

## On Approximation by Post-Widder and Stancu Operators Preserving $x^2$

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ABSTRACT. In the papers [5]-[7] was examined approximation of functions by the modified Szász-Mraky operators and other positive linear operators preserving  $e_2(x) = x^2$ . In this paper we introduce the Post-Widder and Stancu operators preserving  $x^2$  in polynomial weighted spaces. We show that these operators have better approximation properties than classical Post-Widder and Stancu operators.

### 1. Introduction

#### 1.1. The Post-Widder operators

$$(1) \quad P_n(f; x) \equiv P_n(f(t); x) := \int_0^\infty f(t) p_n(x, t) dt, \quad x \in I, \quad n \in N,$$

$$(2) \quad p_n(x, t) := \frac{(n/x)^n t^{n-1}}{(n-1)!} \exp\left(-\frac{nt}{x}\right),$$

$I = (0, \infty)$ ,  $N = \{1, 2, \dots\}$ , were examined in many papers and monographs (e.g. [4]) for real-valued functions  $f$  bounded on  $I$ . It is known ([4], Chapter 9) that  $P_n$  are well defined also for functions  $e_k(x) = x^k$ ,  $k \in N_0 = N \cup \{0\}$ , and

$$(3) \quad P_n(e_0; x) = 1, \quad P_n(e_1; x) = x, \quad P_n(e_2; x) = \frac{n+1}{n} x^2$$

and generally

$$(4) \quad P_n(e_k; x) = \frac{n(n+1) \cdots (n+k-1)x^k}{n^k}, \quad k \in N,$$

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for  $x \in I$  and  $n \in N$ . Denoting by

$$(5) \quad \varphi_x(t) := t - x \quad \text{for } t \in I \text{ and a fixed } x \in I,$$

we have

$$(6) \quad P_n(\varphi_x^2(t); x) = \frac{x^2}{n} \quad \text{for } x \in I, n \in N.$$

From the results given in [4], Chapter 9, we can deduce that for every function  $f$  continuous and bounded on  $I$  there holds

$$(7) \quad |P_n(f; x) - f(x)| \leq M \omega\left(f; \frac{x}{\sqrt{n}}\right), \quad x \in I, n \in N,$$

where  $\omega(f; \cdot)$  is the modulus of continuity of  $f$  and  $M = \text{const.} > 0$  independent on  $x$  and  $n$ .

## 1.2. The Stancu operators

$$(8) \quad L_n(f; x) \equiv L_n(f(t); x) := \int_0^\infty f(t) s_n(x, t) dt, \quad x \in I, n \in N,$$

$$(9) \quad s_n(x, t) := \frac{1}{B(nx, n+1)} \frac{t^{nx-1}}{(1+t)^{nx+n+1}},$$

with the Euler beta function

$$B(a, b) := \int_0^1 t^{a-1} (1-t)^{b-1} dt \equiv \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}} dt, \quad a, b > 0,$$

were introduced in [10] for real-valued functions  $f$  bounded and locally integrable on  $I = (0, \infty)$ . The Stancu operators  $L_n$  are also well defined for functions  $e_k(x) = x^k$ ,  $k \in N_0$ , (see [10], [1], [2]) and

$$(10) \quad \begin{aligned} L_n(e_0; x) &= 1, & L_n(e_1; x) &= x, & \text{for } n \in N, \\ L_n(e_2; x) &= x^2 + \frac{x(x+1)}{n-1} & \text{for } n \geq 2, \end{aligned}$$

and generally

$$(11) \quad L_n(e_k; x) = \frac{nx(nx+1) \cdots (nx+k-1)}{n(n-1) \cdots (n-k+1)}, \quad x \in I, n \geq k \geq 2.$$

In [10] was proved that for every function  $f$  continuous and bounded on  $I$  there holds the following inequality

$$(12) \quad |L_n(f; x) - f(x)| \leq \left(1 + \sqrt{x(x+1)}\right) \omega\left(f; \frac{1}{\sqrt{n-1}}\right)$$

for  $x \in I$  and  $n \geq 2$ , where  $\omega(f; \cdot)$  is the modulus of continuity of  $f$ .

**1.3.** In papers [8] and [9] were examined approximation properties certain modified Post-Widder and Stancu operators for differentiable functions in polynomial weighted spaces. In [5] were investigated modified Szász-Mirakyan operators  $D_n^*$  preserving the function  $e_2(x) = x^2$  and was proved that these operators have better approximation properties than classical Szász-Mirakyan operators. The similar results were given for certain positive linear operators in the papers [6] and [7].

