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ON AN ORDER-BASED CONSTRUCTION OF A TOPOLOGICAL GROUPOID FROM AN INVERSE SEMIGROUP

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Abstract We show how to construct a topological groupoid directly from an inverse semigroup and prove that it is isomorphic to the universal groupoid introduced by Paterson. We then turn to a certain reduction of this groupoid. In the case of inverse semigroups arising from graphs (respectively, tilings), we prove that this reduction is the graph groupoid introduced by Kumjian *et al*. (respectively, the tiling groupoid of Kellendonk). We also study the open invariant sets in the unit space of this reduction in terms of certain order ideals of the underlying inverse semigroup. This can be used to investigate the ideal structure of the associated reduced C^*-algebra.

Keywords: groupoids; inverse semigroups; graphs; tilings

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1. Introduction

This article is concerned with the construction of topological groupoids from inverse semigroups with applications to graphs and tilings. The motivation for our study comes from two sources. The first is the work of Paterson [**12, 13**] and Kellendonk [**4, 5**] (see also [**6, 7**]). Both Paterson and Kellendonk describe constructions assigning groupoids to inverse semigroups. The relationship between their constructions is not clear and it is our aim to present a unified view. The second starting point for this work is given by investigations on graphs and their associated groupoids by Kumjian *et al*. [**9**]. They study (among other topics) the ideal structure of C^*-algebras associated with graphs (see also [**2, 3**] for different approaches to these algebras). Here, it is our aim to present an inverse-semigroup-based approach to this topic.

This paper is organized as follows. In § 2 we review several known facts on groupoids and (the order of) inverse semigroups. In particular, we show that an inverse semigroup Γ gives rise to two groupoids, one arising by restricting the multiplication and the other consisting of minimal elements. In this section, we also introduce a new condition (L) which, while strictly weaker, can replace $E^\star$-unitarity in our context. Section 3 contains the basic constructions showing that the set of directed sets in an inverse semigroup modulo the obvious equivalence relation is again an inverse semigroup. By the considerations in § 2 this new semigroup then gives rise to the two groupoids $G_u(\Gamma)$ and $G_m(\Gamma)$. In § 4 we study certain aspects of $G_u(\Gamma)$ in some detail. In particular, we show that it is an r-discrete ample groupoid. It is Hausdorff if Γ satisfies (L). Moreover, it is shown to be isomorphic to the universal groupoid of Γ introduced by Paterson. Further features of $G_m(\Gamma)$, including compactness features, are studied in §§ 5 and 6. In § 7 we investigate the open invariant subsets of $G_m(\Gamma)$ and characterize them in terms of certain order ideals of Γ. When combined with a theory of Renault, this allows one to characterize the ideals in the reduced groupoid $C^\star$-algebra. Section 8 is devoted to applications to graphs. We show how the material of the preceding sections can be used to recover some results of Kumjian *et al*. Finally, in § 9, we recall the results in [**4, 5**] on tilings and provide a study of ideal theory of the algebras $C^*_{\mathrm{red}}(G_m(\Gamma))$ in this case. This underlines the similarity between the tiling case and the graph case.

2. Preliminaries

In this section we recall basic facts concerning groupoids and inverse semigroups. For further details and the proofs omitted below, we refer the reader to [**10, 13, 15**], for example. We then introduce a new condition, (L), and we describe the two natural groupoids that can be associated with any inverse semigroup.

A groupoid is a set G together with a partially defined associative multiplication $*$ and an involution $x \mapsto x^{-1}$ satisfying the following conditions:

(G1) $(x^{-1})^{-1} = x$;

(G2) if $x * y$ and $y * z$ exist, then $x * y * z$ also exists;

(G3) $x^{-1} * x$ exists and if $x * y$ also exists, then $x^{-1} * x * y = y$;

(G4) $x * x^{-1}$ exists and if $z * x$ also exists, then $z * x * x^{-1} = z$.

Elements of the form xx^{-1} are called units of G and the set of all units of G is denoted by $G^{(0)}$. Each groupoid comes with the maps $r : G \to G^{(0)}$ and $s : G \to G^{(0)}$ defined by $r(x) = xx^{-1}$ and $s(x) = x^{-1}x$. A subset S of G is called a G-set if both r and s are one-to-one on S. If G carries a topology making $*$ and -1 continuous, it is called a topological groupoid. If the topology of G admits a basis of G-sets, then G is called r-discrete. Here, a basis of a topology is a family of open sets such that every open set can be written as a union of sets from this family.

A subset E of the units $G^{(0)}$ of a groupoid G is called *invariant* if for $e \in E$ and $g \in G$ with $e = g^{-1}g$ the element gg^{-1} also belongs to E. If E is invariant, the (set theoretic) *reduction* G_E of G to E is the subgroupoid of G consisting of all elements $g \in G$ with $g^{-1}g \in E$ (which by invariance implies that $gg^{-1} \in E$ as well). In the context of topological groupoids, the invariant set E is further required to be closed in $G^{(0)}$. We will then speak of the *topological reduction* G_E.

Γ is called an inverse semigroup if Γ is a semigroup and for each $x \in \Gamma$ there exists a unique $x^{-1} \in \Gamma$, called the inverse of x with $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The map $x \mapsto x^{-1}$ is an involution. We denote by $\Gamma^{(0)}$ the idempotents of Γ, i.e. the set of p with $p = pp$. Idempotents commute. On Γ we have the relation $\prec$, where $x \prec y$, whenever $xy^{-1} = xx^{-1}$. If $x \prec y$, then x is said to be a *predecessor* of y and y is said to be a *successor* of x. Alternatively, x is said to be *smaller* than y. The relation $\prec$ is an order, i.e. a reflexive, transitive relation such that $x \prec y$ and $y \prec x$ implies $x = y$. It is compatible with multiplication and $x \prec y$ implies that $x^{-1} \prec y^{-1}$. A zero, denoted by 0, is a necessary unique element of Γ with $0x = x0 = 0$ for all $x \in \Gamma$. Γ is said to be an inverse semigroup with zero if it contains such an element. In the following we will sometimes write conditions of the form $0 \neq z \in \Gamma$. This is meant to mean that z is not zero if Γ has a zero and to mean a vacuous condition if Γ does not contain a zero.

An element $x \in \Gamma$ is called *minimal* if $x \neq 0$ and $y \prec x$ and $y \neq 0$ implies $y = x$. The set of minimal elements in Γ is denoted by $\Gamma_{\min}$. One easily obtains the following.

Proposition 2.1. *For $x \in \Gamma$ the following are equivalent:*

(i) *x is minimal;*

(ii) *x^{-1} is minimal;*

(iii) *$x^{-1}x$ is minimal;*

(iv) *xx^{-1} is minimal.*

For $x, y \in \Gamma_{\min}$ the following are equivalent:

(i) *$xy \neq 0$;*

(ii) *$x^{-1}x = yy^{-1}$;*

(iii) *$xy \in \Gamma_{\min}$.*

Now, there is an immediate way of constructing two groupoids from Γ. The first goes back to Ehresman (see, for example, [**13**, Proposition 1.0.1]). In this case, one just considers Γ with its usual inversion and the multiplication defined by $x * y = xy$ if and only if $x^{-1}x = yy^{-1}$. This yields a groupoid, which is denoted by $G(\Gamma)$ in the following. By the previous proposition, the set of minimal elements of Γ is a subgroupoid of $G(\Gamma)$ with involution from Γ and multiplication defined whenever the product is not zero. It will be denoted by $M(\Gamma)$. This construction gives $M(\Gamma) = G(\Gamma)_{\Gamma^{(0)} \cap \Gamma_{\min}}$.

The order $\prec$ will in general not be a semi-lattice. However, one can still wonder about the existence of a largest common predecessor of x and y given that there exists a common predecessor. If such a largest common predecessor exists, it must be unique and will be denoted by $x \wedge y$. As the existence of such largest predecessors will be very important to us, we introduce the following definition.

Definition 2.2. An inverse semigroup is said to satisfy the lattice condition (L) if, for any $x, y \in \Gamma$ with a common predecessor not equal to zero, there exists a largest common predecessor.

There is a well-known criterion for the existence of largest common predecessors, which we give next. Recall that an *order ideal* $\mathcal{I}$ in Γ is a set with $\{y : y \prec x\} \subset \mathcal{I}$ for every $x \in \mathcal{I}$. An *inverse order ideal* is a set $\mathcal{I}$ in Γ with $\{y : x \prec y\} \subset \mathcal{I}$ for every $x \in \mathcal{I}$. An inverse semigroup Γ is called *E-unitary* if $\Gamma^{(0)}$ is an inverse order ideal and *$E^\star$-unitary* if $\Gamma^{(0)} \setminus \{0\}$ is an inverse order ideal. If Γ is such an inverse semigroup and $x, y \in \Gamma$ have a common predecessor z not equal to zero, then there exists a largest such z. It is given by $yx^{-1}y = xy^{-1}x$.

