

On applications of Simons' type formula and reduction of codimension for complete submanifolds in space forms

Kellcio O. Araújo

Departamento de Matemática, Universidade de Brasília - UnB, 70910-900, Brasília, DF, Brazil.

E-mail address: `kellcioaraujo@gmail.com`

João Paulo dos Santos

Departamento de Matemática, Universidade de Brasília - UnB, 70910-900, Brasília, DF, Brazil.

E-mail address: `j.p.santos@mat.unb.br`

Abstract. We provide a survey of applications of Simons' type formula to submanifolds with constant mean curvature or with parallel mean curvature vector in Riemannian space forms. Also, we show a result of reduction of codimension for complete submanifolds such that the normalized mean curvature vector is parallel and the squared norm of the second fundamental form satisfies certain inequality. At the end, we give some open questions to submanifolds in general products of Riemannian space forms.

1. Introduction

The present paper has two main purposes. The first is provide a survey of applications of Simons' type formula to submanifolds with constant mean curvature or with parallel mean curvature vector. Secondly, we prove a result of reduction of codimension for complete n -dimensional submanifolds in space forms, where the normalized mean curvature vector is parallel.

Simons' type formula is a powerful tool in classification results for constant mean curvature submanifolds and also for submanifolds with parallel

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mean curvature vector. For the survey, we collected the principal results regarding the Simons' type formula and its applications, from the fundamental paper due to Simons [29] applied to minimal submanifolds, until generalizations to parallel mean curvature submanifolds in product spaces, as we can see in Fetcu and Rosenberg [17]. Besides, between these results we pointed out some of them that will be important for the second part of the paper.

This second part consist in a theorem of reduction of codimension for complete n -dimensional submanifolds in space forms, that generalizes the main result of Araújo and Barbosa [3]. An important problem in the classification of submanifolds in a space form is the reduction of the codimension. Important results on the reduction of codimension were obtained by Chen and Yano [10], J. Erbacher [16] and Yau [26] in the 1970s. Let M^n , $n \geq 3$, be an n -dimensional complete connected submanifold in the space form $\tilde{M}^{n+p}(c)$. Araújo and Tenenblat [5] studied complete submanifolds M^n with parallel mean curvature vector and showed that for $c \geq 0$ the codimension reduces to 1, if the parallel mean curvature vector does not vanish and the squared norm S of the second fundamental form of M^n satisfies the inequality

$$S \leq \frac{n^2 H^2}{n-1}.$$

As remarked in [5], when $c = 0$ (euclidean case), this result was proved by Cheng and Nonaka [12]. For the case $c < 0$, Araújo and Tenenblat [5] assumed an additional condition in order to reduce the codimension to 1.

We note that the last inequality is true for every surface with nonnegative Gaussian curvature in the euclidean space \mathbb{R}^3 . This is a consequence of Gauss equation. In this case, we have a classification result for complete surfaces with constant mean curvature and nonnegative Gaussian curvature in \mathbb{R}^3 due to Klotz and Osserman [18]. Namely, these surfaces are the plane, the sphere $\mathbb{S}^2(c)$ or the product $\mathbb{S}^1(c) \times \mathbb{R}$. Therefore, as we will see in the survey, the results due to Cheng and Nonaka [12] and Araújo and Tenenblat [5] are natural extensions of Klotz and Osserman's results for the case of submanifolds in space forms.

Submanifolds with nonzero parallel mean curvature vector also have parallel normalized mean curvature vector. The condition to have parallel normalized mean curvature vector is much weaker than the condition to have parallel mean curvature vector. If a submanifold has parallel mean curvature vector, then the its mean curvature is constant. Thus it is natural to study problems concerning this condition much weaker because in this case the mean curvature can be not constant.

Araújo and Barbosa [3] considered submanifolds with bounded mean curvature and applied the generalized maximum principle due to Omori [25] and Yau [27] and showed a result of reduction of codimension that is Theorem 2.18. Now, in this paper, we will show that if $n > 3$, then in Theorem 2.18 is not necessary to assume that the mean curvature is bounded because we can use the generalized maximum principle of Suh [31] instead of the generalized maximum principle of Omori-Yau. Namely, we have the following theorem:

Theorem 1.1. *Let M^n be an n -dimensional, $n > 3$, complete connected submanifold in the space form $\tilde{M}^{n+p}(c)$, $c \in \mathbb{R}$. Suppose the mean curvature H does not vanish and the normalized mean curvature vector is parallel. If*

$$S \leq \frac{n^2 H^2}{n-1} + 2c, \quad (1)$$

where S is the squared norm of the second fundamental form of M^n , then the codimension reduces to 1.

Remark: The hypothesis that the normalized mean curvature vector is parallel in Theorem 1.1 is only used in the proof of Lemma 3.1 in order to obtain the inequality (8). When the normalized mean curvature vector is parallel, the mean curvature is not necessarily constant or bounded. Note that in Theorem 1.1 there is not the condition that the mean curvature is bounded. Therefore this result is more general than the result which appears in [3] and its proof can be obtained from the same ideas, but using another maximum principle as cited above.

We finish the paper with some questions that can be state regarding constant mean curvature submanifolds and parallel mean curvature submanifolds in product spaces. Recently, Lira, Tojeiro and Vitório [21] and Mendonça and Tojeiro [23] have considered immersions of Riemannian manifolds in products of space forms, then it is worth to think about classification results for these submanifolds, and consequently, to generalize the results of Batista [6] and Fetcu and Rosenberg [17].

As a final observation, we would like to point that there are many applications of Simons' type formula in the context of semi-Riemannian geometry for submanifolds with parallel mean curvature vector in semi-Riemannian space forms, as we can see in [8] and [4] and the references therein, or with parallel normalized mean curvature vector, where we refer the paper [7]. However, we have focus our attention in the Riemannian case, mainly because this is the ambient where we have obtained our result.

2. Previous results

In this section we will present some previous results as applications of Simons' type formula. We consider a isometric immersion $\phi : M^n \rightarrow \tilde{M}^{n+p}$ of an n -dimensional Riemannian manifold M^n into an $(n+p)$ -dimensional Riemannian manifold \tilde{M}^{n+p} . Let h be the associated second fundamental form and S the squared norm of h .

