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ON ANNIHILATORS OF BCK-ALGEBRAS

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1. INTRODUCTION

Let X be a commutative BCK-algebra and A an ideal of X. To any subset B of X we associate the set $(A:B) = \{x \in X : x \wedge B \subseteq A\}$, where $x \wedge B = \{x \wedge y : y \in B\}$. We show that $(A:B)$ is an ideal of X and define it as the generalized annihilator of B (relative to A). If $A = \{0\}$, then $(A:B)$ coincides with the usual annihilator of B (see for instance [4]). These and some other properties of generalized annihilators are contained in Section 3 of this paper. Section 4 contains some applications of generalized annihilators in quotient BCK-algebras and in the theory of prime ideals of BCK-algebras. Using the technique of generalized annihilators, we show that the quotient BCK-algebra of an involutory BCK-algebra is again an involutory BCK-algebra. We also obtain a characterization of prime ideals: A categorical ideal A is prime if and only if $(A:B) = A$ (see Proposition 4.9). Section 2 contains some preliminary material for the development of our results.

2. PRELIMINARIES

A BCK-algebra is a system $(X, *, 0, \leqslant)$ (denoted simply by X), satisfying (i) $(x*y)*(x*z) \leqslant z*y$; (ii) $x*(x*y) \leqslant y$; (iii) $x \leqslant x$; (iv) $0 \leqslant x$; (v) $x \leqslant y$, $y \leqslant x$ imply that $x = y$ and (vi) $x \leqslant y$ if and only if $x*y = 0$ for all $x, y, z \in X$. If X contains an element 1 such that $x \leqslant 1$ for all $x \in X$, then X is said to be bounded. X is said to be commutative if $x \wedge y = y \wedge x$ for all $x, y \in X$, where $x \wedge y = y*(y*x)$. A BCK-algebra X is called implicative if $x*(y*x) = x$ for all $x, y \in X$. Every implicative BCK-algebra is commutative and positive implicative.

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In any commutative BCK-algebra X the inequality $(x \wedge y) * (x \wedge z) \leqslant x \wedge (y * z)$ holds for all $x, y, z \in X$ (see [5, 6]). This inequality will be repeatedly used. A proper ideal A of a BCK-algebra X is prime if $x \wedge y \in A$ implies that $x \in A$ or $y \in A$ (see [11]). If X is a BCK-algebra and A is an ideal of X, then we define an equivalence relation $\sim$ on X by $x \sim y$ if and only if $x * y, y * x \in A$. Let C_x denote the equivalence class containing x. Then one can see that $C_0 = A$, $C_x = C_y$ if and only if $x \sim y$. Let X/A denote the set of all equivalence classes C_x, $x \in X$. Then X/A is a BCK-algebra (known as the quotient BCK-algebra) with $C_x * C_y = C_{x*y}$ and $C_x \leqslant C_y$ if and only if $x * y \in A$, and $C_0 = A$ is the zero of X/A (see for instance [13]). If X is a commutative BCK-algebra, then X/A is commutative [2]. Let X be a commutative BCK-algebra, and let A be a subset of X. Then following [4], we define $A^* = \{x \in X : x \wedge a = 0 \text{ for all } a \in A\}$ and call it the annihilator of A; A^* is an ideal of X. If $A = \{a\}$, then we write $(a)^*$ instead of $(\{a\})^*$. In general, for any ideal A, $A \cap A^* = \{0\}$ and $A \subseteq A^{**}$ where $A^{**} = (A^*)^*$ is the double annihilator of A. If $A = A^{**}$, then A is called an involutory ideal. A commutative BCK-algebra all of whose ideals are involutory is called an involutory BCK-algebra. For instance, any finite commutative BCK-algebra or any implicative BCK-algebra is an involutory BCK-algebra (see [4]). For more information on annihilators and involutory ideals we refer to [4]. A commutative BCK-algebra X is cancellative if $x \wedge y = 0$ implies $x = 0$ or $y = 0$ for $x, y \in X$ (see [2]), that is $(x)^* = 0$ for all $x \in X$ with $x \neq 0$. An ideal A of a commutative BCK-algebra X is categorical if $(x \wedge y) \wedge z \in A$ implies that $x \wedge z, y \wedge z \in A$ (see [2]). If the zero ideal is categorical, then X is said to be categorical. Recently, Aslam and Thaheem [5] introduced an ideal $x^{-1}A = \{y \in X : y \wedge x \in A\}$ associated with an element $x \in X$ and an ideal A. It follows from [5] that $A \subseteq x^{-1}A$. An ideal A is prime if and only if $A = x^{-1}A$ for all $x \in X - A$ (see [5, 6]). For an ideal A, $x^{-1}A = X$ if and only if $x \in A$ (see [5]). For the general theory of the BCK-algebra we refer to [13], and for an ideal theory of the BCK-algebra we may refer to [1, 3, 6, 7, 8, 9, 10, 11, 14].

