On analogies between nonlinear difference and differential equations

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Abstract: In this paper, we point out some similarities between results on the existence and uniqueness of finite order entire solutions of the nonlinear differential equations and differential-difference equations of the form

$$f^n + L(z, f) = h.$$

Here n is an integer ≥ 2, h is a given non-vanishing meromorphic function of finite order, and $L(z, f)$ is a linear differential-difference polynomial, with small meromorphic functions as the coefficients.

Key words: Difference-differential polynomial; difference polynomial; difference-differential equation; Nevanlinna theory.

1. Introduction. Nevanlinna value distribution theory of meromorphic functions has been extensively applied to resolve growth (see e.g. [7]), value distribution [7], and solvability of meromorphic solutions of linear and nonlinear differential equations (see e.g. [6, 9–11]). Considering meromorphic functions f in the complex plane, we assume that the reader is familiar with the standard notations and results such as the proximity function $m(r, f)$, counting function $N(r, f)$, characteristic function $T(r, f)$, the first and second main theorems, lemma on the logarithmic derivatives etc. of Nevanlinna theory, see e.g. [5, 7]. Given a meromorphic function f, recall that a meromorphic function α is said to be a small function of f, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ is used to denote any quantity that satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside of a set of r of finite logarithmic measure. A polynomial $P(z, f)$ is called a differential polynomial in f whenever f is a polynomial in f and its derivatives, with small functions of f as the coefficients. Similarly, a polynomial $Q(z, f)$ is called a differential-difference polynomial in f whenever f is a polynomial in $f(z)$, its derivatives and its shifts $f(z + c)$, with small functions of f again as the coefficients.

The following lemma (Clunie [2]) has been extensively applied in studying the value distribution of a differential polynomial $P(z, f)$, as well as the growth estimates of solutions and meromorphic solvability of differential equations in the complex plane:

Lemma 1.1. *Let f denote a transcendental meromorphic function, and $P(z, f), Q(z, f)$ be two differential polynomials of f. If*

$$f^n P(z, f) = Q(z, f)$$

holds and if the total degree of $Q(z, f)$ in f and its derivatives is $\leq n$, then $m(r, P(z, f)) = S(r, f)$.

Remark. The key tool in the proof of this lemma is the core part of value distribution theory, namely the lemma on the logarithmic derivatives.

We now give three results to serve as a background for our considerations in the next section. Recalling [10, Theorem 1], and [6, Theorem 4.2], we have

Theorem 1.2. *Consider a differential equation*

$$p(z)f^n - L(z, f) = h, \tag{1.1}$$

where $p(z)$ is a small function of f of degree n, $L(z, f)$ is a linear differential polynomial in f, and h is a meromorphic function. If $n \geq 4$, then equation (1.1) *may admit at most n distinct entire solutions.*

Remark. As pointed out in [6], there may be either one entire solution, n entire solutions, or none of them.

Lemma 1.3. *Suppose c is a nonzero constant and α is a nonconstant meromorphic function. Then the differential equation $f^2 + (cf^{(n)})^2 = \alpha$ has no transcendental meromorphic solutions satisfying $T(r, \alpha) = S(r, f)$.*

The preceding lemma, see [11], has a key role in proving the following

Theorem 1.4. *Let p be a non-vanishing poly-*

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nomial, and let b, c be nonzero complex numbers. If p is nonconstant, then the differential equation

$$f^3 + p(z)f'' = c\sin bz \tag{1.2}$$

admits no transcendental entire solutions, while if p is constant, then equation (1.2) *admits three distinct transcendental entire solutions, provided $(pb^2/27)^3 = \frac{1}{4}c^2$.*

Proof. We leave the proof as an exercise for the reader. The idea to be applied may be collected from the proof of Theorem 2.5 below. □

Remark. As an example of the case with $p(z)$ constant, recall the nonlinear differential equation

$$4f^3 + 3f'' = -\sin 3z. \tag{1.3}$$

As pointed out by Li and Yang in [9], equation (1.3) admits exactly three distinct transcendental entire solutions: $f_1(z) = \sin z$, $f_2(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$, $f_3(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$. As one may immediately see, the condition given in Theorem 1.4 is satisfied.

This paper aims to present some studies on differential-difference analogues of equation (1.2), showing that similar conclusions follow if one restricts the solutions to be of finite order.

2. Main results and their proofs. Our results below mainly are differential-difference analogues of previous results concerning equation (1.3). A natural tool in studying equations of this type is the difference variant of Nevanlinna theory, and in particular, difference counterpart of the Clunie lemma above, see [8]. For completeness, we recall basic notions to this end: Given a meromorphic functions $f(z)$ and a constant c, $f(z+c)$ is called a shift of f. As for a difference product, we mean a difference monomial of type $\prod_{j=1}^{k} f(z+c_j)^{n_j}$, where $c_1, \dots, c_k$ are complex constants, and $n_1, \dots, n_k$ are natural numbers.