In 1968 Simons [29] has established a formula for the Laplacian of the squared norm of the second fundamental form for a submanifold in a Riemannian manifold and has obtained an important application in the case of a minimal submanifold in the unit sphere \mathbb{S}^{n+p} , for which the formula takes a rather simplest form. Starting with codimension 1, we have the following result:

Theorem 2.1 (Simons [29]). *Let M^n be an n -dimensional minimal variety immersed in \mathbb{S}^{n+1} . Then the squared norm S of the second fundamental form satisfies*

$$\frac{1}{2}\Delta S = nS - S^2 + \|\nabla A\|^2,$$

where A is the shape operator.

For general codimensions, the following fundamental inequality is given:

Theorem 2.2 (Simons [29]). *Let M^n an n -dimensional minimal variety immersed in \mathbb{S}^{n+p} . Then the squared norm S of the second fundamental form of M^n satisfies the inequality*

$$\Delta S \geq \left(n - \left(2 - \frac{1}{p} \right) S \right) S.$$

When M^n is closed, S satisfies

$$\int_M \left(S - \frac{n}{2 - \frac{1}{p}} \right) S \geq 0.$$

As a consequence, we have:

Corollary 2.1 (Simons [29]). *Let M^n be a closed, n -dimensional minimal variety immersed in \mathbb{S}^{n+p} . Then either M^n is the totally geodesic, or $S = \frac{n}{2 - \frac{1}{p}}$, or at some $m \in M^n$, $S(m) > \frac{n}{2 - \frac{1}{p}}$.*

Due to this fundamental paper, other formulae that involve the Laplacian of squared norm of the second fundamental form or other formulae related to the second fundamental form are traditionally called "Simons' type formulae".

In 1969, Nomizu and Smyth obtained a Simons' type formula for a hypersurface M^n immersed with constant mean curvature in a space $\tilde{M}^{n+1}(c)$ of constant sectional curvature c , and then a new formula for S , which involves the sectional curvature of M^n .

Let M^n be a connected hypersurface immersed with constant mean curvature in a space form $\tilde{M}^{n+1}(c)$ of dimension $n + 1$ with constant curvature c . Let A be the shape operator and consider the squared norm of the second fundamental form $S = \text{trace}A^2$, then we have

$$\frac{1}{2}\Delta S = cnS - S^2 - c(\text{trace}A)^2 + (\text{trace}A)(\text{trace}A^3) + \|\nabla A\|^2.$$

Observe that the above equation is a generalization of Theorem 2.1 for an arbitrary hypersurface in space forms. Based on this new formula the main results are the determination of hypersurfaces M^n of non-negative sectional curvature immersed in the Euclidean space \mathbb{R}^{n+1} or in the unit sphere \mathbb{S}^{n+1} with constant mean curvature under the additional assumption that S is constant.

Theorem 2.3 (Nomizu and Smyth [24]). *Let M^n be a complete Riemannian manifold of dimension n with non-negative sectional curvature, and $\phi : M^n \rightarrow \mathbb{R}^{n+1}$ an isometric immersion with constant mean curvature into a Euclidean space \mathbb{R}^{n+1} . If $S = \text{trace}A^2$ is constant on M^n , then $\phi(M^n)$ is of the form $\mathbb{S}^p \times \mathbb{R}^{n-p}$, $0 \leq p \leq n$, where \mathbb{R}^{n-p} is an $(n-p)$ -dimensional subspace of \mathbb{R}^{n+p} and \mathbb{S}^p is a sphere in the Euclidean subspace perpendicular to \mathbb{R}^{n-p} . Except for the case $p = 1$, ϕ is an imbedding.*

Theorem 2.4 (Nomizu and Smyth [24]). *Let M^n be an n -dimensional complete Riemannian manifold with non-negative sectional curvature, and $\phi : M^n \rightarrow \mathbb{S}^{n+1}$ an isometric immersion with constant mean curvature. If S is constant on M^n , then either*

- i) $\phi(M^n)$ is a great or small sphere in \mathbb{S}^{n+1} and ϕ is an imbedding; or*
- ii) $\phi(M^n)$ is a product of spheres $\mathbb{S}^p(r) \times \mathbb{S}^q(s)$ and for $p \neq 1, n - 1$, ϕ is an imbedding.*

Following the ideas of Simons, Chern, do Carmo and Kobayashi in 1970 have also considered M^n as an n -dimensional manifold which is minimally immersed in a unit sphere \mathbb{S}^{n+p} . By Simons, we already know that if $S \leq n/(2 - \frac{1}{p})$ everywhere on M^n , then either $S = 0$ (i.e., M^n is totally geodesic) or $S = n/(2 - \frac{1}{p})$. The purpose of Chern, do Carmo and Kobayashi was to determine all minimal submanifolds M^n of \mathbb{S}^{n+p} satisfying $S = n/(2 - \frac{1}{p})$.

The main result is:

Theorem 2.5 (Chern, do Carmo and Kobayashi [14]). *The Veronese surface in \mathbb{S}^4 and the submanifolds $M_{m,n-m} = \mathbb{S}^m(\sqrt{\frac{m}{n}}) \times \mathbb{S}^{n-m}(\sqrt{\frac{n-m}{n}})$ in \mathbb{S}^{n+1} are the only compact minimal submanifolds of dimension n in \mathbb{S}^{n+p} satisfying $S = n/(2 - \frac{1}{p})$.*

It is important to note that the techniques used by Chern, do Carmo and Kobayashi are quite different from Simons and Nomizu-Smyth [24]. Specifically, the results were obtained by the approach of orthonormal frames and differential forms. This approach is very useful and was used in many of the results cited here as well as in the demonstration of our main result. Besides, Chern, do Carmo and Kobayashi have also obtained a more general inequality than that one obtained by Simons, namely, the inequality is given for a general space form.

