_+$. The Radon transform $\mathcal{R}f(\xi, s)$, for $\xi \in F^n \setminus \{0\}$ and $t \in F^\times$, is defined by the formula

$$\mathcal{R}f(\xi, t) = \int_{F^n} f(x)\delta(\xi \cdot x - t)dx,$$

where $\xi \cdot x = \xi_1 x_1 + \cdots + \xi_n x_n$ and δ is the delta distribution on F. The expression for $\mathcal{R}f(\xi, t)$ can also be written directly as

$$\mathcal{R}f(\xi, t) = \int_{\xi \cdot x = t} f(x) d\mu_\xi$$

where $d\mu_\xi$ is the measure on the hyperplane $\xi \cdot x = t$ such that $d\mu_\xi dt = dx$. We get an operator $R : \mathcal{C}_+ \to \mathcal{C}_-$ by setting

$$Rf(\xi) = \int_{\xi \cdot x = 1} f(x) d\mu_\xi. \tag{3.1}$$

Proposition 3.2.1. *For any number N one has $R(\mathcal{C}_{\geq N}) \subset \mathcal{C}_{\leq -N}$.*

Proof. Let $f \in \mathcal{C}_{\geq N}$ and $\xi \in F^n \setminus \{0\}$ with $v(\xi) > -N$. Then $\xi \cdot x = 1$ implies $v(x) < N$, so $f(x) = 0$. Therefore $Rf \in \mathcal{C}_{\leq -N}$. □

The natural action of G on $F^n \setminus \{0\}$ induces a G-action on $\mathcal{C}$ by $(g \cdot f)(x) := f(g^{-1}x)$ for $g \in G, f \in \mathcal{C}, x \in F^n \setminus \{0\}$. Then

$$R(g \cdot f) = |\det g|^d (g^T)^{-1} Rf \tag{3.2}$$

for $f \in \mathcal{C}_+$ and $g \in G$, where g^T is the transpose matrix, and $d = 1$ if $F \neq \mathbb{C}$ and $d = 2$ if $F = \mathbb{C}$ (i.e., $|\det g|^d$ is the normalized absolute value of $\det g$).

3.3 F non-Archimedean

In this section we consider the case when F is a non-Archimedean local field. Let $\mathcal{O}$ the ring of integers, $\mathfrak{p}$ the maximal ideal, ϖ a uniformizer, and $\mathbb{F}_q$ the residue field of F.

The main result of this section is Theorem 3.3.6. In order to state the theorem, we must first define a new operator $A_\beta : \mathcal{C}_- \to \mathcal{C}_+$, which is done in §3.3.2.

3.3.1 K-finite functions

The action of G on $F^n \setminus \{0\}$ is continuous and transitive. Since F is non-Archimedean, K is an open subgroup of G, and we have the following description of K-finite functions.

Lemma 3.3.1. *A C^∞ function φ on $F^n \setminus \{0\}$ is K-finite if and only if there exists an open subgroup $H \subset K$ such that $\varphi(h\xi) = \varphi(\xi)$ for all $h \in H$ and $\xi \in F^n \setminus \{0\}$.*

Proof. Let W denote the span of the K translates of f. By assumption W is finite dimensional, and this implies that there exists a compact open subset X of $F^n \setminus \{0\}$ such that the restriction map $W \to C^\infty(X)$ is injective. Any locally constant function on X is fixed by an open subgroup of K, which proves the lemma. $\square$

One may sometimes wish to consider the group $\mathrm{SL}_n(F)$ rather than $\mathrm{GL}_n(F)$ acting on $F^n \setminus \{0\}$. The next lemma shows that this does not change the corresponding subspaces of invariant functions in $\mathcal{C}$.

Lemma 3.3.2. *For an integer $r > 0$, set $K_r := \ker(\mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}/\mathfrak{p}^r))$. Then the following properties of a function φ on $F^n \setminus \{0\}$ are equivalent for $n \geq 2$:*

(i) φ is stabilized by $K_r \cap \mathrm{SL}_n(F)$,

(ii) $\varphi(\xi') = \varphi(\xi)$ for $\xi, \xi' \in F^n \setminus \{0\}$ satisfying $v(\xi' - \xi) \geq v(\xi) + r$,

(iii) φ is stabilized by K_r.

Proof. Suppose that φ is stabilized by $K_r \cap G$. Take $\xi, \xi' \in F^n \setminus \{0\}$ with $v(\xi' - \xi) \geq v(\xi) + r$. We can find a basis $v_1, \ldots, v_n$ of $\mathcal{O}^n$ with $v_1 = \varpi^{-v(\xi)}\xi$ and $v_2 = \varpi^{-v(\xi'-\xi)}(\xi' - \xi)$. Let g send v_1 to $v_1 + \varpi^{v(\xi'-\xi)-v(\xi)}v_2$ and v_k to v_k for $k > 1$. Then $g \in K_r \cap G$ and $g\xi = \xi'$. Thus $\varphi(\xi') = \varphi(\xi)$. This proves (i) implies (ii). The other implications are easy. $\square$

3.3.2 The operator $A_\beta : \mathcal{C}_- \to \mathcal{C}_+$

Let $\mathcal{S}'_b(F)$ denote the space of distributions β on F such that for any open subgroup $U \subset \mathcal{O}^\times$, the multiplicative U-average[2] β_U has compact support and $\langle \beta_U, 1 \rangle = 0$. Note that if $\langle \beta_U, 1 \rangle = 0$ for some U, then it is true for all U.

We would like to define $A_\beta : \mathcal{C}_- \to \mathcal{C}_+$ for $\beta \in \mathcal{S}'_b(F)$ by

$$(A_\beta \varphi)(x) = \int_{F^n} \beta(\xi \cdot x)\varphi(\xi) d\xi$$

2. The multiplicative U-average β_U is defined by $\beta_U(t) = \frac{1}{\mathrm{mes}(U)} \int_U \beta(ut) d^\times u$.

but we must explain the meaning of the r.h.s.

Fix $\varphi \in \mathcal{C}_-$ and $x \in F^n \setminus \{0\}$. For any open compact subgroup $\Lambda \subset F^n$ let

$$I(\Lambda) := \int_\Lambda \beta(\xi \cdot x)\varphi(\xi)d\xi.$$

Lemma 3.3.3. *There exists Λ such that $I(\Lambda') = I(\Lambda)$ for any Λ' containing Λ.*

Proof. Choose $\xi_0 \in F^n$ such that $\xi_0 \cdot x = 1$ and $v(\xi_0) = -v(x)$. Then $F^n = F\xi_0 \oplus H$ where H is the hyperplane $\{\xi \mid \xi \cdot x = 0\}$. Lemma 3.3.1 implies that $\varphi \in \mathcal{C}_-$ is fixed by the homothety actions of an open subgroup $U \subset \mathcal{O}^\times$. Therefore we can replace β by the multiplicative average β_U. Let $\mathfrak{p}^i \subset F$ be a fractional ideal containing the support of β_U. Lemma 3.3.2(ii) implies that $\varphi(s\xi_0 + \xi) = \varphi(\xi)$ if $s \in \mathfrak{p}^i$ and $v(\xi) \leq v(\xi_0) + i - r$, where φ is stabilized by the congruence subgroup K_r. Put $a := r - i$. Let $\Lambda := \mathfrak{p}^i\xi_0 \oplus \{\xi \in H \mid v(\xi) \geq -v(x) - a\}$.

Now suppose Λ' is a subgroup containing Λ. Define $\Lambda'' = \{\xi \in \Lambda' \mid \xi \cdot x \in \mathfrak{p}^i\} \supset \Lambda$. Then $I(\Lambda') = I(\Lambda'')$ since $\mathfrak{p}^i$ contains the support of β. Now $\Lambda'' = \mathfrak{p}^i\xi_0 \oplus (\Lambda'' \cap H)$. Thus

$$I(\Lambda'') - I(\Lambda) = \int_{\xi \in (\Lambda'' \setminus \Lambda) \cap H} \int_{\mathfrak{p}^i} \beta_U(s)\varphi(s\xi_0 + \xi)|\xi_0| ds d\mu_x.$$

Note that $\xi \in (\Lambda'' \setminus \Lambda) \cap H$ satisfies $v(\xi) < -v(x) - a$ and hence $\varphi(s\xi_0 + \xi) = \varphi(\xi)$. We conclude that $I(\Lambda'') = I(\Lambda)$ since $\langle \beta_U, 1 \rangle = 0$. $\square$

Put $(A_\beta\varphi)(x) := I(\Lambda)$ where Λ is as in Lemma 3.3.3.

Corollary 3.3.4. *Let N be any number. If $\varphi \in \mathcal{C}_{\leq -N}$, then $A_\beta\varphi \in \mathcal{C}_{\geq N-a}$, where a is an integer depending only on β and the stabilizer of φ in G.*

Proof. We use the notation from the proof of Lemma 3.3.3. Note that the choice of a is independent of $x \in F^n \setminus \{0\}$. It follows from our definition above and the proof of Lemma 3.3.3 that

$$(A_\beta\varphi)(x) = \int_{\substack{\xi \in H \\ v(\xi) \geq -v(x) - a}} \int_{\mathfrak{p}^i} \beta_U(t)\varphi(s\xi_0 + \xi)|\xi_0| ds d\mu_x,$$

which is zero if $v(x) < N - a$. $\square$

Thus we have defined an operator $A_\beta : \mathcal{C}_- \to \mathcal{C}_+$.

Remark 3.3.5. For $\varphi \in \mathcal{C}_-$ we have $A_\beta(g\varphi) = |\det g|(g^T)^{-1}(A_\beta\varphi)$ where g^T is the transpose. In other words, the operator $\mathcal{C}_- \to \{\text{measures on } (F^n)^* \setminus \{0\}\}$ defined by $\varphi \mapsto (A_\beta\varphi)dx$ is equivariant with respect to the action of G.

The goal of this section is to prove the following.

Theorem 3.3.6. *The operator $R : \mathcal{C}_+ \to \mathcal{C}_-$ is an isomorphism. The inverse of R is A_β, where β is the compactly supported distribution on F equal to*

$$\frac{1-q^{n-1}}{1-q^{-n}}(|s-1|^{-n} - |s|^{-n}).$$

The distributions $|s-1|^{-n}$ and $|s|^{-n}$ are defined as in [34, Ch. 2, §2.3], i.e.,

$$\langle |s|^{-n}, f\rangle = \int_F |s|^{-n}(f(s) - f(0))ds$$

for a test function $f \in C_c^\infty(F)$.

We prove Theorem 3.3.6 in §3.3.5.

Remark 3.3.7. Let β be as defined in Theorem 3.3.6. Then the integral of β along any compact open subset of F has value in $\mathbb{Z}[\frac{1}{q}]$. This is not true for the distribution $\frac{1-q^{n-1}}{1-q^{-n}}|s|^{-n}$.

3.3.3 Fourier transform

We assume without loss of generality that E contains all roots of unity. Choose a nontrivial additive character ψ of F which is trivial on $\mathcal{O}$ but nontrivial on $\varpi^{-1}\mathcal{O}$. The Haar measure we chose for F is self-dual with respect to ψ. Note that $\psi \in \mathcal{S}_b'(F)$. Define the Fourier transform $\mathcal{F} : \mathcal{C}_- \to \mathcal{C}_+$ by

$$\mathcal{F} := A_\psi.$$

On the other hand, we also have an operator $\mathcal{F}' : \mathcal{C}_+ \to \mathcal{C}_-$ defined by

$$\mathcal{F}' f(\xi) = \int_{F^n} f(x)(\psi(-\xi \cdot x) - 1)dx. \tag{3.3}$$

Moreover for any number N, one observes that $\mathcal{F}'(\mathcal{C}_{\geq N}) \subset \mathcal{C}_{<-N}$.

Proposition 3.3.8. *The operators $\mathcal{F}$ and $\mathcal{F}'$ are mutually inverse.*

Proof. Proposition 3.2.1 and Corollary 3.3.4 imply that

$$\mathcal{F}\mathcal{F}'(\mathcal{C}_{\geq N}) \subset \mathcal{C}_{\geq N+a} \quad \text{and} \quad \mathcal{F}'\mathcal{F}(\mathcal{C}_{\leq -N}) \subset \mathcal{C}_{\leq -N-a}$$

on functions stabilized by K_r for a fixed $r > 0$. As a consequence, it is enough to check the equalities $\mathcal{F}\mathcal{F}' = \mathrm{id}$ and $\mathcal{F}'\mathcal{F} = \mathrm{id}$ on the subspace $\mathcal{C}_c = \mathcal{C}_+ \cap \mathcal{C}_-$.

Let $f \in \mathcal{C}_c$. Then the usual Fourier transform $\hat{f}$ is a compactly supported function on F^n. Note that $\mathcal{F}' f(\xi) = \hat{f}(\xi) - \hat{f}(0)$. By the definition of $\mathcal{F}$, we have

$$\mathcal{F}\mathcal{F}' f(x) = \int_{\Lambda} (\hat{f}(\xi) - \hat{f}(0))\psi(\xi \cdot x)d\xi$$

for any sufficiently large open compact subgroup $\Lambda \subset F^n$. Since $\hat{f}$ is compactly supported, the usual Fourier inversion formula implies that $\int_{\Lambda} \hat{f}(\xi)\psi(\xi \cdot x)d\xi = f(x)$ if Λ contains the support of $\hat{f}$. Since x is nonzero, $\int_{\Lambda} \psi(\xi \cdot x)d\xi = 0$ for Λ large enough. Therefore $\mathcal{F}\mathcal{F}' f = f$.

In the other direction, let $\varphi \in \mathcal{C}_c$. Then $\mathcal{F}\varphi(x) = \hat{\varphi}(-x)$ is compactly supported on F^n. Again the Fourier inversion formula implies that

$$\mathcal{F}'\mathcal{F}\varphi(\xi) = \int_{F^n} \hat{\varphi}(x)\psi(-\xi \cdot x)dx - \int_{F^n} \hat{\varphi}(x)dx = \varphi(\xi) - \varphi(0) = \varphi(\xi). \qquad \square$$

3.3.4 Actions on $\mathcal{C}_\pm$

For any real number a, let $\mathcal{A}_{\leq a}$ be the space of generalized functions α on $F^\times$ whose support is contained in $\{t \in F^\times \mid v(t) \leq a\}$. Let $\mathcal{A}_-$ denote the union of all $\mathcal{A}_{\leq a}$ for all a. Then $\mathcal{A}_-$ becomes an algebra under convolution using the measure $d^\times t$.

We have an action of $\mathcal{A}_-$ on $\mathcal{C}_-$ defined by

$$(\alpha * \varphi)(\xi) = \int_{F^\times} \alpha(t)\varphi(t^{-1}\xi)d^\times t$$

for $\alpha \in \mathcal{A}_-$, $\varphi \in \mathcal{C}_-$, and $\xi \in F^n \setminus \{0\}$. One similarly defines $\mathcal{A}_{\geq a}$, $\mathcal{A}_+$, and an action of $\mathcal{A}_+$ on $\mathcal{C}_+$. There is an isomorphism $\sigma : \mathcal{A}_{\leq a} \to \mathcal{A}_{\geq -a}$ defined by

$$\sigma(\alpha)(t) = \alpha(t^{-1})|t|^{-n}.$$

We would like to define a multiplicative convolution action of $\mathcal{A}_+$ on $\mathcal{S}'_b(F)$ by

$$(\widetilde{\alpha} * \beta)(s) = \int_{F^\times} \widetilde{\alpha}(t)\beta(t^{-1}s)d^\times t$$

for $\widetilde{\alpha} \in \mathcal{A}_+$ and $\beta \in \mathcal{S}'_b(F)$, but we must explain the meaning of this formula as a distribution on F. Let $\mathcal{S}(F)$ denote the space of locally constant, compactly supported functions on F.

Lemma 3.3.9. *Let $f \in \mathcal{S}(F)$ and $t \in F^\times$. Then $\int_F \beta(t^{-1}s)f(s)ds = 0$ if $v(t)$ is sufficiently large.*

Proof. Since $f \in \mathcal{S}(F)$, there exists an open subgroup $U \subset \mathcal{O}^\times$ that stabilizes f under homotheties. Thus we can replace β by the multiplicative average β_U, which is compactly supported. Then $\int_F \beta(t^{-1}s)f(s)ds = |t| \int_{\operatorname{supp}\beta_U} \beta_U(s)f(ts)ds$. If $v(t)$ is large enough such that f is constant on $t(\operatorname{supp}\beta_U)$, the integral vanishes since $\langle \beta_U, 1\rangle = 0$. $\square$

Define the distribution $\widetilde{\alpha} * \beta \in \mathcal{S}'_b(F)$ by putting the value at $f \in \mathcal{S}(F)$ to be

$$\langle \widetilde{\alpha} * \beta, f \rangle = \int_{F^\times} \widetilde{\alpha}(t) \left(\int_F \beta(t^{-1}s) f(s) ds \right) d^\times t,$$

which is well-defined by Lemma 3.3.9 and the fact that $\widetilde{\alpha} \in \mathcal{A}_+$.

Remark 3.3.10. Observe that $\mathcal{A}_{\leq a} * \mathcal{C}_{\leq N} \subset \mathcal{C}_{\leq N+a}$ and $\mathcal{A}_{\geq a} * \mathcal{C}_{\geq N} \subset \mathcal{C}_{\geq N+a}$ for any numbers a and N. Moreover if $\widetilde{\alpha} \in \mathcal{A}_{\geq a}$ and $\beta \in \mathcal{S}'_b(F)$ has support contained in $\mathfrak{p}^i$, then the support of $\widetilde{\alpha} * \beta$ is contained in $\mathfrak{p}^{a+i}$.

Remark 3.3.11. The convolution action of $\mathcal{A}_+$ on $\mathcal{S}'_b(F)$ is indeed an action, i.e., $\widetilde{\alpha}_1 * (\widetilde{\alpha}_2 * \beta) = (\widetilde{\alpha}_1 * \widetilde{\alpha}_2) * \beta$ for $\widetilde{\alpha}_1, \widetilde{\alpha}_2 \in \mathcal{A}_+$ and $\beta \in \mathcal{S}'_b(F)$. One sees this by restricting β to $F^\times$ and identifying $\mathcal{A}_+$ with the space of distributions on $F^\times$ with bounded support using the measure $d^\times t$.

Lemma 3.3.12. *Let $\alpha \in \mathcal{A}_-$, $\beta \in \mathcal{S}'_b(F)$, and $\varphi \in \mathcal{C}_-$. Then*

$$A_\beta(\alpha * \varphi) = \sigma(\alpha) * A_\beta \varphi = A_{\sigma(\alpha) * \beta}(\varphi)$$

Proof. By Corollary 3.3.4 and Remark 3.3.10, we reduce to the case where $\alpha \in \mathcal{A}_- \cap \mathcal{A}_+$ and $\varphi \in \mathcal{C}_c$. Consequently, $\alpha * \varphi \in \mathcal{C}_c$. Fix $x \in F^n \setminus \{0\}$. We have

$$A_\beta(\alpha * \varphi)(x) = \int_{F^n} \beta(\xi \cdot x) \int_{F^\times} \alpha(t) \varphi(t^{-1}\xi) d^\times t d\xi = \int_{F^\times} \alpha(t) |t|^n \int_{F^n} \beta(\xi \cdot tx) \varphi(\xi) d\xi d^\times t.$$

by a change of variables. Substituting t with t^{-1} in the last integral shows that $A_\beta(\alpha * \varphi) = \sigma(\alpha) * A_\beta \varphi$. One observes that $\sigma(\alpha) * A_\beta \varphi = A_{\sigma(\alpha) * \beta}(\varphi)$ essentially by definition. $\square$

Remark 3.3.13. One easily checks that if $\alpha \in \mathcal{A}_-$ and $f \in \mathcal{C}_+$, then $R(\sigma(\alpha) * f) = \alpha * Rf$.

3.3.5 Relation between Radon and Fourier transforms

Note that $\mathcal{F}'$ and R are both operators $\mathcal{C}_+ \to \mathcal{C}_-$. Comparing formulas (3.1) and (3.3), we deduce the formula

$$\mathcal{F}' f = \alpha * Rf \tag{3.4}$$

where $f \in \mathcal{C}_+$ and $\alpha(t) := \psi(-t) - 1$ for $t \in F^\times$.

Let β be the distribution defined in Theorem 3.3.6.

Lemma 3.3.14. *We have an equality of distributions*

$$\beta = \sigma(\alpha) * \psi.$$

Proof. Let $f \in \mathcal{S}(F)$. Then $\langle \sigma(\alpha) * \psi, f \rangle = \int_{F^\times} |t|^n (\psi(t) - 1) \left(\int_F f(s)\psi(-ts)ds \right) d^\times t$. This is the value at f of the Fourier transform of $|t|^{n-1}(\psi(t) - 1)$ considered as a distribution on F. It is well-known [34, Ch. 2, §2.5-6] that the Fourier transform of $|t|^{n-1}$ is $\frac{1-q^{n-1}}{1-q^{-n}}|s|^{-n}$. Therefore we conclude that $\sigma(\alpha) * \psi = \beta$. □

Observe that $\mathcal{F} = A_\psi$ and A_β are both operators $\mathcal{C}_- \to \mathcal{C}_+$. Let $\varphi \in \mathcal{C}_-$. From Lemmas 3.3.12 and 3.3.14, we deduce the equality

$$A_\beta \varphi = \sigma(\alpha) * \mathcal{F}\varphi. \tag{3.5}$$

Proof of Theorem 3.3.6. We deduce from (3.4) and Proposition 3.3.8 that R has a left inverse sending $\varphi \in \mathcal{C}_-$ to $\mathcal{F}(\alpha * \varphi)$. Lemma 3.3.12 and (3.5) together say that $\mathcal{F}(\alpha * \varphi) = A_\beta \varphi$. Applying R to (3.5) and using Remark 3.3.13, we see that $RA_\beta = \mathcal{F}'\mathcal{F} = \mathrm{id}$. Therefore A_β is both left and right inverse to R. □

3.3.6 Comparison with Černov's Radon inversion formula

Let f be a Schwartz (i.e., compactly supported C^∞) function on F^n. Recall that the Radon transform $\mathcal{R}f(\xi,s)$ is a C^∞ function on $(F^n \setminus \{0\}) \times F$ (in particular it is defined at $s=0$), and $\mathcal{R}f(\xi,s)=0$ if $\|s\xi\|^{-1}$ is sufficiently large. The following "non-archimedean Cavalieri's condition" is also well-known:

Lemma 3.3.15. *The integral $\int_F \mathcal{R}f(\xi,s)ds$ does not depend on ξ.*

Proof. The integral of f over F^n along a pencil of parallel hyperplanes does not depend on the direction of the pencil. □

It was previously known ([20, Theorem 5], [46, formula (8)]) that the following inversion formula holds:

$$f(x) = \frac{1-q^{n-1}}{(1-q^{-1})(1-q^{-n})} \int_{\|\eta\|=1} \langle |s|^{-n}, \mathcal{R}f(\eta, s+\eta\cdot x)\rangle d\eta \tag{3.6}$$

where $x \in F^n \setminus \{0\}$ and η ranges over norm 1 vectors in F^n.

We will deduce formula (3.6) from Theorem 3.3.6. Since f is compactly supported on F^n, we have $f \in \mathcal{C}_+$ and Theorem 3.3.6 implies that

$$f(x) = A_\beta Rf(x) = \int_{v(\xi)\geq N} \beta(\xi\cdot x) Rf(\xi) d\xi$$

for $x \in F^n \setminus \{0\}$ and N a sufficiently large number. We can write $\xi = t^{-1}\eta$ where $t \in F^\times$ and $\eta \in F^n$ with $\|\eta\|=1$. This gives the equality

$$f(x) = \int_{v(t)\leq -N} \int_{\|\eta\|=1} \beta(t^{-1}\eta\cdot x) Rf(t^{-1}\eta)|t|^{-n} d\eta d^\times t.$$

Homogeneity of $\mathcal{R}f$ implies that $|t|^{-1}Rf(t^{-1}\eta) = \mathcal{R}f(\eta,t)$. Therefore we have the formula

$$f(x) = \frac{1-q^{n-1}}{(1-q^{-1})(1-q^{-n})} \int_{v(t)\leq -N} \int_{\|\eta\|=1} (|\eta\cdot x - t|^{-n} - |\eta\cdot x|^{-n}) \mathcal{R}f(\eta,t) d\eta dt. \tag{3.7}$$

Choose $\eta_0 \in F^n$ with $v(\eta_0) = -v(x)$ and $\eta_0 \cdot x = 1$. Then $\eta \cdot x - t = (\eta - t\eta_0) \cdot x$. Note that if $v(t) > v(x)$, then translation by $t\eta_0$ preserves the unit sphere of norm 1 vectors. Moreover smoothness of $\mathcal{R}f$ implies that $\mathcal{R}f(\eta + t\eta_0, t) = \mathcal{R}f(\eta, t)$ if $v(t)$ is sufficiently large. Therefore the inner integral of (3.7) is zero if $v(t)$ is sufficiently large. Thus we may integrate over all $t \in F$ and switch the order of integration.

Lemma 3.3.16. *The integral $\int_{\|\eta\|=1} |\eta \cdot x|^{-n} d\eta$ equals zero.*

Proof. Using the G-action, we may assume that $x = (1, 0, \ldots, 0)$. Then $\eta \cdot x = \eta_1$, the first coordinate of η. One sees that $\int_{\|\eta\|=1} |\eta_1|^{-n} d\eta = (1 - q^{-1}) + \int_{\mathfrak{p}} |\eta_1|^{-n} d\eta_1 (1 - q^{1-n})$. A simple calculation shows that the latter expression vanishes. $\square$

Lemmas 3.3.15 and 3.3.16 imply that $\int_{\|\eta\|=1} |\eta \cdot x|^{-n} \int_F \mathcal{R}f(\eta, t) dt d\eta = 0$, so the $|\eta \cdot x|^{-n}$ term in (3.7) vanishes. After a change of variables $s = t - \eta \cdot x$, the formula (3.7) becomes equal to Černov's formula (3.6).

3.4 F real

In this section we prove the invertibility of R when $F = \mathbb{R}$. Recall that in this case $K = O(n)$. The inversion formula is given in Theorem 3.4.2, and a reformulation using the Mellin transform is given in Theorem 3.4.4. The K-finiteness of $\mathcal{C}$ plays a crucial role in the proofs, so we begin by recalling the classification of the K-isotypic components of $\mathcal{C}$.

The non-K-finite situation is considered in §3.4.7.

3.4.1 Spherical harmonics

Let S^{n-1} denote the unit sphere centered at the origin in $\mathbb{R}^n$, which has a natural action by $O(n)$. Let $\mathcal{C}(S^{n-1})$ be the space of smooth K-finite functions on S^{n-1}. For a nonnegative integer k, let H^k denote the space of harmonic polynomials on $\mathbb{R}^n$ of degree k.

Theorem 3.4.1 ([39, Theorem I.3.1], [42, Theorem 3.1], [47]). *Let $H^k|S^{n-1}$ denote the space of harmonic polynomials restricted to S^{n-1}. Then*

(i) the restriction map $H^k \to H^k|S^{n-1}$ is an isomorphism,

(ii) $\mathcal{C}(S^{n-1}) = \bigoplus_{k\geq 0} H^k|S^{n-1}$ as $O(n)$-representations,

(iii) the $O(n)$-representations H^k are irreducible and not isomorphic to each other.

3.4.2 Decomposing $\mathcal{C}$ into K-isotypes

We have a decomposition $\mathbb{R}^n \setminus \{0\} = \mathbb{R}_{>0} \times S^{n-1}$, with $O(n)$ acting on the S^{n-1} component. Let $\mathcal{C}(\mathbb{R}_{>0})$ denote the space of smooth functions on $\mathbb{R}_{>0}$ and define the subspaces $\mathcal{C}_\pm(\mathbb{R}_{>0}), \mathcal{C}_c(\mathbb{R}_{>0})$ as in §3.2.2.

Theorem 3.4.1 implies that there is a decomposition

$$\mathcal{C} = \bigoplus_{k\geq 0} \mathcal{C}(\mathbb{R}_{>0}) \otimes H^k.$$

For $u \in \mathcal{C}(\mathbb{R}_{>0})$ and $Y \in H^k$, we define $u \otimes Y \in \mathcal{C}$ by $(u \otimes Y)(x) := u(|x|) \cdot Y(\frac{x}{|x|})$.

