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On an n-th Order Linear Ordinary Differential Equation with a Turning-Singular Point

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Dedicated to Professor Toshihiko Nishimoto on his 60th birthday

1. Introduction.

1.1. We consider an n-th order linear ordinary differential equation

$$\varepsilon^{nh}y^{(n)}=\sum_{k=1}^{n}\varepsilon^{(n-k)h}p_k(x,\varepsilon)y^{(n-k)}\quad\left(0<|x|\le x_0,\ 0<\varepsilon\le\varepsilon_0,\ '=\frac{d}{dx}\right),\tag{1.1}$$

where x is a complex variable, and h, x_0 and ε_0 are positive constants.

The coefficients $p_k(x,\varepsilon)$'s are given by

$$p_k(x,\varepsilon):=p_k\cdot(x^m-\varepsilon^l/x^r)^k\qquad(k=1,2,\cdots,n),\tag{1.1)'$$

where m, l and r are positive integers satisfying *the singular perturbation condition*:

$$h>\frac{m+1}{m+r}l,\tag{1.2}$$

and the constants p_k's are supposed to satisfy

$$\begin{cases}p_1:=\sum_{k=1}^{n}a_k\,,\quad p_2:=-\sum_{k_1<k_2}a_{k_1}a_{k_2}\,,\quad p_3:=\sum_{k_1<k_2<k_3}a_{k_1}a_{k_2}a_{k_3}\,,\\ \cdots\cdots\cdots\cdots\\ p_{n-1}:=(-1)^n\sum_{k_1<k_2<\cdots<k_{n-1}}a_{k_1}a_{k_2}\cdots a_{k_{n-1}}\,,\qquad p_n:=(-1)^{n+1}\prod_{k=1}^{n}a_k\,,\end{cases}\tag{1.3}$$

$$a_{k-1}<a_k\ (k=2,3,\cdots,n)\,;\qquad \forall a_k\ne 0\,.\tag{1.4}$$

Accordingly, the characteristic equation of (1.1) is given by

$$L(x,\lambda)=0\,,\qquad L(x,\lambda):=\lambda^n-\sum_{k=1}^{n}p_k\cdot x^{km}\lambda^{n-k}=\prod_{k=1}^{n}(\lambda-a_kx^m)\tag{1.5}$$

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and then the characteristic roots are

(1.6) $$\lambda=\lambda_k(x):=a_k x^m \qquad (k=1,2,\cdots,n).$$

The characteristic roots $\lambda_k(x)$'s coincide at $x=0$, then the origin $x=0$ is, by definition, *a turning point* or *a transition point* of (1.1). The turning point $x=0$ is also a singular point of (1.1) from (1.1)$'$. Thus we call the origin $x=0$ *a turning-singular point* of (1.1).

1.2. The differential equation with a turning point is characterized by its characteristic polygon introduced by Iwano-Sibuya [8]. The characteristic polygon is similar to the Newton polygon for a differential equation with an irregular singular point. We can analyze the asymptotic property of the differential equation near the turning point by its characteristic polygon. The characteristic polygon is, however, not effective for a case of turning-singular points.

In general, the characteristic polygon is composed of several segments. The case of one-segment characteristic polygon are analyzed by Nakano [11], Nishimoto [19], [20] and Wasow [27] et al. for second order, higher order differential equations and systems of differential equations. The cases of two- and three-segment characteristic polygon are analyzed by Nakano-Nishimoto [17] and Roos [23], [24] for second order differential equations, and Nakano [14] analyzes certain third order differential equation with a two-segment characteristic polygon. The second and the third order differential equations with a turning-singular point are analyzed by Nakano [12], [13] and [15].

1.3. Our aim is to analyze the asymptotic property of solutions of (1.1). We use the so-called stretching-matching method and the result is given in the theorem 7.2.

In the second section the domain $0<|x|\leq x_0$ is divided to two circular regions, in each of which the differential equation (1.1) is reduced. They are called the outer and the inner equations. In the third section the outer and the inner WKB solutions are obtained and the inner WKB solutions have the double asymptotic property.

In the fourth and fifth sections topology of Stokes curves are analyzed and the brief sketch of Fedoryuk's theory about canonical regions is given, and the canonical region for the inner equation is obtained in the sixth section. In the last section the matching matrix connecting the outer and the inner solutions is calculated.

2. Reduction of the equation.

2.1. We see that the coefficient $p_k(x,\varepsilon)$ has an asymptotic form

(2.1) $$p_k(x,\varepsilon)=p_k\cdot x^{km}+O((\varepsilon x^{-1/\alpha})^l)$$

for small $\varepsilon x^{-1/\alpha}$ $(\alpha:=l/(m+r)>0)$ and x in the region

(2.2) $$K\varepsilon^{\alpha}\leq|x|\leq x_0,$$

where K is a positive constant.

Thus the differential equation (1.1) is asymptotically equal to

$$\varepsilon^{nh}y^{(n)}=\sum_{k=1}^{n}\varepsilon^{(n-k)h}p_k\cdot x^{km}y^{(n-k)} \tag{2.3}$$

for small $\varepsilon x^{-1/\alpha}$ and x in the region (2.2).