Theorem 2.6 (Chern, do Carmo and Kobayashi [14]). *Let M^n be an n -dimensional compact oriented manifold which is minimally immersed in an $(n+p)$ -dimensional space of constant curvature $\tilde{M}^{n+p}(c)$. Then*

$$\int_M \left(S - \frac{nc}{2 - \frac{1}{p}} \right) S \geq 0.$$

In order to generalize the Simons' type formula, Erbacher [15], in 1971, extended the results of Nomizu and Smyth [24] to isometric immersions of codimension p . Firstly, a Simons' type formula was extended for codimension p .

Let $\phi: M^n \rightarrow \tilde{M}^{n+p}(c)$, where $\tilde{M}^{n+p}(c)$ has constant sectional curvature c . Let ∇^* denote the sum of the tangential and the normal connections. If we consider ξ_1, \dots, ξ_n orthonormal normal vector fields in M^n , we define $-A_\xi$ to be the tangential component of $\tilde{\nabla}_X \xi$ and then we consider $f = \sum_{\alpha} \text{trace} A_{\alpha}^2$. Let η the mean curvature vector defined by

$$\eta = \sum_{\alpha} (\text{trace} A_{\alpha}) \xi_{\alpha}$$

and $D\eta$ the covariant differentiation in the normal bundle. In this context, we have the following Simons' type formula:

Lemma 2.1 (Erbacher [15]). *If $D\eta = 0$, then*

$$\begin{aligned} \frac{1}{2} \Delta f &= cnf - c \sum_{\alpha} (\text{trace} A_{\alpha})^2 + \sum_{\alpha, \beta} \text{trace} [A_{\alpha}, A_{\beta}]^2 \\ &+ \sum_{\alpha, \beta} (\text{trace} A_{\alpha}) (\text{trace} A_{\alpha} A_{\beta})^2 + \sum_{\alpha} \|\nabla^* A_{\alpha}\|^2. \end{aligned}$$

If in addition the normal connection is trivial and we let $\lambda_{i\alpha}$ $1 \leq i \leq n$, $1 \leq \alpha \leq p$, be the eigenvalues of A_α corresponding to eigenvectors E_i (recall if the normal curvature vanishes then the A'_α s are simultaneously diagonalizable), then

$$\frac{1}{2}\Delta f = \sum_{\alpha} \sum_{i < j} (\lambda_{i\alpha} - \lambda_{j\alpha})^2 (c + \sum_{s=1}^p \lambda_{is} \lambda_{js}) + \sum_{\alpha} \|\nabla^* A_\alpha\|^2,$$

where $c + \sum_{s=1}^p \lambda_{is} \lambda_{js} = K(E_i \wedge E_j)$.

Erbacher has used the previous lemma to get another further results and then, has summarized them to get a result regarding n -dimensional submanifolds M^n of non-negative sectional curvature isometrically immersed in the Euclidean space \mathbb{R}^{n+p} or the sphere \mathbb{S}^{n+p} with parallel mean curvature, under the additional assumptions that M^n has constant scalar curvature and the curvature tensor of the connection in the normal bundle is zero.

Theorem 2.7 (Erbacher [15]). *Let ψ be an isometric immersion of an n -dimensional, connected, complete Riemannian manifold M^n of non-negative sectional curvatures into \mathbb{R}^{n+p} or \mathbb{S}^{n+p} . Suppose that the mean curvature normal is parallel with respect to the normal connection and that the curvature tensor of the normal connection is zero. If either M^n is compact or has constant scalar curvature, then*

$$\psi(M^n) = M^{n_1} \times \dots \times M^{n_i},$$

where each M^{n_i} is an n_i -dimensional sphere of some radius contained in some Euclidean space N^{n_i+1} of dimension $n_i + 1$, $N^{n_i+1} \perp N^{n_j+1}$ for $i \neq j$; except possibly one of the M^{n_i} is a Euclidean space $= N^{n_i}$ (this can only occur if $\tilde{M}^{n+p} = \mathbb{R}^{n+p}$). Furthermore, the immersion is an imbedding except possibly when some $M^{n_i} = \mathbb{S}^1(\frac{1}{r^2})$, a circle of radius r in some Euclidean plane. The corresponding local result is true with the assumption of constant scalar curvature.

Isometric immersions of space forms into space forms were also investigated and conditions that imply the vanishing of the curvature tensor of the connection in the normal bundle were obtained. Let $\psi : M^n(c) \rightarrow \tilde{M}^{n+p}(\tilde{c})$ be an isometric immersion of a Riemannian manifold $M^n(c)$ of constant sectional curvature c into a Riemannian manifold $M^{n+p}(\tilde{c})$ of constant sectional curvature \tilde{c} .

Theorem 2.8 (Erbacher [15]). *Let $p = 2$, $n \geq 3$.*

a) *If $c \neq \tilde{c}$, then the curvature tensor of the normal connection is zero.*

b) If $c = \tilde{c}$, then for each $x \in M^n$ the curvature tensor of the normal connection is zero at x or the relatively nullity (see [19]) at x is $n - 2$.

Theorem 2.9 (Erbacher [15]). *If $p = 3$, $n \geq 4$, $D\eta = 0$, $\eta \neq 0$, then we have a) and b) of Theorem 2.8.*

In 1974, Yau has obtained other results regarding applications of Simons' type formula to surfaces with parallel mean curvature and reduction of codimension. Besides a Chern's presentation of Simons' work, that can be found in [13] and it is closer to that one presented in [14], we have on this paper a generalization of Erbacher's work about submanifolds that lies in a totally umbilical submanifold. Yau's results is given in the following context:

Let N_1 be a sub-bundle of the normal bundle. We say that an n -dimensional submanifold M^n of a $(n+p)$ -dimensional manifold \tilde{M}^{n+p} is umbilical (totally geodesic) with respect to N_1 if M^n is umbilical (totally geodesic) with respect to any local section of N_1 . We say that N_1 is parallel in the normal bundle if it is invariant under the parallel translations in the normal bundle.