3.4.3 Radon inversion formula

We have an isomorphism $\mathrm{Inv} : \mathcal{C}_- \to \mathcal{C}_+$ defined by

$$(\mathrm{Inv}\,\varphi)(x) = \|x\|^{-n}\varphi\left(\frac{x}{\|x\|^2}\right).$$

Set $\widetilde{R} := \mathrm{Inv}^{-1} \circ R$. Consider $\mathbb{R}_{>0}$ as a subgroup of diagonal matrices in G. Then it follows from (3.2) that $\widetilde{R}$ is a $K \times \mathbb{R}_{>0}$ equivariant operator from $\mathcal{C}_+$ to $\mathcal{C}_+$.

Let A denote the space of distributions on $\mathbb{R}_{>0}$ supported on $(0, 1]$. Then A is an algebra under the convolution product $*$ induced by the multiplication operation on $\mathbb{R}_{>0}$. The action of $\mathbb{R}_{>0}$ on $\mathcal{C}_+$ induces an action of A.

Theorem 3.4.2. *The operator $R : \mathcal{C}_+ \to \mathcal{C}_-$ is an isomorphism. For $\varphi \in \mathcal{C}_-(\mathbb{R}_{>0}) \otimes H^k$, the inverse $R^{-1} : \mathcal{C}_- \to \mathcal{C}_+$ is given by the formula*

$$R^{-1}\varphi = \beta_k * \mathrm{Inv}(\varphi)$$

where β_k is the distribution on $\mathbb{R}_{>0}$ defined by

$$\beta_k(t) = \frac{1}{2^{n+k-2}\pi^{\frac{n-1}{2}}\Gamma(\frac{n+2k-1}{2})} t^{k-1} \left(-\frac{d}{dt}\right)^{n+k-1} \left(t^{-k+1}(1-t^2)_+^{\frac{n+2k-3}{2}}\right) dt. \tag{3.8}$$

The derivative $\frac{d}{dt}$ is applied in the sense of generalized functions. For $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > -1$, the generalized function $(1-t)_+^\lambda$ is defined by $\langle (1-t)_+^\lambda, f_0(t)dt\rangle = \int_0^1 (1-t)^\lambda f_0(t)dt$ for $f_0 \in \mathcal{C}_c(\mathbb{R}_{>0})$. This generalized function can be analytically continued to all $\lambda \in \mathbb{C}$ not equal to a negative integer [35, §I.3.2]. We define $(1-t^2)_+^\lambda = (1+t)^\lambda \cdot (1-t)_+^\lambda$.

Corollary 3.4.3. *For any number N one has $R^{-1}(\mathcal{C}_{\le -N}) \subset \mathcal{C}_{\ge N}$.*

Proof. Observe that β_k is supported on $(0,1]$ for all k. □

3.4.4 A formula for $\widetilde{R}$ in terms of convolution

For $t \in (-1,1)$, define $A_t : \mathcal{C}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ such that $(A_t f)(x)$ is the average value of f on the $(n-2)$-sphere $\{\omega \in S^{n-1} \mid \omega \cdot x = t\}$. Then A_t is $O(n)$-equivariant, so by Schur's lemma it acts on $H^k|S^{n-1}$ by a scalar $a_k(t)$. Since H^k is stable under complex conjugation, $a_k(t)$ is real valued. One observes that a_k is a smooth function on $(-1,1)$, $|a_k(t)| \le 1$ for all $t \in (-1,1)$, and $\lim_{t\to 1} a_k(t) = 1$.

Suppose that $f \in \mathcal{C}(\mathbb{R}_{>0}) \otimes H^k$ and there exists $C > 0$ and $\sigma > n-1$ such that $|f(x)| \le C\|x\|^{-\sigma}$ for all x with $\|x\| \ge 1$. Since the intersection of S^{n-1} with the hyperplane $\{\omega \mid \omega \cdot x = t\}$ has radius $(1-t^2)^{1/2}$ for a unit vector x, we deduce that

$$\widetilde{R}f = \alpha_k * f \tag{3.9}$$

where α_k is the measure $\mathrm{mes}(S^{n-2})\cdot t^{-n}\cdot a_k(t)(1-t^2)^{\frac{n-3}{2}}dt$ on the interval $(0,1)$ extended by zero to the whole $\mathbb{R}_{>0}$. The convolution $\alpha_k * f$ is well-defined because of the bound on $|f(x)|$, and $\mathrm{mes}(S^{n-2})$ denotes the surface area of the $(n-2)$-sphere. In fact, [33, Proposition 2.11] says that $a_k(t)$ is the scalar multiple of the Gegenbauer polynomial $C_k^{(\frac{n-2}{2})}(t)$ normalized by $a_k(1) = 1$.

The Mellin transform $\mathfrak{M}\alpha_k$ is defined for $s \in \mathbb{C}$ by integrating t^s against α_k if $\mathrm{Re}(s) > n-1$.

Theorem 3.4.4. *The distribution α_k is invertible in A. The inverse β_k is defined by* (3.8)*. The Mellin transforms are given by*

$$\mathfrak{M}\beta_k(s) = \frac{1}{\mathfrak{M}\alpha_k(s)} = 2^{1-n-k}\pi^{\frac{1-n}{2}}\frac{\Gamma(s+k)}{\Gamma(s-n+1)}\cdot\frac{\Gamma(\frac{s-n-k}{2}+1)}{\Gamma(\frac{s+k+1}{2})}. \tag{3.10}$$

Theorem 3.4.4 implies Theorem 3.4.2.

3.4.5 Relation to Fourier transform

Let $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. The Fourier transform is defined for an integrable function f on $\mathbb{R}^n$ by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi\cdot x}dx.$$

This definition can be extended [66, §I.3] to the space of tempered distributions. After this extension, $\mathcal{F}$ becomes an isomorphism $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Let $F : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ denote the 1-dimensional Fourier transform. For $f \in \mathcal{S}(\mathbb{R}^n)$, one gets the Fourier transform from the Radon transform by

$$\mathcal{F}f(r\omega) = F(\mathcal{R}f(\omega,t))(r) \tag{3.11}$$

where F is the Fourier transform with respect to the t variable, $r \in \mathbb{R}$, and $\omega \in S^{n-1}$ is a unit vector.

Lemma 3.4.5. *Let f be a locally integrable function on $\mathbb{R}^n$ for which there exist $C > 0$ and $\sigma > n-1$ such that $|f(x)| \leq C\|x\|^{-\sigma}$ for all x with $\|x\| \geq 1$. Then:*

(i) $\mathcal{R}f$ is a locally integrable function on $S^{n-1} \times \mathbb{R}$.

(ii) $\mathcal{R}f(\omega, t)$ is bounded for $|t| \geq 1$.

(iii) The right hand side of (3.11) *is well-defined as a generalized function on $\mathbb{R} \times S^{n-1}$.*

(iv) Equation (3.11) *holds as an equality between generalized functions on $\mathbb{R}_{>0} \times S^{n-1}$.*

Proof. Since $\mathcal{R}f$ is defined by integrating f on a hyperplane of dimension $n-1$, the bound on $|f(x)|$ implies that $\mathcal{R}f$ is well-defined on $S^{n-1} \times \mathbb{R}$. One also uses this bound and local integrability of f to deduce that $\mathcal{R}f$ is locally integrable. If $\omega \in S^{n-1}$ and $t \in \mathbb{R}$ with $|t| \geq 1$, then integrating in the radial direction on the hyperplane $\omega \cdot x = t$, we see that $|\mathcal{R}f(\omega, t)|$ is bounded by a constant times $\int_0^\infty (r^2 + t^2)^{-\sigma/2} r^{n-2} dr$, which is equal to a constant times $|t|^{n-1-\sigma}$. This proves (ii). Property (iii) follows immediately from properties (i)-(ii).

Let φ be a compactly supported smooth function on $\mathbb{R}^n \setminus \{0\} = \mathbb{R}_{>0} \times S^{n-1}$. Consider f as a tempered distribution on $\mathbb{R}^n$. By the definition of $\mathcal{F}f$,

$$\int_{\mathbb{R}_{>0} \times S^{n-1}} \mathcal{F}f(r\omega)\varphi(r\omega) r^{n-1} dr d\omega = \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}_{>0} \times S^{n-1}} \varphi(r\omega) e^{-2\pi i r(\omega \cdot x)} r^{n-1} dr d\omega dx. \tag{3.12}$$

Since $t \mapsto \int_{\mathbb{R}_{>0}} \varphi(r\omega) e^{-2\pi i r t} r^{n-1} dr$ is a Schwartz function on $\mathbb{R}$, we deduce from the decomposition $dx = d\mu_\omega dt$ and property (ii) applied to $|f|$ that the integral

$$\int_{S^{n-1}} \int_{\mathbb{R}^n} \left| f(x) \int_{\mathbb{R}_{>0}} \varphi(r\omega) e^{-2\pi i r(\omega \cdot x)} r^{n-1} dr \right| dx d\omega$$

converges. Then the Fubini-Tonelli theorem implies that (3.12) is equal to

$$\int_{S^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}_{>0}} \mathcal{R}f(\omega, t) e^{-2\pi i r t} \varphi(r\omega) r^{n-1} dr dt d\omega,$$

which proves (iv). $\square$

3.4.6 Proof of Theorem 3.4.4

Let $Y \in H^k$ and define $f(x) = \|x\|^{-s} \cdot Y(\frac{x}{\|x\|})$ for $s \in \mathbb{C}$. If $n-1 < \mathrm{Re}(s) < n$, then f is locally integrable on $\mathbb{R}^n$ and satisfies the hypothesis of Lemma 3.4.5. Moreover by (3.9) and homogeneity of f we see that

$$\mathcal{R}f(\omega, t) = \mathrm{sgn}(t)^k |t|^{n-1-s} Rf(\omega) = \mathrm{sgn}(t)^k |t|^{n-1-s} \mathfrak{M}\alpha_k(s) Y(\omega)$$

as a locally integrable function on $S^{n-1} \times \mathbb{R}$. Then Lemma 3.4.5 implies that

$$\mathcal{F}f(r\omega) = F(\mathrm{sgn}(t)^k |t|^{n-1-s})(r) \mathfrak{M}\alpha_k(s) Y(\omega).$$

It is well-known [35, §II.2.3] that

$$F(\mathrm{sgn}(t)^k |t|^{n-1-s})(r) = i^k (2\pi)^{s-n+1} \frac{\sin(\pi \frac{(s-n-k+1)}{2})}{\pi} \Gamma(n-s) \, \mathrm{sgn}(r)^k |r|^{s-n}.$$

On the other hand, one can compute the Fourier transform of f directly:

Theorem 3.4.6 ([66, Theorem IV.4.1])**.** *If $0 < \mathrm{Re}(s) < n$, then $\mathcal{F}f(x) = \gamma \|x\|^{s-n} Y(\frac{x}{\|x\|})$, where $\gamma = i^{-k} \pi^{s-\frac{n}{2}} \Gamma(\frac{n+k-s}{2}) / \Gamma(\frac{s+k}{2})$.*

Comparing constant multiples in the two formulas for $\mathcal{F}f$ above and applying Euler's reflection formula, we have

$$\mathfrak{M}\alpha_k(s) = 2^{n-1-s} \pi^{n/2-1} \frac{\Gamma(\frac{n+k-s}{2}) \Gamma(\frac{s-n-k+1}{2}) \Gamma(\frac{n+k+1-s}{2})}{\Gamma(\frac{s+k}{2}) \Gamma(n-s)}$$

for $n-1 < \mathrm{Re}(s) < n$. By analytic continuation, we deduce the equality for all $s \in \mathbb{C}$ away from poles. The duplication formula for the Γ-function implies that

$$\mathfrak{M}\alpha_k(s) = 2^{n+k-1} \pi^{\frac{n-1}{2}} \frac{\Gamma(s-n+1)}{\Gamma(s+k)} \cdot \frac{\Gamma(\frac{s+k+1}{2})}{\Gamma(\frac{s-n-k}{2}+1)}, \tag{3.13}$$

as stated in Theorem 3.4.4. To finish the proof of Theorem 3.4.4, it remains to show that $(\mathfrak{M}\beta_k)^{-1}$ equals the right hand side of (3.13). By considering the Beta function we see that $\Gamma(\frac{s-n-k}{2}+1)/\Gamma(\frac{s+k+1}{2})$ is the Mellin transform of $\nu(t)dt$, where

$$\nu(t) = \frac{2}{\Gamma(\frac{n+2k-1}{2})} t^{1-n-k}(1-t^2)_+^{\frac{n+2k-3}{2}}. \tag{3.14}$$

The generalized function $(1-t^2)_+^{\frac{n+2k-3}{2}}$ is defined in the paragraph after Theorem 3.4.2. Multiplying the right hand side of (3.14) by $\Gamma(s+k)/\Gamma(s-n+1) = (s-n+1)\cdots(s+k-1)$ amounts to replacing ν by $L_k(\nu)$, where $L_k := (-\frac{d}{dt}\cdot t-n+1)\cdots(-\frac{d}{dt}\cdot t+k-1)$. Observe that $L_k = t^{k-1}(-\frac{d}{dt})^{n+k-1}t^n$. Therefore $\mathfrak{M}\beta_k = (\mathfrak{M}\alpha_k)^{-1}$, where β_k is defined by (3.8). This proves Theorem 3.4.4.

In the case $n=2$, the formula (3.13) is well-known (cf. [68, Lemma 7.17], [16, Proposition 2.6.3]).

3.4.7 The non-K-finite situation

In this subsection we consider the situation where we remove K-finiteness from the definitions of $\mathcal{C}_+$ and $\mathcal{C}_-$. Let $\mathscr{C}_+$ be the space of smooth functions on $\mathbb{R}^n \setminus \{0\}$ with bounded support, and let $\mathscr{C}_-$ be the space of smooth functions on $(\mathbb{R}^n)^* \setminus \{0\}$ supported away from a neighborhood of 0.

We have the operator $\mathscr{R} : \mathscr{C}_+ \to \mathscr{C}_-$ defined by

$$\mathscr{R}f(\xi) = \int_{\langle \xi, x\rangle = 1} f(x) d\mu_\xi$$

(cf. formula (3.1)). One can deduce that $\mathscr{R}$ is injective from the injectivity of $R : \mathcal{C}_+ \to \mathcal{C}_-$. However we will show below that $\mathscr{R}$ is not surjective, and hence not an isomorphism.

Let $f \in \mathscr{C}_+$. Define $C_f = \mathrm{supp}(f) \cup \{0\}$, which is a compact subset of $\mathbb{R}^n$. Let $\widehat{C}_f$ denote its convex hull.

Let $C \subset \mathbb{R}^n$ be a convex set containing 0. Define $C^* \subset (\mathbb{R}^n)^*$ to be the set of ξ such that the hyperplane $\langle \xi, x \rangle = 1$ is disjoint from C. By convexity,

$$C^* = \{\xi \mid \langle \xi, x \rangle < 1 \text{ for all } x \in C\}.$$

Observe that C^* is a convex set[3] containing 0. If $\check{C} \subset (\mathbb{R}^n)^*$ is a convex set containing 0, one similarly defines the dual $\check{C}^* \subset \mathbb{R}^n$. Taking duals gives mutually inverse maps between the collection of compact convex subsets of $\mathbb{R}^n$ containing 0 and the collection of open convex subsets of $(\mathbb{R}^n)^*$ containing 0.

Proposition 3.4.7. *The connected component of* $(\mathbb{R}^n)^* \setminus \operatorname{supp}(\mathscr{R}f)$ *containing* 0 *is equal to* $(\widehat{C}_f)^*$*. In particular, it is convex.*

Corollary 3.4.8. *The operator* $\mathscr{R} : \mathscr{C}_+ \to \mathscr{C}_-$ *is not surjective.*

Lemma 3.4.9. *Let* $\xi_0 \in (\mathbb{R}^n)^* \setminus \{0\}$*. If* $\mathscr{R}f$ *vanishes on a neighborhood of the segment* $[0, \xi_0] := \{t\xi_0 \mid 0 \le t \le 1\}$*, then* f *vanishes on the half-space* $\langle \xi_0, x \rangle \ge 1$*.*

Proof. By replacing f by a compactly supported function that is equal to f outside of a small neighborhood of 0, we may assume that f is compactly supported. There exists an open convex neighborhood $\check{C}$ of $[0, \xi_0]$ such that $\mathscr{R}f$ vanishes on $\check{C}$. Then $C = \check{C}^*$ is a compact convex subset of $\mathbb{R}^n$ and $C^* = \check{C}$, so the integral of f along any hyperplane disjoint from C vanishes. Therefore [40, Corollary 2.8] implies that $\operatorname{supp} f \subset C$. Since $\xi_0 \in \check{C}$, one sees that C is contained in the half-space $\langle \xi_0, x \rangle < 1$. □

We have the support function $H : (\mathbb{R}^n)^* \to \mathbb{R}$ associated to C_f, which is defined by

$$H(\xi) = \sup\{\langle \xi, x \rangle \mid x \in C_f\}.$$

3. C^* is called [19] the dual (polar) set of C. Note that if C is compact, then C^* is open. If 0 is an interior point of C, then C^* is bounded.

For $\xi \neq 0$, the set $\{x \mid \langle \xi, x\rangle = H(\xi)\}$ is a supporting hyperplane of $\widehat{C}_f$. The function H uniquely determines the compact convex set $\widehat{C}_f$, and $(\widehat{C}_f)^* = H^{-1}(\mathbb{R}_{<1})$.

Proof of Proposition 3.4.7. It is clear that $H^{-1}(\mathbb{R}_{<1})$ is an open subset of $(\mathbb{R}^n)^* \setminus \mathrm{supp}(\mathscr{R}f)$. Note that since $\mathrm{supp}(\mathscr{R}f)$ is closed, Lemma 3.4.9 implies that if $H(\xi) = 1$ then $\xi \in \mathrm{supp}(\mathscr{R}f)$. Thus $(\widehat{C}_f)^* = H^{-1}(\mathbb{R}_{<1})$ is also closed in $(\mathbb{R}^n)^* \setminus \mathrm{supp}(\mathscr{R}f)$. □

3.5 F complex

In this section we prove the invertibility of R when $F = \mathbb{C}$. Recall that in this case $K = U(n)$. The inversion formula is given in Theorem 3.5.2, and a reformulation using the Mellin transform is given in Theorem 3.5.4. The K-finiteness of $\mathcal{C}$ plays a crucial role in the proofs, so we begin by recalling the classification of the K-isotypic components of $\mathcal{C}$.

3.5.1 Spherical harmonics

Let S^{2n-1} denote unit sphere of norm 1 vectors in $\mathbb{C}^n = \mathbb{R}^{2n}$, which has a natural action by $U(n)$. Let $\mathcal{C}(S^{2n-1})$ be the space of smooth K-finite functions on S^{2n-1}. For nonnegative integers p, q, let $H^{p,q}$ denote the homogeneous polynomials of degree $p+q$ on $\mathbb{R}^{2n}$ that are harmonic and satisfy

$$Y(\lambda z_1, \dots, \lambda z_n) = \lambda^p \overline{\lambda}^q Y(z_1, \dots, z_n)$$

for $\lambda \in \mathbb{C}, (z_1, \dots, z_n) \in \mathbb{C}^n = \mathbb{R}^{2n}$.

Theorem 3.5.1 ([42, Theorem 3.1], [47]). *Let $H^{p,q}|S^{2n-1}$ denote the space of harmonic polynomials restricted to S^{2n-1}. Then*

(i) $\mathcal{C}(S^{2n-1}) = \bigoplus_{p,q\geq 0} H^{p,q}|S^{2n-1}$ as $U(n)$-representations,

(ii) the $U(n)$-representations $H^{p,q}$ are irreducible and not isomorphic to each other.

3.5.2 Decomposing $\mathcal{C}$ into K-isotypes

We have a decomposition $\mathbb{C}^n \setminus \{0\} = \mathbb{R}_{>0} \times S^{2n-1}$, with $O(2n)$ (and hence $U(n)$) acting on the S^{2n-1} component. Let $\mathcal{C}(\mathbb{R}_{>0})$ denote the space of smooth functions on $\mathbb{R}_{>0}$ and define the subspaces $\mathcal{C}_\pm(\mathbb{R}_{>0}), \mathcal{C}_c(\mathbb{R}_{>0})$ as in §3.2.2.

Theorem 3.5.1 implies that there is a decomposition

$$\mathcal{C} = \bigoplus_{p,q \geq 0} \mathcal{C}(\mathbb{R}_{>0}) \otimes H^{p,q}.$$

For $u \in \mathcal{C}(\mathbb{R}_{>0})$ and $Y \in H^{p,q}$, we define $u \otimes Y \in \mathcal{C}$ by $(u \otimes Y)(x) := u(\|x\|) \cdot Y(\frac{x}{\|x\|})$.

3.5.3 Radon inversion formula

We have an isomorphism $\mathrm{Inv} : \mathcal{C}_- \to \mathcal{C}_+$ defined by

$$(\mathrm{Inv}\, \varphi)(x) = \|x\|^{-2n} \varphi\left(\frac{\overline{x}}{\|x\|^2}\right)$$

where $\overline{x}$ is coordinate-wise conjugation. Set $\widetilde{R} := \mathrm{Inv}^{-1} \circ M$. Consider $\mathbb{R}_{>0}$ as a subgroup of diagonal matrices in G. Then it follows from (3.2) that $\widetilde{R}$ is a $K \times \mathbb{R}_{>0}$ equivariant operator from $\mathcal{C}_+$ to $\mathcal{C}_+$.

Let A be the space of distributions on $\mathbb{R}_{>0}$ supported on $(0, 1]$ (see §3.4.3). The action of $\mathbb{R}_{>0}$ on $\mathcal{C}_+$ induces an action of A.

Theorem 3.5.2. *The operator $R : \mathcal{C}_+ \to \mathcal{C}_-$ is an isomorphism. For $\varphi \in \mathcal{C}_-(\mathbb{R}_{>0}) \otimes H^{p,q}$, the inverse $R^{-1} : \mathcal{C}_- \to \mathcal{C}_+$ is given by the formula*

$$R^{-1}\varphi = \beta_{p,q} * \mathrm{Inv}(\varphi)$$

where $\beta_{p,q}$ is the distribution on $\mathbb{R}_{>0}$ defined by

$$\beta_{p,q}(t) = \frac{1}{2^{n+m-2}\pi^{n-1}\Gamma(m)} \prod_{j=1}^{n+m-1} \left(-\frac{d}{dt} \cdot t + p + q - 2j\right) \left(t^{-p-q-2n+1}(1-t^2)_+^{m-1}\right) dt \tag{3.15}$$

where $m = \min(p, q)$.

The derivative $\frac{d}{dt}$ is applied in the sense of generalized functions. The generalized function $(1-t^2)_+^{\lambda}$ is defined by analytic continuation for $\lambda \in \mathbb{C}$ (see the paragraph following the statement of Theorem 3.4.2). In particular, the regularization of $\frac{2}{\Gamma(m)}(1-t^2)_+^{m-1}dt$ at $m = 0$ is equal to $\delta(1-t)$.

Corollary 3.5.3. *For any number N one has $R^{-1}(\mathcal{C}_{\leq -N}) \subset \mathcal{C}_{\geq N}$.*

Proof. Observe that $\beta_{p,q}$ is supported on $(0, 1]$ for all p, q. □

3.5.4 A formula for $\widetilde{R}$ in terms of convolution

We consider the dot product on $S^{2n-1} \subset \mathbb{C}^n$ induced by the dot product on $\mathbb{C}^n$. For $t \in (-1, 1)$, define $A_t : \mathcal{C}(S^{2n-1}) \to \mathcal{C}(S^{2n-1})$ such that $(A_t f)(x)$ is the average value of f on the $(2n-3)$-sphere $\{\omega \in S^{2n-1} \mid \omega \cdot \overline{x} = t\}$. Then A_t is $U(n)$-equivariant, so by Schur's lemma it acts on $H^{p,q}|S^{2n-1}$ by a scalar $a_{p,q}(t)$. One observes that $a_{p,q}$ is a smooth function on $(-1, 1)$, $|a_{p,q}(t)| \leq 1$ for all $t \in (-1, 1)$, and $\lim_{t\to 1} a_{p,q}(t) = 1$.

Suppose that $f \in \mathcal{C}(\mathbb{R}_{>0}) \otimes H^{p,q}$ and there exist $C > 0$ and $\sigma > n-1$ such that $|f(x)| \leq C\|x\|^{-2\sigma}$ for all x with $\|x\| \geq 1$. One can deduce as in the real case that

$$\widetilde{R}f = \alpha_{p,q} * f \tag{3.16}$$

where $\alpha_{p,q}$ is the measure $\mathrm{mes}(S^{2n-3}) \cdot t^{1-2n} \cdot a_{p,q}(t)(1-t^2)^{n-2}dt$ on the interval $(0, 1)$ extended by zero to the whole $\mathbb{R}_{>0}$. The convolution $\alpha_{p,q} * f$ is well-defined due to the bound on $|f(x)|$, and $\mathrm{mes}(S^{2n-3})$ denotes the surface area of the $(2n-3)$-sphere. By considering zonal spherical functions, one can check [69, Lemma 1.2] that $a_{p,q}(t)$ is the scalar multiple of the Jacobi polynomial $P^{(n-2,|p-q|)}_{\min(p,q)}(2t^2-1)$ normalized by $a_{p,q}(1)=1$.

The Mellin transform $\mathfrak{M}\alpha_{p,q}$ is defined for $s\in\mathbb{C}$ by integrating t^s against $\alpha_{p,q}$ if $\mathrm{Re}(s) > 2n-2$.

Theorem 3.5.4. *The distribution $\alpha_{p,q}$ is invertible in A. The inverse $\beta_{p,q}$ is defined by* (3.15)*. The Mellin transforms are given by*

$$\mathfrak{M}\beta_{p,q}(s) = \frac{1}{\mathfrak{M}\alpha_{p,q}(s)} = \pi^{1-n}\frac{\Gamma(\frac{s+p+q}{2})}{\Gamma(\frac{s+|p-q|}{2}-n+1)}\cdot\frac{\Gamma(\frac{s-p-q}{2}-n+1)}{\Gamma(\frac{s-|p-q|}{2}-n+1)}. \tag{3.17}$$

Theorem 3.5.4 implies Theorem 3.5.2.

3.5.5 Relation to the Fourier transform

Let $\mathcal{S}(\mathbb{C}^n)$ denote the space of Schwartz functions on $\mathbb{C}^n$ and $\mathcal{S}'(\mathbb{C}^n)$ the dual space of tempered distributions on $\mathbb{C}^n$. The Fourier transform is defined for an integrable function f on $\mathbb{C}^n$ by

$$\mathcal{F}f(\xi) = \int_{\mathbb{C}^n} f(x)e^{-2\pi i\,\mathrm{Re}(\xi\cdot\overline{x})}dx.$$

This definition coincides with the one from §3.4.5 by identifying $\mathbb{C}^n=\mathbb{R}^{2n}$. The Fourier transform can be extended to an isomorphism of tempered distributions $\mathcal{F}:\mathcal{S}'(\mathbb{C}^n)\to\mathcal{S}'(\mathbb{C}^n)$.

Let $F:\mathcal{S}'(\mathbb{C})\to\mathcal{S}'(\mathbb{C})$ denote the Fourier transform over $\mathbb{C}$. For $f\in\mathcal{S}(\mathbb{C}^n)$, one gets $\mathcal{F}f$ from the Radon transform by

$$\mathcal{F}f(r\omega) = F(\mathcal{R}f(\overline{\omega},t))(r) \tag{3.18}$$

where F is the Fourier transform with respect to the t variable, $r\in\mathbb{C}$, and $\omega\in S^{2n-1}$ is a unit vector. We have the following complex analog of Lemma 3.4.5, which is proved in exactly the same way.

Lemma 3.5.5. *Let f be a locally integrable function on $\mathbb{C}^n$ for which there exist $C > 0$ and $\sigma > n-1$ such that $|f(x)| \leq C\|x\|^{-2\sigma}$ for all x with $\|x\| \geq 1$. Then:*

(i) $\mathcal{R}f$ is a locally integrable function on $S^{2n-1} \times \mathbb{C}$.