3. Generalized annihilators

Throughout this section X denotes a commutative BCK-algebra unless mentioned otherwise explicitly. First, we give the definition of the generalized annihilator.

Definition 3.1. Let X be a commutative BCK-algebra and let A be an ideal of X. Suppose that B is a subset of X. Then we define the set $(A : B) = \{x \in X : x \wedge B \subseteq A\}$ as the generalized annihilator of B (relative to A). We observe that if $A = \{0\}$, then $B^* = (0 : B)$ and $(A : B)$ is non-empty because $0 \in (A : B)$.

Remark 3.2. One can observe that if $x \in (A : B)$, then $x \wedge B \subseteq A$ and hence $B \subseteq x^{-1}A$. This implies that $(A : B) = \{x \in X \colon B \subseteq x^{-1}A\}$.

Proposition 3.3. *Let A be an ideal of X. If $B \subseteq X$, then $(A : B)$ is an ideal of X containing A.*

Proof. Let $x * y, y \in (A : B)$. Then $(x * y) \wedge B \subseteq A$, $y \wedge B \subseteq A$. This implies that $(x*y)\wedge b \in A$, $y \wedge b \in A$ for all $b \in B$. Since $(x\wedge b)*(y\wedge b) \leqslant (x*y)\wedge b$ (cf. Section 2), $(x * y) \wedge b \in A$, and A being an ideal implies that $(x \wedge b) * (y \wedge b) \in A$. Again by the definition of an ideal and the fact that $y \wedge b \in A$, it follows that $x \wedge b \in A$ for all $b \in B$. Thus $x \wedge B \subseteq A$ and consequently $x \in (A : B)$. This proves that $(A : B)$ is an ideal of X. To show that $A \subseteq (A : B)$, let $a \in A$. Then $a \wedge b \leqslant a$ for all $b \in B$ and A being an ideal implies that $a \wedge b \in A$. This shows that $a \wedge B \subseteq A$ and hence $A \subseteq (A : B)$. □

Corollary 3.4 [4, Proposition 3.3]. *Let $B \subseteq X$. Then B^* is an ideal of X. In the following proposition, we collect the properties of generalized annihilators.*

Proposition 3.5. *Let A be an ideal of X, let B and C be subsets of X. Then the following hold:*

(i) *if $B \subseteq C$, then $(A : C) \subseteq (A : B)$,*
(ii) $B \subseteq (A : (A : B))$,
(iii) $(A : B) = (A : (A : (A : B)))$,
(iv) *if B is an ideal of X and $A \subseteq B$, then $(A : B) \cap B = A$,*
(v) $(A : (A : B)) \cap (A : B) = A$,
(vi) $(A : X) = A$.

Proof. (i) If $x \in (A : C)$, then $x \wedge C \subseteq A$. As $B \subseteq C$, we get $x \wedge B \subseteq x \wedge C$ and consequently $(A : C) \subseteq (A : B)$.

(ii) Let $x \in B$ and $y \in (A : B)$. Then $B \subseteq y^{-1}A$ (Remark 3.2) and hence $x \in y^{-1}A$. This implies that $x \wedge y \in A$ for all $y \in (A : B)$ and hence $x \wedge (A : B) \subseteq A$. This proves that $x \in (A : (A : B))$ and consequently, $B \subseteq (A : (A : B))$. This proves (ii).

(iii) By (ii), $(A : B) \subseteq (A : (A : (A : B)))$. The opposite inclusion $(A : (A : (A : B))) \subseteq (A : B)$ can be obtained by combining (i) and (ii). This proves that $(A : B) = (A : (A : (A : B)))$.

(iv) Let $x \in (A : B) \cap B$. Then $B \subseteq x^{-1}A$ and $x \in B$. This implies that $x \in A$ and hence $(A : B) \cap B \subseteq A$. The opposite inclusion follows from the fact that $A \subseteq (A : B)$ (Proposition 3.3) and $A \subseteq B$. This proves that $(A : B) \cap B = A$.