Putting $x=t\varepsilon^{\alpha}$ (*a stretching transformation*), we get

$$p_k(x,\varepsilon)=p_k\cdot p^k(t)\,,\quad p(t):=t^m-1/t^r \qquad (k=1,2,\cdots,n)\,.$$

Thus we can reduce the differential equation (1.1) to

$$\varepsilon^{nh'}y^{(n)}=\sum_{k=1}^{n}\varepsilon^{(n-k)h'}p_k\cdot p^k(t)y^{(n-k)} \quad \left(h':=h-(m+1)\alpha>0,\ '=\frac{d}{dt}\right) \tag{2.4}$$

in the region $0<|x|\leq K\varepsilon^{\alpha}$. By the singular perturbation condition (1.2), the exponent h' of ε is positive, and so (2.4) is a differential equation of singular perturbation type.

We investigate the differential equation (2.4) in the region

$$0<|t|<\infty \tag{2.5}$$

instead of the region: $0<|x|\leq K\varepsilon^{\alpha}$, which is equivalent to a region: $0<|t|\leq K$. Then two regions (2.2) and (2.5) have common interior points.

2.2. We call the differential equation (2.3) and (2.4) *the outer equation* and *the inner equation* of (1.1) respectively, and the regions (2.2) and (2.5) are called *the outer region* and *the inner region* of (1.1) respectively.

Summing up the above consideration we get

THEOREM 2.1. *Consider the differential equation*

$$\varepsilon^{nh}y^{(n)}=\sum_{k=1}^{n}\varepsilon^{(n-k)h}p_k\left(x^m-\frac{\varepsilon^l}{x^r}\right)^k y^{(n-k)} \quad \left(0<|x|\leq x_0,\ 0<\varepsilon\leq\varepsilon_0,\ '=\frac{d}{dx}\right), \tag{1.1}$$

and suppose that positive constants h, l, m and r satisfy the singular perturbation condition:

$$h-(m+1)\alpha>0 \qquad (\alpha:=l/(m+r))\,. \tag{1.2}$$

Then the differential equation (1.1) *is reduced in the outer region*

$$K\varepsilon^{\alpha}\leq|x|\leq x_0 \tag{2.2}$$

to the outer equation

$$\varepsilon^{nh}y^{(n)}=\sum_{k=1}^{n}\varepsilon^{(n-k)h}p_k\cdot x^{km}y^{(n-k)} \qquad \left('=\frac{d}{dx}\right), \tag{2.3}$$

and to the inner equation

$$\varepsilon^{nh'}y^{(n)}=\sum_{k=1}^{n}\varepsilon^{(n-k)h'}p_k\cdot p^k(t)y^{(n-k)} \quad \left(p(t):=t^m-\frac{1}{t^r},\ t:=\frac{x}{\varepsilon^{\alpha}},\ ':=\frac{d}{dt}\right) \tag{2.4}$$

in the inner region

$$0<|t|<\infty\,, \tag{2.5}$$

where $h':=h-(m+1)\alpha$.

3. The WKB solutions.

3.1. *The WKB solution* is, by definition, the leading term of the formal solution, that is obtained by substituting a power series of ε in the differential equation.

LEMMA 3.1. *A linear ordinary differential equation containing a small parameter* ε

$$L[y]=0\,,\qquad L[y]:=\sum_{k=0}^{n}\varepsilon^{(n-k)h}q_k(x)y^{(n-k)}\qquad\left(h>0,\,q_0(x)\equiv 1,\,'=\frac{d}{dx}\right) \tag{3.1}$$

possesses the WKB solutions

$$\tilde{y}_j(x,\varepsilon)=\exp\left(\frac{1}{\varepsilon^h}\int_{x_0}^{x}\lambda_j(t)dt-\sum_{k\neq j}\int_{x_0}^{x}\frac{\lambda_j'(t)}{\lambda_j(t)-\lambda_k(t)}dt\right)\qquad(j=1,2,\cdots,n)\,, \tag{3.2}$$

where $\lambda_j(x)$*'s are characteristic roots of* (3.1) *which are roots of the characteristic equation of* (3.1) *defined by*

$$L(x,\lambda)=0\,,\qquad L(x,\lambda):=\sum_{k=0}^{n}q_k(x)\lambda^{n-k}\,. \tag{3.3}$$

PROOF. Put a formal solution

$$\tilde{y}:=\exp\left(\frac{1}{\varepsilon}S(x)\right)\sum_{j=0}^{\infty}\varepsilon^j a_j(x)\qquad(a_0(x)\neq 0)$$

and substitute this for y of (3.1). By using the Leibniz's formula we get

$$\begin{aligned}L[\tilde{y}]&=\sum_{k=0}^{n}\frac{1}{k!}\frac{d^k}{dx^k}\exp\left(\frac{1}{\varepsilon}S(x)\right)\frac{\partial^k}{\partial\lambda^k}L(x,\lambda)\Big|_{\lambda=\sum\varepsilon^j a_j(x)}\\&=L(x,S'(x))\sum_{j=0}^{\infty}\varepsilon^j a_j(x)\\&\quad+\varepsilon\left\{\frac{1}{2}L_{\lambda\lambda}(x,S'(x))S''(x)\sum_{j=0}^{\infty}\varepsilon^j a_j(x)+L_\lambda(x,S'(x))\left(\sum_{j=0}^{\infty}\varepsilon^j a_j(x)\right)'\right\}+\cdots\,.\end{aligned}$$

By rearranging terms according to powers of ε, the following equation holds:

$$L(x,S'(x))a_0(x)+\varepsilon\{\tfrac{1}{2}L_{\lambda\lambda}(x,S'(x))S''(x)a_0(x)+L_\lambda(x,S'(x))a_0'(x)\}+O(\varepsilon^2)\equiv 0\,.$$