Theorem 2.10 (Yau [26]). *Let \tilde{M}^{n+p} be a conformally flat $(n+p)$ -dimensional manifold. Let N_1 be a sub-bundle of the normal bundle of M^n with fiber dimension k . Suppose that M^n is an n -dimensional submanifold umbilical with respect to N_1 and N_1 is parallel in the normal bundle. Then M^n lies in an $(n+p-k)$ -dimensional umbilical submanifold \bar{M}^{n+p-k} of \tilde{M}^{n+p} such that the fiber of N_1 is everywhere perpendicular to \bar{M}^{n+p-k} . If \tilde{M}^{n+p} has constant curvature, the size of \bar{M}^{n+p-k} can be determined. In particular, if M^n is totally geodesic with respect to N_1 , then M^{n+p-k} is totally geodesic.*

The main result about surfaces with parallel mean curvature vector is the following:

Theorem 2.11 (Yau [26]). *Let M^2 be a surface with parallel mean curvature vector in a constant curved manifold N . Then either M^2 is a minimal surface of an umbilical hypersurface of N or M^2 lies in a three-dimensional umbilical submanifold of N with constant mean curvature.*

In 1994, Alencar and do Carmo [2] have considered M^n as an n -dimensional orientable manifold and $f : M^n \rightarrow \mathbb{S}^{n+1}(1) \subset \mathbb{R}^{n+2}$ an immersion of M^n into the unit $(n+1)$ -sphere. They defined a linear map $\psi : T_p M \rightarrow T_p M$ by

$$\langle \psi X, Y \rangle = H \langle X, Y \rangle - \langle AX, Y \rangle.$$

It is easily checked that $\text{trace } \psi = 0$ and that

$$|\psi|^2 = \frac{1}{2n} \sum_{i,j} (k_i - k_j)^2, \quad i, j = 1, \dots, n,$$

so that $|\psi|^2 = 0$ if and only if M^n is totally umbilic.

To see the main result it is necessary some notation. An $H(r)$ -torus in $\mathbb{S}^{n+1}(1)$ is obtained by considering the standard immersions $\mathbb{S}^{n-1}(r) \subset \mathbb{R}^n$, $\mathbb{S}^1(\sqrt{1-r^2}) \subset \mathbb{R}^2$, $0 < r < 1$, where the value within the parentheses denotes the radius of the corresponding sphere, and taking the product immersion $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2}) \rightarrow \mathbb{R}^n \times \mathbb{R}^2$. By the choices made, the $H(r)$ -torus turns out to be contained in $\mathbb{S}^{n+1}(1)$ and has principal curvatures given, in some orientation, by

$$k_1 = \dots = k_{n-1} = \frac{\sqrt{1-r^2}}{r}, \quad k_n = -\frac{r}{\sqrt{1-r^2}},$$

or the symmetric of these values for the opposite orientation.

Let M^n be compact and orientable, and let $f : M^n \rightarrow \mathbb{S}^{n+1}(1)$ have constant mean curvature H ; choose an orientation for M^n such that $H > 0$. For each H , set

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + 1),$$

and let B_H be the square of the positive root of $P_H(x) = 0$. The main result of the paper is the following:

Theorem 2.12 (Alencar, do Carmo [2]). *Assume that $|\psi|^2 < B_H$ for all $p \in M$. Then:*

- i) Either $|\psi|^2 = 0$ (and M^n is totally umbilic) or $|\psi|^2 \equiv B_H$.*
- ii) $|\psi|^2 = B_H$ if and only if:*
 - a) $H = 0$ and M^n is a Clifford torus in $\mathbb{S}^{n+1}(1)$.*
 - b) $H \neq 0$, $n > 3$, and M^n is an $H(r)$ -torus with $r^2 < \frac{n-1}{n}$.*
 - c) $H \neq 0$, $n = 2$, and M^n is an $H(r)$ -torus with $r^2 \neq \frac{n-1}{n}$.*

It is important to note that, for the proof of the Theorem 2.12, the authors have used Simons' type formula for the application ψ . The formula used was the following:

$$\frac{1}{2} \Delta |\psi|^2 = |\nabla \psi|^2 - |\psi|^2 + n|\psi|^2 + nH^2|\psi|^2 - nH \sum_i \mu_i^3,$$

where μ_i , $1 \leq i \leq n$ are eigenvalues of ψ .

In 1994, Santos [28] generalized [2] for codimension greater than 1. Let us consider an n -dimensional submanifold M^n of the $(n+p)$ -dimensional sphere $\mathbb{S}^{n+p}(c)$ of constant sectional curvature c . Following Erbacher's notation, consider an orthonormal normal frame ξ_1, \dots, ξ_n and the applications $A_{\xi_\alpha} \equiv A_\alpha$, as the tangential components of $\nabla \xi_\alpha$ and η the mean curvature vector, which will be considered in this case as $\eta = \frac{1}{n} \sum_{\alpha} \text{trace}(A_\alpha) \xi_\alpha$. For

each α , define linear maps ψ_α by

$$\langle \psi_\alpha X, Y \rangle = \langle X, Y \rangle \langle \eta, \xi_\alpha \rangle - \langle A_\alpha X, Y \rangle,$$

then consider the application ψ by

$$\psi(X, Y) = \sum_{\alpha=1}^p \langle \psi_\alpha X, Y \rangle \xi_\alpha.$$

Theorem 2.13 (Santos [28]). *Let M^n be a compact orientable submanifold of $\mathbb{S}^{n+p}(c)$. Assume that the mean curvature vector η is parallel with respect to the normal connection. If ψ satisfies*

$$|\psi|^2 \leq B_{p,h} \left\{ n(c + H^2) - \frac{n(n-2)}{\sqrt{n(n-1)}} |\psi_h| \right\},$$

where $\psi_\eta(X, Y) = \langle \psi(X, Y), \eta \rangle$ then either

- i) $|\psi| = 0$ and M^n is totally umbilic or
- ii) the equality holds and one of the following cases occurs:
 - a) $H = 0$, $p = 1$ and M^n is a minimal Clifford hypersurface

$$M^n = \mathbb{S}^m \left(\sqrt{\frac{m}{nc}} \right) \times \mathbb{S}^{n-m} \left(\sqrt{\frac{n-m}{nc}} \right).$$

- b) $H = 0$, $n = p = 2$ and M^2 is a Veronese surface.
- c) $H \neq 0$, $p = 1$ and M^n is an H -torus,

$$M^n = \mathbb{S}^{n-1}(r_1) \times \mathbb{S}^1(r_2),$$

where $r_1^2 + r_2^2 = c^{-1}$. If $n \geq 3$, we have only those H -tori which satisfy $r_1^2 < (n-1)/nc$; if $n = 2$, the only condition is $r_1^2 \neq 1/2c$.