**1.4.** The purpose of this note is to investigate modified Post-Widder and Stancu operators  $P_n^*$  and  $L_n^*$  preserving  $e_2(x) = x^2$  in polynomial weighted spaces. These operators have better approximation properties than  $P_n$  and  $L_n$  given by (1) and (8). The definition and some properties of operators  $P_n^*$  and  $L_n^*$  will be given in Section 2. The main theorems will be given in Section 3.

**1.5.** First we give definition of polynomial weighted space  $C_r$ .

Similarly to [3] let  $r \in N_0$ ,

$$(13) \quad w_0(x) := 1, \quad w_r(x) := (1 + x^r)^{-1} \quad \text{if } r \geq 1, \quad x \in I,$$

and let  $C_r \equiv C_r(I)$  be the set of all real-valued functions  $f$  defined on  $I$ , for which  $w_r f$  is uniformly continuous and bounded on  $I$  and the norm is given by

$$(14) \quad \|f\|_r \equiv \|f(\cdot)\|_r := \sup_{x \in I} w_r(x) |f(x)|.$$

It is obvious that if  $q < r$ , then  $C_q \subset C_r$  and  $\|f\|_r \leq \|f\|_q$  for  $f \in C_q$ . For  $f \in C_r$ ,  $r \in N_0$ , we shall consider the modulus of continuity

$$(15) \quad \omega(f; C_r; t) := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|_r, \quad t \geq 0,$$

where  $\Delta_h f(x) = f(x + h) - f(x)$ .

In this paper we shall apply the following inequalities

$$(16) \quad (w_r(x))^2 \leq w_{2r}(x), \quad (w_r(x))^{-2} \leq 4(w_{2r}(x))^{-1},$$

for  $x \in I$  and  $r \in N_0$ , which immediately result from (13).

We shall denote by  $M_i(r)$ ,  $i \in N$ , suitable positive constants depending only on indicated parameter  $r$ .

## 2. The definition and elementary properties of $P_n^*$ and $L_n^*$

**2.1.** We introduce for  $f \in C_r$ ,  $r \in N_0$ , the following modified Post-Widder operators  $P_n^*$

$$(17) \quad P_n^*(f; x) := \int_0^\infty f(t) p_n(u_n(x), t) dt = P_n(f; u_n(x)), \quad x \in I, \quad n \in N,$$

where  $P_n(f)$  and  $p_n$  are given by (1) and (2) and

$$(18) \quad u_n(x) := \sqrt{\frac{n}{n+1}} x,$$

and modified Stancu operators

$$(19) \quad L_n^*(f; x) := \int_0^\infty f(t) s_n(v_n(x), t) dt = L_n(f; v_n(x))$$

for  $x \in I$  and  $n \geq r \geq 2$  or  $n \geq 2$  if  $r = 0, 1$ , where  $L_n(f)$  and  $s_n$  are given by (8) and (9) and

$$(20) \quad v_n(x) := \frac{-1 + \sqrt{1 + 4n(n-1)x^2}}{2n}.$$

**2.2.** The formulas (18) and (20) imply that

$$(21) \quad 0 < u_n(x) < x, \quad 0 \leq v_n(x) \leq x \quad \text{for } x \in I, \quad n \in N.$$

From (17)-(20) and (1)-(4) and (8)-(11) we immediately obtain the following

**Lemma 1.** *Let  $e_k(x) = x^k$  for  $k \in N_0$  and  $x \in I$ . Then for all  $x \in I$  and  $n \in N$  we have*

$$(22) \quad P_n^*(e_0; x) = 1, \quad P_n^*(e_1; x) = u_n(x), \quad P_n^*(e_2; x) = x^2$$

and

$$P_n^*(e_k; x) = \frac{n(n+1) \cdots (n+k-1) u_n^k(x)}{n^k} \quad \text{if } k \geq 3.$$

Moreover, for  $x \in I$  and  $n \geq 2$  we have

$$(23) \quad L_n^*(e_0; x) = 1, \quad L_n^*(e_1; x) = v_n(x), \quad L_n^*(e_2; x) = x^2$$

and generally

$$L_n^*(e_k; x) = \frac{nv_n(x)(nv_n(x)+1) \cdots (nv_n(x)+k-1)}{n(n-1) \cdots (n-k+1)} \quad \text{for } n \geq k \geq 2.$$

The formulas (22) and (23) show that  $P_n^*$  and  $L_n^*$  preserve the functions  $e_0$  and  $e_2$ .