(v) The proof of (v) follows directly from (iv) and Proposition 3.3.

(vi) Let $x \in X$. Then $x \in (A : X)$ if and only if $x^{-1}A = X$ and only if $x \in A$ (cf. Section 2). This proves that $(A : X) = A$. $\square$

If we take $A = \{0\}$, then we obtain

Corollary 3.6 [4]. *Let B and C be subsets of X. Then the following hold:*
(i) *If $B \subseteq C$ then $C^* \subseteq B^*$,*
(ii) $B \subseteq B^{**}$,
(iii) $B^* = B^{***}$,
(iv) *if B is and ideal of X, then $B \cap B^* = \{0\}$,*
(v) $X^* = \{0\}$,
(vi) *$B^* = X$ if and only if $B = \{0\}$.*

Proposition 3.7. *Let A, B be ideals of X and let C be a subset of X. Then*

$$(A : C) \cap (B : C) = (A \cap B : C).$$

Proof. Let $x \in X$. Then $x \in (A \cap B : C)$ if and only if $x \wedge C \subseteq A \cap B$ if and only if $x \wedge C \subseteq A$ and $x \wedge C \subseteq B$ if and only if $x \in (A : C) \cap (B : C)$. This proves that $(A : C) \cap (B : C) = (A \cap B : C)$. $\square$

Proposition 3.8. *Let A be an ideal of X, and let B, C be subsets of X. Then $(A : B \cup C) = (A : B) \cap (A : C)$.*

Proof. Let $x \in X$. Then $x \in (A : B \cup C)$ if and only if $B \cup C \subseteq x^{-1}A$ if and only if $B \subseteq x^{-1}A$ and $C \subseteq x^{-1}A$ if and only if $x \in (A : B) \cup (A : C)$. This proves that $(A : C) \cap (A : C) = (A : B \cup C)$. $\square$

If we choose $A = \{0\}$. then we obtain

Corollary 3.9 [4, Proposition 3.5]. *Let B and C be subsets of X. Then $(B \cup C)^* = B^* \cap C^*$.*

Proposition 3.10. *If A is a categorical ideal of X and B is any subset of X, then $(A : B)$ is a prime ideal of X.*

Proof. Assume that $x, y \in X$ and $x \wedge y \notin (A : B)$. Then $B \not\subseteq (x \wedge y)^{-1}A$ and hence there exists $b \in B$ such that $b \notin (x \wedge y)^{-1}A$. This means that $b \wedge (x \wedge y) \notin A$. Since A is categorical (cf. Section 2), we have $b \wedge x \notin A$ and $b \wedge y \notin A$. Thus $B \not\subseteq x^{-1}A$ and $B \not\subseteq y^{-1}A$. Consequently $x \notin (A : B)$ and $y \notin (A : B)$. This proves that $(A : B)$ is a prime ideal of X. $\square$

The following example shows that the converse of Proposition 3.10 is not true in general.

Example 3.11. Let $X = \{0, a, b, c, d, e, f, 1\}$. Define the binary operation $*$ in X as in the following table:

$*$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	a	0	a	a	0	0	a	0
b	b	b	0	b	0	b	0	0
c	c	c	c	0	c	0	0	0
d	d	b	a	d	0	b	a	0
e	e	c	e	a	c	0	a	0
f	f	f	c	b	c	b	0	0
1	1	f	e	d	c	b	a	0

Table 1

Then X is a bounded commutative BCK-algebra, and $(c)^* = \{0, a, b, d\}$, $(f)^* = \{0, a\}$ are ideals of X. Let $A = \{0, a\}$ and $B = \{c\}$. Then A is not a categorical ideal because$f \wedge (d \wedge c) = 0 \in A$ but $f \wedge d = b \notin A, f \wedge c = c \notin A$. Also $(A : B) = \{x \in X : B \subseteq x^{-1}A\} = \{x \in X : x \wedge c \in A\} = \{0, a, b, d\}$ (see Table 2), which is a prime ideal.

$\wedge$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	a	0	a
b	0	0	b	0	b	0	b	b
c	0	0	0	c	0	c	c	c
d	0	a	b	0	d	a	b	d
e	0	a	0	c	a	e	c	e
f	0	0	b	c	b	c	f	f
1	0	a	b	c	d	e	f	1

Table 2

4. Some applications

This section is devoted to some applications of generalized annihilators. We prove some properties of the involutory BCK-algebra. We also obtain a characterization for a categorical ideal to be prime.