- d) $H \neq 0$, $p = 2$ and M^n is a minimal Clifford hypersurface in a hypersphere

$$M^n = \mathbb{S}^m \left(\sqrt{\frac{m}{n(c+H^2)}} \right) \times \mathbb{S}^{n-m} \left(\sqrt{\frac{n-m}{n(c+H^2)}} \right).$$

e) $H \neq 0$, $p = 2$ and for all H_2 , $0 < H_2 \leq H$, M^n is an H_1 - torus

$$M^n = \mathbb{S}^{n-1}(r_1) \times \mathbb{S}^1(r_2),$$

where $H_1^2 + H_2^2 = H^2$, $r_1^2 + r_2^2 = (c + H_2^2)$. If $n \geq 3$ we have only those H_1 - tori with $r_1^2 < (n - 1)/n(c + H_2^2)$; if $n = 2$ the only condition is $r_1^2 \neq 1/2(c + H_2^2)$.

f) $H \neq 0$, $n = 2$, $p = 3$ and M^2 is a Veronese surface in a hypersphere

$$M^2 \subset \mathbb{S}_{c+H^2}^4.$$

Regarding submanifolds with parallel mean curvature vector, we have an article due to Cheng and Nonaka [12], where an extension of the following theorem due to Klotz and Osserman [18] was considered:

Theorem 2.14 (Klotz and Osserman [18]). *Let M^2 be a complete and connected surface with constant mean curvature in \mathbb{R}^3 . If the Gauss curvature of M^2 is nonnegative, then M^2 is the plane \mathbb{R}^2 , the sphere $\mathbb{S}^2(c)$ or the cylinder $\mathbb{S}^1(c) \times \mathbb{R}$.*

From the Gauss equation, we know that the Gaussian curvature of a surface M^2 in \mathbb{R}^3 is nonnegative if and only if

$$S \leq \frac{n^2|H|^2}{n-1}$$

with $n = 2$, where S is the squared norm of the second fundamental form of M^2 and H is the mean curvature of M^2 . Then we have the following extension to higher dimensions and codimensions:

Theorem 2.15 (Cheng and Nonaka [12]). *Let M^n be an n -dimensional complete and connected submanifold with parallel mean curvature vector H in \mathbb{R}^{n+p} , $n \geq 3$. If the squared norm S of the second fundamental form of M^n satisfies*

$$S \leq \frac{n^2|H|^2}{n-1},$$

then M^n is the totally geodesic Euclidean space \mathbb{R}^n , the totally umbilical sphere $\mathbb{S}^n(c)$ or the generalized cylinder $\mathbb{S}^{n-1}(c) \times \mathbb{R}$ in \mathbb{R}^{n+1} .

In 2009, Araújo and Tenenblat [5] extended Cheng-Nonaka [12] for submanifolds of the sphere and of the hyperbolic space. The following theorem extend the results of [12], where the Euclidean space was considered:

Theorem 2.16 (Araújo and Tenenblat [5]). *Let M^n , $n \geq 3$, be a complete connected submanifold of a space form $M^{n+p}(c)$, with $c \geq 0$. Suppose the mean curvature vector H does not vanish and it is parallel in the normal*

bundle. If the squared norm S of the second fundamental form of M^n satisfies

$$S \leq \frac{n^2|H|^2}{n-1},$$

then the codimension reduces to 1.

When $c < 0$, we have the following:

Theorem 2.17 (Araújo and Tenenblat [5]). *Let M^n , $n \geq 3$, be a complete and connected submanifold of the hyperbolic space $\mathbb{H}^{n+p}(c)$, with $c < 0$. Suppose the mean curvature vector does not vanish and it is parallel in the normal bundle. Let $\xi_1, \xi_2, \dots, \xi_p$ be orthonormal vector fields normal to M^n such that $H = |H|\xi_1$. Suppose that the squared norm S of the second fundamental form of M^n satisfies*

$$S \leq \frac{n^2|H|^2}{n-1}$$

and

$$cn|T|^2 + \sum_{\alpha=2}^p \|\nabla^* A_\alpha\|^2 \geq 0,$$

where A_α is the second fundamental form associated to ξ_α , S and ∇^* are defined by

$$\nabla^* A_\alpha = \nabla_X A_\alpha - \sum_{\beta=1}^p S_{\alpha\beta}(X) A_\beta$$

and

$$|T|^2 = \sum_{\alpha=2}^p \text{tr} A_\alpha^2.$$

Then the codimension reduces to 1.

We observe that the last condition of the above Theorem is trivially satisfied when $c \geq 0$.

In 2011, Araújo and Barbosa [3] extended the results proved in [5] to submanifolds M^n in a space form $\tilde{M}^{n+p}(c)$, $c \in \mathbb{R}$, whose mean curvature does not vanish and it is only bounded with a parallel normalized mean curvature vector.

Theorem 2.18 (Araújo and Barbosa [3]). *Let M^n be an n -dimensional, $n \geq 3$, complete connected submanifold in the space form $\tilde{M}^{n+p}(c)$, $c \in \mathbb{R}$. If $n = 3$, we assume that $c \leq 0$. Suppose the mean curvature H does not*

vanish and it is bounded with a parallel normalized mean curvature vector. If

$$S \leq \frac{n^2 H^2}{n-1} + 2c,$$

where S is the squared norm of the second fundamental form of M^n , then the codimension reduces to 1.

We finish this section by considering results regarding Simons' type formula for constant mean curvature surfaces or submanifolds with parallel mean curvature vector in product spaces.