(ii) $\mathcal{R}f(\omega, t)$ is bounded for $|t| \geq 1$.

(iii) The right hand side of (3.18) is well-defined as a generalized function on $\mathbb{C} \times S^{2n-1}$.

(iv) Equation (3.18) holds as an equality between generalized functions on $\mathbb{R}_{>0} \times S^{2n-1}$.

3.5.6 Proof of Theorem 3.5.4

Let $Y \in H^{p,q}$ and define $f(x) = \|x\|^{-s} \cdot Y(\frac{x}{\|x\|})$ for $s \in \mathbb{C}$. If $2n-2 < \mathrm{Re}(s) < 2n$, then f is locally integrable on $\mathbb{C}^n$ and satisfies the hypothesis of Lemma 3.5.5. Moreover by (3.16) and homogeneity of f we see that

$$\mathcal{R}f(\overline{\omega}, t) = t^p\bar{t}^q|t|^{2n-2-p-q-s}\widetilde{R}f(\omega) = t^p\bar{t}^q|t|^{2n-2-p-q-s}\mathfrak{M}\alpha_{p,q}(s)Y(\omega)$$

as a locally integrable function on $S^{2n-1} \times \mathbb{C}$. Then Lemma 3.5.5 implies that

$$\mathcal{F}f(r\omega) = F(t^p\bar{t}^q|t|^{2n-2-p-q-s})(r)\mathfrak{M}\alpha_{p,q}(s)Y(\omega)$$

as generalized functions on $\mathbb{R}_{>0} \times S^{2n-1}$.

Lemma 3.5.6. *If $2n-2 < \mathrm{Re}(s) < 2n$, then*

$$F(t^p\bar{t}^q|t|^{2n-2-p-q-s})(r) = \pi^{s-2n+1}i^{-|p-q|}\frac{\Gamma(\frac{2n+|p-q|-s}{2})}{\Gamma(\frac{s-2n+2+|p-q|}{2})}r^p\bar{r}^q|r|^{s-2n-p-q}$$

as locally integrable functions on $\mathbb{C}$.

Proof. Apply Theorem 3.4.6 for $n = 2$, $k = |p-q|$, and $Y(x_1, x_2) = (x_1 + ix_2)^{p-q}$ if $p \geq q$ or $Y(x_1, x_2) = (x_1 - ix_2)^{q-p}$ if $p \leq q$. $\square$

Alternatively, we can use Theorem 3.4.6 to find that $\mathcal{F}f(x) = \gamma\|x\|^{s-2n}Y(\frac{x}{\|x\|})$, where

$\gamma = i^{-p-q}\pi^{s-n}\Gamma(\frac{2n+p+q-s}{2})/\Gamma(\frac{s+p+q}{2})$. Comparing the two formulas we have derived for $\mathcal{F}f$ and applying Euler's reflection formula, we conclude that

$$\mathfrak{M}\alpha_{p,q}(s) = \pi^{n-1}\frac{\Gamma(\frac{s+|p-q|}{2} - n + 1)\Gamma(\frac{s-|p-q|}{2} - n + 1)}{\Gamma(\frac{s+p+q}{2})\Gamma(\frac{s-p-q}{2} - n + 1)}, \tag{3.19}$$

as stated in Theorem 3.5.4. The equation holds *a priori* for $2n-2 < \mathrm{Re}(s) < 2n$, and we deduce by analytic continuation that it holds for all $s \in \mathbb{C}$, away from poles.

To finish the proof of Theorem 3.5.4, it remains to show that $(\mathfrak{M}\beta_{p,q})^{-1}$ is equal to the right hand side of (3.19). By considering the Beta function we see that $\Gamma(\frac{s-p-q}{2} - n + 1)/\Gamma(\frac{s-|p-q|}{2} - n + 1)$ is the Mellin transform of $\nu(t)dt$, where

$$\nu(t) = \frac{2}{\Gamma(m)}t^{-p-q-2n+1}(1-t^2)_+^{m-1} \tag{3.20}$$

for $m = \min(p,q)$. Note that if $m = 0$, then $\nu(t)dt = \delta(1-t)$. Multiplying the right hand side of (3.20) by $\Gamma(\frac{s+p+q}{2})/\Gamma(\frac{s+|p-q|}{2} - n + 1) = \prod_{j=1}^{n+m-1}(\frac{s+p+q}{2} - j)$ amounts to replacing ν by $L_{p,q}(\nu)$, where $L_{p,q}$ is the differential operator

$$2^{1-n-m}\prod_{j=1}^{n+m-1}\left(-\frac{d}{dt}\cdot t + p + q - 2j\right).$$

Theorem 3.5.4 is proved.

In the case $n = 2$, the formula (3.19) is well-known (cf. [68, Lemma 7.23], [28, Proposition III.3.7]).

APPENDIX A

GEOMETRIC REPRESENTATION THEORY

A.1 Substacks of the Hecke stack

To keep the notation consistent throughout this chapter, in this section M denotes an arbitrary connected split reductive group over a perfect field k.

We will attach to any algebraic normal irreducible monoid $\tilde{M}$ with group of units M a substack of a symmetrized version of the Hecke stack. This substack is the global model for the formal arc space of the embedding $M \hookrightarrow \tilde{M}$, and it was also considered in [11, §2]. We are particularly interested in the case when M is the Levi factor of a parabolic subgroup of G and $\tilde{M} = \overline{M}$ is the closure of the M-orbit of the coset U in $\overline{G/U}$. The monoid $\overline{M}$ is studied in detail in Chapter 1.

We recall the relation between the Hecke stack and the Beilinson–Drinfeld Grassmannian. We use a symmetrized factorizable version of the affine Grassmannian. A detailed exposition on the factorizable version of the affine Grassmannian over the Ran space can be found in [71].

A.1.1 Recollections on normal reductive monoids

Let $\tilde{M}$ be an algebraic normal irreducible monoid with group of units M.

Fix a maximal torus $T \subset M$. The *Renner cone* $\check{C} \subset \check{\Lambda}^{\mathbb{Q}}$ of $\tilde{M}$ is the rational convex cone corresponding by [45] to the closure of T in $\tilde{M}$ after base changing to an algebraic closure of k. This cone is stable under the actions of W_M and $\mathrm{Gal}(\bar{k}/k)$. The Renner cone is canonical and only depends on the abstract Cartan of M (which identifies with T after choosing a Borel subgroup). L. Renner showed in [56, Theorem 5.4] that algebraic normal irreducible monoids with group of units M bijectively correspond (via the Renner cone) to convex rational polyhedral cones generating $\check{\Lambda}^{\mathbb{Q}}$ as a vector space and stable under the actions of W_M and $\mathrm{Gal}(\bar{k}/k)$.

Since M is scheme-theoretically dense in $\tilde{M}$, the restriction functor

$$\mathrm{Rep}(\tilde{M}) \to \mathrm{Rep}(M)$$

is fully faithful, so we may consider $\mathrm{Rep}(\tilde{M})$ as a full subcategory of $\mathrm{Rep}(M)$.

We will consider the algebraic stack $M\backslash\tilde{M}/M$ which sends a test scheme S to the groupoid of pairs of M-bundles $\mathcal{F}_M^1, \mathcal{F}_M^2$ on S equipped with a section

$$\beta_M : S \to \tilde{M} \overset{M \times M}{\times} (\mathcal{F}_M^1 \underset{S}{\times} \mathcal{F}_M^2).$$

Such a section β_M will be called an $\tilde{M}$-morphism from $\mathcal{F}_M^2$ to $\mathcal{F}_M^1$. By the Tannakian formalism, giving an $\tilde{M}$-morphism β_M is the same as giving a collection of assignments

$$V \in \mathrm{Rep}(\tilde{M}) \rightsquigarrow \beta_M^V : V_{\mathcal{F}_M^2} \to V_{\mathcal{F}_M^1}$$

where β_M^V is $\mathcal{O}_S$-linear, and the Plücker relations hold. This means that for V being the trivial representation, β_M^V is the identity map $\mathcal{O}_S \to \mathcal{O}_S$, and for an M-morphism $V_1 \otimes V_2 \to V_3$, the diagram

$$\begin{tikzcd}
(V_1 \otimes V_2)_{\mathcal{F}_M^2} \arrow[r, "\beta_M^{V_1} \otimes \beta_M^{V_2}"] \arrow[d] & (V_1 \otimes V_2)_{\mathcal{F}_M^1} \arrow[d] \\
(V_3)_{\mathcal{F}_M^2} \arrow[r, "\beta_M^{V_3}"] & (V_3)_{\mathcal{F}_M^1}
\end{tikzcd}$$

commutes. Observe that the Plücker relations imply that the assignments β_M^V are functorial in V.

Observe that the fiber bundle $M \times^{M \times M} (\mathcal{F}_M^1 \times_S \mathcal{F}_M^2)$ canonically identifies with the scheme of isomorphisms $\mathrm{Isom}(\mathcal{F}_M^2, \mathcal{F}_M^1)$ over S. In other words, an M-morphism is an isomorphism of M-bundles. Therefore the stack $M\backslash\tilde{M}/M$ contains the open substack $M\backslash M/M$, which canonically identifies with the classifying stack $\mathbb{B}M = M\backslash\mathrm{pt}$ of M.

A.1.2 Definition of $\tilde{\mathcal{H}}_M$

Let X be a smooth projective geometrically connected curve over a field k. We consider the mapping stack $\mathrm{Maps}(X, M\backslash\tilde{M}/M)$ whose value on a test scheme S is the groupoid of all maps $X \times S \to M\backslash\tilde{M}/M$. Such a map is said to be non-degenerate if the preimage of $M\backslash M/M \subset M\backslash\tilde{M}/M$ is a subset of $X \times S$ whose projection on S is surjective. We will denote

$$\tilde{\mathcal{H}}_M := \mathrm{Maps}^\circ(X, M\backslash\tilde{M}/M) \subset \mathrm{Maps}(X, M\backslash\tilde{M}/M)$$

the open substack consisting of non-degenerate maps. A map $X \times S \to M\backslash\tilde{M}/M$ is the datum of a pair of M-bundles $\mathcal{F}^1_M, \mathcal{F}^2_M$ on $X \times S$ and an $\tilde{M}$-morphism $\beta_M : \mathcal{F}^2_M \to \mathcal{F}^1_M$ between them. In the Tannakian language, β_M is non-degenerate if and only if for every geometric point $s \to S$, the restriction of β^V_M to the fiber $X \times s$ is generically an isomorphism. The last condition is equivalent to requiring that β^V_M is an embedding of coherent sheaves such that the quotient is S-flat.

Relation to the full Hecke stack. The projection $M \to M/[M,M]$ induces an inclusion $\check{\Lambda}_{M/[M,M]} \subset \check{\Lambda}$. Recall that $\check{C}$ denotes the Renner cone of $\tilde{M}$, which is a W_M-stable convex polyhedral cone that generates $\check{\Lambda}^{\mathbb{Q}}$ as a group. If $M/[M,M]$ is not finite (i.e., M is not semisimple), then there exists a character $\check{\lambda} \in \check{\Lambda}$ that lies in the interior of the cone $\check{C} \cap \check{\Lambda}^{\mathbb{Q}}_{M/[M,M]}$. If M is semisimple, we put $\check{\lambda} = 0$ (in this case $\tilde{M}$ must equal M).

Considering $\check{\lambda}$ as a homomorphism $\tilde{M} \to \mathbb{A}^1$, the open subscheme $\check{\lambda}^{-1}(\mathbb{G}_m)$ coincides with $M \subset \tilde{M}$. The closed subscheme $\check{\lambda}^{-1}(0)$ has the same reduced scheme structure as $(\tilde{M} - M)_{\mathrm{red}}$.

Let Div_+ denote the scheme of relative effective divisors of X. The map $\check{\lambda} : \tilde{M} \to \mathbb{A}^1$ induces a map

$$\tilde{\mathcal{H}}_M \to \mathrm{Div}_+ . \tag{A.1}$$

More explicitly, an S-point $X \times S \to M\backslash\tilde{M}/M$ is sent to the preimage of $M\backslash\check{\lambda}^{-1}(0)/M$, which is a relative effective divisor of $X \times S$.

Define the stack $\mathrm{Hecke}(M)_{\mathrm{Div}_+}$ as follows: its S-points are quadruples $(D, \mathcal{F}^1_M, \mathcal{F}^2_M, \beta_M)$ where D is a relative effective divisor on $X \times S$, $\mathcal{F}^1_M$ and $\mathcal{F}^2_M$ are two M-bundles on $X \times S$, and β_M is an isomorphism of M-bundles on the restrictions

$$\mathcal{F}^2_M|_{X \times S - D} \cong \mathcal{F}^1_M|_{X \times S - D}.$$

From this definition it is evident that we have a closed embedding of stacks

$$\tilde{\mathcal{H}}_M \hookrightarrow \mathrm{Hecke}(M)_{\mathrm{Div}_+}.$$

Remark A.1.1. The above inclusion slightly depends on the choice of $\check{\lambda}$, which is used to define (A.1). This choice goes away if we consider Ran versions of the corresponding Hecke stacks, since a point of $\mathrm{Hecke}(M)_{\mathrm{Div}_+}$ only depends on the reduced structure D_{red} of the divisor D.

We let $\overleftarrow{h}, \overrightarrow{h}$ denote the two forgetful maps $\tilde{\mathcal{H}}_M \to \mathrm{Bun}_M$. We use the convention where $\overleftarrow{h}$ is the map corresponding to $\mathcal{F}^1_M$.

Proposition A.1.2. *The morphism $\tilde{\mathcal{H}}_M \to \mathrm{Bun}_M \times \mathrm{Bun}_M$ is schematic, quasi-affine, and of finite presentation.*

Proof. Consider a test scheme S and a map $S \to \mathrm{Bun}_M \times \mathrm{Bun}_M$ corresponding to M-bundles $\mathcal{F}^1_M, \mathcal{F}^2_M$ on $X \times S$. Then the fiber product $\tilde{\mathcal{H}}_M \underset{\mathrm{Bun}_M \times \mathrm{Bun}_M}{\times} S$ is isomorphic to an open subspace of the space of sections of the map

$$\tilde{M} \overset{M \times M}{\times} (\mathcal{F}^1_M \underset{X \times S}{\times} \mathcal{F}^2_M) \to X \times S.$$

This map is affine and of finite presentation because $\tilde{M}$ is, which implies that the space of sections is representable by a scheme affine and of finite presentation over S. □

The monoid structure on $\tilde{M}$ allows us to compose two $\tilde{M}$-morphisms $\mathcal{F}^2_M \to \mathcal{F}^1_M$ and

$\mathcal{F}_M^3 \to \mathcal{F}_M^2$ to get an $\tilde{M}$-morphism $\mathcal{F}_M^3 \to \mathcal{F}_M^1$. The composition of two non-degenerate morphisms is non-degenerate. This defines a map

$$\mathrm{comp} : \tilde{\mathcal{H}}_M \underset{\overrightarrow{h}, \mathrm{Bun}_M, \overleftarrow{h}}{\times} \tilde{\mathcal{H}}_M \to \tilde{\mathcal{H}}_M.$$

Remark A.1.3. The map (A.1) takes the composition of $\tilde{M}$-morphisms into the sum of effective divisors (which is a proper map $\mathrm{Div}_+ \times \mathrm{Div}_+ \to \mathrm{Div}_+$). Hence Proposition A.1.6 below implies that comp is a proper map.

Lemma A.1.4. *Let $\mathcal{F}_M \in \mathrm{Bun}_M(S)$ for a k-scheme S. Any non-degenerate $\tilde{M}$-morphism $\beta_M : \mathcal{F}_M \to \mathcal{F}_M$ is an M-bundle automorphism.*

Proof. Let $V \in \mathrm{Rep}(\tilde{M})$. Then $\beta_M^V : V_{\mathcal{F}_M} \to V_{\mathcal{F}_M}$ is generically an isomorphism on geometric fibers of $X \times S$. The same is true for $\det(\beta_M^V)$, and $\Gamma(X, \mathcal{O}_X) = k$ implies that β_M^V is an isomorphism. Since M is the group of units of $\tilde{M}$, we conclude that β_M is an automorphism of $\mathcal{F}_M$. □

Corollary A.1.5. *Suppose there exist $\tilde{M}$-morphisms $\beta_M : \mathcal{F}_M^2 \to \mathcal{F}_M^1$ and $\beta'_M : \mathcal{F}_M^1 \to \mathcal{F}_M^2$. Then β_M, β'_M are both isomorphisms.*

A.1.3 Affine Grassmannians

In this section, we explain the relation between the Hecke stack and the Beilinson–Drinfeld affine Grassmannian. We refer the reader to [71] for a far more complete treatment of the affine Grassmannian.

For this purpose we introduce the divisor version of the jet group $M(\mathfrak{o})$. For $D \in \mathrm{Div}_+(S)$, let $\hat{D}$ denote the formal completion of D in $X \times S$ (which is a formal scheme). Define

$$M(\mathfrak{o})_{\mathrm{Div}_+} = \{(D, \gamma) \mid D \in \mathrm{Div}_+(S),\ \gamma \in M(\hat{D})\},$$

which is representable by a scheme affine over Div_+ (cf. [71, Proposition 3.1.6]).

We define the symmetrized version of the Beilinson-Drinfeld affine Grassmannian as

$$\mathrm{Gr}_{M,\mathrm{Div}_+} := \mathrm{Spec}(k) \underset{\mathcal{F}^0_M, \mathrm{Bun}_M, \overleftarrow{h}}{\times} \mathrm{Hecke}(M)_{\mathrm{Div}_+}$$

where $\mathcal{F}^0_M \in \mathrm{Bun}_M(k)$ is the trivial bundle.

The Hecke stack $\mathrm{Hecke}(M)_{\mathrm{Div}_+}$ can be regarded as a twisted product $\mathrm{Bun}_M \tilde{\times} \mathrm{Gr}_{M,\mathrm{Div}_+}$. More precisely, consider the stack

$$\mathcal{Y} = \{(D, \mathcal{F}^1_M, \tilde{\gamma}) \mid D \in \mathrm{Div}_+,\, \mathcal{F}^1_M \in \mathrm{Bun}_M,\, \tilde{\gamma} : \mathcal{F}^0_M|_{\hat{D}} \cong \mathcal{F}^1_M|_{\hat{D}}\}.$$

Then $\mathcal{Y} \to \mathrm{Div}_+ \times \mathrm{Bun}_M$ is a $M(\mathfrak{o})_{\mathrm{Div}_+}$-torsor. Observe that the group scheme $M(\mathfrak{o})_{\mathrm{Div}_+}$ also acts on $\mathrm{Gr}_{M,\mathrm{Div}_+}$ over Div_+. We have a canonical isomorphism

$$\mathrm{Hecke}(M)_{\mathrm{Div}_+} \cong \mathcal{Y} \overset{M(\mathfrak{o})_{\mathrm{Div}_+}}{\times} \mathrm{Gr}_{M,\mathrm{Div}_+},$$

where on the right hand side we take the fiber product $\mathcal{Y} \times_{\mathrm{Div}_+} \mathrm{Gr}_{M,\mathrm{Div}_+}$ and quotient by the anti-diagonal action of $M(\mathfrak{o})_{\mathrm{Div}_+}$.

Proposition A.1.6. *The morphism $\tilde{\mathcal{H}}_M \to \mathrm{Bun}_M \times \mathrm{Div}_+$ is proper, where $\tilde{\mathcal{H}}_M$ maps to Bun_M by either $\overleftarrow{h}$ or $\overrightarrow{h}$.*

Proof. Without loss of generality we will consider the projection $\overleftarrow{h} : \tilde{\mathcal{H}}_M \to \mathrm{Bun}_M$. Let $\widetilde{\mathrm{Gr}}_{M,\mathrm{Div}_+} := \mathrm{Spec}(k) \times_{\mathcal{F}^0_M, \mathrm{Bun}_M} \tilde{\mathcal{H}}_M$, which is a closed subspace of $\mathrm{Gr}_{M,\mathrm{Div}_+}$. Choose a uniformizer $\varpi_v \in \mathfrak{o}_v$ at a place v. Let $C \subset \Lambda^{\mathbb{Q}}$ denote the dual of the Renner cone of $\tilde{M}$. Over a divisor $n_v \cdot v$ supported at a single point, the fiber of $\widetilde{\mathrm{Gr}}_{M,\mathrm{Div}_+}$ is equal to the union of the orbits

$$M(\mathfrak{o}_v) \cdot \theta_v(\varpi_v) \subset (\tilde{M}(\mathfrak{o}_v) \cap M(F_v))/M(\mathfrak{o}_v)$$

for $\theta_v \in C \cap \Lambda^+_M$ satisfying $\langle \check{\lambda}, \theta_v \rangle = n_v$, where $\check{\lambda}$ is the character chosen in §A.1.2. Since $\check{\lambda}$ lies in the interior of $\check{C} \cap \check{\Lambda}^{\mathbb{Q}}_{M/[M,M]}$, there are only finitely many θ_v such that $\langle \check{\lambda}, \theta_v \rangle = n_v$.

We deduce, by factorization, that $\widetilde{\mathrm{Gr}}_{M,\mathrm{Div}_+}$ is representable by a scheme of finite type over Div_+. It is known that $\mathrm{Gr}_{M,\mathrm{Div}_+}$ is ind-proper over Div_+ (cf. [71, Remark 3.1.4]). Thus $\widetilde{\mathrm{Gr}}_{M,\mathrm{Div}_+}$ is proper over Div_+. The Proposition follows by considering $\tilde{\mathcal{H}}_M$ as a twisted product $\mathrm{Bun}_M \tilde{\times} \widetilde{\mathrm{Gr}}_{M,\mathrm{Div}_+}$ as explained in §A.1.3. □

A.1.4 Slope comparisons

Let $\pi_1(M)$ denote the quotient of Λ by the subgroup generated by coroots of M. It is well-known that there is a bijection $\deg_M : \pi_0(\mathrm{Bun}_M) \simeq \pi_1(M)$. Note that $\pi_1(M)\otimes\mathbb{Q} = \Lambda^{\mathbb{Q}}_{M/[M,M]} = \Lambda^{\mathbb{Q}}_{Z_0(M)}$. We call the composition

$$\mathrm{Bun}_M \to \pi_1(M) \to \Lambda^{\mathbb{Q}}_{Z_0(M)}$$

the *slope* map. Its fibers are not necessary connected but have finitely many connected components. The slope map coincides with the composition

$$\mathrm{Bun}_M \to \mathrm{Bun}_{M/[M,M]} \to \pi_0(\mathrm{Bun}_{M/[M,M]}) = \Lambda_{M/[M,M]} \subset \Lambda^{\mathbb{Q}}_{Z_0(M)}.$$

Following the notation of [25, Theorem 7.4.3], let $\mathrm{Bun}_M^{(\lambda)}$, $\lambda \in \Lambda_M^{+,\mathbb{Q}}$ denote the quasi-compact locally closed reduced substack of M-bundles with Harder-Narasimhan coweight λ. We refer the reader to [25, §7] and [61] for statements and proofs of the main results of reduction theory for a general reductive group.

Lemma A.1.7. *Suppose $\mathcal{F}_M^1 \in \mathrm{Bun}_M^{(\lambda_1)}(k)$, $\mathcal{F}_M^2 \in \mathrm{Bun}_M^{(\lambda_2)}(k)$ for $\lambda_1, \lambda_2 \in \Lambda_M^{+,\mathbb{Q}}$, and there exists an isomorphism $\beta_M : \mathcal{F}_M^2|_{X-D} \to \mathcal{F}_M^1|_{X-D}$ for a divisor $D \subset X$. At each closed point $v \in D$, the restriction of β_M to F_v determines a coweight $\lambda_v \in \Lambda_M^+ = K_{M,v}\backslash M(F_v)/K_{M,v}$. Then*

$$w_0^M \sum_v n_v \cdot \lambda_v \leq_M^{\mathbb{Q}} \lambda_1 - \lambda_2 \leq_M^{\mathbb{Q}} \sum_v n_v \cdot \lambda_v, \tag{A.2}$$

where $n_v = \dim_k(k_v)$ is the dimension of the residue field of v.

Proof. Consider the Harder–Narasimhan flag of $\mathcal{F}^1_M$: this is a canonical reduction of $\mathcal{F}^1_M$ to a Q-bundle $\mathcal{F}^1_Q$, where $Q \subset M$ is the parabolic subgroup corresponding to the Harder–Narasimhan coweight λ_1. Recall that a reduction of $\mathcal{F}^2_M$ to Q is the same as a section of the proper map $\mathcal{F}^2_M/Q := \mathcal{F}^2_M \times^M M/Q \to X$. The reduction $\mathcal{F}^1_Q$ and the isomorphism β_M determine a section $X - D \to \mathcal{F}^2_M/Q$. By properness, this extends to a reduction $\mathcal{F}^2_Q$ of $\mathcal{F}^2_M$ with an isomorphism

$$\beta_Q : \mathcal{F}^2_Q|_{X-D} \to \mathcal{F}^1_Q|_{X-D}$$

inducing β_M. Let L denote the Levi quotient of Q, and let $\mathcal{F}^1_L, \mathcal{F}^2_L, \beta_L$ denote the corresponding induced objects. Then the restriction of β_L to $\operatorname{Spec} F_v$ for $v \in D$ determines a coweight $\nu_v \in \Lambda^+_L$ via the quotient map $Q \to L$. Since ν_v and λ_v are both induced by β_Q, we see that $\nu_v(\varpi_v)U_Q(F_v) \cap K_{M,v}\lambda_v(\varpi_v)K_{M,v} \neq \emptyset$ where $\varpi_v \in \mathfrak{o}_v$ is a uniformizer. This implies that ν_v is contained in the convex hull of $W_M \cdot \lambda_v$ (cf. [17, p. 148]).

Let $\nu_2 \in \Lambda^{\mathbb{Q}}_{L/[L,L]}$ be the slope of $\mathcal{F}^2_L$. Then $\lambda_1 - \nu_2$ is equal to the image of $\sum_{v \in D} n_v \cdot \nu_v$ under the projection $\Lambda^{\mathbb{Q}} \to \Lambda^{\mathbb{Q}}_{L/[L,L]} = \Lambda^{\mathbb{Q}}_{Z_0(L)}$. In particular, $\lambda_1 - \nu_2$ lies in the convex hull of the W_M-orbit of $\sum_{v \in D} n_v \cdot \nu_v$, so $\lambda_1 - \nu_2 \leq^{\mathbb{Q}}_M \sum n_v \cdot \lambda_v$. By the comparison theorem [61, Theorem 4.5.1], $\nu_2 \leq^{\mathbb{Q}}_M \lambda_2$. We have shown the second inequality in (A.2). Switching $\mathcal{F}^1_M, \mathcal{F}^2_M$ and considering β_M^{-1} proves the first inequality by symmetry. $\square$

Corollary A.1.8. *Suppose $\mathcal{F}^1_M \in \operatorname{Bun}^{(\lambda_1)}_M(k), \mathcal{F}^2_M \in \operatorname{Bun}^{(\lambda_2)}_M(k)$ for $\lambda_1, \lambda_2 \in \Lambda^{+,\mathbb{Q}}_M$, and there exists a non-degenerate $\tilde{M}$-morphism $\beta_M : \mathcal{F}^2_M \to \mathcal{F}^1_M$. Let $C \subset \Lambda^{\mathbb{Q}}$ be the dual of the Renner cone of $\tilde{M}$. Then $\lambda_1 - \lambda_2$ belongs to the rational convex cone generated by C and $\Lambda^{\mathrm{pos},\mathbb{Q}}_M$.*

Proof. Since β_M is non-degenerate, there exists a divisor $D \subset X$ such that $\beta_M|_{X-D}$ is an isomorphism. For $v \in D$, the coweight λ_v defined in Lemma A.1.7 corresponds to an element in $K_{M,v}\backslash(M(F_v) \cap \tilde{M}(\mathfrak{o}_v))/K_{M,v}$. By the classification of double orbits of $M(F_v) \cap \tilde{M}(\mathfrak{o}_v)$, the latter set identifies with $C \cap \Lambda^+_M$. Therefore the W_M-stable cone C contains $w_0^M \sum n_v \cdot \lambda_v$. The corollary follows from Lemma A.1.7. $\square$

A.1.5 The stack $\mathcal{H}_M^+$

Let P be a standard parabolic subgroup of G with Levi factor M. We recall that G/U is quasi-affine. Let $\overline{G/U}$ denote the affine closure. We embed $M \hookrightarrow G/U$ by $m \mapsto mU$ and define $\overline{M}$ to be the closure of M in $\overline{G/U}$. Then $\overline{M}$ is a monoid acting on $\overline{G/U}$ (see Chapter 1).