Remark 2.3. The condition (L) is strictly weaker than E-unitarity and $E^\star$-unitarity. This can be seen by considering a groupoid G with $G \neq G^{(0)}$ and the inverse semigroup $S(G)$ of its G-sets. Then, $S(G)$ satisfies (L) as it is closed under the usual intersection of sets. However, as $G \neq G^{(0)}$, there exist G-sets S, S' with $S \subset S'$, $S \subset G^{(0)}$ and an S' not contained in $G^{(0)}$. Then, $S \prec S'$ as $S \subset S'$ but $S' \notin S(G)^0$ as S' is not contained in $G^{(0)}$.

While E-unitarity and $E^\star$-unitarity have been used when studying topological properties of groupoids associated with inverse semigroups [**5**,**13**] (see also [**11**]), it turns out that our considerations need only the weaker condition (L).

For arbitrary Γ and $x, y \in \Gamma$ with a common successor $z \in \Gamma$ and a common non-zero predecessor, $x \wedge y$ exists and equals $xx^{-1}yy^{-1}x = xx^{-1}yy^{-1}y$.

3. The basic construction

In this section we consider the set of directed subsets of Γ. This set is equipped with a natural pre-order. Factoring out by the associated equivalence relation leads to an order and in fact to an inverse semigroup with respect to the obvious multiplicative structure. Lemma 3.1 is strongly related to the results in [**4**]. This is discussed at the end of § 6.

A subset A of Γ is called (downward) *directed* if, for any $x, y \in A$, there exists a $z \in A$ with $z \prec x, y$. The set of directed subsets of Γ is denoted by $\mathcal{F}(\Gamma)$.

On $\mathcal{F}(\Gamma)$, we define the relation $\prec$ by $A \prec B$ if, for any $b \in B$, there exists an $a \in A$ with $a \prec b$. Moreover, we define AB by $AB \equiv \{ab : a \in A,\ b \in B\}$ and A^{-1} by $A^{-1} = \{a^{-1} : a \in A\}$. The corresponding sets are indeed directed by the results in § 2. Moreover, we set $A \sim B$, whenever $A \prec B$ and $B \prec A$. Then, $\prec$ is a pre-order and by well-known results $\sim$ is then an equivalence relation on $\mathcal{F}(\Gamma)$. We set $\mathcal{O}(\Gamma) = \mathcal{F}(\Gamma)/{\sim}$. Representatives of $X, Y \in \mathcal{O}(\Gamma)$ will be denoted by $\dot{X}$ and $\dot{Y}$. The class of $A \in \mathcal{F}(\Gamma)$ will be denoted by $[A]$. On $\mathcal{O}(\Gamma)$, we define a multiplication by $XY \equiv [\dot{X}\dot{Y}]$, where $\dot{X}$ and $\dot{Y}$ are arbitrary representatives of X and Y. It is easy to check that this is a well-defined associative multiplication. Moreover, we define a map $i : \mathcal{O}(\Gamma) \to \mathcal{O}(\Gamma)$ by $i(X) \equiv [\dot{X}^{-1}]$, where again $\dot{X}$ is a representative of X and this is well defined. We note in passing that, in an inverse semigroup Γ with a zero, $[B] \neq 0$ holds for every directed set B with $0 \notin B$.

Theorem 3.1. *The set $\mathcal{O}(\Gamma)$ with multiplication and inversion $X^{-1} \equiv i(X)$ is an inverse semigroup. The relation $X \prec Y$ holds for $X, Y \in \mathcal{O}(\Gamma)$ if $\dot{X} \prec \dot{Y}$ holds for some (all) representatives $\dot{X}$ of X and $\dot{Y}$ of Y.*

Proof. We first show that each $X \in \mathcal{O}(\Gamma)$ has a unique inverse given by $i(X)$. Existence follows easily from

$$\dot{X} = \{xx^{-1}x : x \in \dot{X}\} \sim \{x_1x_2^{-1}x_3 : x_1, x_2, x_3 \in \dot{X}\} = \dot{X}\dot{X}^{-1}\dot{X}.$$

To show uniqueness, let X and Y be given with representatives $\dot{X}$ and $\dot{Y}$ and assume that

(i) $XYX = X$, and

(ii) $YXY = Y$.

By (i), we have

$$\dot{X}^{-1} \sim \dot{X}^{-1}\dot{X}\dot{X}^{-1} \sim \dot{X}^{-1}\dot{X}\dot{Y}\dot{X}\dot{X}^{-1} \prec \dot{Y},$$

yielding $i(X) \prec Y$. Similarly, by (ii), we arrive at $Y \prec i(X)$. Putting these together, we obtain the desired uniqueness result. This shows that $\mathcal{O}(\Gamma)$ is indeed an inverse semigroup. Using this, it is not difficult to obtain the statement about the order. □

We now combine this construction with the results of § 2.

Definition 3.2. The groupoid $G_u(\Gamma) \equiv G(\mathcal{O}(\Gamma))$ is called the universal groupoid of Γ. The groupoid $G_m(\Gamma) \equiv M(\mathcal{O}(\Gamma))$ is called the minimal groupoid of Γ.

The considerations of § 2 immediately yield $G_m(\Gamma) = G_u(\Gamma)_{\mathcal{O}(\Gamma)^{(0)}_{\min}}$.

4. The groupoid $G_u(\Gamma)$

In this section we introduce a topology on $G_u(\Gamma)$, making it into a topological r-discrete groupoid. We also show that $G_u(\Gamma)$ with this topology is isomorphic to the universal groupoid introduced by Paterson in [**12**, **13**].

In the following we simply write x instead of $[\{x\}] \in \mathcal{O}(\Gamma)$ for $x \in \Gamma$. In particular, we write $X \prec x$ instead of $X \prec [\{x\}]$ for $X \in \mathcal{O}(\Gamma)$. Note that we have $X = xX^{-1}X = XX^{-1}x$ for $X \prec x$. This will be used several times in the following. For $x \in \Gamma$, we set $U_x \equiv \{X \in G_u(\Gamma) : X \prec x\}$. For $x, x_1, \dots, x_n \in \Gamma$ with $x_1, \dots, x_n \prec x$, we set

$$U_{x;x_1,\dots,x_n} \equiv U_x \cap U_{x_1}^{\mathrm{c}} \cap \dots \cap U_{x_n}^{\mathrm{c}}.$$

Here, U_x^{c} is the complement of U_x in $\mathcal{O}(\Gamma)$. We will show that the family of these $U_{x;x_1,\dots,x_n}$ gives the basis of a topology. To do so, we need the following proposition.

Proposition 4.1. *For $X \in G_u(\Gamma)$, $x_1, \dots, x_n \prec x$ and $y_1, \dots, y_m \prec y$ in Γ with $X \in U_{x;x_1,\dots,x_n} \cap U_{y;y_1,\dots,y_m}$, there exist $z_1, \dots, z_k \prec z$ with $z \prec x, y$ and*

$$X \in U_{z;z_1,\dots,z_k} \subset U_{x;x_1,\dots,x_n} \cap U_{y;y_1,\dots,y_m}.$$

Proof. Let p_j and q_l in $\Gamma^{(0)}$ be given such that $x_j = xp_j$ and $y_l = yq_l$, $j = 1, \dots, n$, $l = 1, \dots, m$. By $X \in U_x \cap U_y$, there exists $z \in \Gamma$ with $X \prec z \prec x, y$. Thus, there exist $p, q \in \Gamma^{(0)}$ with $z = xp = yq = xpq = ypq$. Of course, it suffices to show that

$$X \in U_{z;zp_1,\dots,zp_n,\ zq_1,\dots,zq_m} \subset U_{x;x_1,\dots,x_n} \cap U_{y;y_1,\dots,y_m}.$$

It is straightforward to show that X belongs to $U_{z;zp_1,\ldots,zp_n,zq_1,\ldots,zq_m}$. Therefore, let us now show that $Y \in U_{z;zp_1,\ldots,zp_n,\, zq_1,\ldots,zq_m}$ also belongs to $U_{x;x_1,\ldots,x_n} \cap U_{y;y_1,\ldots,y_m}$. By $Y \prec z$ we have $Y \prec x$ and $Y \prec y$. Thus, it remains to show that Y belongs neither to U_{x_j} nor to U_{y_l} for arbitrary j and l as above. Assume that $Y \prec xp_j$. By $Y \prec z$, this gives the contradiction

$$Y = YY^{-1}Y \prec xp_jz^{-1}xp_j = xp_jz^{-1}zp_j = xz^{-1}zp_j = zp_j,$$

where we have used $z \prec x$ twice. Similarly, we show that $Y \prec yq_l$ cannot hold. □

The proposition implies that the family $\mathcal{T}$ defined next is indeed a topology.

Definition 4.2. The topology $\mathcal{T}$ on $G_u(\Gamma)$ is the family of sets which are unions of sets of the form $U_{x;x_1,\ldots,x_n}$.