Firstly we have the results due to Batista [6]. Lets consider immersions M^2 in $\tilde{M}^2(c) \times \mathbb{R}$, with $c \pm 1$, where $\tilde{M}^2(-1) = \mathbb{H}^2$ and $\tilde{M}^2(1) = \mathbb{S}^2$ and a special tensor E defined by

$$EX = 2HAX - c \langle X, T \rangle T + \frac{c}{2}(1 - \nu^2)X - 2H^2X, \quad (2)$$

where $X \in T_p\Sigma$, A is the Weingarten operator associated to the second fundamental form, H is the mean curvature, T is the tangential component of the parallel field ∂_t , tangent to \mathbb{R} in $M^2(c) \times \mathbb{R}$, and $\nu = \langle N, \partial_t \rangle$. Then we have the following Simons' type formula:

Theorem 2.19 (Batista [6]). *Let $M^2 \subset \tilde{M}^2(c) \times \mathbb{R}$ be an immersed surface with nonzero constant mean curvature H . Consider the special tensor E , given by (2), then*

$$\frac{1}{2}\Delta|E|^2 = |\nabla E|^2 - |E|^4 + E^2 \left(\frac{5c\nu^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H} \langle ET, T \rangle \right) + c|ET|^2 - \frac{1}{4H^2} \langle ET, T \rangle^2.$$

The main result of Batista is a result similar to Theorem 2.12, due to Alencar and do Carmo [2], for immersions of M^2 in $\tilde{M}^2(c) \times \mathbb{R}$.

Lets consider the polynomial $p_H(t) = -t^2 - \frac{1}{H}t + \left(\frac{4H^2-1}{2}\right)$. When $H > \frac{1}{2}$, there is a positive root L_H for p_H . Then we have the following:

Theorem 2.20 (Batista [6]). *Let $M^2 \subset \mathbb{S}^2 \times \mathbb{R}$ be a complete immersed surface with constant mean curvature $H \geq \frac{1}{2}$. Assume that*

$$\sup_{M^2} |E| < L_H$$

then M^2 is a constant mean curvature sphere $S_H^2 \subset \mathbb{S}^2 \times \mathbb{R}$.

Consider now the polynomial $q_H(t) = -t^2 - \frac{1}{H}t + \left(\frac{8H^4-12H^2-1}{4H^2}\right)$. When $H > \sqrt{\frac{12+\sqrt{176}}{16}}$, there is a positive root M_h for q_H , then:

Theorem 2.21 (Batista [6]). *Let $M^2 \subset \mathbb{H}^2 \times \mathbb{R}$ be a complete immersed surface with constant mean curvature $H > \sqrt{\frac{12+\sqrt{176}}{16}}$. Assume that*

$$\sup_{\Sigma} |E| < ML_H$$

then M^2 is an orbit of the 2-dimensional solvable groups of isometries of $\mathbb{H}^2 \times \mathbb{R}$ (See Abresch and Rosenberg [1] for details of this last class of surfaces).

In 2011, Fetcu and Rosenberg [17] computed the Laplacian of the second fundamental form of a pmc submanifold (submanifold with parallel mean curvature vector) in $\tilde{M}^n(c) \times \mathbb{R}$ and then used a Simons' type formula to prove some gap theorems for pmc submanifolds in $\tilde{M}^n(c) \times \mathbb{R}$ when $c > 0$ and the mean curvature vector field H of the submanifold makes a constant angle with the unit vector field ξ tangent to \mathbb{R} , or when $c < 0$ and H is orthogonal to ξ . The main theorems are the following:

Theorem 2.22 (Fetcu and Rosenberg [17]). *Let M^m be an immersed complete non-minimal pmc submanifold in $\tilde{M}^n(c) \times \mathbb{R}$, $n > m \geq 3$, $c > 0$, with mean curvature vector field H and squared norm S of the second fundamental form. If the angle between H and ξ is constant and*

$$S + \frac{2c(2m+1)}{m}|T|^2 \leq 2c + \frac{m^2}{m-1}|H|^2,$$

where T is the tangent part of ξ , then M^m is a totally umbilical cmc hypersurface in $\tilde{M}^{m+1}(c)$.

Theorem 2.23 (Fetcu and Rosenberg [17]). *Let M^m be an immersed complete non-minimal pmc submanifold in $\tilde{M}^n(c) \times \mathbb{R}$, $n > m \geq 3$, $c < 0$, with mean curvature vector field H and squared norm S of the second fundamental form. If H is orthogonal to ξ and*

$$S + \frac{2c(m+1)}{m}|T|^2 \leq 4c + \frac{m^2}{m-1}|H|^2,$$

then M^m is a totally umbilical cmc hypersurface in $\tilde{M}^{m+1}(c)$.

Theorem 2.24 (Fetcu and Rosenberg [17]). *Let M^2 be a complete non-minimal pmc surface in $\tilde{M}^n(c) \times \mathbb{R}$, $n > 2$, $c > 0$, such that the angle between H and ξ is constant and*

$$S + 3c|T|^2 \leq 4|H|^2 + 2c.$$

Then, either

a) M^2 is pseudo-umbilical and lies in $\tilde{M}^n(c)$ or

b) M^2 is a torus $\mathbb{S}^1(r) \times \mathbb{S}^1\left(\sqrt{\frac{1}{c} - r^2}\right)$ in $\tilde{M}^3(c)$, with $r^2 \neq \frac{1}{2c}$.

Theorem 2.25 (Fetcu and Rosenberg [17]). *Let M^2 be a complete non-minimal pmc surface in $\tilde{M}^n(c) \times \mathbb{R}$, $n > 2$, $c < 0$, such that H is orthogonal to ξ and*

$$S + 5c|T|^2 \leq 4|H|^2 + 4c.$$

Then M^2 is pseudo-umbilical and lies in $\tilde{M}^n(c)$.

3. A result of reduction of codimension

This section follows the same steps of Section 2 of [3] and contains some basic facts, notations and classical preliminary results that will be necessary for the proof of our result. We include some equations and inequalities with detailed proofs for the sake of completeness.