We now specialize the above discussion of Hecke stacks to the case $\tilde{M} = \overline{M}$. Let

$$\mathcal{H}_M^+ = \mathrm{Maps}^\circ(X, M\backslash\overline{M}/M) \tag{A.3}$$

denote the stack studied above.

Remark A.1.9. By Lemma 1.3.4, the dual of the Renner cone of $\overline{M}$ equals $\Lambda_U^{\mathrm{pos},\mathbb{Q}}$. Therefore if we are in the setting of Corollary A.1.8, the rational cone generated by $\Lambda_U^{\mathrm{pos},\mathbb{Q}}$ and $\Lambda_M^{\mathrm{pos},\mathbb{Q}}$ is $\Lambda_G^{\mathrm{pos},\mathbb{Q}}$, and the corollary implies that $\lambda_1 - \lambda_2 \in \Lambda_G^{\mathrm{pos},\mathbb{Q}}$, i.e., $\lambda_2 \leq_G^{\mathbb{Q}} \lambda_1$.

The rational cone generated by $\Lambda_U^{\mathrm{pos},\mathbb{Q}}$ and $-\Lambda_M^{\mathrm{pos},\mathbb{Q}}$ is $w_0^M\Lambda_G^{\mathrm{pos},\mathbb{Q}}$, and Lemma A.1.7 also implies that $\lambda_1 - \lambda_2 \in w_0^M\Lambda_G^{\mathrm{pos},\mathbb{Q}}$.

Remarks on G. We assume for the rest of this Appendix that G has a simply connected derived group $[G,G]$. The reader may refer to [61, §7] for how to remove this hypothesis, and the relevant geometry remains the same.

The graded Ran space. Let $\Lambda_{G,P} := \pi_1(M)$ denote the quotient of Λ by the subgroup generated by the coroots of M. We have a natural projection $\Lambda \to \Lambda_{G,P}$. Let $\Lambda_{G,P}^{\mathrm{pos}}$ denote the submonoid of $\Lambda_{G,P}$ generated by the image of the positive coroots of G.

Any $\theta \in \Lambda_{G,P}^{\mathrm{pos}}$ can be uniquely written as a sum $\sum n_j\alpha_j$ for $j \in \Gamma_G - \Gamma_M$. Let X^θ denote $\prod X^{(n_j)}$. Define $\mathrm{Ran}(X, \Lambda_{G,P}^{\mathrm{pos}})$ to be the disjoint union of X^θ for $\theta \in \Lambda_{G,P}^{\mathrm{pos}}$. Here we are using the notation of [29], but we include $0 \in \Lambda_{G,P}^{\mathrm{pos}}$ with $X^0 = \mathrm{Spec}(k)$. We can regard $\mathrm{Ran}(X, \Lambda_{G,P}^{\mathrm{pos}})$ as the scheme of $\Lambda_{G,P}^{\mathrm{pos}}$-colored divisors on X, which is a $\Lambda_{G,P}^{\mathrm{pos}}$-graded version of the Ran space. The grading allows us to use the language of factorization algebras graded by a monoid introduced in [29, §2], which is slightly simpler than the more general set-up of factorization algebras from [8] (the difference is that we can replace the Ran space with genuine schemes).

Let us review the construction of the map $\mathcal{H}_M^+ \to \mathrm{Ran}(X, \Lambda_{G,P}^{\mathrm{pos}})$, following [62, §3.1.7]. Recall from [13] that the quotient $G/[P,P]$ is strongly quasi-affine, and let $\overline{G/[P,P]}$ denote its affine closure. Let $\overline{M/[M,M]}$ denote the closure of $M/[M,M]$ in $\overline{G/[P,P]}$ under the natural embedding

$$M/[M,M] = P/[P,P] \hookrightarrow G/[P,P] \subset \overline{G/[P,P]}.$$

The projection $G/U \to G/[P,P]$ extends to a map of affine closures $\overline{G/U} \to \overline{G/[P,P]}$, and therefore the projection $M \to M/[M,M]$ extends to a map $\overline{M} \to \overline{M/[M,M]}$. This induces a map of stacks

$$\mathcal{H}_M^+ \to \mathrm{Maps}^\circ(X, \overline{M/[M,M]}/(M/[M,M])). \tag{A.4}$$

Consider $\check{\Lambda}_{M/[M,M]}$ as a sub-lattice of $\check{\Lambda}_M$. Then one can check that $k[\overline{M/[M,M]}]$ has a basis consisting of the characters in the submonoid $\check{\Lambda}_G^+ \cap \check{\Lambda}_{M/[M,M]}$. Since $[G,G]$ is assumed to be simply connected, $\Lambda_{G,P}^{\mathrm{pos}}$ is the monoid dual to $\check{\Lambda}_G^+ \cap \check{\Lambda}_{M/[M,M]}$. We deduce that the right hand side of (A.4) is isomorphic to the scheme $\mathrm{Ran}(X, \Lambda_{G,P}^{\mathrm{pos}})$. Thus (A.4) becomes a map of stacks

$$\mathcal{H}_M^+ \to \mathrm{Ran}(X, \Lambda_{G,P}^{\mathrm{pos}}).$$

For $\theta \in \Lambda_{G,P}^{\mathrm{pos}}$, denote the preimage of the connected component $X^\theta \subset \mathrm{Ran}(X, \Lambda_{G,P}^{\mathrm{pos}})$ by $\mathcal{H}_{M,X^\theta}^+$.

Remark A.1.10. We defined the map (A.4) "group-theoretically" following [62, §3.1.7]. One can also define this map using the Tannakian formalism, which is essentially done in [13, 12].

We use the character $2\check{\rho}_P = 2\check{\rho} - 2\check{\rho}_M \in \check{\Lambda}_{M/[M,M]}$ to define the map (A.1), which is then equal to the composition

$$\mathcal{H}_M^+ \to \mathrm{Ran}(X, \Lambda_{G,P}^{\mathrm{pos}}) \to \mathrm{Div}_+,$$

where the last map sends X^θ to the symmetric power $X^{(2|\theta|)} \subset \mathrm{Div}_+$ for $|\theta| := \langle \check{\rho}_P, \theta \rangle$.

As in §A.1.3, we can express $\mathcal{H}^+_{X^\theta}$ as a twisted product

$$\mathcal{H}^+_{M,X^\theta} \cong \mathrm{Bun}_M \tilde{\times} \mathrm{Gr}^+_{M,X^\theta} \tag{A.5}$$

where $\mathrm{Gr}^+_{M,X^\theta} := \mathrm{Spec}(k) \times_{\mathcal{F}^0_M, \mathrm{Bun}_M, \overleftarrow{h}} \mathcal{H}^+_{M,X^\theta}$, using the action of the jet group $M(\mathfrak{o})_{X^{(|\theta|)}}$. We will always consider the twisted product with respect to the projection $\overleftarrow{h}$.

By a partition $\mathfrak{A}$ of θ we mean a decomposition

$$\theta = \sum_{\lambda \in \Lambda^{\mathrm{pos}}_{G,P} - 0} n_\lambda \cdot \lambda,\ n_\lambda \in \mathbb{Z}_+.$$

Let $\mathcal{P}_\theta$ denote the set of all partitions of θ. For a partition $\mathfrak{A} \in \mathcal{P}_\theta$, let $X^{\mathfrak{A}} := \prod_\lambda X^{(n_\lambda)}$. This is a scheme of dimension $|\mathfrak{A}| := \sum n_\lambda$. Note that there is a natural map $X^{\mathfrak{A}} \to X^\theta$.

Let $(X^{\mathfrak{A}})_{\mathrm{disj}} \subset X^{\mathfrak{A}}$ denote the open subscheme with all diagonals removed. A k-point $x^{\mathfrak{A}} \in (X^{\mathfrak{A}})_{\mathrm{disj}}$ is a formal sum $\sum_{v \in |X|} \theta_v \cdot v$ for $\theta_v \in \Lambda^{\mathrm{pos}}_{G,P}$, such that for each $\lambda \in \Lambda^{\mathrm{pos}}_{G,P}$, we have $n_\lambda = \sum_{v | \theta_v = \lambda} \deg(v)$. The composition $(X^{\mathfrak{A}})_{\mathrm{disj}} \hookrightarrow X^{\mathfrak{A}} \to X^\theta$ is a locally closed embedding, and the subschemes $(X^{\mathfrak{A}})_{\mathrm{disj}}$ for $\mathfrak{A} \in \mathcal{P}_\theta$ form a stratification of X^θ. Thus we can stratify $\mathcal{H}^+_{M,X^\theta}$ by the substacks

$$\mathcal{H}^{+,\mathfrak{A}}_M := \mathcal{H}^+_{M,X^\theta} \underset{X^\theta}{\times} (X^{\mathfrak{A}})_{\mathrm{disj}}.$$

This stack $\mathcal{H}^{+,\mathfrak{A}}_M$ is the same as the one defined in [12, §1.8].

The diagonal $X \to X^\theta$ corresponds to the trivial partition $\theta = 1 \cdot \theta$, and we denote by $\mathcal{H}^{+,\theta}_M$ (resp. $\mathrm{Gr}^{+,\theta}_M$) the stack $\mathcal{H}^+_{M,X^\theta} \underset{X^\theta}{\times} X$ (resp. the scheme $\mathrm{Gr}^+_{M,X^\theta} \underset{X^\theta}{\times} X$).

Let $\mathfrak{A} \in \mathcal{P}_\theta$. The stack $\mathcal{H}^{+,\mathfrak{A}}_M$ is fibered over $(X^{\mathfrak{A}})_{\mathrm{disj}} \times \mathrm{Bun}_M$ with respect to $\overleftarrow{h}$. Similar to §A.1.3, one can express $\mathcal{H}_M^{+,\mathfrak{A}}$ as a twisted product

$$\mathcal{H}_M^{+,\mathfrak{A}} \cong \mathrm{Bun}_M \tilde{\times} \mathrm{Gr}_M^{+,\mathfrak{A}}, \tag{A.6}$$

where $\mathrm{Gr}_M^{+,\mathfrak{A}} := \mathrm{Gr}^+_{M,X^\theta} \times_{X^\theta} (X^{\mathfrak{A}})_{\mathrm{disj}}$. To define the twisted product one considers the action of the jet group $M(\mathfrak{o})_{X^{(|\mathfrak{A}|)}}$ on $\mathrm{Gr}_M^{+,\mathfrak{A}}$ over $X^{(|\mathfrak{A}|)}$. The embedding

$$\mathcal{H}_M^{+,\mathfrak{A}} \hookrightarrow \mathcal{H}^+_{M,X^\theta} \tag{A.7}$$

lies over the map of symmetric powers $X^{(2|\mathfrak{A}|)} \to X^{(2|\theta|)}$. The latter map induces a map of jet groups $M(\mathfrak{o})_{X^{(2|\mathfrak{A}|)}} \to M(\mathfrak{o})_{X^{(2|\theta|)}}$. Therefore (A.7) can be thought of as a twisted product of $\mathrm{id}_{\mathrm{Bun}_M}$ with the embedding $\mathrm{Gr}_M^{+,\mathfrak{A}} \hookrightarrow \mathrm{Gr}^+_{M,X^\theta}$, which is equivariant with respect to the actions of the corresponding jet groups.

Let $x^{\mathfrak{A}} \in (X^{\mathfrak{A}})_{\mathrm{disj}}(k)$ be a $\Lambda^{\mathrm{pos}}_{G,P}$-colored divisor $\sum \theta_v \cdot v$, and let $\mathcal{F}^1_M$ be a k-point of Bun_M. Then the fiber of $\mathcal{H}_M^{+,\mathfrak{A}}$ over this k-point $(x^{\mathfrak{A}}, \mathcal{F}^1_M)$ is isomorphic to

$$\prod \mathrm{Gr}^{+,\theta_v}_{M,v}$$

where $\mathrm{Gr}^{+,\theta_v}_{M,v}$ is the closed subscheme of the affine Grassmannian $\mathrm{Gr}_{M,v}$ defined in [12, §1.6].

In terms of loop and jet groups,

$$\mathrm{Gr}^+_{M,v}(k) = (\overline{M}(\mathfrak{o}_v) \cap M(F_v))/M(\mathfrak{o}_v) \subset \mathrm{Gr}_{M,v}(k) = M(F_v)/M(\mathfrak{o}_v).$$

A.1.6 Convolution products

Consider the diagram

$$\mathcal{H}^+_M \xleftarrow{\mathrm{comp}} \mathcal{H}^+_M \underset{\overrightarrow{h},\mathrm{Bun}_M,\overleftarrow{h}}{\times} \mathcal{H}^+_M \xrightarrow{(\mathrm{pr}_1,\mathrm{pr}_2)} \mathcal{H}^+_M \times \mathcal{H}^+_M \tag{A.8}$$

and recall (Remark A.1.3) that the left arrow comp is proper. Using (A.5), one sees that comp is isomorphic to the twisted product of $\mathrm{id}_{\mathrm{Bun}_M}$ with the proper map

$$\mathrm{conv} : \mathrm{Gr}^+_{M,X^{\theta_1}} \tilde{\times} \mathrm{Gr}^+_{M,X^{\theta_2}} \to \mathrm{Gr}^+_{M,X^{\theta}}, \tag{A.9}$$

where $\theta = \theta_1 + \theta_2$ and the left hand side is the convolution Grassmannian (cf. [71, (3.1.21)]).

We give $D(\mathcal{H}^+_M)$ the structure of a monoidal category by the convolution product

$$\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2 \mapsto \tilde{\mathcal{F}}_1 \star \tilde{\mathcal{F}}_2 := \mathrm{comp}_!((\mathrm{pr}_1, \mathrm{pr}_2)^*(\tilde{\mathcal{F}}_1 \boxtimes \tilde{\mathcal{F}}_2)) = \mathrm{comp}_!(\mathrm{pr}_1^*(\tilde{\mathcal{F}}_1) \otimes \mathrm{pr}_2^*(\tilde{\mathcal{F}}_2)),$$

where $\mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{H}^+_M \times_{\mathrm{Bun}_M} \mathcal{H}^+_M \to \mathcal{H}^+_M$ are the projection maps.

Let $\mathsf{Sph}^+_{M,X^\theta} = D(\mathrm{Gr}^+_{M,X^\theta})^{M(\mathfrak{o})_{X^{(|\theta|)}}}$. Define a product

$$\star : \mathsf{Sph}^+_{M,X^{\theta_1}} \otimes \mathsf{Sph}^+_{M,X^{\theta_2}} \to \mathsf{Sph}^+_{M,X^{\theta}},$$

by $\mathcal{F}_1 \star \mathcal{F}_2 := \mathrm{conv}_!(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$, which is a symmetrized version of the "external convolution product" (cf. [71, §5.4]) for $\theta = \theta_1 + \theta_2$.

For $\mathcal{F}_1 \in \mathsf{Sph}^+_{M,X^{\theta_1}}$, $\mathcal{F}_2 \in \mathsf{Sph}^+_{M,X^{\theta_2}}$, we can form the sheaves $(\overline{\mathbb{Q}}_\ell)_{\mathrm{Bun}_M} \tilde{\boxtimes} \mathcal{F}_i$ on $\mathcal{H}^+_{M,X^{\theta_i}}$ using (A.5). By construction, there is a canonical isomorphism

$$(\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{F}_1) \star (\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{F}_2) \cong \overline{\mathbb{Q}}_\ell \tilde{\boxtimes} (\mathcal{F}_1 \star \mathcal{F}_2),$$

We remark that $\star$ commutes with Verdier duality on $\mathsf{Sph}^+_{M,X^\theta}$.

The category

$$\mathsf{Sph}^+_M := \{\theta \mapsto \mathcal{F}^\theta \in \mathsf{Sph}^+_{M,X^\theta}\}$$

has a natural monoidal structure with respect to $\star$: for two families $\{\mathcal{F}_1^\theta\}$ and $\{\mathcal{F}_2^\theta\}$ the value of their product in $\mathsf{Sph}^+_{M,X^\theta}$ is

$$\bigoplus_{\theta=\theta_1+\theta_2} \mathcal{F}_1^{\theta_1} \star \mathcal{F}_2^{\theta_2}.$$

Factorization property. For $\theta = \theta_1 + \theta_2$, let $(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}}$ denote the the open locus of $X^{\theta_1} \times X^{\theta_2}$ consisting of pairs of colored divisors with disjoint supports. We have a natural étale map $(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \to X^\theta$.

The schemes $\mathrm{Gr}^+_{M,X^\theta}$, $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$ factorize in the sense that there exist Cartesian diagrams

$$\begin{tikzcd}
(\mathrm{Gr}^+_{M,X^{\theta_1}} \times \mathrm{Gr}^+_{M,X^{\theta_2}}) \underset{X_1^{\theta} \times X^{\theta_2}}{\times} (X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \arrow[r] \arrow[d] & \mathrm{Gr}^+_{M,X^\theta} \arrow[d] \\
(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \arrow[r] & X^\theta
\end{tikzcd}$$

for $\theta = \theta_1 + \theta_2$.

The internal convolution (i.e., fusion) product (cf. [71, (5.4.4)])

$$\circledast : D(\mathrm{Gr}_M^{+,\theta_1})^{M(\mathfrak{o})_X} \otimes D(\mathrm{Gr}_M^{+,\theta_2})^{M(\mathfrak{o})_X} \to D(\mathrm{Gr}_M^{+,\theta})^{M(\mathfrak{o})_X}$$

for $\theta = \theta_1 + \theta_2$ is related to the $\star$ product as follows: Let $\Delta^\theta : \mathrm{Gr}_M^{+,\theta} \hookrightarrow \mathrm{Gr}^+_{M,X^\theta}$ denote the closed embedding. Then there is a canonical isomorphism

$$\mathcal{F}_1 \circledast \mathcal{F}_2 \cong \Delta^{\theta *}(\Delta_*^{\theta_1}(\mathcal{F}_1) \star \Delta_*^{\theta_2}(\mathcal{F}_2)). \tag{A.10}$$

A.2 Factorization algebras in Sph^+_M

In this section we review some of the objects introduced in [14, 29] and their properties. For our purposes, we only need to work with these objects at a very coarse level (e.g., in the Grothendieck group) so we omit much of the higher categorical nuances.

Let G be a connected split reductive group over a perfect field k. Let P be a standard parabolic subgroup of G. While the results of *loc. cit.* are stated only in the case $P = B$, we state them for arbitrary parabolics. The reader may check that the proofs readily generalize.

We will continue using the notation of Appendix A.1 for the monoid $\overline{M}$, the Hecke stack, the affine Grassmannian, and the graded Ran space.

A.2.1 Remarks on G

For simplicity, we assume throughout this Appendix that G has a simply connected derived group $[G, G]$, so that we may use the same construction of $\widetilde{\mathrm{Bun}}_P$ as in [13, 12]. The reader may refer to [61, §7] for how to remove this hypothesis, and the basic geometry of the Zastava space and Drinfeld's compactification remains the same.

A.2.2 Geometric Satake

For simplicity, we will only use the non-factorizable geometric Satake functor. Let $\check{M}$ denote the Langlands dual group of M over the field $\overline{\mathbb{Q}}_\ell$. Observe that each $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$ defines a central character of $\check{M}$. Let $\mathrm{Rep}(\check{M})_\theta$ denote the subcategory of $\check{M}$-modules with central character θ. Then we have a t-exact (with respect to the perverse t-structures) functor

$$\mathrm{Sat}^{\mathrm{naive}}_X : D(\mathrm{Rep}(\check{M})_\theta) \otimes D(X) \to D(\mathrm{Gr}^{+,\theta}_{M,X})^{M(\mathfrak{o})_X},$$

which is a special case of the factorizable geometric Satake functor $\mathrm{Sat}^{\mathrm{naive}}_{\mathrm{Ran}(X)}$ constructed in [55, §6]. If we allow θ to range over all of $\Lambda^{\mathrm{pos}}_{G,P}$, then $\mathrm{Sat}^{\mathrm{naive}}_X$ is monoidal with respect to the usual tensor structure on the left hand side and the internal convolution product $\circledast$ on the Beilinson-Drinfeld Grassmannian (cf. §A.1.6) on the right hand side.

Remark A.2.1. Suppose $k = \mathbb{F}_q$. Fix a closed point $v \in |X|$ and let $m_v \in \mathrm{Gr}^{+,\theta}_{M,v}(\mathbb{F}_q)$. As explained in [71, §5.6], the geometric Satake functor corresponds to the classical Satake isomorphism $\mathcal{S}_v : H_{M,v} \to \mathbf{K}(\mathrm{Rep}(\check{M})) \otimes \overline{\mathbb{Q}}_\ell$ by Grothendieck's functions–sheaves dictionary:

the trace of the geometric Frobenius at the $*$-fiber at m_v of $\mathrm{Sat}_X^{\mathrm{naive}}(V \otimes (\overline{\mathbb{Q}}_\ell)_X)$ equals

$$\mathcal{S}_v^{-1}([V])(m_v),$$

where $[V]$ is the image of V in the Grothendieck group $\mathbf{K}(\mathrm{Rep}(\check{M}))$. Here $H_{M,v}$ is the spherical Hecke algebra of $M(F_v)$, and $\mathcal{S}_v$ is Langlands' reformulation of the classical Satake isomorphism (cf. §2.3.4).

A.2.3 Factorization algebras

We use the language of factorization algebras graded by the monoid $\Lambda_{G,P}^{\mathrm{pos}}$ introduced in [29, §2]. This is simply a particular case of the general notion of factorization algebra defined in [8].

A $\Lambda_{G,P}^{\mathrm{pos}}$-graded factorization algebra $\mathcal{F} \in \mathsf{Sph}_M^+$ is a family of sheaves $\mathcal{F}^\theta \in \mathsf{Sph}_{M,X^\theta}^+$ such that for $\theta = \theta_1 + \theta_2$, we have an isomorphism

$$\mathcal{F}^\theta|_{(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}}} \cong (\mathcal{F}^{\theta_1} \boxtimes \mathcal{F}^{\theta_2})|_{(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}}},$$

of sheaves on $\mathrm{Gr}_{M,X^\theta}^+ \times_{X^\theta} (X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}}$ satisfying the natural compatibilities. On the left hand side, we are restricting along the étale map $(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \to X^\theta$. On the right hand side, we restrict along the open embedding $(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}} \hookrightarrow X^{\theta_1} \times X^{\theta_2}$ and use the factorization property of $\mathrm{Gr}_{M,X^\theta}^+$ explained in §A.1.6.

Let $\mathcal{F} \in \mathsf{Sph}_M^+$ be a commutative algebra object in the monoidal category. For $\theta = \theta_1 + \theta_2$, the multiplication map $\mathcal{F}^{\theta_1} \star \mathcal{F}^{\theta_2} \to \mathcal{F}^\theta$ induces, by adjunction, a map

$$(\mathcal{F}^{\theta_1} \boxtimes \mathcal{F}^{\theta_2})|_{(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}}} \to \mathcal{F}^\theta|_{(X^{\theta_1} \times X^{\theta_2})_{\mathrm{disj}}}.$$

We say that $\mathcal{F}$ is a *commutative factorization algebra* if these maps are isomorphisms for all $\theta = \theta_1 + \theta_2$.

Let $\mathcal{F} \in \mathsf{Sph}_M^+$ be a cocommutative coalgebra object in the monoidal category. For $\theta = \theta_1 + \theta_2$, the comultiplication map $\mathcal{F}^\theta \to \mathcal{F}^{\theta_1} \star \mathcal{F}^{\theta_2}$ induces, by adjunction, a map

$$\mathcal{F}^\theta|_{(X^{\theta_1} \times X^{\theta_2})_{\text{disj}}} \to (\mathcal{F}^{\theta_1} \boxtimes \mathcal{F}^{\theta_2})|_{(X^{\theta_1} \times X^{\theta_2})_{\text{disj}}}.$$

We say that $\mathcal{F}$ is a *cocommutative factorization algebra* if these maps are isomorphisms for all $\theta = \theta_1 + \theta_2$.

A.2.4 Cocommutative factorization algebras

Let us recall the definition of the cocommutative factorization algebra $\Upsilon(\check{\mathfrak{u}}_P) \in \mathsf{Sph}_M^+$ introduced in [14].

For $\theta \in \Lambda_{G,P}^{\text{pos}}$, let $\Delta^\theta : \text{Gr}_M^{+,\theta} \hookrightarrow \text{Gr}_{M,X^\theta}^+$ denote the closed diagonal. The $\Lambda_{G,P}^{\text{pos}}$-graded $\check{M}$-module

$$\check{\mathfrak{u}}_P = \bigoplus_{\alpha \in \Phi_G^+ - \Phi_M^+} \check{\mathfrak{u}}_\alpha$$

gives a complex

$$\check{\mathfrak{u}}_{P,\mathsf{Sph}_M^+} := \bigoplus_{\alpha \in \Phi^+ - \Phi_M^+} \Delta_*^\alpha(\text{Sat}_X^{\text{naive}}(\check{\mathfrak{u}}_\alpha \otimes (\overline{\mathbb{Q}}_\ell)_X)) \in \mathsf{Sph}_M^+ .$$

The Lie algebra structure on $\check{\mathfrak{u}}_P$ gives a Lie algebra structure to $\check{\mathfrak{u}}_{P,\mathsf{Sph}_M^+}$ with respect to the $\star$ monoidal structure on Sph_M^+. Then

$$\Upsilon(\check{\mathfrak{u}}_P) := C_\bullet(\check{\mathfrak{u}}_{P,\mathsf{Sph}_M^+})$$

is the homological Chevalley complex associated to this Lie algebra, and $\Upsilon(\check{\mathfrak{u}}_P)$ is a cocommutative factorization algebra.

Let $U(\check{\mathfrak{u}}_P)_{\mathsf{Sph}_M^+}$ denote the universal enveloping algebra of the Lie algebra $\check{\mathfrak{u}}_{P,\mathsf{Sph}_M^+}$. This is a cocommutative factorization algebra in Sph_M^+ with a compatible associative algebra structure with respect to $\star$.

Remark A.2.2. If we consider $\check{\mathfrak{u}}_{P,\mathsf{Sph}^+_M}[1]$ as a Lie superalgebra in degree -1, then

$$\Upsilon(\check{\mathfrak{u}}_P) = U(\check{\mathfrak{u}}_{P,\mathsf{Sph}^+_M}[1]).$$

Restriction to strata. For $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$ and $\mathfrak{A} \in \mathcal{P}_\theta$ a partition, let $x^{\mathfrak{A}} \in (X^{\mathfrak{A}})_{\mathrm{disj}}(k)$ be a $\Lambda^{\mathrm{pos}}_{G,P}$-colored divisor $\sum \theta_v \cdot v$. Then the fiber of $\mathrm{Gr}^{+,\mathfrak{A}}_M \to (X^{\mathfrak{A}})_{\mathrm{disj}}$ over $x^{\mathfrak{A}}$ is isomorphic to $\prod \mathrm{Gr}^{+,\theta_v}_{M,v}$ by the factorization property.

Since $\Upsilon(\check{\mathfrak{u}}_P)$ is a factorization algebra and $\mathrm{Sat}^{\mathrm{naive}}_X$ is a monoidal functor, (A.10) implies that the $*$-restriction of $\Upsilon(\check{\mathfrak{u}}_P)$ to this fiber of $\mathrm{Gr}^{+,\mathfrak{A}}_M$ canonically identifies with

$$\boxtimes_v \mathrm{Sat}^{\mathrm{naive}}_v(C_\bullet(\check{\mathfrak{u}}_P)^{\theta_v})$$

where $C_\bullet(\check{\mathfrak{u}}_P)^{\theta_v}$ denotes the θ_v-graded piece of the Chevalley complex of the Lie algebra $\check{\mathfrak{u}}_P$ in $D(\mathrm{Rep}(\check{M}))$, and $\mathrm{Sat}^{\mathrm{naive}}_v$ is the non-relative geometric Satake functor for $\mathrm{Gr}^{+,\theta_v}_{M,v}$.