Proposition 4.3. *$G_u(\Gamma)$ is a topological groupoid with respect to $\mathcal{T}$.*

Proof. Continuity of inversion is obvious. To show that multiplication is continuous, let $Z = X * Y \in U_{z;z_1,\ldots,z_n}$ be given. Let $p_j, q_j \in \Gamma^{(0)}$ be given with $z_j = p_jz = zq_j$ for $j = 1,\ldots,n$. There exist $x, y \in \Gamma$ with $X \prec x$, $Y \prec y$ and $xy \prec z$. As $X * Y$ exists in $G_u(\Gamma)$, we have $X^{-1}X = YY^{-1}$ and we can assume without loss of generality that $x^{-1}x = yy^{-1}$. It is now straightforward to show that $X \in U_{x;p_1x,\ldots,p_nx}$ and $Y \in U_{y,yq_1,\ldots,yq_n}$. Thus, it remains to show that for $A \in U_{x;p_1x,\ldots,p_nx}$ and $B \in U_{y,yq_1,\ldots,yq_n}$ the product $A * B$ belongs to $U_{z;z_1,\ldots,z_n}$ (if it exists). Clearly, $A * B$ belongs to $U_{xy} \subset U_z$. Assume that $A * B \prec z_j$ for some j. Then, there exist $a, b \in \Gamma$ with $A \prec a$ and $B \prec b$ and $ab \prec z_j$. Again, as AB exists in $G_u(\Gamma)$, we can assume without loss of generality that $a^{-1}a = bb^{-1}$. Moreover, we can assume without loss of generality that $a \prec x$ and $b \prec y$ as $A \in U_x$ and $B \in U_y$. This gives

$$a =\prec ax^{-1}x = abb^{-1}x^{-1}x \prec z_jb^{-1}x^{-1}x \prec z_jy^{-1}x^{-1}x \prec p_jzz^{-1}x \prec p_jx.$$

This gives a contradiction, as A does not belong to U_{p_jx}. □

Proposition 4.4.

(a) *For arbitrary $X \neq Y \in G_u(\Gamma)$, there exists $z \prec x \in \Gamma$ with $X \in U_{x;z}$ and $Y \notin U_{x;z}$.*

(b) *The set $G_u(\Gamma)^{(0)}$ is closed in $G_u(\Gamma)$.*

Proof. (a) Consider first the case $Y \prec X$ (and $X \neq Y$). Let $x \in \Gamma$ with $X \prec x$ be given. Then, there exists an $y \prec x$ with $Y \prec y$ and not $X \prec y$. This gives $X \in U_{x;y}$ and $Y \notin U_{x;y}$. On the other hand, if $Y \prec X$ does not hold, then there exists an $x \in \Gamma$ with $X \prec x$ and not $Y \prec x$ and we infer that $X \in U_x$ and $Y \notin U_x$.

(b) It suffices to show that, for every converging net (P_i) in $G_u(\Gamma)^{(0)}$, the limit P belongs to $G_u(\Gamma)^{(0)}$, i.e. it satisfies $P = PP^{-1}$. But this is immediate from (a) and continuity of multiplication. □

Proposition 4.5. *Let $p_1, \dots, p_n \prec p \in \Gamma^{(0)}$ be given. Let $x \in \Gamma$ with $p \prec x^{-1}x$ be given. Then $U_{xp;xp_1,\dots,xp_n} = xU_{p;p_1,\dots,p_n}$.*

Proof. This follows easily from $X = xX^{-1}X$ and the fact that $X^{-1}X \prec q$ if and only if $xX^{-1}X \prec xq$ for $q \prec x^{-1}x$ and $X \prec x$. □

Proposition 4.6. *$G_u(\Gamma)$ is r-discrete.*

Proof. It suffices to show that the maps

$$s : U_{x;x_1,\dots,x_n} \to U_{x^{-1}x;x_1^{-1}x_1,\dots,x_n^{-1}x_n}, \quad X \mapsto X^{-1}X$$

and

$$r : U_{x;x_1,\dots,x_n} \to U_{xx^{-1};x_1x_1^{-1},\dots,x_nx_n^{-1}}, \quad X \mapsto XX^{-1}$$

are homeomorphisms. We show only the statement about s. The statement about r follows similarly. By Proposition 4.5, the map

$$s^* : U_{x^{-1}x;x_1^{-1}x_1,\dots,x_n^{-1}x_n} \to U_{x;x_1,\dots,x_n}, \quad P \mapsto xP$$

is surjective. By

$$(xP)^{-1}xP = Px^{-1}xP = PP = P,$$

we have that s^* is also injective. Moreover, we see that s and s^* are inverse to each other and s is therefore a bijection. By Proposition 4.3, the map s is continuous. Using Propositions 4.1 and 4.5, one can also infer that s^* is continuous. □

Proposition 4.7. *For arbitrary $x_1, \dots, x_n \prec x \in \Gamma$, the set $U_{x;x_1,\dots,x_n}$ is compact.*

Proof. By Proposition 4.6, it suffices to consider $U_{p;p_1,\dots,p_n}$ with $p_1, \dots, p_n \prec p \in \Gamma^{(0)}$. This, however, is just a reformulation of the well-known properties of the maximal ideal space of the commutative Banach algebra $l^1(\Gamma^{(0)})$ (see also [**13**]). We include a short sketch for completeness. Clearly, the map $j : G_u(\Gamma)^{(0)} \to \{0,1\}^{\Gamma^{(0)}}$ with $j(P)(q) = 1$ if $P \prec q$ and $j(P)(q) = 0$ otherwise is injective (see also Lemma 4.10). Moreover, if $\{0,1\}$ carries the discrete topology and $\{0,1\}^{\Gamma^{(0)}}$ is given the product topology, then the topology in $G_u(\Gamma)^{(0)}$ is easily seen to be the topology induced by this injection. Thus, it remains to show that $j(G_u(\Gamma)^{(0)})$ is closed in $\{0,1\}^{\Gamma^{(0)}}$. Therefore, assume that the net $(j(P_i))$ converges to $f \in \{0,1\}^{\Gamma^{(0)}}$. Then, it is not difficult to see that $\{q \in \Gamma^{(0)} : f(q) = 1\}$ is a directed inverse order ideal and $f = j(P)$ with $P = [\{q \in \Gamma^{(0)} : f(q) = 1\}]$. □

Proposition 4.8. *The map $U : \Gamma \to S(G_u(\Gamma))$, $x \mapsto U_x$ is an injective homomorphism of inverse semigroups.*

Proof. By Proposition 4.6, the sets U_x are indeed $G_u(\Gamma)$-sets. Thus, U maps into $S(G_u(\Gamma))$. Clearly, U preserves the involution. Thus, it only remains to show that $U_{xy} = U_xU_y$. The inclusion $\supset$ is obvious. Let now $Z \in U_{xy}$ be given. By $Z \prec xy$, we have $x^{-1}Z \prec y$ and $Zy^{-1} \prec x$ as well as $Zy^{-1}x^{-1} = ZZ^{-1}$. This implies that $Z = ZZ^{-1}Z = Zy^{-1}x^{-1}Z = XY$ with $X \equiv Zy^{-1}$ and $Y \equiv x^{-1}Z$. It remains to show that X and Y are composable in the sense of the groupoid $G_u(\Gamma)$, i.e. that $X^{-1}X = YY^{-1}$. But this follows from

$$X^{-1}X = yZ^{-1}Zy^{-1} = x^{-1}Zy^{-1} = x^{-1}ZZ^{-1}x = YY^{-1},$$

where we have used $Zy^{-1} \prec x$ and $x^{-1}Z \prec y$. Injectivity is simple. □

We summarize our considerations in the following theorem.

Theorem 4.9. *The groupoid $G_u(\Gamma)$ is a topological groupoid with basis of topology given by the family of sets $U_{x;x_1,\dots,x_n}$ for arbitrary $x_1,\dots,x_n \prec x \in \Gamma$. These sets are compact $G_u(\Gamma)$-sets on which r and s are homeomorphisms. The map $U : \Gamma \to S(G_u(\Gamma))$ is an injective homomorphism of inverse semigroups.*

Let us now consider the Hausdorff properties of $G_u(\Gamma)$. By the proof of Proposition 4.7, its unit space is Hausdorff. However, in general $G_u(\Gamma)$ will not be Hausdorff. We will show that, for Γ satisfying condition (L), there is a simple alternative description of the topology of $G_u(\Gamma)$. This will then give the result that $G_u(\Gamma)$ is Hausdorff if Γ satisfies (L) (cf. Corollary 4.11, below).

Consider the map $j : G_u(\Gamma) \to \{0,1\}^\Gamma$ with $j(X)(x) = 1$ if $X \prec x$ and $j(X)(x) = 0$ otherwise. Let $\{0,1\}$ be equipped with the discrete topology and let $\{0,1\}^\Gamma$ be given the product topology. We have the following lemma.

Lemma 4.10. *The map j is injective. If Γ satisfies (L), the topology induced on $G_u(\Gamma)$ from $\{0,1\}^\Gamma$ agrees with $\mathcal{T}$.*

Proof. It is not difficult to show that $X = [\{y : X \prec y\}]$, Thus, if $X \neq Y$, there exists without loss of generality an $x \in \Gamma$ with $X \prec x$ but not $Y \prec x$. This gives $j(X)(x) = 1$ and $j(Y)(x) = 0$ and injectivity follows.