Let $\tilde{M}^{n+p}(c)$ be an $(n + p)$ -dimensional connected space form with constant sectional curvature c and $\phi : M^n \rightarrow \tilde{M}^{n+p}(c)$ be an isometric immersion of an n -dimensional connected differential manifold M^n in $\tilde{M}^{n+p}(c)$. We choose a local field of orthonormal frames $\{e_1, \dots, e_{n+p}\}$ adapted to the Riemannian metric of $\tilde{M}^{n+p}(c)$ and the dual coframes $\{\omega_1, \dots, \omega_{n+p}\}$ in such a way that, restricted to the submanifold M^n , $\{e_1, \dots, e_n\}$ are tangent to M^n . We have that $\{e_1, \dots, e_n\}$ is a local field of orthonormal frames adapted to the induced Riemannian metric on M^n and $\{\omega_1, \dots, \omega_n\}$ is a local field of its dual coframes on M^n . We then have

$$\omega_\alpha = 0, \quad \alpha = n + 1, \dots, n + p.$$

From Cartan's Lemma it follows that,

$$\omega_{\alpha i} = \sum_{j=1}^n h_{ij}^\alpha \omega_j \quad \text{and} \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

We use the following standard convention for indexes:

$$1 \leq A, B, C, \dots \leq n + p, \quad 1 \leq i, j, k, \dots \leq n, \quad n + 1 \leq \alpha, \beta, \gamma, \dots \leq n + p.$$

The second fundamental form II and the mean curvature vector h of M^n are defined by

$$II = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^n h_{ij}^\alpha \omega_i \omega_j e_\alpha$$

and

$$h = \frac{1}{n} \sum_{\alpha=n+1}^{n+p} \left(\sum_{i=1}^n h_{ii}^\alpha \right) e_\alpha, \tag{3}$$

respectively. The mean curvature H and the squared norm of the second fundamental form S of M^n are defined by

$$H = \frac{1}{n} \sqrt{\sum_{\alpha=n+1}^{n+p} \left(\sum_{i=1}^n h_{ii}^\alpha \right)^2}$$

and

$$S = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^n (h_{ij}^\alpha)^2, \quad (4)$$

respectively. The connection form of M^n are characterized by the structure equations

$$\begin{aligned} d\omega_i &= - \sum_{j=1}^n \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0 \\ d\omega_{ij} &= - \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l=1}^n R_{ijkl} \omega_k \wedge \omega_l \end{aligned}$$

and

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{\alpha=n+1}^{n+p} (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \quad (5)$$

where R_{ijkl} are the components of the curvature tensor of M^n . Denote by R_{ij} and $n(n-1)r$ the components of the Ricci curvature and the scalar curvature of M^n , respectively. So, we have from (4) and (5) that

$$R_{jk} = (n-1)c\delta_{jk} + \sum_{\alpha=n+1}^{n+p} \left(\sum_{i=1}^n h_{ii}^\alpha h_{jk}^\alpha - \sum_{i=1}^n h_{ik}^\alpha h_{ji}^\alpha \right)$$

and

$$n(n-1)r = n(n-1)c + n^2 H^2 - S. \quad (6)$$

We denote by D^\perp the connection of the normal bundle. A vector field ξ normal to M^n is parallel if $D_X^\perp \xi = 0$, for all C^∞ vector field X tangent to M^n . Thus M^n is a submanifold with parallel normalized mean curvature vector if $D^\perp \frac{h}{H} \equiv 0$. Note that if the mean curvature vector h is parallel then H is constant. In the case of the mean curvature vector $h \neq 0$ on M^n , we have that $e_{n+1} = H^{-1}h$ is a normal vector field defined globally on M^n . We define φ and ψ by

$$\varphi = \sum_{i,j=1}^n (h_{ij}^{n+1} - H\delta_{ij})^2$$

and

$$\psi = \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^n (h_{ij}^\alpha)^2,$$

respectively. Then φ and ψ are functions defined on M^n globally, which do not depend on the choice of the orthonormal frame $\{e_1, \dots, e_n\}$. Note that

$$S - nH^2 = \varphi + \psi. \tag{7}$$

From (3), we obtain that $nH = \sum_{i=1}^n h_{ii}^{n+1}$ and $\sum_{i=1}^n h_{ii}^\alpha = 0$ for $n+2 \leq \alpha \leq n+p$ on M^n . Setting $H_\alpha = (h_{ij}^\alpha)$ and defining $N(A) = \text{trace}({}^tAA)$ for $n \times n$ -matrix A , by making use of a direct computation we have

$$\begin{aligned} \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k,l=1}^n h_{ij}^\alpha h_{kl}^\alpha R_{lijk} &= c \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^n (h_{ij}^\alpha)^2 + \sum_{\alpha=n+2}^{n+p} \text{trace}(H_{n+1}H_\alpha)^2 \\ &\quad - \sum_{\alpha=n+2}^{n+p} (\text{trace}(H_{n+1}H_\alpha))^2 \\ &\quad + \sum_{\alpha,\beta=n+2}^{n+p} \text{trace}(H_\alpha H_\beta)^2 \\ &\quad - \sum_{\alpha,\beta=n+2}^{n+p} (\text{trace}(H_\alpha H_\beta))^2, \end{aligned}$$

$$\begin{aligned} \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k,l=1}^n h_{ij}^\alpha h_{li}^\alpha R_{lkjk} &= (n-1)c \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^n (h_{ij}^\alpha)^2 \\ &\quad + nH \sum_{\alpha=n+2}^{n+p} \text{trace}(H_{n+1}H_\alpha^2) \\ &\quad - \sum_{\alpha=n+2}^{n+p} \text{trace}(H_{n+1}^2 H_\alpha^2) \\ &\quad - \sum_{\alpha,\beta=n+2}^{n+p} \text{trace}(H_\alpha H_\beta H_\beta H_\alpha) \end{aligned}$$