**Lemma 2.** *For function  $\varphi_x$  given by (5) there hold the following analogies of (6):*

$$(24) \quad P_n^*(\varphi_x^2(t); x) = 2x(x - u_n(x)) \leq \frac{x^2}{n} \quad \text{for } x \in I, \quad n \in N,$$

and

$$(25) \quad L_n^*(\varphi_x^2(t); x) = 2x(x - v_n(x)) \leq \frac{x(x+1)}{n-1} \quad \text{for } x \in I, \quad n \geq 2.$$

*Proof.* We shall prove only (25) because the proof of (24) is analogous. By linearity of  $L_n^*$  and (5) and (23) we have

$$\begin{aligned} L_n^*(\varphi_x^2(t); x) &= L_n^*(e_2; x) - 2xL_n^*(e_1; x) + x^2L_n^*(e_0; x) \\ &= 2x(x - v_n(x)) \quad \text{for } x > 0, \quad n \geq 2. \end{aligned}$$

Next, by (20) we get

$$\begin{aligned} 0 < x - v_n(x) &= \frac{2nx + 1 - \sqrt{1 + 4n(n-1)x^2}}{2n} \\ &= \frac{2x(x+1)}{2nx + 1 + \sqrt{1 + 4n(n-1)x^2}} \leq \frac{2x(x+1)}{2nx + 1 + 2(n-1)x} \\ &\leq \frac{2x(x+1)}{4(n-1)x} = \frac{x+1}{2(n-1)} \quad \text{for } x > 0, \quad n \geq 2. \end{aligned}$$

This completes the proof of (25). □

**Lemma 3.** *Let  $r \in N_0$  and let  $w_r$  be the weighted function given by (13). Then for  $n \in N$  the following inequalities*

$$(26) \quad \|P_n^*(1/w_r)\|_r \leq 1, \quad \|L_n^*(1/w_r)\|_r \leq 1 \quad \text{if } r = 0, 1,$$

$$(27) \quad \|P_n^*(1/w_r)\|_r \leq 1 + (r-1)!, \quad \text{if } r \geq 2,$$

and

$$(28) \quad \|L_n^*(1/w_r)\|_r \leq 1 + 2^{2r-1}(1 + r^{r-1}) \quad \text{for } n \geq r \geq 2,$$

hold. Moreover, for every  $f \in C_r$  we have

$$(29) \quad \|P_n^*(f)\|_r \leq \|f\|_r \|P_n^*(1/w_r)\|_r, \quad n \in N,$$

$$(30) \quad \|L_n^*(f)\|_r \leq \|f\|_r \|L_n^*(1/w_r)\|_r, \quad n \geq r.$$

The formulas (17)-(20) and inequalities (29) and (30) show that  $P_n^*$ ,  $n \in N$ , and  $L_n^*$  with  $n \geq r$  are positive linear operators acting from the space  $C_r$  to  $C_r$ ,  $r \in N_0$ .

*Proof.* Similarly to Lemma 2 we shall consider only operators  $L_n^*$ . The inequality (26) is obvious by (13), (23), (21) and (14). If  $r \geq 2$ , then by linearity of  $L_n^*$  and

(13), Lemma 1 and (21) we get

$$\begin{aligned}
L_n^*(1/w_r; x) &= L_n^*(e_0; x) + L_n^*(e_r; x) \\
&\leq 1 + \frac{nx(nx+1)\cdots(nx+r-1)}{n(n-1)\cdots(n-r+1)} \\
&\leq 1 + \frac{n^{r-1}x(x+1/n)\cdots(x+(r-1)/n)}{(n-1)(n-2)\cdots(n-r+1)} \\
&\leq 1 + \frac{2^{r-1}\{(n-r+1)^{r-1} + r^{r-1}\}(x+1)^r}{(n-r+1)^{r-1}} \\
&\leq 1 + 2^{2r-1}(1+r^{r-1})(1+x^r)
\end{aligned}$$

for  $x \in I$  and  $n \geq r$ . This inequality and (14) imply (28).