To show that the induced topology agrees with $\mathcal{T}$, we have to show that, for arbitrary $X \in G_u(\Gamma)$ and $x_1,\dots,x_n,y_1,\dots,y_m$ with $X \in U_{x_1} \cap \cdots \cap U_{x_n} \cap U^{\mathrm{c}}_{y_1} \cap \cdots \cap U^{\mathrm{c}}_{y_m}$, there exist $z_1,\dots,z_k \prec z$ with

$$X \in U_{z;z_1,\dots,z_k} \subset U_{x_1} \cap \cdots \cap U_{x_n} \cap U^{\mathrm{c}}_{y_1} \cap \cdots \cap U^{\mathrm{c}}_{y_m}. \tag{4.1}$$

By $X \in U_{x_1} \cap \cdots \cap U_{x_n}$, there exists an $x \in \Gamma$ with $X \prec x \prec x_1,\dots,x_n$. By (L), we can then set $z \equiv x_1 \wedge \cdots \wedge x_n$. Clearly, we have $X \in U_z \subset U_{x_1} \cap \cdots \cap U_{x_n}$. Similarly, we can define $z_j = z \wedge y_j$ for every j with $U_z \cap U_{y_j} \neq \emptyset$. Assume without loss of generality that the set of these j is given by $\{1,\dots,k\}$. By construction, (4.1) holds. □

Corollary 4.11. *If Γ satisfies (L), then $G_u(\Gamma)$ is Hausdorff.*

Remark 4.12. In [**13**] it is shown that $G_u(\Gamma)$ is Hausdorff if Γ is E-unitary. As E-unitary implies (L), Corollary 4.11 strengthens this result.

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We close this section with a discussion of the isomorphism between $G_u(\Gamma)$ and the universal groupoid, $H_u(\Gamma)$, constructed by Paterson [**12, 13**]. His construction proceeds in three steps:

(i) a certain inverse semigroup Γ' containing Γ is shown to act on the space X of semi-characters such that (X, Γ') is a localization;

(ii) by Paterson's extension of Kumjian's theory of localizations [**8, 13**], there exists a groupoid $G(X, \Gamma')$ for this localization;

(iii) the groupoid $H_u(\Gamma) = G(X, \Gamma')$ can be expressed in terms of X and Γ only.

We refrain from discussing the theory of localizations here and just give the description of $H_u(\Gamma)$ in terms of X and Γ according to [**13**, Theorem 4.3.1]. In our discussion we will identify the space of semi-characters used in [**13**] with $\mathcal{O}(\Gamma)^{(0)}$ (see the proof of Proposition 4.7 above and discussion in [**13**, § 4.3]). Moreover, we will use the notation introduced above. In particular, the action of Γ on X will be written accordingly. Using these adjustments to our setting, the groupoid $H_u(\Gamma)$ can be described as follows: it consists of equivalence classes $[P, x]$, of pairs (P, x) with $P \in \mathcal{O}(\Gamma)^{(0)}$, $x \in \Gamma$ with $P \prec xx^{-1}$. Here, two pairs (P, x) and $(\tilde{P}, \tilde{x})$ are identified if $P = \tilde{P}$ and there exists a $p \in \Gamma$ with $P \prec p$ and $px = p\tilde{x}$. The involution is given by $[P, x]^* \equiv [x^{-1}Px, x^{-1}]$ and the multiplication is given by $[P, x] * [x^{-1}Px, y] \equiv [P, xy]$. A basis of the topology is given by sets of the form $\{[P, x] : P \in U_{p;p_1,\dots,p_n}\}$.

Given these reformulations of Paterson's construction, the proof of the following theorem is a simple exercise.

Theorem 4.13. *The map $J : G_u(\Gamma) \to H_u(\Gamma)$, $J(X) \equiv [X^{-1}X, x]$ with an arbitrary x with $X \prec x$, is an isomorphism of topological groupoids with inverse map K given by $K : H_u(\Gamma) \to G_u(\Gamma)$, $K([P, x]) \equiv Px$.*

5. The inverse semigroup $\tilde{\Gamma}$

In this section we introduce and investigate a certain quotient of Γ, which we call $\tilde{\Gamma}$. The relevance of this quotient will become apparent in the subsequent sections when we deal with $G_m(\Gamma)$.

Definition 5.1. For $n \in \mathbb{N}$ and $x, x_1, \dots, x_n \in \Gamma$, we set $x < (x_1, \dots, x_n)$ if for every $y \prec x$, $y \neq 0$, there exist $z \in \Gamma$, $z \neq 0$ and $j \in \{1, \dots, n\}$ with $z \prec y, x_j$. If $n = 1$, we write $x < x_1$ instead of $x < (x_1)$.

While the relation $<$ is not an order, it induces one, as investigated next. The relation $x \gtrless y$ if and only if $x < y$ and $y < x$ can easily be seen to give an equivalence relation on Γ. The quotient $\tilde{\Gamma}$ is then defined by $\tilde{\Gamma} \equiv \Gamma/\gtrless$. Let $\pi : \Gamma \to \tilde{\Gamma}$ be the canonical projection. Direct arguments then give the following.

Proposition 5.2. *$x < y$ implies $x^{-1} < y^{-1}$ as well as $xz < yz$ and $zx < zy$.*

Proposition 5.3. *There exists a unique inverse semigroup structure on $\tilde{\Gamma}$ making π into a homomorphism of inverse semigroups. The relation $x < y$ holds for $x, y \in \Gamma$ if and only if $\pi(x) \prec \pi(y)$.*

Proof. The uniqueness statement is obvious. Let us now show existence of the desired semigroup structure. Using Proposition 5.2, we infer that the sets $\pi(\pi^{-1}(a)^{-1})$ and $\pi(\pi^{-1}(a)\pi^{-1}(b))$ contain exactly one element. Thus, we can define a^{-1} and ba by $\pi(\pi^{-1}(a)^{-1})$ and $\pi(\pi^{-1}(a)\pi^{-1}(b))$, respectively. Let us now show that the inverse is unique. Let $x, y \in \Gamma$ and $a, b \in \tilde{\Gamma}$ with $a = \pi(x)$ and $b = \pi(y)$ and let $aba = a$ and $bab = b$ be given. By $aba = a$, we have $x < xyx$, implying that $x^{-1} = x^{-1}xx^{-1} < x^{-1}xyxx^{-1} < y$. Similarly, we infer that $y^{-1} < x$, and $y^{-1} \gtrless x$ follows. Thus, $\tilde{\Gamma}$ is indeed an inverse semigroup and π is a homomorphism of inverse semigroups.

It remains to prove the statement about the order. Let $x, y \in \Gamma$ with $x < y$ be given. We have to show that $\pi(x)\pi(x^{-1}) = \pi(x)\pi(y)^{-1}$, i.e. $xx^{-1} \gtrless xy^{-1}$. By $x < y$ and Proposition 5.2, we infer that $xx^{-1} < xy^{-1}$, $xx^{-1} < yx^{-1}$. Thus, it remains to show that $xy^{-1} < xx^{-1}$. However, this follows from

$$xy^{-1} = xx^{-1}xx^{-1}xy^{-1} < xx^{-1}yx^{-1}xy^{-1} \prec xx^{-1}.$$

Conversely, assume $\pi(x) \prec \pi(y)$. This easily gives $x < yx^{-1}x \prec y$. □

The following proposition is our main tool in studying $\tilde{\Gamma}$ for Γ satisfying (L).

Proposition 5.4. *Let Γ satisfy (L). If $x, y, z \in \Gamma$ satisfy $0 \neq z < x, y$, then $x \wedge y \wedge z$ exists and is not equal to zero.*

Proof. By $z \prec z$ and $z < x$, we derive from (L) that $0 \neq z \wedge x$ exists. By $0 \neq z \wedge x \prec z$ and $z < y$, we infer, again by (L), that $0 \neq z \wedge x \wedge y$ exists. □

Proposition 5.5.

(a) *If Γ satisfies (L), then $\tilde{\Gamma}$ also satisfies (L).*

(b) *Let Γ be an inverse semigroup with zero satisfying (L). Then $\tilde{\Gamma} = (\tilde{\Gamma})\tilde{}$*

Proof. (a) Let $0 \neq c \prec a, b \in \tilde{\Gamma}$ be given. Choose $x, y, z \in \Gamma$ with $\pi(x) = a$, $\pi(y) = b$ and $\pi(z) = c$. By Proposition 5.3, we then have $0 \neq z < x, y$. By Proposition 5.4, $0 \neq x \wedge y$ exists. By $x \wedge y < x, y$ (even $x \wedge y \prec x, y$) and Proposition 5.3, we then have $\pi(x \wedge y) \prec \pi(x), \pi(y)$. Moreover, a straightforward argument shows that $z < x \wedge y$ holds, yielding $c = \pi(z) \prec \pi(x \wedge y)$. Combining these estimates, we infer that $\pi(x \wedge y) = \pi(x) \wedge \pi(y)$.

(b) It suffices to show that $x < y$ whenever $\pi(x) < \pi(y)$ for $x, y \in \Gamma$. So, we assume that $\pi(x) < \pi(y)$. Without loss of generality we can assume that $0 \neq \pi(x)$. Let $0 \neq z \prec x$ be given. Then we have $\pi(z) \prec \pi(x)$ and, by $\pi(x) < \pi(y)$, there exists $r \in \Gamma$ with $0 \neq \pi(r) \prec \pi(z), \pi(y), \pi(x)$. This gives $0 \neq r < z, y, x$ by Proposition 5.3. By Proposition 5.4, we then infer that $0 \neq z \wedge y \wedge x \wedge r$ and $x < y$ follows. □

For later use we also note the following proposition.