Let A be an ideal of a BCK-algebra X. Consider the quotient BCK-algebra X/A. If J/A is a subset of X/A, then we have

$$\begin{aligned}(J/A)^* &= \{C_x : C_x \wedge J/A = A\} \quad \text{(cf. Section 2)}\\ &= \{C_x : C_x \wedge C_y = A \quad \text{for all} \quad C_y \in J/A\}\\ &= \{C_x : C_{x\wedge y} = A \quad \text{for all} \quad C_y \in J/A\}\\ &= \{C_x : x \wedge y \in A \quad \text{for all} \quad y \in J\}\\ &= \{C_x : x \wedge J \subseteq A\} = \{C_x : x \in (A : J)\}\\ &= \{C_x : J \subseteq x^{-1}A\} \quad \text{(by Remark 3.2).}\end{aligned}$$

Now we discuss the annihilator of an element of X/A. Let $C_x \in X/A$. Then

$$\begin{aligned}(C_x)^* &= \{C_y : C_x \wedge C_y = A\} = \{C_y : C_{x\wedge y} = A\}\\ &= \{C_y : x \wedge y \in A\} = \{C_y : y \in x^{-1}A\}.\end{aligned}$$

If $x \in A$, then $x^{-1}A = X$ (cf. Section 2) and hence $(C_x)^* = X/A$. If A is a prime ideal of X and C_x is a non-zero element of J/A, then $x \notin A$ and hence $x^{-1}A = A$ (cf. Section 2). This implies that $(C_x)^* = A$ (the zero element of X/A). All these observations lead to

Proposition 4.1. *Let A be an ideal of a BCK-algebra X, let J/A be a subset of X/A and C_x an element of X/A. Then the following statements hold:*
(i) $(J/A)^* = \{C_x : x \in (A : J)\} = \{C_x : J \subseteq x^{-1}A\}$,
(ii) $(C_x)^* = \{C_y : y \in x^{-1}A\}$,
(iii) *if A is a prime ideal of X and $C_x \neq A$ (non-zero element of X/A), then $(C_x)^* = A$ (zero of X/A),*
(iv) $(J/A)^{**} = \{C_x : x \in (A : (A : J))\}$.

Part (iii) of the above proposition can be reformulated as

Corollary 4.2 [3, Proposition 3.2]. *If A is a prime ideal of a BCK-algebra X, then X/A is cancellative.*

Observe that if X is a cancellative involutory BCK-algebra, then it is simple. Indeed, if A is an ideal of X, then

$$A = A^{**} = \bigcap_{x\in A} *(x)^*.$$

Since X is cancellative, therefore $(x)^* = \{0\}$ for all non-zero elements $x \in X$ and hence $A = \{0\}$ or $A = X$. This proves that X is simple. In fact, we have

Proposition 4.3. *Let X be an involutory BCK-algebra. Then X is cancellative if and only if X is simple.*

If A and B are ideals of a BCK-algebra X, then we have seen that $B \subseteq (A : (A : B))$ (Proposition 3.5 (ii)). The following theorem says that for certain classes of BCK-algebras the equality may occur.

Theorem 4.4. *Let X be an involutory BCK-algebra, and let A, B be ideals in X such that $A \subseteq B$. Then $B = (A : (A : B))$.*

Proof. $B \subseteq (A : (A : B))$ follows from Proposition 3.5 (part (ii)). To prove that $(A : (A : B)) \subseteq B$, assume that $x \notin B$. If we show that $x \notin (A : (A : B))$, then the proof is complete. Since X is an involutory BCK-algebra, therefore $B = B^{**}$ (cf. [5]). This implies that $x \wedge y \neq 0$ for some $y \in B^*$ and hence $x \wedge y \in B^*$, because B^* is an ideal of X (cf. Section 2). Since $B \cap B^* = \{0\}$, therefore $x \wedge y \notin B$ and consequently $x \wedge y \notin A (A \subseteq B)$. The expression $(x \wedge y) \wedge B = \{0\} \subseteq A$ follows from the fact that $x \wedge y \in B^*$. This implies that $B \subseteq (x \wedge y)^{-1} A$ and hence $x \wedge y \in (A : B)$. Since $(A : (A : B)) \cap (A : B) = A$ (Proposition 3.5 (v)), therefore $x \wedge y \notin (A : (A : B))$ because $x \wedge y \notin A$ and $x \wedge y \in (A : B)$. It follows that $x \notin (A : (A : B))$, because if $x \in (A : (A : B))$, then $(A : (A : B))$ being an ideal implies that $x \wedge y \in (A : (A : B))$, a contradiction. Thus we have shown that $x \notin B$ implies that $x \notin (A : (A : B))$. In other words, $(A : (A : B)) \subseteq B$ and hence $(A : (A : B)) = B$.