Hence

$$L(x, S'(x))a_0(x)=0\,,\quad \tfrac{1}{2}L_{\lambda\lambda}(x, S'(x))S''(x)a_0(x)+L_\lambda(x, S'(x))a_0'(x)=0\,,\quad \cdots .$$

Since $L(x, S'(x))=0$, $S'(x)$ is a characteristic root. Putting $S'(x)=\lambda_j(x)$ ($j=1, 2, \cdots, n$), we get $S(x)=\int_{x_0}^{x}\lambda_j(t)dt$, and then

$$a_0(x)=\exp\left(-\tfrac{1}{2}\int_{x_0}^{x}L_{\lambda\lambda}(t, \lambda_j(t))/L_\lambda(t, \lambda_j(t))\lambda_j'(t)dt\right).$$

Since the characteristic polynomial of (3.3) is $L(x, \lambda)=\prod_{k=1}^{n}(\lambda-\lambda_k(x))$, we get

$$\frac{L_{\lambda\lambda}(x, \lambda)}{L_\lambda(x, \lambda)}=\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{k\neq i\neq j}(\lambda-\lambda_k(x))}{\sum_{j=1}^{n}\prod_{k\neq j}(\lambda-\lambda_k(x))}=\sum_{k\neq j}\frac{2}{\lambda_j(x)-\lambda_k(x)}\,.$$

Thus we can get the formula (3.2). Q.E.D.

The WKB solutions are asymptotic expansions of the true solutions of (3.1), and they have *the double asymptotic property* (Evgrafov-Fedoryuk [1], Федорюк [3], Leung [10]) such that

LEMMA 3.2. *Let D be a canonical region of* (3.1). *Then the WKB solution* $\tilde{y}_j(x, \varepsilon)$ ($j=1, 2, \cdots, n$) *has the double asymptotic property*:

$$y_j(x, \varepsilon)\sim\tilde{y}_j(x, \varepsilon)\qquad\begin{cases}\textit{as}\quad \varepsilon\to 0\,, & x\in D\,,\\ \textit{as}\quad x\to\infty & \textit{in } D\,,\quad 0<\varepsilon\leq\varepsilon_0\,,\end{cases}\tag{3.4}$$

where $y_j(x, \varepsilon)$ *is the true solution of* (3.1).

D is a maximal region in which there exist n independent solutions of (3.1) with property (3.4). We will give the definition of a canonical region in §6.1 and construct it for (2.4) in §6.2.

The proof is essentially same as Nakano et al. [16] and omitted here.

3.2. The solution of the outer equation (2.3) is called *the outer solution* of (1.1). The solution of the inner equation (2.4) is called *the inner solution* of (1.1). The differential equations (2.3), (2.4) and (3.1) are very similar. Therefore we can obtain the leading terms of formal outer and inner solutions from the lemma 3.1, which are called *the outer* and *the inner WKB solutions* of (1.1) respectively.

THEOREM 3.1. *The differential equation* (1.1) *has the outer WKB solutions*

$$\tilde{y}_j^{out}(x, \varepsilon):=x^{-m\mu_j}\exp\left(\frac{a_j}{\varepsilon^h}\frac{x^{m+1}}{m+1}\right)\qquad (j=1, 2, \cdots, n)\tag{3.5}$$

and the inner WKB solutions

$$\tilde{y}_j^{in}(t, \varepsilon):=p(t)^{-\mu_j}\exp\left(\frac{a_j}{\varepsilon^{h'}}\int^t p(s)ds\right)\qquad (j=1, 2, \cdots, n)\,,\tag{3.6}$$

where $\mu_j=\sum_{k\neq j}a_j/(a_j-a_k)$.

PROOF. The characteristic equation and the characteristic roots of (2.3) are also given by (1.5) and (1.6) respectively.

Then we have

$$\int \frac{\lambda_j'(x)}{\lambda_j(x)-\lambda_k(x)}dx = \frac{ma_j \log x}{a_j - a_k}.$$

By applying the formula (3.2) we get the outer WKB solutions (3.5).

In the same way we can get (3.6). Q.E.D.

4. Topology of Stokes curves (local property).

4.1. In this and the following two sections we sketch the Fedoryuk's theory about the canonical region to apply it to our differential equation. The Fedoryuk's theory is explained in Evgrafov-Fedoryuk [1], Fedoryuk [4], [5] and Wasow [29].

It is essential to analyze the maximal existence region of n independent inner solutions of (1.1) with double asymptotic property which is called *the canonical region* of (2.4), because the existence region of the outer solutions must be induced from the canonical region to apply the matching method (§7). The outer solutions' existence region is an angular sector whose boundaries correspond to the boundaries of the canonical region as $t \to \infty$ (Nakano et al. [17]).

We are constructing canonical regions of (2.4) in the section six. To do it we have to study topology of Stokes curves.

The canonical region for the inner equation (2.4) is bounded by *Stokes curves* defined by the equation

$$\Re \xi_{jk}(t_0, t) = 0 \qquad (\lambda_j(t_0) = \lambda_k(t_0),\ j \neq k), \tag{4.1}$$

$$\xi_{jk}(t_0, t) := \xi_j(t_0, t) - \xi_k(t_0, t), \quad \xi_j(t_0, t) := \int_{t_0}^{t} \lambda_j(s)ds \qquad (k, j = 1, 2, \cdots, n), \tag{4.2}$$

and a curve defined by the equation

$$\Im \xi_{jk}(t_0, t) = 0, \qquad (\lambda_j(t_0) = \lambda_k(t_0),\ j \neq k) \tag{4.3}$$

is called *an anti-Stokes curve* of (2.4), where $\lambda_j(t)$'s are characteristic roots of the inner equation (2.4).