Proposition 5.6.

(a) *The relation* $<$ *is transitive, i.e. for* $x < (x_1, \dots, x_k)$ *with* $x_j < (x_{j,1}, \dots, x_{j,n(j)})$, $j = 1, \dots, k$, $x_{j,l} \in \Gamma$ *suitable, the relation*

$$x < (x_{1,1}, \dots, x_{1,n(1)}, \dots, x_{k,1}, \dots, x_{k,n(k)})$$

holds.

(b) *If* $p < (p_1, \dots, p_n)$ *and* $p \prec x^{-1}x$ *for suitable* $p, p_1, \dots, p_n \in \Gamma^{(0)}$ *and* $x \in \Gamma$, *then* $xpx^{-1} < (xp_1x^{-1}, \dots, xp_nx^{-1})$.

Proof. Part (a) is straightforward. As for (b), let $0 \neq q \prec xpx^{-1}$ be given. By $xpx^{-1} = xx^{-1}xpx^{-1}xx^{-1}$, this implies that $q = xx^{-1}qxx^{-1}$, yielding $x^{-1}qx \neq 0$. Furthermore, we have $x^{-1}qx \prec x^{-1}xpx^{-1}x = p$. Thus, there exist $r \in \Gamma^{(0)} \setminus \{0\}$ and $j \in \{1, \dots, n\}$ with $r \prec x^{-1}qx$ and $r \prec p_j$. This implies that $0 \neq xrx^{-1} \prec xx^{-1}qxx^{-1} \prec q$ and $xrx^{-1} \prec xp_jx^{-1}$, completing the proof of (b). □

6. The groupoid $G_m(\Gamma)$

By the general theory presented in § 2, the groupoid $G_m(\Gamma)$ is a reduction in the set-theoretical sense. Thus, it inherits the topology from $G_u(\Gamma)$ and is a topological r-discrete groupoid. A basis of the topology is given by the sets

$$V_{x;x_1,\dots,x_n} \equiv U_{x;x_1,\dots,x_n} \cap G_m(\Gamma)$$

for arbitrary $x_1, \dots, x_n \prec x \in \Gamma$. In this section, we study compactness of the $V_{x;x_1,\dots,x_n}$, non-emptiness of V_x and the algebraic properties of the map $x \mapsto V_x$.

Proposition 6.1. *If* Γ *contains a zero, then, for every* $Y \in \mathcal{O}(\Gamma)$, $Y \neq 0$, *there exists an* $X \in \mathcal{O}(\Gamma)_{\min}$ *with* $X \prec Y$. *In particular,* $V_x \neq \emptyset$ *for every* $x \neq 0$.

Proof. Let $\dot{Y}$ be a representative of Y. By $Y \neq 0$, we have $0 \notin \dot{Y}$. Consider the family of directed sets containing $\dot{Y}$ but not containing 0. The usual inclusion gives a partial order on this family. Application of Zorn's lemma then gives a maximal element B in this family. This element does not contain zero and, as Γ contains a zero, we see that $[B] \neq 0$. By construction, $[B]$ is minimal and precedes Y. □

Proposition 6.2.

(a) *Let* Γ *be an inverse semigroup with zero satisfying* (L). *The following are then equivalent:*

(i) $x < (x_1, \dots, x_n)$;

(ii) $V_x \subset V_{x_1} \cup \dots \cup V_{x_n}$.

In particular, $V_x = V_y$ *if and only if* $x < y$ *and* $y < x$.

(b) *For arbitrary* Γ *with zero (not necessarily satisfying (L)), the equivalence of (i) and (ii) holds, whenever* $x, x_1, \dots, x_n$ *all belong to* $\Gamma^{(0)}$.

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Proof. (a) (i) $\Longrightarrow$ (a) (ii). Let $X \in V_x$ be given. Then, $A \equiv \{y : y \prec x, X \prec y\}$ is a representative of X. Set $A_j \equiv \{y \wedge x_j : y \in A$ such that $0 \neq y \wedge x_j$ exists$\}$. By (i) and (L), there exists a j with $A_j \prec A$. This gives $[A_j] \prec X$. As Γ has a zero, we have $[A_j] \neq 0$ and, by the minimality of X, we infer that $X = [A_j]$. As $[A_j]$ belongs to V_{x_j}, statement (ii) is proven.

(a) (ii) $\Longrightarrow$ (a) (i). Let $y \prec x$, $y \neq 0$, be given. As Γ has a zero, by Proposition 6.1 there exists a $Y \in \mathcal{O}(\Gamma)_{\min}$ with $Y \prec y$. This implies that $Y \in V_y \subset V_x$. By (ii), we infer that $Y \in V_{x_j}$, i.e. $Y \prec x_j$ for a suitable j. Thus, y and x_j have a common predecessor not equal to zero.

(b) This follows as existence of largest predecessors is always valid on $\Gamma^{(0)}$. □

We can now study compactness properties of the V_x, $x \in \Gamma$.

Proposition 6.3. *The following are equivalent:*

(i) *for arbitrary $x_1, \dots, x_n \prec x \in \Gamma$ the set $V_{x;x_1,\dots,x_n}$ is compact;*

(ii) *the set $G_m(\Gamma)^{(0)}$ is closed in $G_u(\Gamma)$;*

(iii) *$G_m(\Gamma)$ is a topological reduction of $G_u(\Gamma)$.*

Proof. The equivalence of (ii) and (iii) is immediate from the considerations of the second section. The implication (ii) $\Longrightarrow$ (i) is immediate from Propositions 4.5 and 4.7. Thus, it remains to show (i) $\Longrightarrow$ (ii). Let (P_i) be a net in $G_m(\Gamma)^{(0)}$ converging in $G_u(\Gamma)$ to $P \in G_u(\Gamma)$. By Proposition 4.4 (b), P belongs to $G_u(\Gamma)^{(0)}$. Thus, P belongs to U_p for a suitable $p \in \Gamma^{(0)}$. Then P_i belongs to V_p for large i. As V_p is compact, $P = \lim P_i$ must also belong to $V_p \subset G_m(\Gamma)^{(0)}$. □

Let us now give a simple criterion, which can be checked for certain concrete semigroups, for example, those arising in the context of tilings and graphs. A function $R : \Gamma \to I$ with $I = [0, \infty)$ or $I = [0, \infty]$ is called a *radius function* if it satisfies

$$R(x^{-1}) = R(x), \quad R(xy) \geqslant \min\{R(x), R(y)\} \quad \text{and} \quad R(u) \leqslant R(v)$$

for all $x, y, u, v \in \Gamma$ with $u \prec v$. A radius function R on Γ gives rise to a radius function on $\mathcal{O}(\Gamma)$, again denoted by $R(X) \equiv \sup\{R(x) : X \prec x\}$. A radius function is called *admissible* if $R(X) = \infty$ if and only if $X \in \mathcal{O}(\Gamma)_{\min}$.

Proposition 6.4. *If R is an admissible and continuous radius function on $G_u(\Gamma)$, then $V_{x;x_1,\dots,x_n}$ is compact for arbitrary $x_1, \dots, x_n \prec x \in \Gamma$.*

Proof. By Proposition 6.3, we must show that $G_m(\Gamma)^{(0)}$ is closed in $G_u(\Gamma)^{(0)}$. This follows from the continuity of R as

$$G_m(\Gamma)^{(0)} = G_u(\Gamma)^{(0)} \cap \{X : R(X) = \infty\}.$$

□

Remark 6.5.

(a) Any radius function must be lower semicontinuous.

(b) The radius functions arising in the context of graphs or tilings are admissible and have strong additional properties. These can be used to show that $G_m(\Gamma)$ can be considered as a kind of metric completion of $\mathcal{O}(\Gamma)$ (see [**5**] for the tiling case).

Let us give another condition for the closedness of $G_m(\Gamma)^{(0)}$ in $G_u(\Gamma)^{(0)}$. This condition is local in that it can be checked by considering only Γ (and not $\mathcal{O}(\Gamma)$).

Definition 6.6. The inverse semigroup Γ is said to satisfy the trapping condition (T) if Γ contains a zero and, for every $p, q \in \Gamma^{(0)}$ with $q \prec p$, there exist $p_1, \dots, p_n \in \Gamma^{(0)}$ such that $p_j \prec p$, $j = 1, \dots, n$, $p < (p_1, \dots, p_n, q)$ and, for every $j \in \{1, \dots, n\}$, either $p_j \prec q$ or $p_j q = 0$.

Proposition 6.7. *Let Γ satisfy (T). Then $G_m(\Gamma)^{(0)}$ is closed in $G_u(\Gamma)^{(0)}$.*

Proof. Let (P_i) be a net in $G_m(\Gamma)^{(0)}$ converging in $G_u(\Gamma)$ to P. As $G_u(\Gamma)^{(0)}$ is closed in $G_u(\Gamma)$, the element P belongs to $G_u(\Gamma)^{(0)}$. Next, we show that P is not zero. Assume the contrary. As Γ contains a zero, this implies that $P \in U_0$, yielding the contradiction $0 \neq P_i \in U_0$ for large i.