The $*$-restriction of $U(\check{\mathfrak{u}}_P)_{\mathsf{Sph}^+_M}$ to the fiber $\prod \mathrm{Gr}^{+,\theta_v}_{M,v}$ canonically identifies with

$$\boxtimes_v \mathrm{Sat}^{\mathrm{naive}}_v(U(\check{\mathfrak{u}}_P)^{\theta_v})$$

where $U(\check{\mathfrak{u}}_P)^{\theta_v}$ is the θ_v-graded piece of the universal enveloping algebra of $\check{\mathfrak{u}}_P$.

Koszul resolution. Let $\mathbf{1}$ denote the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\mathrm{Spec}(k) = \mathrm{Gr}^+_{M,X^0}$. Then $\mathbf{1}$ is the unit in the monoidal category Sph^+_M.

Consider the acyclic complex $\check{\mathfrak{u}}_{P,\mathsf{Sph}^+_M} \to \check{\mathfrak{u}}_{P,\mathsf{Sph}^+_M}$ as a Lie superalgebra in degrees $-1, 0$. The universal enveloping algebra of this Lie superalgebra is quasi-isomorphic to $\mathbf{1}$. This resolution endows $\mathbf{1}$ with the structure of a comodule with respect to $\Upsilon(\check{\mathfrak{u}}_P)$ and of a module with respect to $U(\check{\mathfrak{u}}_P)_{\mathsf{Sph}^+_M}$.

A.2.5 Commutative factorization algebras

We define the commutative factorization algebra $\Omega(\check{\mathfrak{u}}_P^-) \in \mathsf{Sph}_M^+$ as the Verdier dual of $\Upsilon(\check{\mathfrak{u}}_P^-)$. (Here we use the opposite Lie algebra $\check{\mathfrak{u}}_P^-$ so that $\Omega(\check{\mathfrak{u}}_P^-)$ is still $\Lambda_{G,P}^{\mathrm{pos}}$-graded.)

One can also define $\Omega(\check{\mathfrak{u}}_P^-)$ from scratch by considering

$$(\check{\mathfrak{u}}^-_{P,\mathsf{Sph}_M^+})^\vee := \mathbb{D}(\check{\mathfrak{u}}^-_{P,\mathsf{Sph}_M^+}) \cong \bigoplus_{\alpha\in\Phi_G^+-\Phi_M^+} \Delta_*^\alpha(\mathrm{Sat}_X^{\mathrm{naive}}(\check{\mathfrak{u}}_{-\alpha}^\vee \otimes (\overline{\mathbb{Q}}_\ell)_X(1)[2])).$$

Then $(\check{\mathfrak{u}}^-_{P,\mathsf{Sph}_M^+})^\vee$ is a Lie coalgebra in Sph_M^+ with respect to the $\star$ monoidal structure. There is a canonical isomorphism

$$\Omega(\check{\mathfrak{u}}_P) \cong C^\bullet((\check{\mathfrak{u}}^-_{P,\mathsf{Sph}_M^+})^\vee),$$

where the right hand side is the cohomological Chevalley complex associated to this Lie coalgebra.

Let $U^\vee(\check{\mathfrak{u}}_P^-)_{\mathsf{Sph}_M^+}$ denote the universal co-enveloping coalgebra of $(\check{\mathfrak{u}}^-_{P,\mathsf{Sph}_M^+})^\vee$. This is a commutative factorization algebra in Sph_M^+ with a compatible coalgebra structure with respect to $\star$.

Restriction to strata. For $\theta \in \Lambda_{G,P}^{\mathrm{pos}}$ and $\mathfrak{A} \in \mathcal{P}_\theta$ a partition, let $x^{\mathfrak{A}} \in (X^{\mathfrak{A}})_{\mathrm{disj}}(k)$ be a $\Lambda_{G,P}^{\mathrm{pos}}$-colored divisor $\sum \theta_v \cdot v$. Then the fiber of $\mathrm{Gr}_M^{+,\mathfrak{A}} \to (X^{\mathfrak{A}})_{\mathrm{disj}}$ over $x^{\mathfrak{A}}$ is isomorphic to $\prod \mathrm{Gr}_{M,v}^{+,\theta_v}$.

Since $\Omega(\check{\mathfrak{u}}_P^-)$ is a factorization algebra and $\mathrm{Sat}_X^{\mathrm{naive}}$ is a monoidal functor, the $*$-restriction of $\Omega(\check{\mathfrak{u}}_P^-)$ to this fiber of $\mathrm{Gr}_M^{+,\mathfrak{A}}$ canonically identifies with

$$\boxtimes_v \mathrm{Sat}_v^{\mathrm{naive}}(C^\bullet((\check{\mathfrak{u}}_P^-)^\vee(1)[2])^{\theta_v})$$

where $C^\bullet((\check{\mathfrak{u}}_P^-)^\vee(1)[2])^{\theta_v}$ denotes the θ_v-graded piece of the cohomological Chevalley complex, and $\mathrm{Sat}_v^{\mathrm{naive}}$ is the non-relative geometric Satake functor for $\mathrm{Gr}_{M,v}^{+,\theta_v}$.

The $*$-restriction of $U^\vee(\check{\mathfrak{u}}_P^-)_{\mathsf{Sph}_M^+}$ to the fiber $\prod \mathrm{Gr}_{M,v}^{+,\theta_v}$ canonically identifies with

$$\boxtimes_v \mathrm{Sat}_v^{\mathrm{naive}}(U((\check{\mathfrak{u}}_P^-)^\vee(1)[2])^{\theta_v})$$

where $U((\check{\mathfrak{u}}_P^-)^\vee(1)[2])^{\theta_v}$ is the θ_v-graded piece of the universal co-enveloping coalgebra.

Koszul resolution. Applying Verdier duality to the Koszul resolution in §A.2.4 gives $\mathbf{1} \in \mathsf{Sph}_M^+$ the structure of a module with respect to $\Omega(\check{\mathfrak{u}}_P^-)$ and of a comodule with respect to $U^\vee(\check{\mathfrak{u}}_P)_{\mathsf{Sph}_M^+}$.

A.2.6 Eisenstein series

Let

$$\jmath : \mathrm{Bun}_P \hookrightarrow \widetilde{\mathrm{Bun}}_P$$

denote the open embedding into Drinfeld's compactification of Bun_P.

Action on Drinfeld's compactifications. For every $\theta \in \Lambda_{G,P}^{\mathrm{pos}}$ there corresponds a proper map

$$\bar{\iota}^\theta : \widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_{M,X^\theta}^+ \to \widetilde{\mathrm{Bun}}_P.$$

Using (A.5), we can express $\widetilde{\mathrm{Bun}}_P \times_{\mathrm{Bun}_M} \mathcal{H}_{M,X^\theta}^+$ as a twisted product $\widetilde{\mathrm{Bun}}_P \tilde{\times} \mathrm{Gr}_{M,X^\theta}^+$. Given $\mathcal{E} \in D(\widetilde{\mathrm{Bun}}_P)$ and $\mathcal{F} \in \mathsf{Sph}_{M,X^\theta}^+$, we can form a sheaf

$$\mathcal{E} \mathbin{\tilde{\boxtimes}} \mathcal{F} \in D(\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_{M,X^\theta}^+).$$

We define an action of $\mathsf{Sph}_{M,X^\theta}^+$ on $D(\widetilde{\mathrm{Bun}}_P)$ by

$$\mathcal{E}, \mathcal{F} \mapsto \mathcal{E} \star \mathcal{F} := \bar{\iota}_*^\theta(\mathcal{E} \mathbin{\tilde{\boxtimes}} \mathcal{F}),$$

which is compatible with the monoidal product $\star$ on Sph_M^+.

It was established in [14, Theorem 4.2] (cf. [29, Theorem 5.2.2]) that there exists a map

$$\jmath_*(\mathrm{IC}_{\mathrm{Bun}_P}) \to \jmath_*(\mathrm{IC}_{\mathrm{Bun}_P}) \star \Upsilon(\check{\mathfrak{u}}_P)$$

that gives $\jmath_*(\mathrm{IC}_{\mathrm{Bun}_P})$ the structure of an $\Upsilon(\check{\mathfrak{u}}_P)$-comodule.

By [14, Theorem 6.6] (cf. [29, Theorem 5.2.4]), we have a quasi-isomorphism

$$\jmath_*(\mathrm{IC}_{\mathrm{Bun}_P}) \underset{\Upsilon(\check{\mathfrak{u}}_P)}{\square} \mathbf{1} \to \mathrm{IC}_{\widetilde{\mathrm{Bun}}_P} \tag{A.11}$$

where $\square$ denotes the homotopy cotensor product over $\Upsilon(\check{\mathfrak{u}}_P)$ (i.e., the coBar construction), which can be computed using the Koszul resolution of $\mathbf{1}$ from §A.2.5.

Recall that $\mathbf{1}$ is also a $U(\check{\mathfrak{u}}_P)$-module. Thus applying the homotopy tensor product over $U(\check{\mathfrak{u}}_P)$ to (A.11), we get a quasi-isomorphism

$$\jmath_*(\mathrm{IC}_{\mathrm{Bun}_P}) \cong \jmath_*(\mathrm{IC}_{\mathrm{Bun}_P}) \underset{\Upsilon(\check{\mathfrak{u}}_P)}{\square} \mathbf{1} \underset{U(\check{\mathfrak{u}}_P)}{\otimes} \mathbf{1} \cong \mathrm{IC}_{\widetilde{\mathrm{Bun}}_P} \underset{U(\check{\mathfrak{u}}_P)}{\otimes} \mathbf{1}, \tag{A.12}$$

where the first quasi-isomorphism follows from the Koszul duality $\Upsilon(\check{\mathfrak{u}}_P) \cong \mathbf{1} \otimes_{U(\check{\mathfrak{u}}_P)} \mathbf{1}$.

A.2.7 The factorization algebra $\tilde{\Upsilon}(\check{\mathfrak{u}}_P)$

Let ι^θ denote the composition

$$\bar{\iota}^\theta \circ (\jmath \times \mathrm{id}_{\mathcal{H}^+_{M,X^\theta}}) : \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}^+_{M,X^\theta} \to \widetilde{\mathrm{Bun}}_P.$$

The maps ι^θ are locally closed embeddings and their images define a stratification of $\widetilde{\mathrm{Bun}}_P$ (cf. [13, §6.2], [12]).

Following [29, Proposition 6.1.3], there exists a canonically defined factorization algebra

$\tilde{\Upsilon}(\check{\mathfrak{u}}_P)$ equipped with the structure of a coassociative coalgebra in Sph_M^+ such that

$$\iota^{\theta *}(\jmath_*(\mathrm{IC}_{\mathrm{Bun}_P})) \cong \mathrm{IC}_{\mathrm{Bun}_P} \tilde{\boxtimes}\, \tilde{\Upsilon}(\check{\mathfrak{u}}_P)^{\theta}.$$

We now describe the image of $\tilde{\Upsilon}(\check{\mathfrak{u}}_P)$ in the Grothendieck group of Sph_M^+. Taking the image of (A.12) in the Grothendieck group gives the equality $[\jmath_*(\mathrm{IC}_{\mathrm{Bun}_P})] = [\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P} \star \Upsilon(\check{\mathfrak{u}}_P)]$. For $\mu \in \Lambda_{G,P}$, let Bun_M^{μ} denote the corresponding connected component consisting of M-bundles of degree μ. Let $\mathrm{Bun}_P^{\mu} := \mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathrm{Bun}_M^{\mu}$. We have a Cartesian square

$$\begin{tikzcd}
\mathrm{Bun}_P^{\mu} \underset{\mathrm{Bun}_M}{\times} \mathcal{H}^+_{M,X^{\theta_1}} \underset{\mathrm{Bun}_M}{\times} \mathcal{H}^+_{M,X^{\theta_2}} \arrow[r] \arrow[d, "\mathrm{id}_{\mathrm{Bun}_P} \times \mathrm{comp}"'] & \widetilde{\mathrm{Bun}}_P^{\mu-\theta_1} \underset{\mathrm{Bun}_M}{\times} \mathcal{H}^+_{M,X^{\theta_2}} \arrow[d, "\bar{\iota}^{\theta_2}"] \\
\mathrm{Bun}_P^{\mu} \underset{\mathrm{Bun}_M}{\times} \mathcal{H}^+_{M,X^{\theta}} \arrow[r, "\iota^{\theta}"] & \widetilde{\mathrm{Bun}}_P^{\mu-\theta}
\end{tikzcd}$$

where $\theta = \theta_1 + \theta_2$. Therefore pulling back by $\iota^{\theta *}$ gives

$$[\iota^{\theta *}\jmath_*(\mathrm{IC}_{\mathrm{Bun}_P})] = \sum_{\theta_1+\theta_2=\theta} [\iota^{\theta_1 *}(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}) \star \Upsilon(\check{\mathfrak{u}}_P)^{\theta_2}] \tag{A.13}$$

in the Grothendieck group of $\mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathcal{H}^+_{M,X^{\theta}}$. The following result is proved in [12, Theorem 1.12] after passing to the Grothendieck group, and it is proved in the derived category in [14, Proposition 4.4]:

Proposition A.2.3. *There exists a canonical isomorphism in* $D(\mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathcal{H}_M^{+,\theta})$*:*

$$\iota^{\theta *}(\mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}) \cong \mathrm{IC}_{\mathrm{Bun}_P} \tilde{\boxtimes}\, U^{\vee}(\check{\mathfrak{u}}_P^-)^{\theta}_{\mathsf{Sph}_M^+}.$$

Combining (A.13) and Proposition A.2.3, we deduce that

$$[\tilde{\Upsilon}(\check{\mathfrak{u}}_P)^{\theta}] = \sum_{\theta_1+\theta_2=\theta} [U^{\vee}(\check{\mathfrak{u}}_P^-)^{\theta_1}_{\mathsf{Sph}_M^+} \star \Upsilon(\check{\mathfrak{u}}_P)^{\theta_2}] \tag{A.14}$$

in the Grothendieck group of Sph_M^+.

Proposition A.2.4. *Suppose $k = \mathbb{F}_q$. The trace of the geometric Frobenius on $*$-stalks of $\tilde{\Upsilon}(\check{\mathfrak{u}}_P)^\theta$ is equal to the function*

$$(\mathcal{F}_M, \beta_M) \in \mathrm{Gr}^+_{M,X^\theta}(\mathbb{F}_q) \mapsto q^{-\langle \check{\rho}_P, \theta \rangle} \prod_v \nu_{M,v}(m_v),$$

where $m_v \in \mathrm{Gr}^+_{M,v}(\mathbb{F}_q) \subset M(F_v)/M(\mathfrak{o}_v)$ is determined by β_M, and $\nu_{M,v}$ is the $K_{M,v}$-bi-invariant measure on $M(F_v)$ defined in §2.3.3.

Proof. Let $x^{\mathfrak{A}} \in (X^{\mathfrak{A}})_{\mathrm{disj}}(\mathbb{F}_q)$, $\mathfrak{A} \in \mathcal{P}_\theta$, denote the image of $(\mathcal{F}_M, \beta_M)$ under $\mathrm{Gr}^+_{M,X^\theta} \to X^\theta$. Then the fiber of $x^{\mathfrak{A}}$ is isomorphic to $\prod_v \mathrm{Gr}^{+,\theta_v}_{M,v}$, and the point $(\mathcal{F}_M, \beta_M)$ corresponds to the collection $\{m_v \in \mathrm{Gr}^{+,\theta_v}_{M,v}(\mathbb{F}_q)\}$. Since $\tilde{\Upsilon}(\check{\mathfrak{u}}_P)$ is a factorization algebra, it suffices to consider the case when β_M is an isomorphism on $X - v$ for a fixed closed point v, i.e., $x^{\mathfrak{A}} = \theta_v \cdot v$ is supported at a single point v.

The trace of geometric Frobenius on $*$-stalks of a complex only depends on the image of the complex in the Grothendieck group. Therefore (A.14), the discussion in §A.2.4 and §A.2.5, and the compatibility of geometric Satake with the classical Satake transform (Remark A.2.1) together imply that the trace of geometric Frobenius at the $*$-stalk of $(\mathcal{F}_M, \beta_M)$ equals

$$\mathcal{S}_v^{-1}\left(\left(\sum_{n=0}(-1)^n[\wedge^n \check{\mathfrak{u}}_P]\right) \otimes \left(\sum_{n=0}^{\infty}[\mathrm{Sym}^n\, \check{\mathfrak{u}}_P] \cdot q_v^{-n}\right)\right)(m_v).$$

Comparing with (2.20), we deduce the proposition. □

A.2.8 Geometric proof

We prove Proposition A.2.4 above using (2.19), which is essentially the classical Gindikin–Karpelevich formula. However, the Satake transform does not appear in the statement of Proposition A.2.4. In this subsection we give a more direct proof of Proposition A.2.4 using derived algebraic geometry.

Zastava spaces. Let $\mathcal{Z}^{P,\theta}$ denote the Zastava space defined in [12] corresponding to the parabolic P, and let $\overset{\circ}{\mathcal{Z}}{}^{P,\theta}$ denote the open Zastava space (called $\mathcal{Z}^{P,\theta}_{\max}$ in *loc. cit.*). We have a map $\pi_{\mathcal{Z}} : \mathcal{Z}^{P,\theta} \to \mathrm{Gr}^+_{M,X^\theta}$. Let $\overset{\circ}{\pi}_{\mathcal{Z}} : \overset{\circ}{\mathcal{Z}}{}^{P,\theta} \to \mathrm{Gr}^+_{M,X^\theta}$ denote the restriction.

Let $\tilde{\Omega}(\check{\mathfrak{u}}_P^-)$ denote the Verdier dual of $\tilde{\Upsilon}(\check{\mathfrak{u}}_P^-)$. Then $\tilde{\Omega}(\check{\mathfrak{u}}_P^-)$ is a factorization algebra on Sph^+_M with the defining equation

$$\iota^{\theta!}(\jmath_!(\mathrm{IC}_{\mathrm{Bun}_P})) \cong \mathrm{IC}_{\mathrm{Bun}_P} \boxtimes \tilde{\Omega}(\check{\mathfrak{u}}_P^-)^\theta$$

for $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$. Using the local model of [12, §3] and the contraction principle ([12, Proposition 5.2]), one sees that there is a canonical isomorphism

$$\tilde{\Omega}(\check{\mathfrak{u}}_P^-)^\theta \cong (\overset{\circ}{\pi}_{\mathcal{Z}})_!(\mathrm{IC}_{\overset{\circ}{\mathcal{Z}}{}^{P,\theta}}). \tag{A.15}$$

From this equation, factorization of $\tilde{\Omega}(\check{\mathfrak{u}}_P^-)$ follows from the factorization property of $\overset{\circ}{\mathcal{Z}}{}^{P,\theta}$.

Lemma A.2.5. *Suppose $k = \mathbb{F}_q$. The trace of geometric Frobenius on $*$-stalks of $\tilde{\Omega}(\check{\mathfrak{u}}_P^-)^\theta$ is equal to the function*

$$(\mathcal{F}_M, \beta_M) \in \mathrm{Gr}^+_{M,X^\theta}(\mathbb{F}_q) \mapsto q^{-\langle \check{\rho}_P, \theta\rangle} \prod_v \mu_{M,v}(m_v),$$

where $m_v \in \mathrm{Gr}^+_{M,v}(\mathbb{F}_q) \subset M(F_v)/M(\mathfrak{o}_v)$ is determined by β_M, and $\mu_{M,v}$ is the $K_{M,v}$-bi-invariant measure on $M(F_v)$ defined in §2.3.1.

Proof. Let $x^{\mathfrak{A}} \in (X^{\mathfrak{A}})_{\mathrm{disj}}(\mathbb{F}_q)$, $\mathfrak{A} \in \mathcal{P}_\theta$, denote the image of $(\mathcal{F}_M, \beta_M)$ under $\mathrm{Gr}^+_{M,X^\theta} \to X^\theta$. Then the fiber of $x^{\mathfrak{A}}$ is isomorphic to $\prod_v \mathrm{Gr}^{+,\theta_v}_{M,v}$, and the point $(\mathcal{F}_M, \beta_M)$ corresponds to the collection $\{m_v \in \mathrm{Gr}^{+,\theta_v}_{M,v}(\mathbb{F}_q)\}$. Since $\tilde{\Omega}(\check{\mathfrak{u}}_P)$ is a factorization algebra, it suffices to consider the case when β_M is an isomorphism on $X - v$ for a fixed closed point v, i.e., $x^{\mathfrak{A}} = \theta_v \cdot v$.

Therefore we can restrict our attention to the central fiber

$$\mathring{\mathfrak{Z}}_v^\theta := \mathring{\mathcal{Z}}^{P,\theta} \underset{X^\theta}{\times} \operatorname{Spec}(k)$$

where $\operatorname{Spec}(k) \to X^\theta$ is the point $\theta_v \cdot v$. Let $\operatorname{Gr}_{P,v}^{\theta_v}$ denote the preimage under $\operatorname{Gr}_{P,v} \to \operatorname{Gr}_{M/[M,M],v}$ of the point corresponding to θ_v. By [12, Proposition 2.6], there is a natural identification $\mathring{\mathfrak{Z}}_v^\theta \cong \operatorname{Gr}_{P,v}^{\theta_v} \cap \operatorname{Gr}_{U^-,v}$ such that $\mathring{\pi}_{\mathcal{Z}}$ corresponds to the map

$$\operatorname{Gr}_{P,v}^{\theta_v} \cap \operatorname{Gr}_{U^-,v} \hookrightarrow \operatorname{Gr}_{P,v} \to \operatorname{Gr}_{M,v}.$$

In other words, the central fiber of the open Zastava space is an intersection of semi-infinite orbits in the affine Grassmannian. Recall from §2.3.1 that $\mu_{M,v}$ is defined as the measure of certain semi-infinite orbits. By Grothendieck's trace formula, we deduce that the trace of geometric Frobenius on $*$-stalks of $(\mathring{\pi}_{\mathcal{Z}})_!(\overline{\mathbb{Q}}_\ell)_{\mathring{\mathcal{Z}}^{P,\theta}}$ equals the function $m_v \mapsto \mu_{M,v}(m_v)$. Since $\mathring{\mathcal{Z}}^{P,\theta}$ is a smooth scheme of dimension $\langle 2\check{\rho}_P, \theta\rangle$, we have proved the lemma. $\square$

By [29, Proposition 6.2.2], we have a Koszul duality

$$\mathbf{1} \underset{\tilde{\Omega}(\check{\mathfrak{u}}_P^-)}{\otimes} \mathbf{1} \cong \tilde{\Upsilon}(\check{\mathfrak{u}}_P).$$

At the level of Grothendieck groups, this tells us that we have an equality $[\tilde{\Omega}(\check{\mathfrak{u}}_P^-)] \star [\tilde{\Upsilon}(\check{\mathfrak{u}}_P)] = [\mathbf{1}]$. Therefore the Grothendieck function of $\tilde{\Omega}(\check{\mathfrak{u}}_P^-)$ is the inverse, with respect to convolution, of the Grothendieck function of $\tilde{\Upsilon}(\check{\mathfrak{u}}_P)$. Since $\nu_{M,v}$ is defined to be the convolution inverse of $\mu_{M,v}$, Lemma A.2.5 implies Proposition A.2.4.

A.3 The Drinfeld–Lafforgue–Vinberg compactification

Let k be a perfect base field. In this section we review the definition and properties of the stack VinBun_G introduced in [62]. In §A.3.6, we mention an alternate definition of

VinBun_G in the general case when $[G, G]$ is not simply connected. For $k = \mathbb{F}_q$, we use results from *loc. cit.* to compute the trace of the geometric Frobenius acting on the $*$-stalks of the $*$-pushforward of the constant sheaf $\overline{\mathbb{Q}}_\ell$ under the diagonal morphism Δ of Bun_G (see Theorem A.3.12). This is done by using a certain compactification of Δ, which we construct in §A.3.8.

A.3.1 The Deconcini–Procesi–Vinberg semigroup

Set $T_{\mathrm{adj}} := T/Z(G)$, where $Z(G)$ is the center of G. The simple roots identify T_{adj} with $\mathbb{G}_m^{|\Gamma_G|}$.

Let $\overline{G_{\mathrm{enh}}}$ denote the Vinberg semigroup of G, which admits a homomorphism

$$\bar{\pi} : \overline{G_{\mathrm{enh}}} \to \overline{T_{\mathrm{adj}}}$$

where $\overline{T_{\mathrm{adj}}} := (\mathbb{A}^1)^{|\Gamma_G|}$. Any representation of G_{enh} decomposes into ones of the form $V \otimes k_{\check{\lambda}}$ where $V \in \mathrm{Rep}(G)$ and $\check{\lambda} \in \check{\Lambda}$ is a weight of T, such that for all weights $\check{\mu}$ of V, the difference $\check{\lambda} - \check{\mu}$ belongs to the root lattice. By definition, $V \otimes k_{\check{\lambda}} \in \mathrm{Rep}(\overline{G_{\mathrm{enh}}})$ if and only if $\check{\lambda} - \check{\mu} \in \check{\Lambda}_G^{\mathrm{pos}}$ for all weights $\check{\mu}$ of V.

We recall some facts whose proofs can be found in [67, §8] in the characteristic zero case and in [58] in general. Let $V(\check{\lambda})$ denote the irreducible G-module of highest weight $\check{\lambda} \in \check{\Lambda}_G^+$.

Following [24, Lemma D.4.2, Definition D.4.3], we let $\overset{\circ}{\overline{G_{\mathrm{enh}}}}$ denote the non-degenerate locus of $\overline{G_{\mathrm{enh}}}$. By definition, this is the open subscheme of $\overline{G_{\mathrm{enh}}}$ whose $\bar{k}$-points are the elements $g \in \overline{G_{\mathrm{enh}}}(\bar{k})$ with nonzero action on $V(\check{\lambda}) \otimes \bar{k}_{\check{\lambda}}$ for all dominant weights $\check{\lambda} \in \check{\Lambda}_G^+$.

It is known that $\overset{\circ}{\overline{G_{\mathrm{enh}}}}$ is smooth over $\overline{T_{\mathrm{adj}}}$. The choice of Cartan subgroup $T \subset G$ defines a section $\mathfrak{s} : T_{\mathrm{adj}} \to G_{\mathrm{enh}}$ by $\mathfrak{s}(t) = (t^{-1}, t)$, which extends to a homomorphism of monoids

$$\bar{\mathfrak{s}} : \overline{T_{\mathrm{adj}}} \to \overline{G_{\mathrm{enh}}}$$

with image contained in $\overset{\circ}{\overline{G_{\mathrm{enh}}}}$ (cf. [24, Lemma D.5.2]). The $G \times G$-action on $\overset{\circ}{\overline{G_{\mathrm{enh}}}}$ gives an equality ([24, Corollary D.5.4])

$$\overset{\circ}{\overline{G_{\mathrm{enh}}}} = G \cdot \bar{\mathfrak{s}}(\overline{T_{\mathrm{adj}}}) \cdot G.$$

For a standard parabolic P with Levi subgroup M, let $\mathbf{c}_P \in \overline{T_{\mathrm{adj}}}$ be the point defined by the condition that $\check{\alpha}_i(\mathbf{c}_P) = 1$ for simple roots $\check{\alpha}_i$ contained inside M, and $\check{\alpha}_i(\mathbf{c}_P) = 0$ for all other simple roots.

There is a canonical T-stable stratification of $\overline{T_{\mathrm{adj}}}$ indexed by standard parabolics, with the point $\mathbf{c}_P$ contained in the stratum $(\overline{T_{\mathrm{adj}}})_P$ corresponding to the parabolic P. In other words,

$$(\overline{T_{\mathrm{adj}}})_P = \{t \in \overline{T_{\mathrm{adj}}} \mid \check{\alpha}_i(t) \neq 0,\, i \in \Gamma_M \text{ and } \check{\alpha}_j(t) = 0,\, j \in \Gamma_G - \Gamma_M\}.$$

The T-action on $\mathbf{c}_P$ induces an isomorphism $T/Z(M) \cong (\overline{T_{\mathrm{adj}}})_P$. Define

$$(\overline{T_{\mathrm{adj}}})_{\geq P} = \{t \in \overline{T_{\mathrm{adj}}} \mid \check{\alpha}_i(t) \neq 0,\, i \in \Gamma_M\}$$

to be the open locus of $\overline{T_{\mathrm{adj}}}$ obtained by removing all strata corresponding to parabolic subgroups not containing P.