The inequality (30) immediately follows from (19) and (14).  $\square$

Applying the Hölder inequality and Lemma 2, Lemma 3 and (16), we easily obtain the following

**Lemma 4.** *Let  $r \in N_0$  and let  $\varphi_x$  be given by (5). Then there exist  $M_i(r) = \text{const.} > 0$ ,  $i = 1, 2$ , such that for  $x \in I$  and  $n \in N$*

$$(31) \quad w_r(x)P_n^*\left(\frac{|\varphi_x(t)|}{w_r(t)}; x\right) \leq M_1(r) \sqrt{2x(x - u_n(x))}$$

and

$$(32) \quad w_r(x)L_n^*\left(\frac{|\varphi_x(t)|}{w_r(t)}; x\right) \leq M_2(r) \sqrt{2x(x - v_n(x))}, \quad \text{for } n \geq 2r.$$

### 3. Theorems

**3.1.** Denote by  $C_r^1 \equiv C_r^1(I)$ , with a fixed  $r \in N_0$ , the set of all functions  $f \in C_r$  which the first derivative belonging also to  $C_r$ .

**Theorem 1.** *Let  $r \in N_0$ . Then there exist  $M_i(r) = \text{const.} > 0$ ,  $i = 3, 4$ , such that for every  $f \in C_r^1$ ,  $x \in I$  and  $n \in N$  the following inequalities*

$$(33) \quad w_r(x)|P_n^*(f; x) - f(x)| \leq M_3(r) \|f'\|_r \sqrt{2x(x - u_n(x))}$$

and

$$(34) \quad w_r(x)|L_n^*(f; x) - f(x)| \leq M_4(r) \|f'\|_r \sqrt{2x(x - v_n(x))}, \quad n \geq 2r,$$

hold.

*Proof.* From (17), (18) and Lemma 1 we deduce that

$$|P_n^*(f(t); x) - f(x)| = |P_n^*(f(t) - f(x); x)| \leq P_n^*\left(\left|\int_x^t f'(y)dy\right|; x\right)$$

for every  $f \in C_r^1$ ,  $x \in I$  and  $n \in N$ . Next by (13) and (14) we have

$$\begin{aligned} \left| \int_x^t f'(y) dy \right| &\leq \|f'\|_r \left| \int_x^t \frac{dy}{w_r(y)} \right| \\ &\leq \|f'\|_r \left( \frac{1}{w_r(t)} + \frac{1}{w_r(x)} \right) |t - x|, \quad x, t \in I. \end{aligned}$$

Consequently, we get

$$w_r(x) |P_n^*(f(t); x) - f(x)| \leq \|f'\|_r \left\{ P_n^* \left( \frac{|\varphi_x(t)|}{w_r(t)}; x \right) + P_n^* \left( \frac{|\varphi_x(t)|}{w_0(t)}; x \right) \right\},$$

for  $x \in I$ ,  $n \in N$ , where  $\varphi_x$  is defined by (5). Now using (31), we obtain the desired estimation (33).

Similarly, applying (32), we obtain (34).  $\square$

**Theorem 2.** Let  $r \in N_0$ . Then there exist  $M_i(r) = \text{const.} > 0$ ,  $i = 5, 6$ , such that for every  $f \in C_r$ ,  $x \in I$  and  $n \in N$  we have

$$(35) \quad w_r(x) |P_n^*(f; x) - f(x)| \leq M_5(r) \omega(f; C_r; \sqrt{2x(x - u_n(x))})$$

and

$$(36) \quad w_r(x) |L_n^*(f; x) - f(x)| \leq M_6(r) \omega(f; C_r; \sqrt{2x(x - v_n(x))}), \quad n \geq 2r,$$

where  $\omega(f; C_r)$  is the modulus of continuity of  $f$  defined by (15).

*Proof.* Because the proofs of (35) and (36) are analogous, we shall prove only (35). We shall use the Stiecklov function  $f_h$  of  $f \in C_r$ , i.e.

$$(37) \quad f_h(x) := \frac{1}{h} \int_0^h f(x+t) dt, \quad x, h > 0.$$

From (37) and (15) it follows that

$$(38) \quad \|f_h - f\|_r \leq \omega(f; C_r; h),$$

$$(39) \quad \|f'_h\|_r \leq h^{-1} \omega(f; C_r; h),$$

for every  $f \in C_r$  and  $h > 0$ . These inequalities show that if  $f \in C_r$  with a fixed  $r \in N_0$ , then  $f_h \in C_r^1$  for every  $h > 0$ . Hence for  $f \in C_r$  and  $h > 0$  we can write

$$(40) \quad \begin{aligned} P_n^*(f(t); x) - f(x) &= P_n^*(f(t) - f_h(t); x) + P_n^*(f_h(t); x) - f_h(x) \\ &\quad + f_h(x) - f(x) \quad \text{for } x \in I, \quad n \in N. \end{aligned}$$