Thus, it suffices to show that every $Q \neq 0$, $Q \prec P$, agrees with P. Let such a Q be given and assume that $P \neq Q$. Then there exist $p, q \in \Gamma^{(0)}$ with $q \prec p$ and $Q \prec q$, $P \prec p$ but not $P \prec q$. Choose $p_1, \dots, p_n$ according to (T) for $q \prec p$. Then we have $V_p \subset V_{p_1} \cup V_{p_n} \cup V_q$ by (T) and Proposition 6.2 (b). Then, it is not difficult to see that there exists a subnet (P_k) of (P_i) also converging to P and $(P_k) \subset V_{p_j}$ for a suitable j. By $P_k \in V_{p_j} \subset U_{p_j}$ and compactness of U_{p_j}, we infer that $P \in U_{p_j}$, i.e. $P \prec p_j$. There are two cases.

Case 1 ($p_j \prec q$). In this case we arrive at the contradiction $P \prec p_j \prec q$.

Case 2 ($p_j q = 0$). We have $Pq = Pp_j q = P0 = 0$, contradicting $0 \neq Q = Qq \prec Pq$. □

If Γ satisfies (L), the topology of $G_m(\Gamma)$ has a particularly nice basis.

Lemma 6.8. *If Γ satisfies (L) and has a zero, then the family of sets V_x, $x \in \Gamma$, is a basis of the topology of $G_m(\Gamma)$.*

Proof. It suffices to show that for arbitrary $X \in G_m(\Gamma)$ and $z_1, \dots, z_n \prec z \in \Gamma$ with $X \in V_{z;z_1,\dots,z_n}$, we have $X \in V_x \subset V_{z;z_1,\dots,z_n}$ for a suitable x. Assume the contrary. Thus, there exists $X \in G_u(\Gamma)$ such that, for every x with $X \prec x$, the set $V_x \cap (V_{z_1} \cup \dots \cup V_{z_n})$ is not empty. We must therefore have $X = [A_j]$ with a suitable j for $A_j \equiv \{x : X \prec x,\ V_x \cap V_{z_j} \neq \emptyset\}$. Assume without loss of generality that $j = 1$. By (L), the minimum $x \wedge z_1$ exists for arbitrary $X \prec x$ and is not zero. Moreover, the construction gives $[\{x \wedge z_1 : X \prec x\}] \prec X$ and $[\{x \wedge z_1 : X \prec x\}]$ is not zero, as Γ has a zero. By minimality of X, this gives $X = [\{x \wedge z_1 : x \in \dot{X}\}]$ and the contradiction $X \prec z_1$ follows. □

Proposition 6.9. *The map $V : \Gamma \to S(G_m(\Gamma))$, $x \mapsto V_x$ is a homomorphism of inverse semigroups. If Γ satisfies (L) and contains a zero, $V(\Gamma)$ is canonically isomorphic to $\tilde{\Gamma}$ by $V_x \mapsto \pi(x)$.*

Proof. The first statement can be proved using the same proof as Proposition 4.8. The second statement then follows from Proposition 6.2. □

Remark 6.10. The proposition shows, in particular, that the map V on Γ is, unlike U, not necessary injective. Nevertheless, it is still possible to show that $G_m(\Gamma)$ is isomorphic to $G_m(V(\Gamma))$, whenever Γ satisfies (L).

Recall now that the *ample semigroup* of a groupoid G is the inverse semigroup consisting of all compact open G-sets. A groupoid is called *ample* if this semigroup is a basis of the topology.

Theorem 6.11. *Let Γ be a subsemigroup of the inverse semigroup of an ample Hausdorff groupoid G. Assume that Γ is closed under intersections (which implies (L)) and that Γ is a basis of the topology of G. Then $G_m(\Gamma) \simeq G$.*

Proof. This is the analogue of our setting to a result of [**5**]. Thus, we only briefly sketch the idea. To each point $g \in G$ we associate the set A_g, consisting of all $x \in \Gamma$ with $g \in x$. This set is directed, i.e. belongs to $\mathcal{O}(\Gamma)$, as Γ is closed under intersections. Using the fact that G is Hausdorff, one easily sees that $[A_g]$ must be minimal, i.e. must belong to $G_m(\Gamma)$. The map $g \mapsto [A_g]$ is the desired isomorphism. □

The preceding considerations suggest the distinction of the class of inverse semigroups defined next. It covers the graph case and the tiling case (see below).

Definition 6.12. The inverse semigroup Γ is said to satisfy condition (LC) if it contains a zero and satisfies (L) and $G_m(\Gamma)^{(0)}$ is closed in $G_u(\Gamma)^{(0)}$.

The considerations of this section extend the corresponding considerations of [**4, 5**] in some ways. There, a topological groupoid $H_m(\Gamma)$ is constructed from an inverse semigroup with zero by considering directed sequences. Its topology is generated by V_x, $x \in \Gamma$ (in our notation). Here, we make the relationship between $H_m(\Gamma)$ and $G_u(\Gamma)$ explicit. More precisely, the groupoid $H_m(\Gamma)$ clearly agrees with $G_m(\Gamma)$ as a set, but the topology might be different. Lemma 6.8 then yields that $G_m(\Gamma)$ and $H_m(\Gamma)$ agree as topological groupoids if Γ satisfies (L). Moreover, we study whether $G_m(\Gamma)$ has a basis of compact open sets for general Γ and provide criteria applying to both the tiling and the graph cases.

7. Open invariant subsets of $G_m(\Gamma)^{(0)}$

In this section we relate the open invariant subsets of $G_m(\Gamma)^{(0)}$ to certain order ideals in $\Gamma^{(0)}$, and use this and results of [**15**] to study the ideals in $C^*_{\mathrm{red}}(G_m(\Gamma))$.

We start with a discussion of invariance. Let $X \in G_m(\Gamma)$ be given. Let $x \in \Gamma$ with $X \prec x$ be given. We then have $X = XX^{-1}X = Px = xQ$ with $P = XX^{-1}$, $Q = X^{-1}X$ in

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$G_m(\Gamma)^{(0)}$. This shows that $X^{-1}X = x^{-1}Px$ and $XX^{-1} = xQx^{-1}$. These considerations easily imply the following proposition.

Proposition 7.1.

(a) *A subset E of $G_m(\Gamma)^{(0)}$ is invariant if and only if, for every $P \in E$ and $x \in \Gamma$ with $P \prec xx^{-1}$, the element $x^{-1}Px$ belongs to E.*

(b) *$G_m(\Gamma)_P^P \equiv \{X : XX^{-1} = X^{-1}X = P\} = \{P\}$ if and only if every $x \in \Gamma$ with $x^{-1}Px = P$ and $P \prec xx^{-1}$ satisfies $P \prec x$.*

Definition 7.2. An element $P \in G_m(\Gamma)^{(0)}$ is called aperiodic if $G_m(\Gamma)_P^P = \{P\}$.

Using this definition and Proposition 7.1, we can reformulate the definition of (essentially) principality for $G_m(\Gamma)$ given in [**15**] as follows: $G_m(\Gamma)$ is principal if and only if every $P \in G_m(\Gamma)^{(0)}$ is aperiodic. $G_m(\Gamma)$ is essentially principal if and only if in every closed invariant set F the set of aperiodic points is dense.

Definition 7.3. A subset I of $\Gamma^{(0)}$ is called $<$-closed if $p \in \Gamma^{(0)}$ belongs to I whenever $p < (p_1, \dots, p_n)$ for $p_1, \dots, p_n \in I$.

A subset I of $\Gamma^{(0)}$ is called invariant if xpx^{-1} belongs to I for every $p \in I$ and $x \in \Gamma$ with $p \prec x^{-1}x$.

Note that a $<$-closed set is in particular an order ideal, as $p \prec q$ implies $p < q$. Invoking Proposition 5.6, we can now easily infer the following two results.

Proposition 7.4.

(a) *Let I be an arbitrary subset of $\Gamma^{(0)}$. Then*

$$Cl(I) \equiv \{p : p < (p_1, \dots, p_n) \text{ for suitable } p_1, \dots, p_n \in I\}$$

is the smallest $<$-closed subset of $\Gamma^{(0)}$ containing I.

(b) *If I is an invariant order ideal in $\Gamma^{(0)}$, then $Cl(I)$ is the smallest $<$-closed invariant subset of $\Gamma^{(0)}$ containing I.*

Proposition 7.5.

(a) *The set of $<$-closed invariant subsets of $\Gamma^{(0)}$ with the usual inclusion as partial order is a lattice with $I \vee J \equiv Cl(I \cup J)$ and $I \wedge J \equiv I \cap J$.*

(b) *The set of open invariant subsets of $G_m(\Gamma)^{(0)}$ with the usual inclusion as order is a lattice with $U \vee V \equiv U \cup V$ and $U \wedge V \equiv U \cap V$.*

Definition 7.6.

(a) The lattice in Proposition 7.5 (a) will be denoted by $\mathcal{I}(\Gamma)$.

(b) The lattice in Proposition 7.5 (b) will be denoted by $\mathcal{V}(\Gamma)$.