Definition 2.1. A difference polynomial, resp. a differential-difference polynomial, in f is a finite sum of difference products of f and its shifts, resp. of products of f, derivatives of f and of their shifts, with all the coefficients of these monomials being small functions of f.

Remark. As far as Clunie type lemmas are concerned, same conclusions hold as long as the proximity functions of the coefficients $\alpha(z)$ satisfy $m(r,\alpha) = S(r,f)$. The next lemma is a rather general variant of difference counterpart of the Clunie Lemma 1.1 above, see [8], for the corresponding results on differential polynomials, see [12].

Lemma 2.2. *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$H(z,f)P(z,f) = Q(z,f), \tag{2.1}$$

where $H(z,f), P(z,f), Q(z,f)$ are difference polynomials in f such that the total degree of $H(z,f)$ in f and its shifts is n, and that the corresponding total degree of $Q(z,f)$ is $\leq n$. If $H(z,f)$ contains just one term of maximal total degree, then for any $\varepsilon > 0$,

$$m(r, P(z,f)) = O(r^{\rho-1+\varepsilon}) + S(r,f), \tag{2.2}$$

possibly outside of an exceptional set of finite logarithmic measure.

Remark. If in the above lemma, $H(z,f) = f^n$, then a similar conclusion holds, if $P(z,f), Q(z,f)$ are differential-difference polynomials in f.

The final element in our preparations is the following lemma on quotients of shifts, see [1] and [3], which may be understood as the difference counterpart of the lemma on the logarithmic derivatives:

Lemma 2.3. *Let f be a transcendental meromorphic function of finite order ρ. Then for any given complex numbers c_1, c_2, and for each $\varepsilon > 0$,*

$$m\left(r, \frac{f(z+c_1)}{f(z+c_2)}\right) = O(r^{\rho-1+\varepsilon}). \tag{2.3}$$

Remark. Since the preceding lemma fails for meromorphic functions of infinite order, see e.g. $f(z) = \exp(e^z)$, we have been forced to restrict ourselves to finite order solutions of nonlinear difference equations, resp. differential-difference equations in what follows.

Theorem 2.4. *Let p, q be polynomials. Then a nonlinear difference equation*

$$f^2(z) + q(z)f(z+1) = p(z) \tag{2.4}$$

has no transcendental entire solutions of finite order.

Proof. Suppose that f is a transcendental entire solution of finite order ρ to equation (2.4). Without loss of generality, we may assume that $q(z)$ does not vanish identically. From (2.4), we readily conclude by Lemma 2.3 that

$$\begin{aligned} 2m(r,f) &= m(r, p(z) - q(z)f(z+1)) \\ &\leq m(r, f(z+1)) + O(\log r) \\ &\leq m(r,f) + m(r, f(z+1)/f(z)) + O(\log r) \\ &\leq m(r,f) + O(r^{\rho-1+\varepsilon}) + O(\log r). \end{aligned}$$

Therefore,

$$T(r,f) = m(r,f) = O(r^{\rho-1+\varepsilon}) + O(\log r),$$

hence $\rho(f) < \rho$, a contradiction. □

Remark. It seems likely that a similar result holds for difference equations of type

$$f^2(z) + L(z,f) = p(z),$$

where $L(z,f)$ is a linear differential-difference polynomial of f with polynomial coefficients.

Theorem 2.5. *A nonlinear difference equation*

$$f^3(z) + q(z)f(z+1) = c\sin bz, \tag{2.5}$$

where $q(z)$ is a nonconstant polynomial and $b, c \in \mathbf{C}$ are nonzero constants, does not admit entire solutions of finite order. If $q(z) = q$ is a nonzero constant, then equation (2.5) *possesses three distinct entire solutions of finite order, provided $b = 3\pi n$ and $q^3 = (-1)^{n+1}\frac{27}{4}c^2$ for a nonzero integer n.*

Proof. Let f be an entire solution of equation (2.5). Without loss of generality, we may assume that f is transcendental entire.