It is well-known that the quotient algebra of a commutative BCK-algebra is commutative [3]. If A is an ideal of X then there is a one to one correspondence between ideals of X containing A and ideals of X/A (see [3, Theorem 2.3]). Thus an ideal X/A is of the form B/A for an ideal B of X and such that $A \subseteq B$. By Proposition 4.1 (iv), we have $(B/A)^{**} = (B : (B : A))/A$. This observation and the above theorem lead to □

Corollary 4.5. *If X is an involutory BCK-algebra, then every quotient BCK-algebra of X is an involutory BCK-algebra.*

The following proposition gives a characterization of the prime ideal.

Corollary 4.6. *Let A be an ideal of involutory BCK-algebra X. Then X/A is simple if and only if A is prime.*

Proof. Let A be a prime ideal of X. Then X/A is a cancellative (Corollary 4.2) and involutory BCK-algebra (Proposition 4.5). This implies that X/A is simple (Proposition 4.3). Conversely, assume that X/A is simple. This implies that X/A is cancellative (Proposition 4.3). Let $x \wedge y \in A$. Then $C_{x \wedge y} = A$, $C_x \wedge C_y = A$. Since

X/A is cancellative, therefore $C_x = A$ or $C_y = A$. Consequently, $x \in A$ or $y \in A$ and this implies that A is prime. This completes the proof. $\square$

Now, we obtain another characterization of prime ideals by using the notion of generalized annihilators. First, we prove

Proposition 4.7. *Let X be a BCK-algebra, let A be an ideal in X and $B \subseteq X$. Then $(A : B) = X$ if and only if $B \subseteq A$.*

Proof. Let $B \subseteq A$. Since A is an ideal of X, therefore $x \wedge B \subseteq A$ for all $x \in X$. This proves that $X \subseteq (A : B)$ and consequently $(A : B) = X$. Conversely, assume that $(A : B) = X$. We will show that $B \subseteq A$. Suppose that $B \nsubseteq A$. Then there exists $b \in B$ such that $b \notin A$. Since $(A : B) = X$, therefore $x \wedge B \subseteq A$ for all $x \in X$. In particular, $b \wedge B \subseteq A$. This implies that $b \wedge b = b \in A$, which is a contradiction, and hence $B \subseteq A$. $\square$

Proposition 4.8. *If A is a prime ideal and $(A : B)$ is a proper ideal of a BCK-algebra X, then $(A : B) = A$.*

Proof. Assume on the contrary that $(A : B) \neq A$. Since $A \subseteq (A : B)$ (Proposition 3.3) therefore there exists $x \in (A : B)$ such that $x \notin A$ and hence $B \subseteq x^{-1}A$. A being prime ideal implies that $A = x^{-1}A$ (Proposition 3.4). This shows that $B \subseteq A$ and hence by Proposition 4.7, $(A : B) = X$, which is a contradiction because $(A : B)$ is a proper subset of X. This proves that $(A : B) = A$. $\square$

Proposition 4.9. *Let A be a categorical ideal of a BCK-algebra X. Then A is prime if and only if $(A : B) = A$ for $B \subseteq X$.*

Proof. Let A be a prime ideal of X. Then $(A : B) = A$ follows from Proposition 4.8. Conversely, assume that $(A : B) = A$. We shall show that A is prime. Suppose that $x, y \in X$ and $x \wedge y \notin A$. Since $(A : B) = A$, therefore $x \wedge y \notin (A : B)$. This implies that $B \nsubseteq (x \wedge y)^{-1}A$ and there exists $b \in B$ such that $b \notin (x \wedge y)^{-1}A$. This means that $b \wedge (x \wedge y) \notin A$. Since A is a categorical ideal, therefore $b \wedge x \notin A$, $b \wedge y \notin A$. As $b \wedge x \leqslant x$ if $x \in A$, A being an ideal implies that $b \wedge x \in A$, which is not possible, and hence $x \notin A$. Similarly $y \notin A$ and this proves that A is a prime ideal. $\square$

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