4.2. The characteristic roots of the inner equation (2.4) are given by $\lambda_k(t) = a_k \cdot p(t)$ $(k = 1, 2, \cdots, n)$, where a_k's differ each other from the condition (1.4). The characteristic roots $\lambda_k(t)$'s coincide at zeros of $p(t)$ which are turning points of (2.4) and are called *secondary turning points* of the differential equation (1.1) (Nakano [14], Nakano-Nishimoto [17], Wasow [29]).

They are the $(m+r)$-th roots of 1: $t = e^{2j\pi i/(m+r)}$ $(j = 0, 1, 2, \cdots, m+r-1)$. The origin $t = 0$ is a singular point of (2.4) and it corresponds to the turning point $x = 0$ of (1.1) because of the definition of t: $x = t\varepsilon^{\alpha}$.

Since all a_k's are different each other, all $\lambda_k(t)$'s are different each other except secondary turning points. Every difference of arbitrary two characteristic roots contains $p(t)$, and it vanishes only at secondary turning points.

We get the local property of Stokes and anti-Stokes curves of (2.4).

THEOREM 4.1. *If $r \geq 1$, four Stokes curves emerge from a secondary turning point $t_0 = e^{2j\pi i/(m+r)}$ $(j = 0, 1, 2, \cdots, m+r-1)$ in the directions*

$$\arg(t - t_0) = \frac{2k+1}{4}\pi - \frac{m-1}{m+r} j\pi \qquad (j = 0, 1, 2, \cdots, m+r-1;\ k = 0, 1, 2, 3) \tag{4.4}$$

and they tend to other secondary turning points, the origin or the point at infinity.

Stokes curves approach the origin in $2r-2$ $(r \geq 1)$ directions given by

$$\arg t = \frac{2k+1}{2(r-1)}\pi \qquad (r > 1,\ k \in \mathbf{Z}) \tag{4.5}$$

or approach ∞ in $2m+2$ directions given by

$$\arg t = \frac{2k+1}{2(m+1)}\pi \qquad (r \geq 1,\ k \in \mathbf{Z}). \tag{4.6}$$

PROOF. Let t_0 be a secondary turning point. We have to analyze the integral $\xi := \int_{t_0}^{t} p(s)ds$, $p(s) = s^m - 1/s^r$. Put $t = t_0 + \tau$. Then, for small τ the values of $p(t)$ are approximated such that $p(t) \sim (m+r)t_0^{m-1}\tau$ $(\tau \sim 0)$. By integrating this, we get $\xi \sim (m+r)t_0^{m-1}\tau^2/2$ $(\tau \sim 0)$. Near $t = t_0$ the Stokes curves $\Re\xi = 0$ have their arguments

$$\arg\tau = \frac{2k+1}{4}\pi - \frac{m-1}{2}\arg t_0 \quad (\tau \sim 0,\ k \in \mathbf{Z}).$$

From this equation we get four different arguments of Stokes curves near a secondary turning point $t_0 = e^{2j\pi i/(m+r)}$:

$$\arg\tau = \frac{2k+1}{4}\pi - \frac{m-1}{m+r} j\pi \qquad (\tau \sim 0;\ j = 0, 1, 2, \cdots, m+r-1;\ k = 0, 1, 2, 3).$$

Near the origin $t = 0$, $p(t)$ is approximated such that $p(t) \sim -t^{-r}(t \sim 0)$. Then $\xi \sim t^{1-r}/(r-1)$ $(r > 1, t \sim 0)$. From the equation $\Re\xi = 0$, we get arguments $\arg t = (2k+1)\pi/2(r-1)$ $(k \in \mathbf{Z})$ near $t = 0$. Therefore there are $2r-2$ different directions in which Stokes curves tend to the origin (cf. Theorem 5.2 for $r = 1$).

Near $t = \infty$, $p(t)$ is approximated so that $p(t) \sim t^m$. Then $\xi \sim t^{m+1}/(m+1)$ $(t \sim \infty)$. From the equation $\Re\xi = 0$, we get arguments $\arg t = (2k+1)\pi/2(m+1)$ $(k \in \mathbf{Z})$. Therefore there exist $2m+2$ different directions in which Stokes curves tend to the point at infinity. Q.E.D.

5. Topology of Stokes curves (global property).

5.1. By the definition, every Stokes curve of (2.4) emerges from a secondary turning point (cf. (4.1)). Thus there exist no Stokes curves which emerge from the origin, tend to the point at infinity and do not pass through any secondary turning point. Any Stokes curve can not cross other Stokes curves emerging from other secondary turning points and they can cross only at the secondary turning points (Evgrafov-Fedoryuk [1], Fedoryuk [2]).

We are precisely analyzing the global property of Stokes curves and anti-Stokes curves of (2.4).