Differentiating (2.5) results in

$$3f^2(z)f'(z) + q'(z)f(z+1) + q(z)f'(z+1) = bc\cos bz. \tag{2.6}$$

Combining (2.6) and (2.5), we get

$$(bf^3(z) + bq(z)f(z+1))^2 + (3f^2(z)f'(z) + q'(z)f(z+1) + q(z)f'(z+1))^2 = b^2c^2.$$

This means that

$$f^4(z)(b^2f^2(z) + 9f'^2(z)) = T_4(z,f), \tag{2.7}$$

where $T_4(z,f)$ is a differential-difference polynomial of f, of total degree at most 4. If now $T_4(z,f)$ vanishes identically, then $f' = \pm i\frac{b}{3}f$, and therefore,

$$f'' + (b/3)^2 f = 0. \tag{2.8}$$

Otherwise, the Clunie lemma applied to a differential-difference equation, see Remark after Lemma 2.2, implies that

$$T(r, b^2f^2 + 9f'^2) = m(r, b^2f^2 + 9f'^2) = S(r,f). \tag{2.9}$$

Therefore, $\alpha := b^2f^2 + 9f'^2$ is a small function of f, not vanishing identically. By Lemma 1.3, α must be a constant. Differentiating $b^2f^2 + 9f'^2 = \alpha$, we immediately conclude that (2.8) holds in this case as well. Solving (2.8) shows that f must be of the form

$$f(z) = c_1e^{ibz/3} + c_2e^{-ibz/3}. \tag{2.10}$$

Substituting the preceding expression of f into the original difference equation (2.5), expressing $\sin bz$ in terms of exponential functions, and denoting $\omega(z) := e^{ibz/3}$, an elementary computation results in

$$a_6\omega^6 + a_4\omega^4 + a_2\omega^2 + a_0 = 0,$$

where

$$\begin{cases} a_6 = c_1^3 + \frac{1}{2}ic, \\ a_4 = 3c_1^2c_2 + c_1e^{ib/3}q(z), \\ a_2 = 3c_1c_2^2 + c_2e^{-ib/3}q(z), \\ a_0 = c_2^3 - \frac{1}{2}ic. \end{cases}$$

Since $\omega(z)$ is transcendental, we must have $a_0 = a_2 = a_4 = a_6 = 0$. Therefore, $c_1 \neq 0$, $c_2 \neq 0$, and the condition $a_4 = 0$ implies that $q(z)$ is a constant, say $q \neq 0$. Combining now the conditions $a_4 = 0$ and $a_2 = 0$ we conclude that $e^{2ib/3} = 1 = e^{2\pi in}$, hence $b = 3\pi n$. The connection between q and c now follows from $3c_1c_2 + (-1)^n q = 0$ and $(c_1c_2)^3 = \frac{1}{4}c^2$. Finally, observe that from the nine combinations of possible values of c_1, c_2, three only satisfy the requirement that $3c_1c_2 + (-1)^n q = 0$. □

Remark. In the special case of

$$f^3(z) + \frac{3}{4}f(z+1) = -\frac{1}{4}\sin 3\pi z,$$

a finite order entire solution is

$$f_1(z) = \sin\pi z = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z}).$$

The other two immediately follow from the conditions above:

$$f_2(z) = \frac{1}{2i}(\varepsilon e^{i\pi z} - \varepsilon^2 e^{-i\pi z}) = -\frac{1}{2}\sin\pi z + \frac{\sqrt{3}}{2}\cos\pi z,$$

$$f_3(z) = \frac{1}{2i}(\varepsilon^2 e^{i\pi z} - \varepsilon e^{-i\pi z}) = -\frac{1}{2}\sin\pi z - \frac{\sqrt{3}}{2}\cos\pi z,$$

where $\varepsilon := -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a cubic root of unity.

Concerning more general differential-difference equations, we obtain

Theorem 2.6. *Let $n \geq 4$ be an integer, $M(z,f)$ be a linear differential-difference polynomial of f, not vanishing identically, and h be a meromorphic function of finite order. Then the differential-difference equation*

$$f^n + M(z,f) = h \tag{2.11}$$

possesses at most one admissible transcendental entire solution of finite order such that all coefficients of $M(z,f)$ are small functions of f. If such a solution f exists, then f is of the same order as h.

Proof. The arguments here are somewhat similar to those in [10]. We first observe that

$\rho(h) = \rho(f)$ for all entire solutions of finite order of (2.11). Since the inequality $\rho(h) \le \rho(f)$ trivially holds, suppose for a while that $\rho(h) < \sigma < \rho(f) =: \rho$, and write (2.11) in the form

$$f^{n-1} = \frac{h}{f} - \frac{M(z,f)}{f}.$$

From Lemma 2.3 and the lemma on the logarithmic derivatives we conclude that

$$\begin{aligned}(n-1)T(r,f) &= (n-1)m(r,f)\\ &\le T(r,h) + T(r,f) + O(r^{\rho-1+\varepsilon}) + S(r,f)\\ &\le T(r,f) + r^{\sigma} + O(r^{\rho-1+\varepsilon}) + S(r,f)\end{aligned}$$

for all r sufficiently large, outside of an exceptional set of finite logarithmic measure. Provided ε has been chosen small enough, and removing the exceptional set by standard reasoning, see [7, Chapter 1.1], we obtain

$$\rho(f) \le \max(\rho - 1 + 2\varepsilon, \sigma + \varepsilon) < \rho,$$

a contradiction.