and

$$\begin{aligned} \sum_{\alpha, \beta=n+1}^{n+p} \sum_{i, j, k=1}^n h_{ij}^\alpha h_{ki}^\beta R_{\beta\alpha jk} &= \sum_{\alpha, \beta=n+1}^{n+p} \text{trace}(H_\alpha H_\beta)^2 \\ &- \sum_{\alpha, \beta=n+1}^{n+p} \text{trace}(H_\alpha H_\beta H_\beta H_\alpha). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{2} \Delta \psi &= \sum_{\alpha=n+2}^{n+p} \sum_{i, j, k=1}^n (h_{ijk}^\alpha)^2 + \sum_{\alpha=n+2}^{n+p} \sum_{i, j=1}^n h_{ij}^\alpha \Delta h_{ij}^\alpha \\ &= \sum_{\alpha=n+2}^{n+p} \sum_{i, j, k=1}^n (h_{ijk}^\alpha)^2 + \sum_{\alpha=n+2}^{n+p} \sum_{i, j, k=1}^n h_{ij}^\alpha h_{kkij}^\alpha \\ &\quad \sum_{\alpha=n+2}^{n+p} \text{trace}(H_{n+1} H_\alpha)^2 - \sum_{\alpha=n+2}^{n+p} \text{trace}(H_\alpha H_{n+1} H_{n+1} H_\alpha) \\ &\quad + n c \sum_{\alpha=n+2}^{n+p} \sum_{i, j=1}^n (h_{ij}^\alpha)^2 \\ &\quad + n H \sum_{\alpha=n+2}^{n+p} \text{trace}(H_{n+1} H_\alpha^2) - \sum_{\alpha=n+2}^{n+p} (\text{trace}(H_{n+1} H_\alpha))^2 \\ &\quad - \sum_{\alpha, \beta=n+2}^{n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha, \beta=n+2}^{n+p} (\text{trace}(H_\alpha H_\beta))^2 \\ &\quad + \sum_{\alpha=n+2}^{n+p} \text{trace}(H_{n+1} H_\alpha)^2 - \sum_{\alpha=n+2}^{n+p} \text{trace}(H_{n+1}^2 H_\alpha^2). \end{aligned}$$

According to a result in [20], and the definition of ψ , we obtain

$$- \sum_{\alpha, \beta=n+2}^{n+p} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_{\alpha, \beta=n+2}^{n+p} (\text{trace}(H_\alpha H_\beta))^2 \geq -\frac{3}{2} \psi^2.$$

Now, using the same arguments as in the proof of Proposition 3.6 in [11], we can prove the following lemma:

Lemma 3.1. *Let M^n be a submanifold in $\tilde{M}^{n+p}(c)$ with mean curvature $H \neq 0$ such that the normalized mean curvature vector is parallel. Then*

$$\frac{1}{2}\Delta\psi \geq \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^n (h_{ijk}^\alpha)^2 + nc\psi + \left(nH^2 - \sqrt{\frac{n}{n-1}}(n-2)H\sqrt{\varphi} - \varphi - \frac{3}{2}\psi \right) \psi \tag{8}$$

This lemma is the key for the proof of Theorem 1.1. We end this section by recalling the following result which we shall use later to obtain this proof.

Theorem 3.1. (*Generalized Maximum Principle, Suh [31] or [30]*)

Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded from below. If a C^2 -nonnegative function f satisfies

$$\Delta f \geq kf^s, \tag{9}$$

where k is any positive constant and s is a real number greater than 1, then f vanishes identically.

4. Proof of Theorem 1.1

Since

$$\left(\sqrt{\frac{n(n-2)}{n-1}}H - \sqrt{(n-2)\varphi} \right)^2 \geq 0,$$

we obtain

$$-\sqrt{\frac{n}{n-1}}(n-2)H\sqrt{\varphi} \geq -\frac{n}{2} \left(\frac{n-2}{n-1} \right) H^2 - \frac{(n-2)}{2}\varphi. \tag{10}$$

Therefore, from (8) and (10), we have the inequality $\frac{1}{2}\Delta\psi \geq$

$$\sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^n (h_{ijk}^\alpha)^2 + \left(nc + nH^2 - \frac{n}{2} \left(\frac{n-2}{n-1} \right) H^2 - \frac{(n-2)}{2}\varphi - \varphi - \frac{3}{2}\psi \right) \psi. \tag{11}$$

Observe that

$$-\frac{(n-2)}{2}\varphi - \varphi = \frac{n^2}{2}H^2 - \frac{n}{2}S + \frac{n}{2}\psi. \tag{12}$$

Hence, by (11), (12) and the hypothesis (1), we obtain

$$\begin{aligned} \frac{1}{2}\Delta\psi &\geq \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^n (h_{ijk}^\alpha)^2 + \left(cn + \frac{n}{2} \left(\frac{n^2 H^2}{n-1} - S \right) + \frac{n-3}{2} \psi \right) \psi \\ &\geq \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^n (h_{ijk}^\alpha)^2 + \left(\frac{n-3}{2} \right) \psi^2. \end{aligned}$$

Thus we have

$$\Delta\psi \geq (n-3)\psi^2. \quad (13)$$

On the other hand, from the inequality (1) and the relation (6), we obtain that

$$n(n-1)r \geq (n-2)S + (n-1)(n-2)c. \quad (14)$$

So, it follows from the Theorem 4.1 in [9] that the sectional curvature of M^n is non-negative. Consequently, the Ricci curvature of M^n is non-negative. Thus M^n is a complete Riemannian manifold whose Ricci curvature is bounded from below and from (13) we have a C^2 - non-negative function ψ satisfying $\Delta\psi \geq k\psi^s$, where $k = n-3$ is a positive constant because $n > 3$ and $s = 2 > 1$. Therefore, the Generalized Maximum Principle of Suh (Theorem 3.1), applied to the function ψ , implies that

$$\psi = 0 \quad \text{and} \quad \sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^n (h_{ijk}^\alpha)^2 = 0.$$

We conclude from Erbacher's theorem [16] (or the Theorem 1 in [26]) that the codimension reduces to 1.

5. Open questions

Recently Lira, Tojeiro and Mendonça [21] and Mendonça and Tojeiro [23] have considered immersions in products of space forms. In this context, Manfio and Vitória [22], has considered minimal immersions of riemannian manifolds in this kind of riemannian product. Following this approach and motivated by the results of Fetcu and Rosenberg [17] and Batista [6], one can ask:

Question 1 *How does look like a Simons' type formula in Riemannian products of space forms $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$?*

Question 2 *As an application of Question 1, how we can characterize parallel mean curvature submanifolds of products $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$?*

Question 2' *One can state a similiar, but more general Question 2: how we can characterize parallel normalized mean curvature submanifolds of products $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$?*

Note that question 2' is a natural generalization of Fetcu and Rosenberg's results. Nevertheless, it is important to find examples of submanifolds that have parallel normalized mean curvature submanifolds where the mean curvature is not parallel. These examples are still unknown.

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