THEOREM 5.1. (a) *For arbitrary positive integers m and r, the point $t=1$ is a secondary turning point and two intervals on the positive real axis ($t\geq 1, 0<t\leq 1$) are anti-Stokes curves.*

(b) *When $m+r$ is even, the point $t=-1$ is a secondary turning point and two intervals on the negative real axis ($t\leq -1$, $-1\leq t<0$) are anti-Stokes curves.*

(c) *When $m+r$ is odd, there exists a Stokes curve connecting two secondary turning points neighboring $t=-1$.*

(d) *When $m+r=4k$ ($k\in\mathbf{N}$), the points $t=\pm i$ are secondary turning points, and moreover*

(1) *if both m and r are odd, then four intervals on the imaginary axes ($|\mathfrak{I}t|\geq 1$, $0<|\mathfrak{I}t|\leq 1$) are anti-Stokes curves, and*

(2) *if both m and r are even, then four intervals on the imaginary axes ($|\mathfrak{I}t|\geq 1, 0<|\mathfrak{I}t|\leq 1$) are Stokes curves.*

(e) *When $r=m+2$,*

(1) *every radial line, which passes through a secondary turning point and tends to the origin and the point at infinity, is an anti-Stokes curve, and*

(2) *the unit circle $|t|=1$ is composed of $2m+2$ anti-Stokes curves.*

PROOF. (a), (b) Since the function $p(t)$ ($=t^m-t^{-r}$) and its indefinite integral take real values for all real $t>0$ and the point $t=1$ is a secondary turning point for any m and r, a part ($t\geq 1$) and a part ($0<t\leq 1$) of the positive real axis are anti-Stokes curves for any m and r, and the negative real axis also consists of two anti-Stokes curves when $m+r$ is even because the point $t=-1$ is a secondary turning point.

(c) Putting $m+r=2u+1$ ($u\in\mathbf{Z}$), secondary turning points are $t=e^{2j\pi i/(2u+1)}$ ($j=0, 1, 2, \cdots, 2u$) and neighboring secondary turning points near $t=-1$ are given by $t_1:=e^{2u\pi i/(2u+1)}$ and $t_2:=e^{(2u+2)\pi i/(2u+1)}$. Then, integrating $p(t)$ from t_1 to t_2 we get

$$\xi=\int_{t_1}^{t_2} p(t)dt=\frac{m+r}{(m+1)(r-1)}\,\frac{t_1^{r-1}-t_2^{r-1}}{(t_1t_2)^{r-1}}\qquad (r>1),$$

which is pure imaginary, because $t_2=\overline{t_1}$ and so $t_1^{r-1}-t_2^{r-1}=t_1^{r-1}-\overline{t_1^{r-1}}\in i\mathbf{R}$.

If $r=1$, we get

$$\xi=\int_{t_1}^{t_2} p(t)dt=-\log\frac{t_2}{t_1}=-\frac{2}{2m+1}\pi i\,.$$

Thus there exists a Stokes curve connecting t_1 and t_2.

(d) When $m+r=4k$ $(r>1, k\in\mathbf{Z})$, secondary turning points are zeros of $t^{4k}-1$ and so two points $t=\pm i$ are secondary turning points, and we have

$$\xi=\int_{\pm i}^{\pm it} p(s)ds=\frac{i^{m+1}}{m+1}\{(\pm t)^{m+1}-(\pm 1)^{m+1}\}+\frac{i^{1-r}}{r-1}\left\{\frac{1}{(\pm t)^{r-1}}-\frac{1}{(\pm 1)^{r-1}}\right\}\qquad(t>0)\,.$$

If $r=1$, we have

$$\xi=\int_{\pm i}^{\pm it} p(s)ds=\frac{i^{m+1}}{m+1}\{(\pm t)^{m+1}-(\pm 1)^{m+1}\}-\log t\qquad(t>0)\,.$$

Thus, ξ is real if both $m+1$ and $r+1$ are even. Therefore both positive and negative imaginary axes are anti-Stokes curves if both m and r are odd. They are Stokes curves if both m and r are even.

(e) When $p(t)=t^m-1/t^{m+2}$, the secondary turning points are expressed by $t_0=e^{j\pi i/(m+1)}$ $(j=0, 1, 2, \cdots, 2m+1)$ and we get the integral

$$\xi=\int_{t_0}^{\tau t_0} p(t)dt=\frac{(-1)^j}{m+1}\left(\tau^{m+1}+\frac{1}{\tau^{m+1}}-2\right)\qquad(0<\tau<\infty)\,,$$

and so ξ takes only real values on the line $t=\tau t_0$ $(0<\tau<\infty)$ for any j. This line is an anti-Stokes curve.

If $t_0=e^{j\pi i/(m+1)}$ and $t_1=e^{(j+1)\pi i/(m+1)}$, we see that

$$\xi=\int_{t_0}^{t_1} p(t)dt=\left[\frac{1}{m+1}\left(e^{(m+1)\theta i}+e^{-(m+1)\theta i}\right)\right]_{(j/(m+1))\pi}^{((j+1)/(m+1))\pi}$$

$$=\frac{1}{m+1}\{(e^{(j+1)\pi i}+e^{-(j+1)\pi i})-(e^{j\pi i}+e^{-j\pi i})\}\,.$$

Both the indefinite and the definite integrals are real. Then an arc of the unit circle between arbitrary two secondary turning points is an anti-Stokes curve. Since there exist $2m+2$ secondary turning points on the unit circle, there exist $2m+2$ anti-Stokes curves on the unit circle. Q.E.D.

5.2. When $r=1$, the origin is a regular singular point, and Stokes curve configuration is fairly different from the case $r>1$ near the origin though the Stokes curve configuration is similar as the case $r>1$ for large t.

THEOREM 5.2. *Let m be a positive integer and $r=1$. Then there exists a Stokes curve passing through all secondary turning points. It is homotopic to a circle.*

The unit circle $|t|=1$ is neither a Stokes curve nor an anti-Stokes curve. A radial line from 0 to ∞ passing through a secondary turning point is an anti-Stokes curve.

Near the origin, level curves defined by $\Re\xi = const.$ (>0) are closed curves around the origin and they are homotopic to a circle, and level curves defined by $\Im\xi = const.$ are radial lines emerging from the origin.