Assume now, contrary to the assertion, that f, g are two distinct finite order transcendental entire solutions of (2.11), and write

$$f^n + M(z,f) = g^n + M(z,g). \tag{2.12}$$

Clearly, $\rho(f) = \rho(g)$. From (2.12), we now obtain

$$f^n - g^n = M(z,g) - M(z,f) = M(z, g-f).$$

Therefore,

$$F := \frac{f^n - g^n}{f-g} = \prod_{j=1}^{n-1}(f - \eta_j g) = -\frac{M(z, f-g)}{f-g}. \tag{2.13}$$

is an entire function; here $\eta_1, \dots, \eta_{n-1}$ are the distinct roots $\ne 1$ of equation $z^n = 1$. From this, Lemma 2.3 and the lemma on the logarithmic derivatives, we conclude that

$$\begin{aligned}T(r,F) = m(r,F) &= m\left(\frac{M(z,g-f)}{f-g}\right)\\ &= O(r^{\rho(f-g)-1+\varepsilon}) + S(r,f) + S(r,g)\\ &\le O(r^{\rho(f)-1+\varepsilon}) + S(r,f) =: S_\rho(r,f).\end{aligned}$$

Here $\varepsilon > 0$ is arbitrary and sufficiently small.

An immediate observation now results in

$$\sum_{j=1}^{n-1} N\left(r, \frac{1}{f - \eta_j g}\right) = N(r, 1/F) = S_\rho(r,f),$$

and therefore,

$$N\left(r, \frac{1}{f - \eta_j g}\right) = S_\rho(r,f) \tag{2.14}$$

holds for all $j = 1, \dots, n-1$. Since

$$\frac{1}{f/g - \eta_j} = g\frac{1}{f - \eta_j g},$$

we conclude that

$$N\left(r, \frac{1}{f/g - \eta_j}\right) = S_\rho(r,f)$$

for all $j = 1, \dots, n-1$. Assuming now that $n \ge 4$, the second main theorem implies for $\psi := f/g$ that

$$T(r,\psi) = T(r, f/g) = S_\rho(r,f)$$

and

$$T(r,f) = T(r,g) + S_\rho(r,f).$$

Making use of (2.13), we infer that

$$F = \prod_{j=1}^{n-1}(f - \eta_j g) = g^{n-1}\prod_{j=1}^{n-1}(\psi - \eta_j).$$

Provided ψ is not identically equal to η_j, $j = 1, \dots, n-1$, then

$$\begin{aligned}(n-1)T(r,f) &= (n-1)T(r,g) + S_\rho(r,f)\\ &\le T(r,F) + T\left(r, \prod_{j=1}^{n-1}(\psi - \eta_j)^{-1}\right) + S_\rho(r,f)\\ &= S_\rho(r,f),\end{aligned}$$

a contradiction. Therefore, we must have $\psi = \eta_j$ for some $j = 1, \dots, n-1$. But then $f = \eta_j g$, $f^n = g^n$ and $M(z,f) = M(z,g)$. By the linearity of the differential-difference polynomial M, we obtain $M(z,f) = \eta_j M(z,g)$. Since $\eta_j \ne 1$, a contradiction again follows. □

3. Discussion. By a recent result due to Halburd *et al.*, see [4, Theorem 5.1], the key tool in our argument, Lemma 2.3, extends to the case of hyper-order $\rho_2(f) < 1$. Therefore, it seems apparent that at least some of our results above for the non-existence of entire solutions of finite order may be extended to the case of hyper-order less than one as well. Independently of this, we would like to pose the following

Conjecture 1. There exists no entire functions of infinite order that satisfies a difference equation of type

$$f^n(z) + q(z)f(z+1) = c\sin bz,$$

where q is a nonconstant polynomial, b, c are nonzero constants and $n \ge 2$ is an integer.

More generally, we propose

Conjecture 2. Let f be an entire function of infinite order and $n \geq 2$ be an integer. Then a differential-difference polynomial of the form $f^n + P_{n-1}(z, f)$ cannot be a nonconstant entire function of finite order. Here $P_{n-1}(z, f)$ is a differential-difference polynomial in f of total degree at most $n-1$ in f, its derivatives and its shifts, with entire functions of finite order as coefficients. Moreover, we assume that all terms of $P_{n-1}(z, f)$ have total degree ≥ 1.

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