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Lemma 7.7. *Let Γ satisfy (LC). For V in $\mathcal{V}(\Gamma)$ the set $S_i(V) \equiv \{q \in \Gamma^{(0)} : V_q \subset V\}$ belongs to $\mathcal{I}(\Gamma)$. For $I \in \mathcal{I}(\Gamma)$ the set $S_u(I) \equiv \bigcup_{q\in I} V_q$ belongs to $\mathcal{V}(\Gamma)$. The maps $S_u : \mathcal{I}(\Gamma) \to \mathcal{V}(\Gamma)$, $I \mapsto S_u(I)$ and $I : \mathcal{V}(\Gamma) \to \mathcal{I}(\Gamma)$, $U \mapsto S_i(U)$ are lattice isomorphisms which are inverse to each other.*

Proof. It is easy (and does not use any assumptions on Γ) to show that $S_u(I)$ belongs to $\mathcal{V}(\Gamma)$. Moreover, using (L), $0 \in \Gamma$ and Proposition 6.2, it is not difficult to see that $S_i(V)$ belongs to $\mathcal{I}(\Gamma)$. Let us now show that S_i and S_u are inverse to each other, i.e. that (i) $S_i(S_u(I)) = I$ and (ii) $S_u(S_i(V)) = V$.

(i) By $S_i(S_u(I)) = \{q : V_q \subset \bigcup_{p\in I} V_p\}$, we have $S_i(S_u(I)) \supset I$. Conversely, let q with $V_q \subset \bigcup_{p\in I} V_p$ be given. By the compactness of V_q, we have $V_q \subset V_{p_1} \cup \cdots \cup V_{p_n}$ for suitable $p_1, \dots, p_n \in I$. By Proposition 6.2, this gives $q < (p_1, \dots, p_n)$. As I is $<$-closed, we infer that $q \in I$ and the proof of (i) is finished.

(ii) $S_u(S_i(V)) = \bigcup_{q\in S_i(V)} V_q = \bigcup_{q:V_q\subset V} V_q = V$. Here, we have used in the last equality that the V_x, $x \in \Gamma$, give a basis of the topology of $G_m(\Gamma)$ by (L).

Clearly, the maps S_i and S_u respect the order. Therefore, it remains to show that they respect $\vee$ and $\wedge$ as well. This will be shown next. In fact, $S_i(U \wedge V) = S_i(U) \cap S_i(V)$ is immediate and $S_u(I \wedge J) = S_u(I) \cap S_u(J)$ follows easily, as $p \wedge q = pq$ exists for $p, q \in \Gamma^{(0)}$. Thus, it remains to show $S_u(I \vee J) = S_u(I) \vee S_u(J)$ and $S_i(U \vee V) = S_i(U) \vee S_i(V)$. We have

$$S_u(I \vee J) \equiv S_u(\mathrm{Cl}(I \cup J)) = \bigcup_{q\in \mathrm{Cl}(I\cup J)} V_q = \bigcup_{q\in I} V_q \cup \bigcup_{p\in J} V_p = S_u(I) \cup S_u(J),$$

where we used Proposition 6.2 combined with Proposition 7.4 in the penultimate equality. Also, as the V_q are compact, by Proposition 6.2 we have

$$\begin{aligned} S_i(U \vee V) &= \{q : V_q \subset U \cup V\} \\ &= \{q : V_q \subset V_{q_1} \cup \cdots \cup V_{q_n} \cup V_{p_1} \cdots V_{p_k},\ V_{p_j} \subset U,\ V_{q_l} \subset V\} \\ &= \mathrm{Cl}(\{q : V_q \subset U\} \cup \{q : V_q \subset V\}) \\ &= S_i(U) \vee S_i(V). \end{aligned}$$

This finishes the proof of the lemma. □

Let us now turn to the question of whether there actually exist non-trivial invariant open subsets of $G_m(\Gamma)^{(0)}$.

Lemma 7.8. *Let Γ satisfy (LC). Then the following are equivalent.*

(i) *There do not exist non-trivial $<$-closed invariant subsets of $\Gamma^{(0)}$.*

(ii) *For every $p, q \in \Gamma^{(0)}$, there exist $x_1, \dots, x_n$ with $x_j^{-1}x_j \prec p$, $j = 1, \dots, n$, and $q < (x_1x_1^{-1}, \dots, x_nx_n^{-1})$.*

(iii) *$G_m(\Gamma)$ is minimal, i.e. any non-empty invariant set in $G_m(\Gamma)^{(0)}$ is dense.*

Proof. It is well known that an r-discrete topological groupoid G is minimal if and only if there do not exist any non-trivial invariant open subsets of $G^{(0)}$. Thus, the equivalence of (iii) and (i) is immediate from Lemma 7.7.

It remains to show the equivalence of (i) and (ii). Obviously, (i) is equivalent to the statement that any non-empty $<$-closed invariant subset of $\Gamma^{(0)}$ contains every unit. This means that for every $p \in \Gamma^{(0)}$, $p \neq 0$, the set

$$I_p \equiv \mathrm{Cl}(\{xrx^{-1} : r \prec p, r \prec x^{-1}x\})$$

contains every $q \in \Gamma^{(0)}$. This is the case if and only if, for every $q \in \Gamma^{(0)}$, there exist $y_1, \dots, y_n$ and $r_1, \dots, r_n \in \Gamma^{(0)}$ with $r_j \prec y_j^{-1}y_j, p$ and $q < (y_1 r_1 y_1^{-1}, \dots, y_n r_n y_n^{-1})$. This is equivalent to (ii) with $x_j \equiv y_j r_j$, (respectively, $r_j = x_j^{-1}x_j$, and $y_j = x_j$). □

In our setting, these reductions of $G_m(\Gamma)$ to open invariant sets can be described directly in terms of certain subsemigroups of Γ. Note that, for each invariant $I \subset \Gamma^{(0)}$, the set $\Gamma_I \equiv \{x : xx^{-1} \in I\} = \{x : x^{-1}x \in I\}$ with multiplication and involution from Γ is an inverse subsemigroup of Γ.

Proposition 7.9. *Let Γ be an inverse semigroup with zero satisfying (L). Let I be an invariant order ideal in $\Gamma^{(0)}$. Then $V(I) \equiv \bigcup_{q\in I} V_q$ is an invariant open subset of $G_m(\Gamma)^{(0)}$. The canonical embedding $j : \Gamma_I \to \Gamma$, $x \mapsto x$, induces an isomorphism $J : G_m(\Gamma_I) \to G_m(\Gamma)_{V(I)}$, $X \mapsto [\{j(y) : X \prec y\}]$ of topological groupoids.*

Proof. As in the proof of Lemma 7.7 we infer that $V(I)$ is open and invariant. Direct calculations show that $J : G_m(\Gamma_I) \to G_m(\Gamma)$ and $P : G_m(\Gamma)_{V(I)} \to G_m(\Gamma_I)$, $X \mapsto [\{x \in \Gamma_I : X \prec x\}]$ are continuous groupoid homomorphism which are inverse to each other. □

This proposition allows one to identify $C^*_{\mathrm{red}}(G_m(\Gamma_I))$ with $C^*_{\mathrm{red}}(G_m(\Gamma)_{V(I)})$, which in turn can canonically be considered as an ideal in $C^*_{\mathrm{red}}(G_m(\Gamma))$ by the results of [**15**] mentioned at the beginning of this section. Using this identification and denoting the lattice of ideals of $C^*_{\mathrm{red}}(\Gamma)$ by $\mathcal{I}(C^*_{\mathrm{red}}(\Gamma))$ we obtain the following.

Theorem 7.10. *Let Γ satisfy (LC). Assume that $G_m(\Gamma)$ is essentially principal. Then the map $J : \mathcal{I}(\Gamma) \to \mathcal{I}(C^*_{\mathrm{red}}(\Gamma))$, $J(I) \equiv C^*_{\mathrm{red}}(G_m(\Gamma_I)) \subset C^*_{\mathrm{red}}(G_m(\Gamma))$ is a bijection of lattices. In particular, $C^*_{\mathrm{red}}(G_m(\Gamma))$ is simple if and only if, for every $p, q \in \Gamma^{(0)}$, there exist $x_1, \dots, x_n$ with $x_j^{-1}x_j \prec p$, $j = 1, \dots, n$, and $q < (x_1x_1^{-1}, \dots, x_nx_n^{-1})$.*

Proof. The first statement follows from Lemma 7.7 and the results of [**15**, Chapter II, § 4] (see also [**16**, Corollary 4.9]). The second statement follows from the first statement and Lemma 7.8. □

8. Application to graphs

In this section we present an inverse-semigroup-based approach to the groupoid $G(\boldsymbol{g})$ associated with a graph $\boldsymbol{g}$ in [**9**]. This will provide semigroup-based proofs for some results of [**9**] concerning the structure of the open invariant subsets of $G(\boldsymbol{g})^{(0)}$ (see [**14**] for a recent extension to a non-locally-finite situation).

Let $\boldsymbol{g} = (E, V, f, i)$ be a directed graph [**9**] with the set of edges E and set of vertices V and the range and source map $f, i : E \to V$. We assume that f is onto and that $i^{-1}(v)$ is not empty for each $v \in V$. Moreover, we assume that the graph $\boldsymbol{g}$ is row finite, i.e. that $i^{-1}(v) \subset E$ is finite for all $v \in V$.