PROOF. If we put $t_1 := e^{2j\pi i/(m+1)}$ and $t_2 := e^{(2j+2)\pi i/(m+1)}$ and integrate $p(t)$ from t_1 to t_2 along the unit circle $|t| = 1$, then we see that

$$\xi = \int_{t_1}^{t_2} p(t)dt = \left[\frac{1}{m+1} e^{(m+1)\theta i} - \theta i\right]_{(2i/(m+1))\pi}^{((2j+2)/(m+1))\pi} = -\frac{2\pi}{m+r} i .$$

Thus there exists a Stokes curve connecting neighboring secondary turning points t_1 and t_2, but the unit circle $|t| = 1$ is neither a Stokes curve nor an anti-Stokes curve because the indefinite integral is neither only real nor only imaginary on the whole circle.

If we integrate $p(t)$ from 0 to ∞ passing through a secondary turning point $t_1 = e^{2j\pi i/(m+1)}$ along the radial line $t = \tau t_1$ $(0 < \tau < \infty)$, then we see that the integral

$$\xi = \int_0^{\infty} \left(t^m - \frac{1}{t}\right) dt = \int_{+0}^{+\infty} \left(\tau^m - \frac{1}{\tau}\right) d\tau$$

takes real values only. Then the radial lines $t = \tau t_1$ $(0 < \tau \leq 1)$ and $t = \tau t_1$ $(\tau \geq 1)$ are anti-Stokes curves proceeding from the secondary turning point t_1 (cf. Theorem 5.1 (e)).

Near the origin the function $p(t)$ is approximated such that $p(t) \sim -1/t$, and we have $\xi \sim -\log|t| - i\arg t$ $(t \sim 0)$. Then $\Re\xi$ takes positive values near the origin, and a level curves defined by $\Re\xi = \text{const.}$ is a circle $|t| = \text{const.}$ around the origin. Level curves defined by $\Im\xi = \text{const.}$ are radial lines defined by $\arg t = \text{const.}$ and they emerge from the origin. By the way, a Stokes and an anti-Stokes curves are level lines of level 0. Q.E.D.

When m and r are given, we can draw outline of Stokes and anti-Stokes curves by the theorems above. Several cases are shown in Fig. 6.

6. The canonical region.

6.1. *A Stokes region* of (2.4) is defined to be a simply connected region bounded by Stokes curves of (2.4) and it does not contain any Stokes curve as an interior points. There are two types of Stokes regions, the one is *a half-plane type* and the other is *a strip type*.

If we consider $\xi = \xi(t) := \xi_{jk}(t_0, t)$ (t_0 is a secondary turning point) as a mapping from the t-plane to the ξ-plane it is conformal except secondary turning points and singular points. Level lines $\Re\xi =$ const. and $\Im\xi = $const. on the t-plane are mapped vertical and horizontal lines on the ξ-plane respectively. A Stokes region of half-plane type is mapped one-to-one onto a region $\Re\xi > C$ or $\Re\xi < C$, a Stokes region of strip type is mapped one-to-one onto a region $C < \Re\xi < C'$.

For example, D_2 is a Stokes region of strip type and other D_j's are Stokes regions of half-plane type in Fig. 6-1.