Example A.3.1. Let $G = \mathrm{SL}(2)$. Then $G_{\mathrm{enh}} = \mathrm{GL}(2)$ and $\overline{G_{\mathrm{enh}}} = \mathrm{Mat}(2)$, the monoid of 2×2 matrices. In this case $T_{\mathrm{adj}} = \mathbb{G}_m$, $\overline{T_{\mathrm{adj}}} = \mathbb{A}^1$, and $\bar{\pi} : \mathrm{Mat}(2) \to \mathbb{A}^1$ is the determinant map. The non-degenerate locus $\overset{\circ}{\overline{G_{\mathrm{enh}}}} = \mathrm{Mat}(2) - \{0\}$ is the open subset of nonzero matrices. Let B equal the Borel of upper triangular matrices and identify T with the subgroup of diagonal matrices in G. The section $\bar{\mathfrak{s}}$ corresponding to (B, T) is the map $\mathbb{A}^1 \to \mathrm{Mat}(2)$ sending $x \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)$. The idempotent $\mathbf{c}_G$ equals $1 \in \mathbb{A}^1$, and $\mathbf{c}_B$ equals 0.

Note that the projection $T_{\mathrm{adj}} = T/Z(G) \to T/Z(M) = \mathbb{G}_m^{|\Gamma_M|}$ has a natural splitting $T/Z(M) \hookrightarrow T/Z(G)$ corresponding to the inclusion $\mathbb{G}_m^{|\Gamma_M|} \times \{1\} \hookrightarrow \mathbb{G}_m^{|\Gamma_G|}$. Set $\overline{T/Z(M)} := (\mathbb{A}^1)^{|\Gamma_M|}$. We have a decomposition $T_{\mathrm{adj}} = (T/Z(M)) \times (Z(M)/Z(G))$, which extends to a decomposition

$$\overline{T_{\mathrm{adj}}} = \overline{T/Z(M)} \times \overline{Z(M)/Z(G)},$$

where $\overline{Z(M)/Z(G)} \cong (\mathbb{A}^1)^{|\Gamma_G|-|\Gamma_M|}$ is the closure of $Z(M)/Z(G) \subset T_{\mathrm{adj}}$ in $\overline{T_{\mathrm{adj}}}$. Under the above decomposition, the point $(1,0)$ corresponds to $\mathbf{c}_P$, the stratum $(\overline{T_{\mathrm{adj}}})_P$ corresponds to $T/Z(M) \times \{0\} = \mathbb{G}_m^{|\Gamma_M|} \times \{0\}$, and

$$(\overline{T_{\mathrm{adj}}})_{\geq P} = T/Z(M) \times \overline{Z(M)/Z(G)} = \mathbb{G}_m^{|\Gamma_M|} \times (\mathbb{A}^1)^{|\Gamma_G|-|\Gamma_M|}.$$

A.3.2 The stack VinBun_G

Following [62], the stack

$$\mathrm{VinBun}_G \subset \mathrm{Maps}(X, G\backslash\overline{G_{\mathrm{enh}}}/G)$$

is the open substack[1] of maps generically landing in the non-degenerate locus $\overset{\circ}{\overline{G_{\mathrm{enh}}}}$. For a test scheme S, an S-point of VinBun_G is a datum of $(\mathcal{F}_G^1, \mathcal{F}_G^2, \beta)$, where $\mathcal{F}_G^1, \mathcal{F}_G^2$ are G-bundles on $X \times S$ and

$$\beta : X \times S \to \overline{G_{\mathrm{enh}}} \overset{G\times G}{\times} (\mathcal{F}_G^1 \underset{X\times S}{\times} \mathcal{F}_G^2)$$

is a section over $X \times S$ that generically lands in $\overset{\circ}{\overline{G_{\mathrm{enh}}}}$ over every geometric point of S. We call such a section β a $\overline{G_{\mathrm{enh}}}$-morphism $\mathcal{F}_G^2 \to \mathcal{F}_G^1$.

The map $\bar{\pi}$ induces a map

$$\bar{\pi}_{\mathrm{Bun}} : \mathrm{VinBun}_G \to \mathrm{Maps}(X, \overline{T_{\mathrm{adj}}}) = \overline{T_{\mathrm{adj}}},$$

where the last equality holds because X is proper and geometrically connected.

1. The definition of VinBun_G is deceptively similar to that of $\tilde{\mathcal{H}}_M$ in §A.1.2. We note the differences: in the definition of VinBun_G, we consider the quotient stack by $G \times G$ and not $G_{\mathrm{enh}} \times G_{\mathrm{enh}}$. More importantly, the non-degenerate locus $\overset{\circ}{\overline{G_{\mathrm{enh}}}}$ is larger than the subgroup G_{enh}.

Let $\mathrm{VinBun}_{G,P}$ (resp. $\mathrm{VinBun}_{G,\geq P}$) denote the preimage of $(\overline{T_{\mathrm{adj}}})_P$ (resp. $(\overline{T_{\mathrm{adj}}})_{\geq P}$) under $\bar{\pi}_{\mathrm{Bun}}$. Note that $\mathrm{VinBun}_{G,\geq P}$ contains the open stratum $\mathrm{VinBun}_{G,G}$.

A.3.3 The T_{adj}-action on VinBun_G

In what follows, we will define a canonical action of T_{adj} on VinBun_G which is equivariant with respect to $\bar{\pi}_{\mathrm{Bun}}$ and the identity action on T_{adj}.

Suppose we have an exact sequence of algebraic groups

$$1 \to H' \to H \to H'' \to 1$$

and an action of H on a k-scheme Y. Then the stack $H'\backslash Y$ is an H''-torsor over the stack $H\backslash Y$: indeed, the morphism $H'\backslash Y \to H\backslash Y$ is obtained by base change from the H''-torsor $H'\backslash \mathrm{pt} \to H\backslash \mathrm{pt}$, where $\mathrm{pt} = \mathrm{Spec}(k)$.

In particular, H'' acts on $H'\backslash Y$ over $H\backslash Y$. One can think of this action as follows. An S-point of $H\backslash Y$ is an H-torsor $\mathcal{F}_H \to S$ equipped with an H-equivariant morphism $\mathcal{F}_H \to Y$. Lifting these data to a morphism $S \to H'\backslash Y$ is the same as specifying an H'-structure on $\mathcal{F}_H$, which is the same as specifying an H-equivariant morphism $\mathcal{F}_H \to H''$. The set of all such morphisms $\mathcal{F}_H \to H''$ is equipped with an action of $H''(S)$ (by right translations).

Applying the discussion above to $Y = \overline{G_{\mathrm{enh}}}$ and the exact sequence

$$1 \to G \times G \to G_{\mathrm{enh}} \times G_{\mathrm{enh}} \to T_{\mathrm{adj}} \times T_{\mathrm{adj}} \to 1$$

(so $H = G_{\mathrm{enh}} \times G_{\mathrm{enh}}$, $H' = G \times G$, and $H'' = T_{\mathrm{adj}} \times T_{\mathrm{adj}}$), one gets a canonical action of $T_{\mathrm{adj}} \times T_{\mathrm{adj}}$ on $G\backslash\overline{G_{\mathrm{enh}}}/G$. We will be considering only the action of $T_{\mathrm{adj}} = T_{\mathrm{adj}} \times \{1\} \subset T_{\mathrm{adj}} \times T_{\mathrm{adj}}$ (which comes from the action of G_{enh} on $\overline{G_{\mathrm{enh}}}$ by left translations).

The T_{adj}-action on $G\backslash\overline{G_{\mathrm{enh}}}/G$ preserves $G\backslash\overset{\circ}{\overline{G_{\mathrm{enh}}}}/G$, so it induces an action of T_{adj} on $\mathrm{VinBun}_G \subset \mathrm{Maps}(X, G\backslash\overline{G_{\mathrm{enh}}}/G)$.

This action can be described explicitly as follows. A G-bundle $\mathcal{F}_G$ on $X \times S$ is equivalent to a G_{enh}-bundle $\mathcal{F}_{G_{\mathrm{enh}}}$ on $X \times S$ together with a trivialization of the induced T_{adj}-bundle $\pi(\mathcal{F}_{G_{\mathrm{enh}}})$. The group $T_{\mathrm{adj}}(S)$ acts on the space of such trivializations, so T_{adj} acts on VinBun_G by leaving the G_{enh}-bundle $\mathcal{F}^1_{G_{\mathrm{enh}}}$ induced by $\mathcal{F}^1_G$ fixed and changing the trivialization of $\pi(\mathcal{F}^1_{G_{\mathrm{enh}}})$.

A.3.4 Fiber bundles

Fix a standard parabolic subgroup P, and consider the open locus $\mathrm{VinBun}_{G,\geq P}$ lying over

$$(\overline{T_{\mathrm{adj}}})_{\geq P} = T/Z(M) \times \overline{Z(M)/Z(G)}.$$

Let $\mathrm{VinBun}_{G,\geq P,\mathrm{strict}} = \bar{\pi}^{-1}_{\mathrm{Bun}}(\{1\} \times \overline{Z(M)/Z(G)})$. Since we have the splitting $T/Z(M) \hookrightarrow T_{\mathrm{adj}}$ (see §A.3.1), the T_{adj}-action on VinBun_G defined in §A.3.3 restricts to a $T/Z(M)$-action on $\mathrm{VinBun}_{G,\geq P}$. This action induces an isomorphism

$$\mathrm{VinBun}_{G,\geq P} \cong \mathrm{VinBun}_{G,\geq P,\mathrm{strict}} \times (T/Z(M)), \tag{A.16}$$

i.e., $\mathrm{VinBun}_{G,\geq P}$ is a trivial fiber bundle over the projection to $T/Z(M)$.

Note that $\mathbf{c}_P$ is the zero element in $\overline{Z(M)/Z(G)}$. Let $(\overline{G_{\mathrm{enh}}})_{\mathbf{c}_P}$ denote the fiber of $\bar{\pi}$ over $\mathbf{c}_P$. Then $\mathrm{VinBun}_{G,\mathbf{c}_P} := \bar{\pi}^{-1}_{\mathrm{Bun}}(\mathbf{c}_P)$ is equal to the stack

$$\mathrm{Maps}^\circ(X, G\backslash(\overline{G_{\mathrm{enh}}})_{\mathbf{c}_P}/G)$$

where the superscript $^\circ$ denotes the open substack of maps generically landing in the non-degenerate locus. Intersecting (A.16) with the P-locus gives

$$\mathrm{VinBun}_{G,P} \cong \mathrm{VinBun}_{G,\mathbf{c}_P} \times (T/Z(M)). \tag{A.17}$$

In the case $P = G$, the G-locus $\mathrm{VinBun}_{G,G}$ is isomorphic to $\mathrm{Bun}_G \times T_{\mathrm{adj}}$.

It is known (cf. [24, Appendix C]) that the $G \times G$-action on $\bar{\mathfrak{s}}(\mathbf{c}_P) \in \overset{\circ}{\overline{G_{\mathrm{enh}}}}$ induces an isomorphism

$$\mathbb{X}_P := (G \times G)/(P \underset{M}{\times} P^-) \cong (\overset{\circ}{\overline{G_{\mathrm{enh}}}})_{\mathbf{c}_P}.$$

We learned of the following lemma from [62, Lemma 2.1.11].

Lemma A.3.2. *The variety $(\overline{G_{\mathrm{enh}}})_{\mathbf{c}_P}$ is isomorphic to $\overline{\mathbb{X}}_P$.*

Proof. By [58, Theorem 7], the irreducible affine variety $(\overline{G_{\mathrm{enh}}})_{\mathbf{c}_P}$ is normal. Since $(\overline{G_{\mathrm{enh}}})_{\mathbf{c}_P}$ contains the non-degenerate locus $(\overset{\circ}{\overline{G_{\mathrm{enh}}}})_{\mathbf{c}_P} \cong \mathbb{X}_P$ as a dense open subscheme, the extension property of regular functions on normal varieties implies that it suffices to show that the degenerate locus in $(\overline{G_{\mathrm{enh}}})_{\mathbf{c}_P}$ has codimension at least 2. This can be checked by considering the combinatorial description of $G \times G$-orbits in $\overline{G_{\mathrm{enh}}}$ from [58, Theorem 6]. In characteristic 0, this is the same combinatorial description as in [67]. □

Lemma A.3.2 implies that $\mathrm{VinBun}_{G,\mathbf{c}_P}$ is isomorphic to $\mathrm{Maps}^\circ(X, G\backslash\overline{\mathbb{X}}_P/G)$, where the superscript $^\circ$ denotes the open locus of maps generically landing in $G\backslash\mathbb{X}_P/G$.

A.3.5 Defect stratification

Define the closed embedding $M \hookrightarrow \mathbb{X}_P$ as the composition of the closed embeddings $M \hookrightarrow G/U : m \mapsto mU$ and $G/U \hookrightarrow \mathbb{X}_P : g \mapsto (g, 1)$. From Corollary 1.4.3 we know that $M \hookrightarrow \mathbb{X}_P$ extends to a closed embedding $\overline{M} \hookrightarrow \overline{\mathbb{X}}_P$. This induces a map

$$\mathrm{Maps}^\circ(X, P\backslash\overline{M}/P^-) \to \mathrm{Maps}^\circ(X, G\backslash\overline{\mathbb{X}}_P/G) = \mathrm{VinBun}_{G,\mathbf{c}_P},$$

where $\mathrm{Maps}^\circ(X, P\backslash\overline{M}/P^-)$ is the stack of maps $X \times S \to P\backslash\overline{M}/P^-$ that generically land in $P\backslash M/P^-$ over every geometric point of S.

Let $\mathcal{H}_M^+ = \mathrm{Maps}^\circ(X, M\backslash\overline{M}/M)$ denote the stack introduced in §A.1.5. Recall that there are two maps $\overleftarrow{h}, \overrightarrow{h} : \mathcal{H}_M^+ \to \mathrm{Bun}_M$. Observe that there is a canonical isomorphism

$$\mathrm{Maps}^\circ(X, P\backslash\overline{M}/P^-) \cong \mathrm{Bun}_P \underset{\mathrm{Bun}_M, \overleftarrow{h}}{\times} \mathcal{H}_M^+ \underset{\overrightarrow{h}, \mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}.$$

Thus we have a map of stacks

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_M^+ \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-} \to \mathrm{VinBun}_{G,\mathbf{c}_P}. \tag{A.18}$$

Let $\Lambda_{G,P} = \pi_1(M)$ denote the quotient of Λ by the subgroup generated by the coroots of M. Recall that there is a bijection $\pi_0(\mathrm{Bun}_M) \cong \Lambda_{G,P}$. Let Bun_M^μ, $\mu \in \Lambda_{G,P}$ denote the corresponding connected component consisting of M-bundles of degree μ.

Let Bun_P^μ (resp. $\mathrm{Bun}_{P^-}^\lambda$) denote the preimage of Bun_M^μ (resp. Bun_M^λ) under the projection $\mathrm{Bun}_P \to \mathrm{Bun}_M$ (resp. $\mathrm{Bun}_{P^-} \to \mathrm{Bun}_M$).

For $\mu, \lambda \in \Lambda_{G,P}$, let $\mathcal{H}_M^{+,\mu,\lambda}$ denote the preimage of $\mathrm{Bun}_M^\mu \times \mathrm{Bun}_M^\lambda$ under $(\overleftarrow{h}, \overrightarrow{h})$. One can check using Remark A.1.9 that $\mathcal{H}_M^{+,\mu,\lambda}$ is nonempty if and only if $\mu - \lambda$ lies in the image of $\Lambda_U^{\mathrm{pos},\mathbb{Q}} \cap \Lambda$ under the projection $\Lambda \to \Lambda_{G,P}$.

Proposition A.3.3 ([62, Proposition 3.2.2])**.** *(i) For $\mu, \lambda \in \Lambda_{G,P}$, the restriction of* (A.18) *to the corresponding substack*

$$\mathrm{Bun}_P^\mu \underset{\mathrm{Bun}_M^\mu}{\times} \mathcal{H}_M^{+,\mu,\lambda} \underset{\mathrm{Bun}_M^\lambda}{\times} \mathrm{Bun}_{P^-}^\lambda \to \mathrm{VinBun}_{G,\mathbf{c}_P} \tag{A.19}$$

is a locally closed embedding. Let $\mathrm{VinBun}_{G,\mathbf{c}_P}^{\mu,\lambda}$ *denote the corresponding locally closed substack.*

(ii) The locally closed substacks $\mathrm{VinBun}_{G,\mathbf{c}_P}^{\mu,\lambda}$ *form a stratification of* $\mathrm{VinBun}_{G,\mathbf{c}_P}$*. In particular, the map* (A.18) *is a bijection at the level of k-points.*

We call the stratification above the *defect stratification* of $\mathrm{VinBun}_{G,\mathbf{c}_P}$. By (A.17), we have an identical stratification of $\mathrm{VinBun}_{G,P}$ with strata

$$\mathrm{VinBun}_{G,P}^{\mu,\lambda} \cong \mathrm{VinBun}_{G,\mathbf{c}_P}^{\mu,\lambda} \times (T/Z(M)).$$

A.3.6 Remarks on the case when $[G, G]$ is not simply connected

Note that we have not assumed that $[G, G]$ is simply connected in this Appendix (unlike in Appendices A.1–A.2). Without this assumption, it is possible for the image of $\Lambda_U^{\mathrm{pos},\mathbb{Q}} \cap \Lambda$ in $\Lambda_{G,P}$ to be larger than $\Lambda_{G,P}^{\mathrm{pos}}$. Then Lemma A.3.11 below shows that the G-stratum $\mathrm{VinBun}_{G,G}$ is not necessarily dense in VinBun_G when $[G, G]$ is not simply connected.

We define a slightly different stack $\mathrm{VinBun}_G^{\mathrm{true}}$. We suggest that $\mathrm{VinBun}_G^{\mathrm{true}}$ is the more "philosophically correct" definition for VinBun_G when $[G, G]$ is not simply connected.

Remark A.3.4. We will continue using the original stack VinBun_G in the rest of this chapter (for arbitrary G) as it suffices for our purposes, but it is also possible to work directly with $\mathrm{VinBun}_G^{\mathrm{true}}$ everywhere.

The idea is to replace the Vinberg semigroup $\overline{G_{\mathrm{enh}}}$, which is an algebraic monoid, by its stacky version, which is an algebraic monoidal stack[2].

Let G^{ssc} denote the universal cover of $[G, G]$ (as algebraic groups). Then $Z(G^{\mathrm{ssc}})$ is a finite group scheme containing $\ker(G^{\mathrm{ssc}} \to G)$. Since $G = Z(G) \cdot [G, G]$, we have an isomorphism

$$G \cong (G^{\mathrm{ssc}} \times Z(G))/Z(G^{\mathrm{ssc}}), \tag{A.20}$$

where $Z(G^{\mathrm{ssc}})$ is embedded in $G^{\mathrm{ssc}} \times Z(G)$ anti-diagonally. Furthermore, the following is well-known (cf. [67, 57]):

Lemma A.3.5. *The isomorphism* (A.20) *extends to an isomorphism of the Vinberg semigroup $\overline{G_{\mathrm{enh}}}$ with the the GIT quotient of $\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}} \times Z(G)$ by the anti-diagonal action of $Z(G^{\mathrm{ssc}})$.*

2. An algebraic monoidal stack $\mathcal{Y}$ is an algebraic stack $\mathcal{Y}$ with a coherently associative composition law $\mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$ and a morphism $\mathrm{Spec}(k) \to \mathcal{Y}$ such that for any scheme S, the composition $S \to \mathrm{Spec}(k) \to \mathcal{Y}$ is a unit object of $\mathcal{Y}(S)$.

Lemma A.3.5 motivates us to define the stacky version of the Vinberg semigroup as the quotient stack

$$(\overline{G_{\mathrm{enh}}})^{\mathrm{true}} := (\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}} \times Z(G))/Z(G^{\mathrm{ssc}}). \tag{A.21}$$

This is an algebraic monoidal stack. By Lemma A.3.5, we have a canonical map

$$(\overline{G_{\mathrm{enh}}})^{\mathrm{true}} \to \overline{G_{\mathrm{enh}}} \tag{A.22}$$

from the stack quotient to the GIT quotient, and we see that $\overline{G_{\mathrm{enh}}}$ is the coarse moduli space of $(\overline{G_{\mathrm{enh}}})^{\mathrm{true}}$.

By (A.20), we see that the homomorphism (A.22) restricts to an isomorphism of groups $G^{\mathrm{ssc}}_{\mathrm{enh}} \times^{Z(G^{\mathrm{ssc}})} Z(G) \cong G_{\mathrm{enh}}$, so we have an open embedding

$$G_{\mathrm{enh}} \hookrightarrow (\overline{G_{\mathrm{enh}}})^{\mathrm{true}}.$$

Moreover, $Z(G^{\mathrm{ssc}}_{\mathrm{enh}})$ acts freely[3] on the non-degenerate locus $\overset{\circ}{\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}}$. Thus the open substack

$$(\overset{\circ}{\overline{G_{\mathrm{enh}}}})^{\mathrm{true}} := (\overset{\circ}{\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}} \times Z(G))/Z(G^{\mathrm{ssc}})$$

is representable by a scheme, and Lemma A.3.5 implies that (A.22) restricts to an isomorphism $(\overset{\circ}{\overline{G_{\mathrm{enh}}}})^{\mathrm{true}} \cong \overset{\circ}{\overline{G_{\mathrm{enh}}}}$ on non-degenerate loci.

Lemma A.3.6. *The map* (A.22) *is an isomorphism if and only if* $[G,G]$ *is simply connected.*

Proof. Recall that $\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}$ is an algebraic monoid with zero. Thus the action of $\ker(G^{\mathrm{ssc}} \to [G,G])$ on $\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}$ is free if and only if the kernel is trivial, i.e., $[G,G]$ is simply connected. Therefore if $[G,G]$ is not simply connected, the stack quotient $(\overline{G_{\mathrm{enh}}})^{\mathrm{true}}$ cannot be representable by a scheme.

In the other direction, suppose that $[G,G]$ is simply connected. Then $Z(G^{\mathrm{ssc}}) \subset Z(G)$

3. The quotient $\overset{\circ}{\overline{G_{\mathrm{enh}}}}/Z(G_{\mathrm{enh}})$ is the *wonderful compactification* of $G/Z(G)$.

acts freely on $\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}} \times Z(G)$, so (A.22) is an isomorphism by Lemma A.3.5. $\square$

Definition of $\mathrm{VinBun}_G^{\mathrm{true}}$**.** As explained in §A.3.3, we have an action of $G \times G$ on the stack $(\overline{G_{\mathrm{enh}}})^{\mathrm{true}}$, where we are using the identification (A.20). Define

$$\mathrm{VinBun}_G^{\mathrm{true}} = \mathrm{Maps}^{\circ}(X, G\backslash(\overline{G_{\mathrm{enh}}})^{\mathrm{true}}/G), \tag{A.23}$$

where the superscript $\circ$ denotes the locus of maps that generically land in $G\backslash\overset{\circ}{\overline{G_{\mathrm{enh}}}}/G$ over every geometric point of a test scheme S.

The map (A.22) induces a canonical map of stacks

$$\mathrm{VinBun}_G^{\mathrm{true}} \to \mathrm{VinBun}_G \tag{A.24}$$

over $\mathrm{Bun}_G \times \mathrm{Bun}_G$.

The open embedding $G_{\mathrm{enh}} \hookrightarrow (\overline{G_{\mathrm{enh}}})^{\mathrm{true}}$ induces an open embedding

$$\mathrm{VinBun}_{G,G} \hookrightarrow \mathrm{VinBun}_G^{\mathrm{true}}$$

over VinBun_G.

The projection $\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}} \times Z(G) \to Z(G)$ induces a homomorphism of monoidal stacks $(\overline{G_{\mathrm{enh}}})^{\mathrm{true}} \to Z(G)/Z(G^{\mathrm{ssc}})$. Note that $G^{\mathrm{ab,st}} := Z(G)/Z(G^{\mathrm{ssc}})$ is the stacky abelianization of G defined in [43]. By (A.20), we also get a homomorphism of group stacks $G \to G^{\mathrm{ab,st}}$. The $G \times G$-action on $(\overline{G_{\mathrm{enh}}})^{\mathrm{true}}$ is compatible with these maps to $G^{\mathrm{ab,st}}$.

Consider the map $\mathrm{Spec}(k) \to G^{\mathrm{ab,st}}$ corresponding to $1 \in Z(G)$. We have Cartesian squares

$$\begin{tikzcd} \overline{G^{\mathrm{ssc}}_{\mathrm{enh}}} \arrow[r] \arrow[d] & (\overline{G_{\mathrm{enh}}})^{\mathrm{true}} \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & G^{\mathrm{ab,st}} \end{tikzcd} \qquad \begin{tikzcd} G \underset{G^{\mathrm{ab,st}}}{\times} G \arrow[r] \arrow[d] & G \times G \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & G^{\mathrm{ab,st}} \end{tikzcd}$$

where $G \times G$ maps to $G^{\mathrm{ab,st}}$ by $(g_1, g_2) \mapsto g_1 g_2^{-1}$. Note that $G \times_{G^{\mathrm{ab,st}}} G$ is isomorphic to a semidirect product of G and G^{ssc}. Since $G^{\mathrm{ab,st}}$ is a group stack, it follows formally that

$$(\overline{G_{\mathrm{enh}}})^{\mathrm{true}}/(G\times G)\cong \overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}/(G\underset{G^{\mathrm{ab,st}}}{\times}G).$$

We can also repeat the above discussion at the coarse level: by Lemma A.3.5, we have $\overline{G_{\mathrm{enh}}}=(\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}\times Z(G))/\!\!/Z(G^{\mathrm{ssc}})$, where $/\!\!/$ denotes the GIT quotient. Thus the projection to the second factor induces a homomorphism of monoids $\overline{G_{\mathrm{enh}}}\to Z(G)/\!\!/Z(G^{\mathrm{ssc}})=Z(G)/Z([G,G])=G/[G,G]=:G^{\mathrm{ab}}$ such that the $G\times G$-action on $\overline{G_{\mathrm{enh}}}$ lies over G^{ab}. Let $(\overline{G_{\mathrm{enh}}})_1$ denote the fiber $\overline{G_{\mathrm{enh}}}\times_{G^{\mathrm{ab}}}\mathrm{Spec}(k)$ over $1\in G^{\mathrm{ab}}(k)$. Then it again follows formally that

$$\overline{G_{\mathrm{enh}}}/(G\times G)\cong(\overline{G_{\mathrm{enh}}})_1/G\underset{G^{\mathrm{ab}}}{\times}G.$$

We have a group homomorphism $G\times_{G^{\mathrm{ab,st}}}G\to G\times_{G^{\mathrm{ab}}}G$ whose kernel is isomorphic to $\ker(G^{\mathrm{ssc}}\to[G,G])$. We also have a finite homomorphism of monoids $\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}\to(\overline{G_{\mathrm{enh}}})_1$. Thus we deduce that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{VinBun}_G^{\mathrm{true}} \arrow[d] \arrow[dr] & & \\
\mathrm{Maps}^\circ(X,(\overline{G_{\mathrm{enh}}})_1/(G\times_{G^{\mathrm{ab,st}}}G)) \arrow[r] \arrow[d] & \mathrm{Bun}_{G\times_{G^{\mathrm{ab,st}}}G} \arrow[d] & \\
\mathrm{VinBun}_G \arrow[r] & \mathrm{Bun}_{G\times_{G^{\mathrm{ab}}}G} \arrow[r] & \mathrm{Bun}_G\times\mathrm{Bun}_G
\end{tikzcd}$$

where the square is Cartesian, the superscript $\circ$ denotes the substack of maps generically landing in $(\overset{\circ}{\overline{G_{\mathrm{enh}}}})_1/(G\times_{G^{\mathrm{ab,st}}}G)$, and the composition of the left vertical maps equals the map (A.24). Here we have factored the map (A.24) into a "change of space" map and a "change of group" map. The following lemma is well-known.