By (29), (26), (27) and (38) we see that there exists  $M_7(r) = \text{constant} > 0$  such that

$$(41) \quad \begin{aligned} w_r(x) |P_n^*(f(t) - f_h(t); x)| &\leq M_7(r) \|f - f_h\|_r \\ &\leq M_7(r) \omega(f; C_r; h). \end{aligned}$$

Applying Theorem 1 for  $f_h$  and (39), we get

$$(42) \quad w_r(x)|P_n^*(f_h(t); x) - f_h(x)| \leq M_3(r)\|f_h'\|_r\sqrt{2x(x - u_n(x))} \\ \leq M_3(r)h^{-1}\omega(f; C_r; h)\sqrt{2x(x - u_n(x))}.$$

Using (41), (42) and (38), we deduce from (40)

$$(43) \quad w_r(x)|P_n^*(f; x) - f(x)| \leq M_8(r)\omega(f; C_r; h) \times \left\{ 1 + h^{-1}\sqrt{2x(x - u_n(x))} \right\}$$

for  $x > 0$ ,  $h > 0$  and  $n \in N$ . Now, for given  $x$  and  $n$  setting  $h = \sqrt{2x(x - u_n(x))}$  to (43), we obtain desired inequality (35) and we complete the proof.  $\square$

From Theorem 2 and Lemma 2 results the following

**Corollary.** *For every  $f \in C_r$ ,  $r \in N_0$ , we have  $\lim_{n \rightarrow \infty} P_n^*(f; x) = f(x)$ ,  $x \in I$ , and this convergence is uniform on every interval  $[a, b]$ ,  $a > 0$ .*

*The above statement is also true for Stancu operators  $L_n^*$ .*

**3.2.** Considering the Stancu operators  $L_n$  in polynomial weighted spaces  $C_r$  and using methods of proofs of Theorem 1 and Theorem 2, we can obtain the following estimation

$$(44) \quad w_r(x)|L_n(f; x) - f(x)| \leq M_9(r)\omega\left(f; C_r; \sqrt{\frac{x(x+1)}{n-1}}\right),$$

for every  $f \in C_r$ ,  $r \in N_0$ ,  $x > 0$  and  $n \geq 2r + 2$ .

The inequalities (44), (36) and (12) show that the Stancu operators  $L_n^*$  have better approximation properties than  $L_n$  for functions  $f \in C_r$ ,  $r \in N_0$ , and  $n \geq 2r + 2$ . Moreover, by (20) and Lemma 2 we get for arguments of moduli of continuity of  $f$  given in (36) and (44)

$$\begin{aligned} & \sqrt{\frac{x(x+1)}{n-1}} - \sqrt{2x(x - v_n(x))} \\ = & \sqrt{\frac{x(x+1)}{n-1}} - \frac{\sqrt{4x^2(x+1)}}{\sqrt{2nx+1 + \sqrt{1+4n(n-1)x^2}}} \\ = & \sqrt{\frac{x(x+1)}{n-1}} \left( 1 - \frac{\sqrt{4(n-1)x}}{\sqrt{2nx+1 + \sqrt{1+4n(n-1)x^2}}} \right) \end{aligned}$$

$$\begin{aligned}
&= \sqrt{\frac{x(x+1)}{\sqrt{n-1}}} \frac{\sqrt{1+4n(n-1)x^2} - 2(n-1)x + 2x + 1}{\sqrt{2nx+1} + \sqrt{1+4n(n-1)x^2}} \\
&\quad \times \frac{1}{\left[ \sqrt{2nx+1} + \sqrt{1+4n(n-1)x^2} + \sqrt{4(n-1)x} \right]} \\
&> \sqrt{\frac{x(x+1)}{n-1}} \frac{2x+1}{\sqrt{4nx+2} \left[ \sqrt{4nx+2} + \sqrt{4(n-1)x} \right]} \\
&> \sqrt{\frac{x(x+1)}{n-1}} \frac{2x+1}{2(4nx+2)} > \sqrt{\frac{x(x+1)}{n-1}} \frac{1}{4n},
\end{aligned}$$

for all  $x > 0$  and  $n \geq 2r + 2$ .

Analogously, estimations (7), (35) and (24) show that  $P_n^*$ ,  $n \in N$ , have better approximation properties than  $P_n$  for functions  $f \in C_r$  (see [8]). Moreover, by (7), (35) and (18) we can obtain

$$\frac{x}{\sqrt{n}} - \sqrt{2x(x - u_n(x))} \geq \frac{x}{4(n+1)\sqrt{n}} \quad \text{for } x > 0 \text{ and } n \in N.$$

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