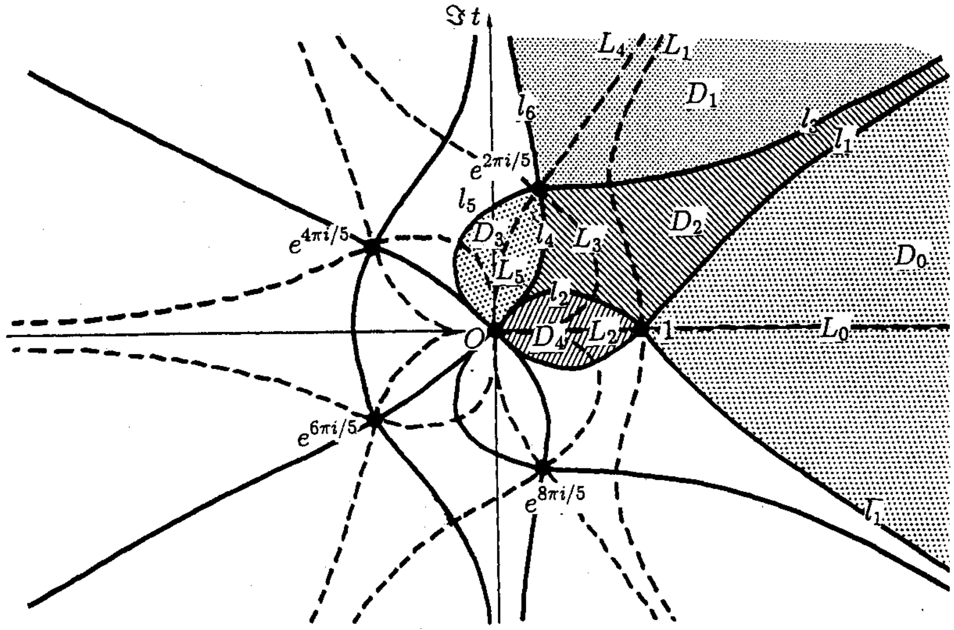

FIGURE 6-1. Stokes curve configuration for $p(t)=t^2-1/t^3$

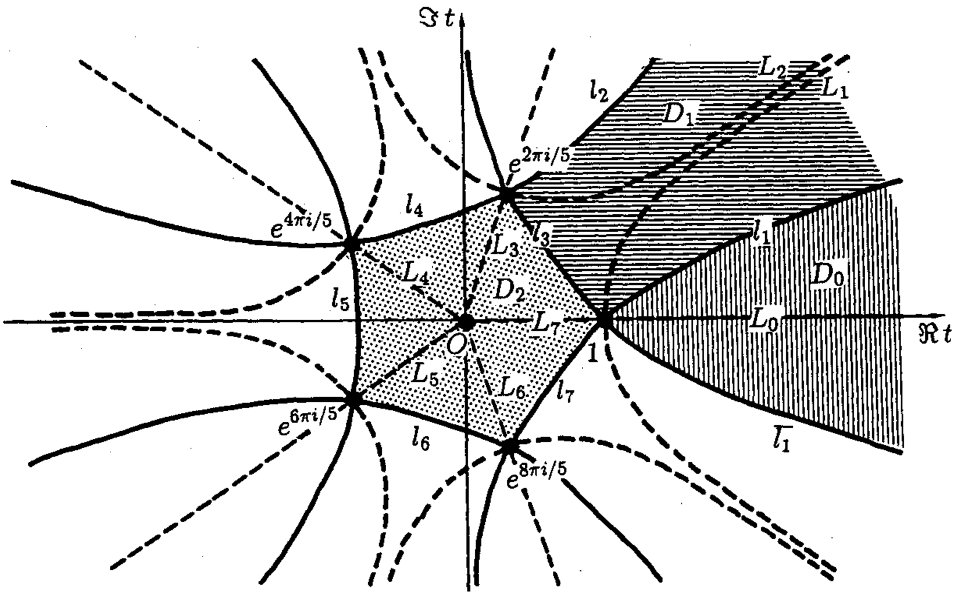

FIGURE 6-2. Stokes curve configuration for $p(t)=t^4-1/t$

A maximal existence region of each solution $y_j(x, \varepsilon)$ in the lemma 3.2 is called *a λ_j-admissible region* of (3.1), which is defined to consist of adjacent several Stokes regions such that there exist paths $|\Re\xi_{jk}|\to\infty$ for any $k\neq j$ in their mapped region. In order to prove the existence theorem (lemma 3.2), we have to show an existence of

λ_j-admissible region (Nakano et al. [16]). This proving technique is very similar to the case of the local thoery (Fukuhara [7], Wasow [28] et al.), and in order to get a λ_j-admissible region it is sufficient to show the existence of routes along which $\Re\xi \to +\infty, -\infty$.

A maximal existence region of n independent solutions with double asymptotic property, which is called *a canonical region*, is defined to be an intersection of all λ_j-admissible regions for $j=1, 2, \cdots, n$ (Evgrafov-Fedoryuk [1], Fedoryuk [5], cf. Kelly [10]).

6.2. Since the origin and the point at infinity are irregular singular points of the inner equation for $r>1$, there are two directions or paths tending to them along which $\Re\xi \to \pm\infty$ as $t \to 0, \infty$. Those routes are given by anti-Stokes curves, for example, L_2, L_5 and L_0, L_4 in Fig. 6-1.

A canonical region is mapped one-to-one onto one or more sheets of the ξ-plane together with slits emerging from the images of the secondary turning points.

For example, D_0 and D_2 in Fig. 4-1 is mapped onto a right half plane ($\Re\xi>0$) and a strip region ($\frac{5}{6}(\cos(4\pi/5)-1)<\Re\xi<0$) of the ξ-plane by $\xi=\xi(t):=\int_1^t p(s)ds$ respectively, and other D_j's are mapped onto a right or a left half plane of the ξ-plane.

The region $(\bigcup_{j=0}^4 D_j) \cup (\bigcup_{j=1}^4 l_j)$ is a canonical region for (2.4) and it is mapped onto a region consisting of two left half planes, two right half planes and a strip region between them with slits of the ξ-plane.

In Fig. 6-2, D_0 and D_1 are Stokes regions of half-plane type and their images are $\xi(D_0)=\{\xi : \Re\xi>0\}$ and $\xi(D_1)=\{\xi : \Re\xi<0\}$ by the mapping $\xi(t):=\int_1^t p(s)ds$. D_2 is a special Stokes region of strip type, because it is mapped onto a strip region $\mathbf{D}_k=\{\xi : C_k<\Im\xi<C_{k+1}, \Re\xi>0\}$ and the image of the sum of infinitely many D_2's, i.e., $\bigcup_{k=-\infty}^{\infty} \mathbf{D}_k$ is a half plane ($\Re\xi>0$). In Fig. 6 the solid lines denote Stokes curves and the broken lines denote anti-Stokes curves.

7. The matching matrix.

7.1. The outer region (2.2) and the inner region (2.5) are overlapped for small ε, and there exist the true outer and the true inner solutions of (1.1) in them respectively.

Taking an appropriate point belonging to both regions, we can compute a linear relation between the outer and the inner solutions. This relation can be represented by a matrix and it is called *the matching matrix* (Wasow [27]).

Using n independent outer solutions $y_j^{out}(x, \varepsilon)$'s, we get a vector form solution

$$Y^{out} := {}^t[y_1^{out}(x, \varepsilon), y_2^{out}(x, \varepsilon), \cdots, y_n^{out}(x, \varepsilon)]\,, \tag{7.1}$$

and we call Y^{out} *the outer solution* of (1.1), too.