Lemma A.3.7. *Let H be a connected reductive group, and let $A\subset Z(H)$ be a finite central subgroup. Then $\mathrm{Bun}_H\to\mathrm{Bun}_{H/A}$ is a torsor by the group stack Bun_A over an open and* *closed substack of* $\mathrm{Bun}_{H/A}$*. More specifically, this substack is the union of the connected components in* $\pi_0(\mathrm{Bun}_{H/A}) = \pi_1(H/A)$ *corresponding to* $\pi_1(H) \subset \pi_1(H/A)$*.*

Proof. Let B be a Borel subgroup of H. Then $\mathrm{Bun}_B \to \mathrm{Bun}_H$ and $\mathrm{Bun}_{B/A} \to \mathrm{Bun}_{H/A}$ are surjective by [26]. Thus to prove the statement about the image of Bun_H in $\mathrm{Bun}_{H/A}$, it suffices to consider $\pi_0(\mathrm{Bun}_B) \to \pi_0(\mathrm{Bun}_{B/A})$. This reduces to an analogous statement in the case where H is a torus, which is straightforward.

It is a standard fact that the action of Bun_A on Bun_H defines an isomorphism between $\mathrm{Bun}_A \times \mathrm{Bun}_H$ and the fiber product $\mathrm{Bun}_H \times_{\mathrm{Bun}_{H/A}} \mathrm{Bun}_H$. Therefore to prove that the map from Bun_H to its image in $\mathrm{Bun}_{H/A}$ is a torsor, we must show that this map is flat. The map is flat because it is a morphism between smooth stacks of the same dimension with 0-dimensional fibers. $\square$

We now consider the "change of space" map

$$\mathrm{VinBun}_G^{\mathrm{true}} = \mathrm{Maps}^\circ\left(X, \overline{G_{\mathrm{enh}}^{\mathrm{ssc}}}/(G \underset{G^{\mathrm{ab,st}}}{\times} G)\right) \to \mathrm{Maps}^\circ\left(X, (\overline{G_{\mathrm{enh}}})_1/(G \underset{G^{\mathrm{ab,st}}}{\times} G)\right). \tag{A.25}$$

Lemma A.3.8. *Let* $\mathcal{Y}_1 \to \mathcal{Y}_2$ *be a finite schematic morphism of stacks. Then the induced morphism* $\mathrm{Maps}(X, \mathcal{Y}_1) \to \mathrm{Maps}(X, \mathcal{Y}_2)$ *is also finite schematic.*

Proof. Fix a test scheme S and a map $X_S := X \times S \to \mathcal{Y}_2$. Let Y denote the fiber product $X_S \times_{\mathcal{Y}_2} \mathcal{Y}_1$, which is representable by a finite scheme over X_S. Then the corresponding fiber product of S and $\mathrm{Maps}(X, \mathcal{Y}_1)$ over $\mathrm{Maps}(X, \mathcal{Y}_2)$ is representable by the S-scheme $\mathrm{Sect}(X_S, Y)$ of sections of $Y \to X_S$. It is well-known that since $Y \to X_S$ is affine, $\mathrm{Sect}(X_S, Y) \to S$ is also affine. Therefore to show that $\mathrm{Sect}(X_S, Y) \to S$ is finite, it suffices to show that it is proper. We use the valuative criterion of properness:

Let R be a discrete valuation ring with field of fractions K, and suppose that we have a map $\mathrm{Spec}(R) \to S$ and a section $X_K := X \times \mathrm{Spec}(K) \to Y$ over X_S. This section and the natural map $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$ define a section $X_K \to Y_R := Y \times_S \mathrm{Spec}(R)$. Let Z_K

denote the image of $X_K \to Y_R$, and let Z_R denote the scheme-theoretic closure of Z_K in Y_R. Extending the section $X_K \to Y$ to a section $X_R := X \times \mathrm{Spec}(R) \to Y$ is equivalent to showing that the projection $Z_R \to X_R$ is an isomorphism. Note that Z_R is an integral scheme (because Z_K is) and the map $Z_R \to X_R$ is birational (because the map $Z_K \to X_K$ is an isomorphism). On the other hand, the map $Z_R \to X_R$ is finite since $Y_R \to X_R$ is finite. Lastly, smoothness of X implies that X_R is a regular scheme. Hence X_R is normal, and $Z_R \to X_R$ is an isomorphism. This checks the condition of the valuative criterion and hence proves the lemma. $\square$

Let A denote the finite abelian group scheme $\ker(G^{\mathrm{ssc}} \to [G, G])$.

Corollary A.3.9. *The map* (A.25) *is schematic and finite. More specifically, it is the composition of an A-torsor followed by a closed embedding.*

Proof. The map $\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}} \to (\overline{G_{\mathrm{enh}}})_1$ is finite, and the preimage of $(\overset{\circ}{\overline{G_{\mathrm{enh}}}})_1$ equals $\overset{\circ}{\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}}$. Then Lemma A.3.8 implies that the map (A.25) is schematic and finite. Let

$$\mathcal{Y} \subset \mathrm{Maps}^{\circ}(X, (\overline{G_{\mathrm{enh}}})_1 / (G \underset{G^{\mathrm{ab,st}}}{\times} G))$$

denote the (scheme-theoretic) image, which is a closed substack. Take $(\mathcal{P}, \beta) \in \mathcal{Y}(S)$ for a test scheme S, where $\mathcal{P}$ is a $G \times_{G^{\mathrm{ab,st}}} G$-torsor over $X_S := X \times S$ and β is a section $X_S \to ((\overline{G_{\mathrm{enh}}})_1)_{\mathcal{P}}$ over X_S such that $\beta|_{\mathrm{Spec}(F) \times S}$ lands in $((\overset{\circ}{\overline{G_{\mathrm{enh}}}})_1)_{\mathcal{P}}$. Since $\overset{\circ}{\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}} \to (\overset{\circ}{\overline{G_{\mathrm{enh}}}})_1$ is an A-torsor, the set of sections $\tilde{\beta}|_{\mathrm{Spec}(F) \times S} : \mathrm{Spec}(F) \times S \to ((\overset{\circ}{\overline{G_{\mathrm{enh}}}})_1)_{\mathcal{P}}$ lifting β has a simply transitive action by $A(\mathrm{Spec}(F) \times S)$. Since A is finite over k and X is geometrically connected, we deduce that $A(\mathrm{Spec}(F) \times S) = A(S)$. Thus the canonical $A(S)$-action on the set of sections $\tilde{\beta} : X_S \to (\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}})_{\mathcal{P}}$ lifting β is simply transitive. By definition of $\mathcal{Y}$, a lift $\tilde{\beta}$ exists after restricting along some fppf covering $S' \to S$. We conclude that $\mathrm{VinBun}_G^{\mathrm{true}} \to \mathcal{Y}$ is an A-torsor. $\square$

Proposition A.3.10. *The map* $\mathrm{VinBun}_G^{\mathrm{true}} \to \mathrm{VinBun}_G$ *is finite schematic.*

Proof. We first show that the map $\mathrm{VinBun}_G^{\mathrm{true}} \to \mathrm{VinBun}_G$ is proper. By Lemma A.3.7, the map $\mathrm{Bun}_{G \times_{G^{\mathrm{ab,st}}} G} \to \mathrm{Bun}_{G \times_{G^{\mathrm{ab}}} G}$ is proper. Thus by base change, the map

$$\mathrm{Maps}^{\circ}\left(X, (\overline{G_{\mathrm{enh}}})_1 / (G \underset{G^{\mathrm{ab,st}}}{\times} G)\right) \to \mathrm{Maps}^{\circ}\left(X, (\overline{G_{\mathrm{enh}}})_1 / (G \underset{G^{\mathrm{ab}}}{\times} G)\right) = \mathrm{VinBun}_G$$

is proper. Composing this map with (A.25), which is finite by Corollary A.3.9, we conclude that $\mathrm{VinBun}_G^{\mathrm{true}} \to \mathrm{VinBun}_G$ is proper.

Next we prove that the map $\mathrm{VinBun}_G^{\mathrm{true}} \to \mathrm{VinBun}_G$ is schematic. Let S be an affine scheme. A map $S \to \mathrm{VinBun}_G$ is the datum of a $G \times_{G^{\mathrm{ab}}} G$-torsor $\mathcal{P}$ on $X \times S$ and a $G \times_{G^{\mathrm{ab}}} G$-equivariant map $\mathcal{P} \to (\overline{G_{\mathrm{enh}}})_1$. Moreover, there is an open subset $\overset{\circ}{X} \subset X$ such that $\mathcal{P}|_{\overset{\circ}{X} \times S}$ is sent to $(\overset{\circ}{\overline{G_{\mathrm{enh}}}})_1$. Recall that we have a surjective homomorphism $G \times_{G^{\mathrm{ab,st}}} G \to G \times_{G^{\mathrm{ab}}} G$ with kernel $A := \ker(G^{\mathrm{ssc}} \to [G, G])$. Then an S'-point of the fiber product $\mathcal{Y} := S \times_{\mathrm{VinBun}_G} \mathrm{VinBun}_G^{\mathrm{true}}$ parametrizes a $G \times_{G^{\mathrm{ab,st}}} G$-torsor $\tilde{\mathcal{P}}$ on $X \times S'$ and a $G \times_{G^{\mathrm{ab,st}}} G$-equivariant map $\tilde{\beta} : \tilde{\mathcal{P}} \to \overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}$ such that the $G \times_{G^{\mathrm{ab}}} G$-torsor induced by $\tilde{\mathcal{P}}$ is isomorphic to $\mathcal{P}|_{X \times S'}$, and the diagram

$$\begin{tikzcd}
\tilde{\mathcal{P}} \arrow[r, "\tilde{\beta}"] \arrow[d] & \overline{G^{\mathrm{ssc}}_{\mathrm{enh}}} \arrow[d] \\
\mathcal{P}|_{X \times S'} \arrow[r] & (\overline{G_{\mathrm{enh}}})_1
\end{tikzcd}$$

commutes. This implies that $\tilde{\mathcal{P}}|_{\overset{\circ}{X} \times S'}$ lands in $\overset{\circ}{\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}}$. Since $\overset{\circ}{\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}} \to (\overset{\circ}{\overline{G_{\mathrm{enh}}}})_1$ is an A-torsor, we get an isomorphism

$$\tilde{\mathcal{P}}|_{\overset{\circ}{X} \times S'} \cong \mathcal{P}|_{\overset{\circ}{X} \times S'} \underset{(\overset{\circ}{\overline{G_{\mathrm{enh}}}})_1}{\times} \overset{\circ}{\overline{G^{\mathrm{ssc}}_{\mathrm{enh}}}},$$

where the r.h.s. only depends on the map $S \to \mathrm{VinBun}_G$. Thus $(\tilde{\mathcal{P}}, \tilde{\beta})$ are determined by their restrictions to the formal completion of $(X - \overset{\circ}{X}) \times S'$ in $X \times S'$. Using twisted versions of the affine Grassmannian, we deduce that the fiber product $\mathcal{Y}$ is a closed subscheme of a projective ind-scheme over S. Since $\mathcal{Y}$ is of finite type, we conclude that $\mathcal{Y}$ is a scheme.

We have shown that $\mathrm{VinBun}_G^{\mathrm{true}} \to \mathrm{VinBun}_G$ is a proper schematic map. One observes from Lemma A.3.7 and Corollary A.3.9 that $\mathrm{VinBun}_G^{\mathrm{true}} \to \mathrm{VinBun}_G$ is also quasi-finite. Therefore this map is finite schematic. $\square$

Lemma A.3.11. *The closure of the open substack* $\mathrm{VinBun}_{G,G}$ *in* VinBun_G *intersects the stratum* $\mathrm{VinBun}_{G,P}^{\mu,\lambda}$ *only if* $\mu - \lambda \in \Lambda_{G,P}^{\mathrm{pos}}$.

Proof. The image of the proper map $\mathrm{VinBun}_G^{\mathrm{true}} \to \mathrm{VinBun}_G$ is a closed substack containing the G-stratum. Let P be a standard parabolic subgroup of G with Levi factor M. Let $\tilde{G}$ denote $G^{\mathrm{ssc}} \times Z(G)$, and let $\tilde{M}, \tilde{P}$ denote the preimages of M, P under the isogeny $\tilde{G} \to G$. We have the corresponding boundary degeneration $\mathbb{X}_{\tilde{P}} \subset \overline{\tilde{G}_{\mathrm{enh}}}$ and its affine closure $\overline{\mathbb{X}}_{\tilde{P}}$. Define the closed embedding $\tilde{M} \hookrightarrow \mathbb{X}_{\tilde{P}}$ as in §A.3.5, and let $\overline{\tilde{M}}$ denote the closure of $\tilde{M}$ in $\overline{\mathbb{X}}_{\tilde{P}}$. For a place v of X, Remark A.1.9 implies that

$$\tilde{M}(\mathfrak{o}_v)\backslash(\overline{\tilde{M}}(\mathfrak{o}_v) \cap \tilde{M}(F_v))/\tilde{M}(\mathfrak{o}_v) = \Lambda_U^{\mathrm{pos},\mathbb{Q}} \cap \Lambda_{\tilde{G}},$$

and $\Lambda_U^{\mathrm{pos},\mathbb{Q}} \cap \Lambda_{\tilde{G}} = \Lambda_U^{\mathrm{pos}}$ since $[\tilde{G}, \tilde{G}]$ is simply connected. Thus we deduce from the construction of the defect stratification in §A.3.5 that the image of $\mathrm{VinBun}_G^{\mathrm{true}} \to \mathrm{VinBun}_G$ intersects $\mathrm{VinBun}_{G,P}^{\mu,\lambda}$ if and only if $\mu - \lambda \in \Lambda_{G,P}^{\mathrm{pos}}$. This implies the lemma. $\square$

A.3.7 The function b

Suppose $k = \mathbb{F}_q$. Let

$$\Delta : \mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$$

denote the diagonal morphism. Given G-bundles $\mathcal{F}_G^1, \mathcal{F}_G^2 \in \mathrm{Bun}_G(\mathbb{F}_q)$, let $b(\mathcal{F}_G^1, \mathcal{F}_G^2)$ denote the trace of the geometric Frobenius acting on the $*$-stalk of the complex $\Delta_*(\overline{\mathbb{Q}}_\ell)$ over the point $(\mathcal{F}_G^1, \mathcal{F}_G^2) \in (\mathrm{Bun}_G \times \mathrm{Bun}_G)(\overline{\mathbb{F}}_q)$.

Recall from §2.4.2 that $\mathrm{Asymp}_P(\delta_K)$ is a $K \times K$-invariant function in $C_b^\infty(\mathbb{X}_P(\mathbb{A}))$, where δ_K is the characteristic function of K on $G(\mathbb{A})$. Let

$$\beta : X \to (\overline{\mathbb{X}}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2}$$

denote a section that generically lands in the non-degenerate locus $(\mathbb{X}_P)_{\mathcal{F}_G^1,\mathcal{F}_G^2}$.

Then for any $v \in |X|$, choosing trivializations of $\mathcal{F}_G^i \times_X \operatorname{Spec}(\mathfrak{o}_v)$ defines an isomorphism $(\overline{\mathbb{X}}_P)_{\mathcal{F}_G^2,\mathcal{F}_G^2}(\mathfrak{o}_v) \cong \overline{\mathbb{X}}_P(\mathfrak{o}_v)$. This defines an element $\beta_v \in \mathbb{X}_P(F_v) \cap \overline{\mathbb{X}}_P(\mathfrak{o}_v)$, and the $K_v \times K_v$-orbit of β_v does not depend on the choice of trivializations. Non-degeneracy of β implies that $(\beta_v) \in \mathbb{X}_P(\mathbb{A})$. We define $\operatorname{Asymp}_P(\delta_K)(\beta)$ to be the value of $\operatorname{Asymp}_P(\delta_K)$ at this adelic point.

Theorem A.3.12. *Let $E = \overline{\mathbb{Q}}_\ell$. We have an equality*

$$b(\mathcal{F}_G^1, \mathcal{F}_G^2) = \sum_P (-1)^{\dim Z(M)} \sum_\beta \operatorname{Asymp}_P(\delta_K)(\beta),$$

where P ranges over the standard parabolic subgroups of G, and β ranges over the non-degenerate sections $\beta : X \to (\overline{\mathbb{X}}_P)_{\mathcal{F}_G^1,\mathcal{F}_G^2}$.

The strategy for proving Theorem A.3.12 was suggested by Drinfeld, and it consists of compactifying the diagonal morphism of Bun_G. The geometry of the compactification then reduces to a theorem of [62], and the corresponding Grothendieck functions are computed using the facts reviewed in Appendix A.2. The proof of Theorem A.3.12 is given at the end of §A.3.9.

A.3.8 Compactifications of the diagonal morphism of Bun_G

The diagonal morphism Δ is in general not proper, and one would like to compactify it (e.g., to compute $*$-restrictions of Δ_*). We first review the definition of the stack $\overline{\operatorname{Bun}}'_G$ (denoted by $\overline{\operatorname{Bun}}_G$ in [62]), which is a compactification of the morphism $\operatorname{Bun}_G \times \mathbb{B}Z(G) \to \operatorname{Bun}_G \times \operatorname{Bun}_G$, which Δ factors through. For the purposes of this thesis, we define a slightly different stack $\overline{\operatorname{Bun}}_G$, which is a compactification of Δ when G is semisimple. When G is not semisimple, $\overline{\operatorname{Bun}}_G$ is not quite a compactification of Δ, but it is equally good for our purposes.

$\overline{\operatorname{Bun}}'_G$**.** The action of $Z(G_{\mathrm{enh}}) = T$ on $\overline{G_{\mathrm{enh}}}$ induces a T-action on VinBun_G. Define $\overline{\operatorname{Bun}}'_G = \operatorname{VinBun}_G / T$. There is an open embedding $\operatorname{VinBun}_{G,G} / T = \operatorname{Bun}_G \times \mathbb{B}Z(G) \hookrightarrow$

$\overline{\mathrm{Bun}}'_G$. Observe that Δ factors as

$$\mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathbb{B}Z(G) \hookrightarrow \overline{\mathrm{Bun}}'_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G,$$

where $\mathbb{B}Z(G)$ is the classifying stack of $Z(G)$-bundles. The following lemma is well-known:

Lemma A.3.13. *The map $\overline{\mathrm{Bun}}'_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$ is schematic and projective*[4].

Proof. Let $\mathcal{F}^1_G, \mathcal{F}^2_G \in \mathrm{Bun}_G(S)$ for a test scheme S. Let $\mathrm{Sect}(X_S, (\overline{G_{\mathrm{enh}}})_{\mathcal{F}^1_G, \mathcal{F}^2_G})$ denote the S-scheme of sections for the fiber bundle $(\overline{G_{\mathrm{enh}}})_{\mathcal{F}^1_G, \mathcal{F}^2_G} \to X_S := X \times S$. Then

$$\overline{\mathrm{Bun}}'_G \underset{\mathrm{Bun}_G \times \mathrm{Bun}_G}{\times} S \cong \mathrm{Sect}^\circ(X_S, (\overline{G_{\mathrm{enh}}})_{\mathcal{F}^1_G, \mathcal{F}^2_G})/T, \tag{A.26}$$

where the superscript $^\circ$ denotes the open locus of sections generically landing in $(\overset{\circ}{\overline{G_{\mathrm{enh}}}})_{\mathcal{F}^1_G, \mathcal{F}^2_G}$. We wish to show (A.26) is a projective scheme over S.

Let $\Delta(\check{\lambda})$ denote the Weyl G-module of highest weight $\check{\lambda} \in \check{\Lambda}^+_G$. It is known from the general theory of reductive monoids (cf. proof of Proposition 1.2.2) that there exists a finite[5] map

$$\overline{G_{\mathrm{enh}}} \to \prod \mathrm{End}(\Delta(\check{\lambda}) \otimes k_{\check{\lambda}}) \times \overline{T_{\mathrm{adj}}}, \tag{A.27}$$

where the product ranges over any *finite* set of generators for the monoid $\check{\Lambda}^+_G$. The image of (A.27) satisfies the Plücker relations (cf. [13, 12]). Therefore by considering the G_{enh}-modules $\det(\Delta(\check{\lambda}) \otimes k_{\check{\lambda}}) = \det(\Delta(\check{\lambda})) \otimes k_{\dim(\Delta(\check{\lambda}))\check{\lambda}}$, we see that the composition of (A.27) with the projection to $\prod \mathrm{End}(\Delta(\check{\lambda}) \otimes k_{\check{\lambda}})$ is a finite map. By Lemma A.3.8, we can reduce to showing that

$$\mathrm{Sect}^\circ(X_S, \prod \mathrm{End}(\Delta(\check{\lambda}) \otimes k_{\check{\lambda}})_{\mathcal{F}^1_G, \mathcal{F}^2_G})/T \tag{A.28}$$

4. The proof shows that there exists a coherent sheaf $\mathcal{F}$ on the stack $\mathrm{Bun}_G \times \mathrm{Bun}_G$ and a closed embedding $\overline{\mathrm{Bun}}'_G \hookrightarrow \mathbb{P}(\mathcal{F})$.

5. In the particular case of the Vinberg semigroup, one can deduce that this map is a closed embedding using [15, Exercises 6.1E, 6.2E].

is representable by a projective scheme over S. Here the superscript $^\circ$ denotes the locus of maps generically landing in $\prod(\mathrm{End}(\Delta(\check{\lambda}) \otimes k_{\check{\lambda}}) - \{0\})_{\mathcal{F}_G^1, \mathcal{F}_G^2}$.

For a test scheme $S' \to S$, an S'-point of the stack (A.28) is the data of $((\mathcal{F}_T)_{S'}, \beta_{\check{\lambda}})$ where $(\mathcal{F}_T)_{S'}$ is a T-bundle over S' and $\beta_{\check{\lambda}}$ is an $\mathcal{O}_{X \times S'}$-module map

$$\Delta(\check{\lambda})_{\mathcal{F}_G^2} \underset{\mathcal{O}_S}{\otimes} \mathcal{L}_{\check{\lambda}} \to \Delta(\check{\lambda})_{\mathcal{F}_G^1} \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_{S'}$$

that is generically nonzero over all geometric points of S', where $\mathcal{L}_{\check{\lambda}} = (k_{\check{\lambda}})_{(\mathcal{F}_T)_{S'}}$ is the corresponding line bundle on S'. Observe that $\beta_{\check{\lambda}}$ is equivalent to an S'-fiberwise nonzero map

$$\pi'^*(\mathcal{L}_{\check{\lambda}}) \to (\Delta(\check{\lambda})^*_{\mathcal{F}_G^2} \otimes \Delta(\check{\lambda})_{\mathcal{F}_G^1}) \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_{S'}, \tag{A.29}$$

where π' is the projection $X \times S' \to S'$. Set $\mathcal{E} = \Delta(\check{\lambda})^*_{\mathcal{F}_G^2} \otimes \Delta(\check{\lambda})_{\mathcal{F}_G^1}$, which is a locally free $\mathcal{O}_{X \times S}$-module. Let π denote the projection $X \times S \to S$, and observe that $\pi_*(\mathcal{E})$ is a perfect complex that commutes with base change (here and elsewhere, π_* denotes the derived direct image functor). Then by adjunction, (A.29) is equivalent to a map in the derived category of coherent sheaves on S'

$$\mathcal{L}_{\check{\lambda}} \to \pi_*(\mathcal{E}) \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_{S'}$$

that is nonzero on every fiber of S' (here the tensor product is derived). Applying derived $\mathcal{H}om(?, \mathcal{O}_{S'})$, this map is equivalent to a fiberwise nonzero map

$$\mathcal{H}om(\pi_*(\mathcal{E}), \mathcal{O}_S) \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_{S'} \to \mathcal{L}^{-1}_{\check{\lambda}}. \tag{A.30}$$

Since $\pi_*(\mathcal{E}) \otimes_{\mathcal{O}_S} k_s$ lives in cohomological degrees $0, 1$ for any point $\mathrm{Spec}(k_s) \to S$, we deduce that $\pi_*(\mathcal{E})$ is locally quasi-isomorphic to a complex of locally free $\mathcal{O}_S$-modules living in degrees $0, 1$. Therefore $\mathcal{H}om(\pi_*(\mathcal{E}), \mathcal{O}_S)$ lives in cohomological degrees $-1, 0$. We conclude that (A.30) is equivalent to a surjection of $\mathcal{O}_{S'}$-modules

$$H^0\mathcal{H}om(\pi_*(\mathcal{E}),\mathcal{O}_S)\underset{\mathcal{O}_S}{\otimes}\mathcal{O}_{S'}\to\mathcal{L}_{\check{\lambda}}^{-1},$$

and $H^0\mathcal{H}om(\pi_*(\mathcal{E}),\mathcal{O}_S)$ commutes with base change.

We have shown that for fixed $\check{\lambda}$, the data $(\mathcal{L}_{\check{\lambda}}^{-1},\beta_{\check{\lambda}})$ defines an S'-point of the projective S-scheme $\mathbf{Proj}_S\,\mathrm{Sym}_{\mathcal{O}_S}(H^0\mathcal{H}om(\pi_*\mathcal{E},\mathcal{O}_S))$. By the Plücker relations, we conclude that the stack (A.28) is representable by a closed subscheme of a projective S-scheme. $\square$

Let $Z_0(G)$ denote the neutral connected component of the center of G. Then we have a finite map $T/Z_0(G)\to T/Z(G)=T_{\mathrm{adj}}$. The character lattice of T_{adj} corresponds to the root lattice in $\check{\Lambda}$. If the Langlands dual group $\check{G}$ does not have a simply connected derived group, then the root lattice is not saturated in $\check{\Lambda}$, so $T/Z_0(G)\neq T/Z(G)$ in general.

Recall that $k[\overline{T_{\mathrm{adj}}}]$ is the semigroup algebra of $\check{\Lambda}_G^{\mathrm{pos}}$. Define $\overline{T/Z_0(G)}$ so that $k[\overline{T/Z_0(G)}]$ is the semigroup algebra of $\check{\Lambda}_G^{\mathrm{pos},\mathbb{Q}}\cap\check{\Lambda}$. There is a natural finite map

$$\overline{T/Z_0(G)}\to\overline{T_{\mathrm{adj}}}$$

extending the map $T/Z_0(G)\to T_{\mathrm{adj}}$.

$\overline{\mathrm{Bun}}_G$**.** Consider the base change

$$(\mathrm{VinBun}_G)_{\overline{T/Z_0(G)}}:=\mathrm{VinBun}_G\underset{\overline{T_{\mathrm{adj}}}}{\times}\overline{T/Z_0(G)}$$

over $\overline{T/Z_0(G)}$. Then T acts diagonally on $(\mathrm{VinBun}_G)_{\overline{T/Z_0(G)}}$, and we define

$$\overline{\mathrm{Bun}}_G=(\mathrm{VinBun}_G)_{\overline{T/Z_0(G)}}/T.$$

Since $\bar{\pi}_{\mathrm{Bun}}^{-1}(T_{\mathrm{adj}})=\mathrm{VinBun}_{G,G}=\mathrm{Bun}_G\times T_{\mathrm{adj}}$, we see that there is an open embedding $\mathrm{Bun}_G\times\mathbb{B}Z_0(G)\hookrightarrow\overline{\mathrm{Bun}}_G$. There is a natural finite map $\overline{\mathrm{Bun}}_G\to\overline{\mathrm{Bun}}'_G$, and we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Bun}_G \arrow[r] & \mathrm{Bun}_G \times \mathbb{B}Z_0(G) \arrow[r, hook] \arrow[d] & \overline{\mathrm{Bun}}_G \arrow[d] & \\
 & \mathrm{Bun}_G \times \mathbb{B}Z(G) \arrow[r, hook] & \overline{\mathrm{Bun}}'_G \arrow[r] & \mathrm{Bun}_G \times \mathrm{Bun}_G
\end{tikzcd}$$

factoring the diagonal Δ. Then the composite map $\overline{\Delta} : \overline{\mathrm{Bun}}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$ is also proper, so $\overline{\mathrm{Bun}}_G$ is a "compactification" of Δ. This is the compactification that we will use to prove Theorem A.3.12.