A path α of length $|\alpha| = n \in \mathbb{N}$ is a sequence $\alpha = (\alpha_1, \dots, \alpha_n)$ of edges $\alpha_1, \dots, \alpha_n$ in E with $i(\alpha_{j+1}) = f(\alpha_j)$, $j = 1, \dots, n-1$. For such an α we set $i(\alpha) \equiv i(\alpha_1)$ and $f(\alpha) \equiv f(\alpha_k)$. A path of length 0 is just a vertex and will also be called a degenerate path. For such a path v, we set $f(v) \equiv v$ and $i(v) \equiv v$. The set of all paths of finite length is denoted by $F(\boldsymbol{g})$. The set of all infinite paths $\alpha = (\alpha_1, \dots)$ is denoted by $P(\boldsymbol{g})$. The concatenation $\alpha\mu$ of two finite paths α and μ with $f(\alpha) = i(\mu)$ is defined in the obvious way. By a slight abuse of language we write $\alpha \prec \beta$ if $\beta = \alpha\mu$.

Let the set $\varGamma \equiv \varGamma(\boldsymbol{g})$ be given by $\varGamma \equiv \{(\alpha, \beta) \in F(\boldsymbol{g}) \times F(\boldsymbol{g}) : r(\alpha) = r(\beta)\} \cup \{0\}$ and define a multiplication on $\varGamma$ by

$$(\alpha, \beta)(\gamma, \delta) \equiv \begin{cases} (\alpha\mu, \delta) & \text{if } \gamma = \beta\mu, \\ (\alpha, \delta\mu) & \text{if } \beta = \gamma\mu, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\varGamma$ is indeed an inverse semigroup, where the inverse of (α, β) is given by $(\alpha, \beta)^{-1} \equiv (\beta, \alpha)$ (see also [**1**, **14**]). Thus, $\varGamma$ gives rise to a groupoid $G_m(\varGamma)$. Now, the function $R : \varGamma \to [0, \infty)$ given by

$$R(\alpha, \beta) \equiv \begin{cases} 0, & \alpha_{|\alpha|} \neq \beta_{|\beta|}, \\ \sup\{j \in \mathbb{N}_0 : \alpha_{|\alpha|-i} = \beta_{|\beta|-i},\ i = 0, \dots, j\}, & \text{otherwise} \end{cases}$$

can easily be seen to be a radius function in the sense of § 6. It is admissible and continuous. Thus, $G_m(\varGamma)$ is a groupoid with a basis consisting of compact sets. We refrain from giving details, but briefly sketch the connection between $G_m(\varGamma)$ and the graph groupoids which were introduced in [**9**]: direct arguments show that the relation $(\gamma, \delta) \prec (\alpha, \beta)$ holds if and only if there exists a (possibly degenerate) $\mu \in F(\boldsymbol{g})$ with $\gamma = \alpha\mu$ and $\delta = \beta\mu$. This immediately shows that for $x, y \in \varGamma$ with a common predecessor either $x \prec y$ or $y \prec x$ holds. Thus, for every $X \in \mathcal{O}(\varGamma)$, we can find $(\alpha, \beta) \in \varGamma$, $I = [0, a] \subset \mathbb{Z}$, $a \in \mathbb{N} \cup \{\infty\}$, and edges e_n, $n \in I$, such that $\{(\alpha e_1 \cdots e_n, \beta e_1 \cdots e_n) : n \in I\}$ is a representative of X. Putting this together, we see that minimal elements in $\mathcal{O}(\varGamma)$ can be identified with double paths of infinite length which agree from a certain point onwards. But this is exactly the way the graph groupoid in [**9**] is constructed.

This allows us to apply the theory of the preceding sections to the study of graph groupoids. In particular, we can rephrase the ideal theory of [**9**] (namely, the characterization of open invariant subsets of $G(\boldsymbol{g})^{(0)}$) in terms of inverse semigroups using § 7. This is done next. Following [**9**], for vertices $v, w \in V$ we write $v \geqslant w$ if there exists a path in P from v to w, and call a subset H of V *hereditary* if $v \in H$ and $v \geqslant w$ implies $w \in H$

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and call it *saturated* if $r(e) \in H$, for all $e \in E$ with $s(e) = v$, implies $v \in H$. Direct arguments then yield that the map

$$I \mapsto \{r(p) : p \in I\}$$

is a lattice isomorphism between the invariant $<$-closed ideals in $\Gamma^{(0)}$ and the hereditary saturated subsets of V. The inverse is given by $H \mapsto \{p \in \Gamma^{(0)} : r(p) \in H\}$.

9. Application to tilings

In this section, we briefly discuss inverse semigroups associated with tilings introduced by Kellendonk [**4**, **5**] (see [**6**, **7**] for recent work on this) and the general theory developed above to describe the ideal structure of $C^*_{\mathrm{red}}(G_m(\Gamma))$ for Γ. While this is essentially known, it serves as a good example for our theory.

A tiling in $\mathbb{R}^d$ is a (countable) cover T of $\mathbb{R}^d$ by compact sets which are homeomorphic to the unit ball in $\mathbb{R}^d$ and which overlap at most at their boundaries. The elements of T are called tiles. A pattern P in T is a finite subset of T. For patterns P and tilings T and $x \in \mathbb{R}^d$, we define $P + x$ and $T + x$ in the obvious way. The set of all patterns which belong to $T + x$ for some $x \in \mathbb{R}^d$ will be denoted by $P(T)$. All patterns will be assumed to be patterns in $P(T)$ if not stated otherwise.

A doubly pointed pattern (a, P, b) (over T) consists of a pattern $P \in P(T)$ together with two tiles $a, b \in P$. We say that (a, P, b) is contained in (c, Q, d), written as $(a, P, b) \subset (c, Q, d)$, if $a = c$, $b = d$ and $P \subset Q$. On the set of doubly pointed patterns over T we introduce an equivalence relation by defining $(a, P, b) \sim (c, Q, d)$ if and only if there exists an $r \in \mathbb{R}^d$ such that $c = a + r$, $d = b + r$ and $Q = P + r$. The class of (a, P, b) will be denoted by $\overline{(a, P, b)}$. Obviously, the relation $\subset$ can be extended to these classes. Similarly, one can introduce an equivalence relation on the set of all patterns in $P(T)$. Denote the class of the pattern P up to translation by $\bar{P}$ and the set of all classes of patterns in $P(T)$ by $\overline{P(T)}$. Following [**4**, **13**], we will assume two finite type conditions, namely that the diameters of the tiles a bounded and that the set $\{\bar{P} \in \overline{P(T)} : \mathrm{diam}(\bigcup_{t \in P} t) \leqslant R\}$ is finite for every R.

Following [**4**], one can make $\Gamma \equiv \{\overline{(a, P, b)};\ P \in P(T),\ a, b \in P\} \cup \{0\}$ into an inverse semigroup with zero such that the relation $\prec$ coincides with the relation $\subset$ defined above. Let $\mathcal{T} = \mathcal{T}(T)$ be the set of all tilings S of $\mathbb{R}^d$ with $\overline{P(S)} \subset \overline{P(T)}$. Then, Γ yields a groupoid $G_m(\Gamma)$ with unit space $\mathcal{T}$. The groupoid itself equals the groupoid $G(T)$ defined in [**4**]. Γ admits a complete radius function defined by $R(\overline{(a, P, b)}) \equiv [\mathrm{dist}(\partial P, \{a, b\})]$ with ∂ denoting the boundary.

Let us now study the open invariant sets in $G_m(\Gamma)$. A subset S of $\overline{P(T)}$ is called *saturated* if $\bar{P} \in \overline{P(T)}$ and $\bar{Q} \in S$ with $\bar{Q} \subset \bar{P}$ implies $\bar{P} \in S$. A subset S of $\overline{P(T)}$ is called *hereditary* if $\bar{P}$ belongs to S, whenever there exist $\bar{P}_1, \ldots, \bar{P}_n$ in S satisfying the condition that, for every pattern Q with $R(Q)$ sufficiently large and $Q \supset P$, there exists $j \in \{1, \ldots, n\}$ with $Q \supset P_j$. Then it is easy to see that the map

$$I \mapsto \{\bar{P} : \overline{(a, P, b)} \in I\}$$

is a lattice isomorphism between the invariant $<$-closed subsets of $\Gamma^{(0)}$ and the saturated hereditary subsets of $\overline{P(T)}$. It remains to study principality of $G_m(\Gamma)$. Recall that a tiling S is called *periodic* if there exists an $x \in \mathbb{R}^d$ with $S + x = S$. Now, $\mathcal{T}$ is called *aperiodic* if it does not contain a periodic tiling.

Theorem 9.1. *$G_m(\Gamma)$ is principal if and only if $\mathcal{T}$ is aperiodic.*

Proof. $G_m(\Gamma)$ is principal if every P in $G_m(\Gamma)^{(0)}$ is aperiodic in the sense of § 7. But this can easily be seen to be equivalent to $\mathcal{T}$ being aperiodic. □

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