Similarly, we get *the inner solution* of (1.1) of vector form

$$Y^{in} := {}^t[y_1^{in}(x, \varepsilon), y_2^{in}(x, \varepsilon), \cdots, y_n^{in}(x, \varepsilon)]\,. \tag{7.2}$$

Since the matching matrix M relates Y^{out} and Y^{in} linearly, M has to satisfy the relation

$$MY^{out}=Y^{in}\,. \tag{7.3}$$

The matching matrix $M=[m_{ij}]$ is an $n\times n$ matrix.

By using the WKB solutions (3.5) and (3.6), we can compute the matching matrix.

THEOREM 7.1. *The asymptotic representation of the matching matrix* $M=[m_{ij}]$ *defined by* (7.3) *is given by*

$$M\sim\varepsilon^{am\cdot \mathrm{diag}[\mu_1,\,\mu_2,\cdots,\,\mu_n]}\qquad(\varepsilon\to 0)\,. \tag{7.4}$$

PROOF. Let $\tilde{Y}^{out}$ be the outer WKB solution of vector form which is defined by substituting the outer WKB solutions $\tilde{y}_j^{out}$'s for y_j^{out}'s in (7.1). Similarly the inner WKB solution of vector form $\tilde{Y}^{in}$ is defined.

The matching relation (7.3) is asymptotically represented by

$$M\tilde{Y}^{out}\sim\tilde{Y}^{in}\qquad(\varepsilon\to 0)\,. \tag{7.5}$$

Elements of the matrix relation (7.5) satisfy

$$\sum_{j'=1}^{n} m_{jj'}\tilde{y}_{j'}^{out}\sim\tilde{y}_j^{in}\qquad(\varepsilon\to 0;\,j=1,2,\cdots,n)$$

or

$$\sum_{j'=1}^{n} m_{jj'}\frac{\tilde{y}_{j'}^{out}}{\tilde{y}_j^{in}}\sim 1\qquad(\varepsilon\to 0;\,j=1,2,\cdots,n)\,. \tag{7.6}$$

Put

$$x=\eta\rho\,,\quad t=\eta\rho^{-1}\,,\quad \rho=\varepsilon^{\alpha/2}\,,\quad |\eta|=1\,. \tag{7.7}$$

Then, x belongs to the outer region ($K\varepsilon^{\alpha}\le|x|\le x_0$) and t $(=x\rho^{-2})$ belongs to the inner region ($0<|t|<\infty$) where $\varepsilon=\rho^{2/\alpha}$ is small. A new complex parameter η will be defined soon later.

By substituting $x=\eta\rho$ and $\varepsilon=\rho^{2/\alpha}$ in the outer WKB solutions (3.5), we get

$$\tilde{y}_j^{out}=(\eta\rho)^{-m\mu_j}\exp\left(\frac{a_j\eta^{m+1}}{m+1}\rho^{m+1-2h/\alpha}\right). \tag{7.8}$$

Similarly, we get from (3.6) and (7.7)

$$\tilde{y}_j^{in}\sim(\eta\rho^{-1})^{-m\mu_j}\exp\left(\frac{a_j\eta^{m+1}}{m+1}\rho^{-m-1-2h'/\alpha}\right)\qquad(\rho\to 0)\,. \tag{7.9}$$

The exponent of ρ in the exp-term of (7.9) is

$$-m-1-\frac{2}{\alpha}h'=-m-1-\frac{2}{\alpha}(h-(m+1)\alpha)=m+1-\frac{2}{\alpha}h\,.$$

The last term is equal to the exponent of ρ in the exp-term of (7.8).

Then, from (7.8) and (7.9) we get

$$\frac{\tilde{y}_j^{out}}{\tilde{y}_j^{in}} \sim \rho^{-2m\mu_j} = \varepsilon^{-\alpha m\mu_j} \qquad (\rho \to 0) \tag{7.10}$$

and

$$\frac{\tilde{y}_{j'}^{out}}{\tilde{y}_j^{in}} \sim \rho^{-m\mu_j - m\mu_{j'}} \exp\left(\frac{a_{j'}-a_j}{m+1} \eta^{m+1} \rho^{m+1-2h/\alpha}\right) \qquad (j \neq j';\ \rho \to 0)\,. \tag{7.11}$$

There exist two routes along which $\Re t$ ($t=\eta\rho^{-1}$, $\rho>0$) is either positive or negative. Those routes are anti-Stokes curves L_1 and L_0 emarging from the secondary turning points $t=1$ in Fig. 6-1, for example.

If we choose a parameter η such as $\Re\eta^{m+1}>0$ for $a_{j'}-a_j>0$, and if we choose η such as $\Re\eta^{m+1}<0$ for $a_{j'}-a_j<0$, then the magnitude of the exp-term of (7.11) tends to $+\infty$ as $\rho \to 0$. From (7.6) and (7.10), (7.11) we get

$$m_{jj} \sim \varepsilon^{\alpha m\mu_j}\,, \quad m_{jj'} \sim 0\ (j \neq j') \qquad (\varepsilon \to 0)\,, \tag{7.12}$$

then the matching matrix (7.4) follows. Q.E.D.

7.2. Thus we could analyze the asymptotic property of the solutions of the differential equation (1.1) in a region of $0<|x|\le x_0$. We conclude our analysis as

THEOREM 7.2. *We suppose the singular perturbation condition* (1.2). *Then the differential equation* (1.1) *is reduced to the outer and inner equations* (2.3) *and* (2.4) *in the outer and inner regions* (2.2) *and* (2.5) *respectively.*

The outer and the inner WKB solutions (3.5) and (3.6) *are asymptotic expansions of the true solutions of the outer and the inner equations of* (1.1) *respectively.*

The outer and the inner solutions are related by the matching matrix (7.4).

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