Example A.3.14. Let $G = \mathrm{SL}(2)$. Then $Z_0(G) = \{1\}$, and the map $\overline{T/Z_0(G)} \to \overline{T_{\mathrm{adj}}}$ corresponds to the map $\mathbb{A}^1 \to \mathbb{A}^1 : \epsilon \mapsto \epsilon^2$. An S-point of $\overline{\mathrm{Bun}}_G$ is a collection $(\mathcal{L}_1, \mathcal{L}_2, l, \beta, \epsilon)$, where

(a) $\mathcal{L}_1, \mathcal{L}_2$ are rank 2 vector bundles on $X \times S$ with trivializations of their determinants,

(b) l is a line bundle on S,

(c) $\beta \in \mathrm{Hom}(\mathcal{L}_2, \mathcal{L}_1) \otimes l$ is not equal to 0 on $X \times s$ for every geometric point $s \to S$,

(d) $\epsilon \in l$, and

(e) the equation $\det \beta = \epsilon^2$ holds.

In comparison, an S-point of $\overline{\mathrm{Bun}}'_G$ is a collection $(\mathcal{L}_1, \mathcal{L}_2, l, \beta)$ satisfying (a)–(c) above.

We have a Cartesian square

$$\begin{tikzcd}
\mathrm{Bun}_G \times T \arrow[r] \arrow[d] & \mathrm{Bun}_G \arrow[d] \\
\mathrm{Bun}_G \times (T/Z_0(G)) \arrow[r] \arrow[d, hook, "j"'] & \mathrm{Bun}_G \times \mathbb{B}Z_0(G) \arrow[d, hook] \\
(\mathrm{VinBun}_G)_{\overline{T/Z_0(G)}} \arrow[r] & \overline{\mathrm{Bun}}_G
\end{tikzcd} \tag{A.31}$$

where the horizontal maps are T-torsors, and the lower vertical maps are open embeddings. Since $Z_0(G)$ is connected, $T \to T/Z_0(G)$ is a trivial $Z_0(G)$-bundle. Therefore the pushforward of the constant sheaf $(\overline{\mathbb{Q}}_\ell)_T$ to $T/Z_0(G)$ equals $(\overline{\mathbb{Q}}_\ell)_{T/Z_0(G)} \otimes H^*(Z_0(G), \overline{\mathbb{Q}}_\ell)$. Thus by smooth base change, to compute the function b it suffices to compute the trace of the geometric Frobenius acting on the $*$-stalks of $\jmath_* \overline{\mathbb{Q}}_\ell$.

A.3.9 The $*$-extension of the constant sheaf

Let

$$\jmath : \mathrm{Bun}_G \times (T/Z_0(G)) \hookrightarrow (\mathrm{VinBun}_G)_{\overline{T/Z_0(G)}}$$

denote the open embedding. We want to compute the $*$-restriction of $\jmath_* \overline{\mathbb{Q}}_\ell$ to the strata $\mathrm{VinBun}_{G,P}^{\mu,\lambda} \times_{\overline{T_{\mathrm{adj}}}} \overline{T/Z_0(G)}$ for $\mu, \lambda \in \Lambda_{G,P}$.

Recall that the P-locus $(\overline{T_{\mathrm{adj}}})_P$ is isomorphic to $T/Z(M)$.

Lemma A.3.15. *The reduced part of $(\overline{T_{\mathrm{adj}}})_P \times_{\overline{T_{\mathrm{adj}}}} \overline{T/Z_0(G)}$ is isomorphic to $T/Z_0(M)$.*

Proof. The locally closed embedding $(\overline{T_{\mathrm{adj}}})_P = \mathbb{G}_m^{|\Gamma_M|} \times \{0\} \hookrightarrow \overline{T_{\mathrm{adj}}} = (\mathbb{A}^1)^{|\Gamma_G|}$ identifies with the spectrum of the algebra map

$$k[\check{\alpha}_j, j \in \Gamma_G] \to k[\check{\alpha}_i^{\pm 1}, i \in \Gamma_M] \tag{A.32}$$

sending $\check{\alpha}_i \mapsto \check{\alpha}_i$ for $i \in \Gamma_M$ and $\check{\alpha}_j \mapsto 0$ for $j \in \Gamma_G - \Gamma_M$ (here we consider $\check{\alpha}_i$ as a character and use the multiplicative notation). Note that $k[\check{\alpha}_j, j \in \Gamma_G]$ is the semigroup algebra of $\check{\Lambda}_G^{\mathrm{pos}}$.

The projection $\overline{T/Z_0(G)} \to \overline{T/Z(G)}$ is the spectrum of the inclusion of semigroup algebras

$$k[\check{\Lambda}_G^{\mathrm{pos}}] \hookrightarrow k[\check{\Lambda}_G^{\mathrm{pos},\mathbb{Q}} \cap \check{\Lambda}].$$

Therefore $(\overline{T_{\mathrm{adj}}})_P \times_{\overline{T_{\mathrm{adj}}}} \overline{T/Z_0(G)}$ is the spectrum of the algebra

$$k[\check{\alpha}_i^{\pm 1}, i \in \Gamma_M] \underset{k[\check{\Lambda}_G^{\mathrm{pos}}]}{\otimes} k[\check{\Lambda}_G^{\mathrm{pos},\mathbb{Q}} \cap \check{\Lambda}]. \tag{A.33}$$

Since the map (A.32) sends $\check{\alpha}_j \mapsto 0$ for $j \in \Gamma_G - \Gamma_M$, the reduced algebra of (A.33) equals

$$k[\check{\alpha}_i^{\pm 1}, i \in \Gamma_M] \underset{k[\check{\Lambda}_M^{\mathrm{pos}}]}{\otimes} k[\check{\Lambda}_M^{\mathrm{pos},\mathbb{Q}} \cap \check{\Lambda}]. \tag{A.34}$$

Since the non-negative integral span of $\check{\Lambda}_M^{\mathrm{pos},\mathbb{Q}} \cap \check{\Lambda}$ and $-\check{\alpha}_i$, $i \in \Gamma_M$ is equal to the lattice $\check{\Lambda}_{T/Z_0(M)} \subset \check{\Lambda}$, the algebra (A.34) equals $k[T/Z_0(M)]$. $\square$

Recall from (A.17) that we have an isomorphism $\mathrm{VinBun}_{G,P} \cong \mathrm{VinBun}_{G,\mathbf{c}_P} \times (T/Z(M))$. Lemma A.3.15 implies that we have embeddings

$$\iota_P : \mathrm{VinBun}_{G,\mathbf{c}_P} \times (T/Z_0(M)) \hookrightarrow (\mathrm{VinBun}_G)_{\overline{T/Z_0(G)}} \tag{A.35}$$

that form a stratification as P ranges over all standard parabolic subgroups.

For $\mu, \lambda \in \Lambda_{G,P}$, let

$$\iota_P^{\mu,\lambda} : \mathrm{VinBun}_{G,\mathbf{c}_P}^{\mu,\lambda} \times (T/Z_0(M)) \hookrightarrow (\mathrm{VinBun}_G)_{\overline{T/Z_0(G)}}$$

denote the locally closed embedding defined by (A.35) and the defect stratification from §A.3.5.

The following is proved in [62, Theorem B] in the case $P = B$. We give a proof using Proposition A.2.4 at the end of this Appendix.

Theorem A.3.16. *Suppose $k = \mathbb{F}_q$. The trace of geometric Frobenius on $*$-stalks of $\iota_P^*(\jmath_*(\overline{\mathbb{Q}}_\ell))$ sends*

$$(\mathcal{F}_G^1, \mathcal{F}_G^2, \beta, t) \in (\mathrm{VinBun}_{G,\mathbf{c}_P} \times T/Z_0(M))(\mathbb{F}_q) \mapsto (1-q)^{|\Gamma_G| - |\Gamma_M|} \, \mathrm{Asymp}_P(\delta_K)(\beta). \tag{A.36}$$

Here we use Lemma A.3.2 to identify β with a section $X \to (\overline{\mathbb{X}}_P)_{\mathcal{F}_G^1, \mathcal{F}_G^2}$, and $\mathrm{Asymp}_P(\delta_K)(\beta)$ is defined in §A.3.7.

Assuming Theorem A.3.16, we prove Theorem A.3.12.

Proof of Theorem A.3.12. Let $\overline{\mathrm{Bun}}_G$ denote the compactification of Δ defined in §A.3.8. Factor Δ into $\bar{\jmath} : \mathrm{Bun}_G \to \overline{\mathrm{Bun}}_G$ and the proper map $\overline{\Delta} : \overline{\mathrm{Bun}}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$. For a sheaf $\mathcal{F}$, we will use $f_{\mathcal{F}}$ to denote its Grothendieck function, i.e., the trace of the geometric Frobenius acting on the $*$-stalks over the $\mathbb{F}_q$-points. By the Grothendieck-Lefschetz trace formula, $b(\mathcal{F}^1_G, \mathcal{F}^2_G)$ equals the sum of the values of $f_{\bar{\jmath}_* \overline{\mathbb{Q}}_\ell}$ at the points of $\overline{\mathrm{Bun}}_G(\mathbb{F}_q)$ lying over $(\mathcal{F}^1_G, \mathcal{F}^2_G) \in (\mathrm{Bun}_G \times \mathrm{Bun}_G)(\mathbb{F}_q)$.

We have the T-torsor $(\mathrm{VinBun}_G)_{\overline{T/Z_0(G)}} \to \overline{\mathrm{Bun}}_G$. Let

$$\jmath : \mathrm{Bun}_G \times T/Z_0(G) \hookrightarrow (\mathrm{VinBun}_G)_{\overline{T/Z_0(G)}}$$

denote the open embedding. Recall that $T \to T/Z_0(G)$ is a trivial $Z_0(G)$-bundle, so the pushforward of $(\overline{\mathbb{Q}}_\ell)_T$ to $T/Z_0(G)$ equals $(\overline{\mathbb{Q}}_\ell)_{T/Z_0(G)} \otimes H^*(Z_0(G), \overline{\mathbb{Q}}_\ell)$. The trace of the geometric Frobenius acting on $H^*(Z_0(G), \overline{\mathbb{Q}}_\ell)$ equals $(1-q)^{\dim(Z_0(G))}$. Since any T-torsor over $\mathbb{F}_q$ is trivial, we deduce from the Cartesian square (A.31) and smooth base change that

$$b(\mathcal{F}^1_G, \mathcal{F}^2_G) = (-1)^{\dim T} (1-q)^{-|\Gamma_G|} \sum_{\tilde{\beta}} f_{\jmath_* \overline{\mathbb{Q}}_\ell}(\tilde{\beta}),$$

where the sum is over $\tilde{\beta} \in (\mathrm{VinBun}_G)_{\overline{T/Z_0(G)}}(\mathbb{F}_q)$ mapping to $(\mathcal{F}^1_G, \mathcal{F}^2_G)$. From (A.35), we have a stratification of $(\mathrm{VinBun}_G)_{\overline{T/Z_0(G)}}$ by $\mathrm{VinBun}_{G,\mathbf{c}_P} \times (T/Z_0(M))$, where P ranges over all standard parabolic subgroups. Theorem A.3.16 implies that

$$f_{\iota_P^* \jmath_* \overline{\mathbb{Q}}_\ell}(\mathcal{F}^1_G, \mathcal{F}^2_G, \beta, t) = (1-q)^{|\Gamma_G| - |\Gamma_M|} \,\mathrm{Asymp}_P(\delta_K)(\beta)$$

for $\beta : X \to (\overline{\mathbb{X}}_P)_{\mathcal{F}^1_G, \mathcal{F}^2_G}$ a non-degenerate section and $t \in (T/Z_0(M))(\mathbb{F}_q)$. Putting it all together, we prove the theorem. □

The remainder of this Appendix works towards setting up the proof of Theorem A.3.16, which is given at the end.

A.3.10 Reduction to the Hecke stack

The isomorphism

$$\mathrm{Bun}_P^{\mu} \underset{\mathrm{Bun}_M^{\mu}}{\times} \mathcal{H}_M^{+,\mu,\lambda} \underset{\mathrm{Bun}_M^{\lambda}}{\times} \mathrm{Bun}_{P^-}^{\lambda} \cong \mathrm{VinBun}_{G,\mathbf{c}_P}^{\mu,\lambda}$$

induced by (A.19) allows us to define the projection map

$$\mathrm{pr}_M^{\mu,\lambda} : \mathrm{VinBun}_{G,\mathbf{c}_P}^{\mu,\lambda} \to \mathcal{H}_M^{+,\mu,\lambda},$$

which is smooth with equidimensional fibers.

For $(\mathcal{F}_G^1, \mathcal{F}_G^2, \beta) \in \mathrm{VinBun}_{G,\mathbf{c}_P}^{\mu,\lambda}(\mathbb{F}_q)$, let

$$(\mathcal{F}_M^1, \mathcal{F}_M^2, \beta_M) := \mathrm{pr}_M^{\mu,\lambda}(\beta) \in \mathcal{H}_M^{+,\mu,\lambda}(\mathbb{F}_q).$$

Choosing trivializations $\mathcal{F}_M^1, \mathcal{F}_M^2$ over $\mathrm{Spec}(\mathfrak{o}_v)$, the $\overline{M}$-morphism β_M defines an element (m_v) in the restricted product $\prod_v(\overline{M}(\mathfrak{o}_v) \cap M(F_v))$ with respect to the open subgroups $M(\mathfrak{o}_v) \subset M(F_v)$. The $M(\mathfrak{o}_v) \times M(\mathfrak{o}_v)$-orbit of m_v does not depend on the choice of trivializations. One deduces from (2.21) that

$$\mathrm{Asymp}_{P,v}(\delta_{K_v})(\beta_v) = \nu_{M,v}(m_v), \tag{A.37}$$

where $\nu_{M,v}$ is the $K_{M,v}$-bi-invariant measure on $M(F_v)$ defined in §2.3.3. Thus the function (A.36) reduces to a function on $\mathcal{H}_M^{+,\mu,\lambda}(\mathbb{F}_q)$.

On the other hand, we have a similar reduction of $\iota_P^{\mu,\lambda *}(\jmath_*(\overline{\mathbb{Q}}_\ell))$ by a modified version of [62, Theorem 4.3.1]:

Recall from Lemma A.3.11 that $\iota_P^{\mu,\lambda *}(\jmath_*(\overline{\mathbb{Q}}_\ell)) = 0$ unless $\mu - \lambda \in \Lambda_{G,P}^{\mathrm{pos}}$. Assuming that $\mu - \lambda \in \Lambda_{G,P}^{\mathrm{pos}}$, the relevant definitions and results from §A.2.7 still hold, without the assumption that $[G, G]$ is simply connected.

Theorem A.3.17. *Let $\mu, \lambda \in \Lambda_{G,P}$ with $\mu - \lambda \in \Lambda_{G,P}^{\mathrm{pos}}$. The $*$-restriction $\iota_P^{\mu,\lambda *}(\jmath_*(\overline{\mathbb{Q}}_\ell))$ to the stratum $\mathrm{VinBun}_{G,\mathbf{c}_P}^{\mu,\lambda} \times (T/Z_0(M))$ is equal to*

$$\mathrm{pr}_M^{\mu,\lambda *}\left(\overline{\mathbb{Q}}_\ell \,\tilde{\boxtimes}\, \tilde{\Upsilon}(\check{\mathfrak{u}}_P)^{\mu-\lambda}\right) \boxtimes (\overline{\mathbb{Q}}_\ell(\tfrac{1}{2})[1])^{-\langle 2\check{\rho}_P, \mu-\lambda\rangle} \otimes H^*(Z_0(M)/Z_0(G), \overline{\mathbb{Q}}_\ell).$$

Here $\tilde{\Upsilon}(\check{\mathfrak{u}}_P)^{\mu-\lambda}$ is the factorization algebra on $\mathrm{Gr}^+_{M,X^{\mu-\lambda}}$ defined in §A.2.7, and we can form the sheaf $(\overline{\mathbb{Q}}_\ell)_{\mathrm{Bun}_M} \,\tilde{\boxtimes}\, \tilde{\Upsilon}(\check{\mathfrak{u}}_P)^{\mu-\lambda} \in D(\mathcal{H}^+_{M,X^{\mu-\lambda}})$ using (A.5).

The proof of Theorem A.3.17 follows the same reasoning as the proof of [62, Theorem 4.3.1], using the local models defined in *loc. cit*, which we now review.

A.3.11 Local models

Recall that $\bar{\pi} : \overline{G_{\mathrm{enh}}} \to \overline{T_{\mathrm{adj}}}$ denotes the projection and $\bar{\mathfrak{s}} : \overline{T_{\mathrm{adj}}} \to \overline{G_{\mathrm{enh}}}$ is a section. Let $(\overline{G_{\mathrm{enh}}})_{\geq P} := \bar{\pi}^{-1}((\overline{T_{\mathrm{adj}}})_{\geq P})$ denote the open submonoid.

Define $\mathcal{Y}^P$ to be the scheme representing the substack

$$\mathrm{Maps}^\circ(X, U^- \backslash (\overline{G_{\mathrm{enh}}})_{\geq P}/P) \subset \mathrm{Bun}_{U^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq P} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_P$$

of maps generically landing in $U^- \cdot \bar{\mathfrak{s}}((\overline{T_{\mathrm{adj}}})_{\geq P}) \cdot P$. Then $\bar{\pi}$ induces a map

$$\mathcal{Y}^P \to (\overline{T_{\mathrm{adj}}})_{\geq P}.$$

For $\theta \in \Lambda_{G,P}$, let $\mathcal{Y}^{P,\theta}$ denote the preimage of $\mathrm{Bun}_M^{-\theta}$ under the projection $\mathrm{Bun}_P \to \mathrm{Bun}_M$.

One can define both left and right actions of T_{adj} on $\mathcal{Y}^P$ in a similar way as in §A.3.3.

Then the action of $T/Z(M) \hookrightarrow T_{\text{adj}}$ defines an isomorphism

$$\mathcal{Y}^P \cong \mathcal{Y}^P_{\text{strict}} \times (T/Z(M)),$$

where $\mathcal{Y}^P_{\text{strict}} := \mathcal{Y}^P \times_{\overline{T_{\text{adj}}}} \overline{Z(M)/Z(G)}$ is the local model for $\text{VinBun}_{G,\geq P,\text{strict}}$ considered in [62, §6.1].

Let $e_P = \bar{\mathfrak{s}}(\mathbf{c}_P)$ and $(\overline{G_{\text{enh}}})_{\geq P,\text{strict}} = \bar{\pi}^{-1}(\{1\} \times \overline{Z(M)/Z(G)})$. By Theorem 1.4.8, we have

$$e_P \cdot (\overline{G_{\text{enh}}})_{\geq P,\text{strict}} \cdot e_P = e_P \cdot (\overline{G_{\text{enh}}})_{\mathbf{c}_P} \cdot e_P = \overline{M}.$$

The map $(\overline{G_{\text{enh}}})_{\geq P,\text{strict}} \to \overline{M}$ factors through $U^- \backslash (\overline{G_{\text{enh}}})_{\geq P,\text{strict}}/U$. Therefore we have a map

$$\pi_{\mathcal{Y}} : \mathcal{Y}^{P,\theta} \to \text{Gr}^+_{M,X^\theta} \times (T/Z(M)).$$

The embedding $\overline{M} = e_P \cdot (\overline{G_{\text{enh}}})_{\mathbf{c}_P} \cdot e_P \hookrightarrow (\overline{G_{\text{enh}}})_{\mathbf{c}_P}$ induces a section

$$\sigma_{\mathcal{Y}} : \text{Gr}^+_{M,X^\theta} \times (T/Z(M)) \to \mathcal{Y}^{P,\theta}$$

of $\pi_{\mathcal{Y}}$. Both $\pi_{\mathcal{Y}}$ and $\sigma_{\mathcal{Y}}$ are compatible with the projection

$$\mathcal{Y}^P \to (\overline{T_{\text{adj}}})_{\geq P} = \mathbb{G}_m^{|\Gamma_M|} \times (\mathbb{A}^1)^{|\Gamma_G| - |\Gamma_M|} \to \mathbb{G}_m^{|\Gamma_M|} = T/Z(M).$$

The scheme $\mathcal{Y}^{P,\theta}$ is a local model for $\text{VinBun}_{G,\geq P}$. We will need to consider

$$\tilde{\mathcal{Y}}^{P,\theta} := \mathcal{Y}^{P,\theta} \underset{\overline{T_{\text{adj}}}}{\times} \overline{T/Z_0(G)},$$

which is a local model for $\text{VinBun}_{G,\geq P} \times_{\overline{T_{\text{adj}}}} \overline{T/Z_0(G)}$. Let $(\overline{T/Z_0(G)})_{\geq P}$ and $(\overline{T/Z_0(G)})_P$ denote the base changes of the corresponding loci in $\overline{T_{\text{adj}}}$, so that $\tilde{\mathcal{Y}}^{P,\theta}$ lies over $(\overline{T/Z_0(G)})_{\geq P}$.

By base change, we get maps

$$\mathrm{Gr}^+_{M,X^\theta} \times (\overline{T/Z_0(G)})_P \xrightarrow{\tilde{\sigma}_{\mathcal{Y}}} \tilde{\mathcal{Y}}^{P,\theta} \xrightarrow{\tilde{\pi}_{\mathcal{Y}}} \mathrm{Gr}^+_{M,X^\theta} \times (\overline{T/Z_0(G)})_P.$$

At the level of reduced schemes, Lemma A.3.15 implies that we have maps

$$(\mathrm{Gr}^+_{M,X^\theta})_{\mathrm{red}} \times (T/Z_0(M)) \xrightarrow{\tilde{\sigma}_{\mathcal{Y}}} \tilde{\mathcal{Y}}^{P,\theta}_{\mathrm{red}} \xrightarrow{\tilde{\pi}_{\mathcal{Y}}} (\mathrm{Gr}^+_{M,X^\theta})_{\mathrm{red}} \times (T/Z_0(M)).$$

A.3.12 Contracting action on $\tilde{\mathcal{Y}}^{P,\theta}_{\mathrm{red}}$

Fix a cocharacter $\nu_M : \mathbb{G}_M \to Z_0(M) \subset T$ which contracts U^- to the element $1 \in U^-$. Then ν_M defines a $\mathbb{G}_m$-action on $\mathcal{Y}^{P,\theta}$ that contracts $\mathcal{Y}^{P,\theta}$ onto the section $\sigma_{\mathcal{Y}}$ by [62, Lemma 6.5.6].

By definition, the induced $\mathbb{G}_m$-action on $(\overline{T_{\mathrm{adj}}})_{\geq P}$ is via the composition $\mathbb{G}_m \xrightarrow{-2\nu_M} T$ and the usual T-action on $\overline{T_{\mathrm{adj}}}$. Therefore if we consider the $\mathbb{G}_m$-action on $\overline{T/Z_0(G)}$ via the composition $\mathbb{G}_m \xrightarrow{-2\nu_M} T$ and the usual T-action on $\overline{T/Z_0(G)}$, we get a $\mathbb{G}_m$-action on $\tilde{\mathcal{Y}}^{P,\theta}$, and hence on $\tilde{\mathcal{Y}}^{P,\theta}_{\mathrm{red}}$. Moreover from Lemma A.3.15 we deduce that this $\mathbb{G}_m$-action contracts $\tilde{\mathcal{Y}}^{P,\theta}_{\mathrm{red}}$ onto the section $(\mathrm{Gr}^+_{M,X^\theta})_{\mathrm{red}} \times (T/Z_0(M))$.

We are now ready to prove Theorem A.3.17.

Proof of Theorem A.3.17. Since the open locus $\mathrm{VinBun}_{G,\geq P}$ contains the P- and G-loci, to compute $\iota_P^{\mu,\lambda *} \jmath_* \overline{\mathbb{Q}}_\ell$, we may restrict to $\mathrm{VinBun}_{G,\geq P} \times_{\overline{T_{\mathrm{adj}}}} \overline{T/Z_0(G)}$.

Let $\mathcal{Y}^{P,\theta}_G \subset \mathcal{Y}^{P,\theta}$ denote the G-locus: the preimage of T_{adj} under the projection to $\overline{T_{\mathrm{adj}}}$. Set $\tilde{\mathcal{Y}}^{P,\theta}_G = \mathcal{Y}^{P,\theta}_G \times_{T_{\mathrm{adj}}} (T/Z_0(G))$. The T_{adj}-action on $\mathcal{Y}^{P,\theta}$ induces an isomorphism

$$\mathcal{Y}^{P,\theta}_G \cong \overset{\circ}{\mathcal{Z}}{}^{P,\theta} \times T_{\mathrm{adj}},$$

where $\overset{\circ}{\mathcal{Z}}^{P,\theta}$ is the open Zastava space (see §A.2.8). Hence there is an isomorphism

$$\tilde{\mathcal{Y}}_G^{P,\theta} \cong \overset{\circ}{\mathcal{Z}}^{P,\theta} \times (T/Z_0(G)).$$

Let $\jmath_G : \tilde{\mathcal{Y}}_G^{P,\theta} \hookrightarrow \tilde{\mathcal{Y}}^{P,\theta}$ denote the open embedding. Then the assertion of the theorem reduces, as explained in [12, §3, §8], to proving that

$$\tilde{\sigma}_{\mathcal{Y}}^*(\jmath_{G*}(\overline{\mathbb{Q}}_\ell)) \cong \tilde{\Upsilon}(\check{\mathfrak{u}}_P)^\theta \boxtimes (\overline{\mathbb{Q}}_\ell(\tfrac{1}{2})[1])^{-\langle 2\check{\rho}_P, \theta\rangle} \otimes H^*(Z_0(M)/Z_0(G), \overline{\mathbb{Q}}_\ell). \tag{A.38}$$

Since we are working with étale sheaves, we can work at the level of reduced schemes. Then we can apply the contraction principle (cf. [12, §5], [62, Lemma 7.2.1]) to the $\mathbb{G}_m$-action on $\tilde{\mathcal{Y}}_{\mathrm{red}}^{P,\theta}$ defined in §A.3.12. This gives an isomorphism $\tilde{\sigma}_{\mathcal{Y}}^*(\jmath_{G*}(\overline{\mathbb{Q}}_\ell)) \cong \tilde{\pi}_{\mathcal{Y}*}(\jmath_{G*}(\overline{\mathbb{Q}}_\ell))$. At the level of reduced schemes,

$$\tilde{\pi}_{\mathcal{Y}} \circ \jmath_G : \mathcal{Y}_{G,\mathrm{red}}^{P,\theta} \cong \overset{\circ}{\mathcal{Z}}^{P,\theta} \times (T/Z_0(G)) \to (\mathrm{Gr}_{M,X^\theta}^+)_{\mathrm{red}} \times (T/Z_0(M))$$

is the product of $\overset{\circ}{\pi}_{\mathcal{Z}}$ and the natural projection $T/Z_0(G) \to T/Z_0(M)$. Recall from (A.15) that there is a canonical isomorphism $\tilde{\Upsilon}(\check{\mathfrak{u}}_P)^\theta \cong (\overset{\circ}{\pi}_{\mathcal{Z}})_*(\mathrm{IC}_{\overset{\circ}{\mathcal{Z}}^{P,\theta}})$. Noting that $\overset{\circ}{\mathcal{Z}}^{P,\theta}$ is smooth of dimension $\langle 2\check{\rho}_P, \theta\rangle$, we get the identification

$$(\overset{\circ}{\pi}_{\mathcal{Z}})_*(\overline{\mathbb{Q}}_\ell) \cong \tilde{\Upsilon}(\check{\mathfrak{u}}_P)^\theta \otimes (\overline{\mathbb{Q}}_\ell(\tfrac{1}{2})[1])^{-\langle 2\check{\rho}_P, \theta\rangle}.$$

Since $Z_0(M)/Z_0(G)$ is a torus, we observe that $T/Z_0(G) \to T/Z_0(M)$ is a trivial torsor. Equation (A.38), and hence the theorem, now follows. □

Proof of Theorem A.3.16. The trace of the geometric Frobenius on $H^*(\mathbb{G}_m, \overline{\mathbb{Q}}_\ell)$ equals $1-q$, and $Z_0(M)/Z_0(G)$ is a product of $|\Gamma_G| - |\Gamma_M|$ copies of $\mathbb{G}_m$. Therefore the trace of geometric Frobenius on $H^*(Z_0(M)/Z_0(G), \overline{\mathbb{Q}}_\ell)$ equals $(1-q)^{|\Gamma_G|-|\Gamma_M|}$. Theorem A.3.16 now follows from Theorem A.3.17, Proposition A.2.4, and (A.37). □

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