On Approximation of the Best Case Optimal Value in Interval Linear Programming

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Motivation

Interval data are used to model:

- real life uncertainties
- measurement errors
- sensitivity analysis

Notation

An interval matrix

$$\mathbf{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} \mid \underline{A} \leq A \leq \overline{A}\}.$$

The center and radius matrices

$$A_c := \frac{1}{2}(\overline{A} + \underline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

Interval linear equations

Interval linear equations

Let $\mathbf{A} \in \mathbb{IR}^{m \times n}$ and $\mathbf{b} \in \mathbb{IR}^{m}$. The family of systems

$$Ax = b, \quad A \in \mathbf{A}, \ b \in \mathbf{b}.$$

is called interval linear equations and abbreviated as $\mathbf{A}x = \mathbf{b}$.

Solution set

The solution set is defined

$$\{x \in \mathbb{R}^n : \exists A \in \mathbf{A} \exists b \in \mathbf{b} : Ax = b\}.$$

Enclosure

$\mathbf{x} \in \mathbb{IR}^n$ containing the solution set.

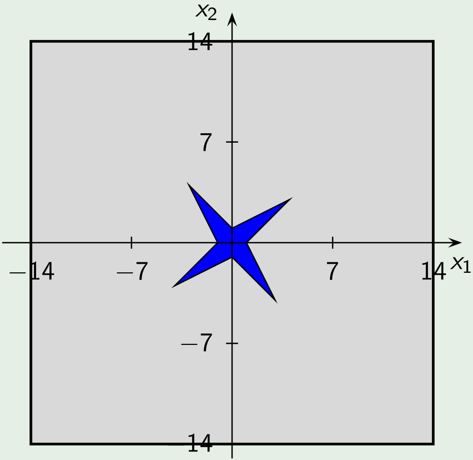
Methods

Interval Gaussian elimination, interval Gauss–Seidel, Krawczyk method, Hansen–Bliek–Rohn method, ...

Interval linear equations

Example (Barth & Nuding, 1974))

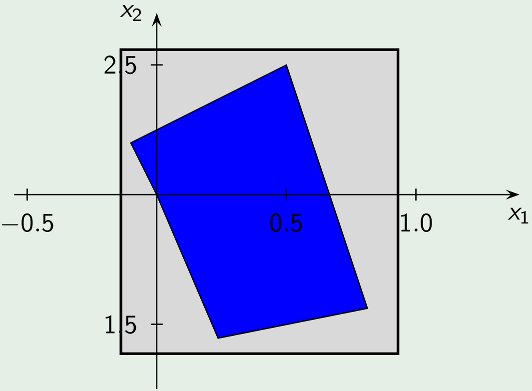
$$\begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}$$



Interval linear equations

Example (typical case)

$$\begin{pmatrix} [6,7] & [2,3] \\ [1,2] & -[4,5] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} [6,8] \\ -[7,9] \end{pmatrix}$$



Interval linear programming

Interval linear programming

Consider a family of linear programming problems

$$\min c^T x \;\text{ subject to }\; Ax \le b, \qquad (\star)$$

where $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Remark

- There is loss of generality assuming the form $(\star)$.
- For instance, transformation of

$$\min c^T x \;\text{ subject to }\; Ax = b,\; x \ge 0$$

to

$$\min c^T x \;\text{ subject to }\; Ax \le b,\; -Ax \le -b,\; x \ge 0$$

causes dependencies.

Problem statement

State of the art

- optimal value range (Chinneck & Ramadan, 2000, Hladík, 2009, Jansson, 2004, Mráz, 1998, Rohn, 2006, etc.)
- duality (Gabrel & Murat, 2010, Rohn, 1980, Serafini, 2005)
- basis stability (Beeck, 1978, Koníčková, 2001, Hladík, 2012, Rohn, 1993)
- optimal solution set (Beeck, 1978, Jansson, 1988, Machost, 1970)

The best and worst case optimal values

$$\underline{f} := \min f(A, b, c) \ \text{ subject to } \ A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c}$$
$$\overline{f} := \max f(A, b, c) \ \text{ subject to } \ A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c}.$$

Algorithm (The worst case optimal value)

1. Compute
$$\overline{\varphi} = \sup\ \underline{b}^T y \ \text{ subject to }\ \overline{A}^T y \le \overline{c},\ -\underline{A}^T y \le -\underline{c},\ y \le 0.$$
2. If $\overline{\varphi} = \infty$, then set $\overline{f} := \infty$ and stop.
3. If the system
$$\overline{A}x^1 - \underline{A}x^2 \le \underline{b},\ x^1 \ge 0,\ x^2 \ge 0$$
is feasible (i.e., each realization of the original system is feasible), then set $\overline{f} := \overline{\varphi}$; otherwise set $\overline{f} := \infty$.

Corollary

We compute $\overline{f}$ by solving two linear programs.

The best case optimal value

Theorem (Gabrel & Murat, 2010)

Computing the best case optimal value

$$\underline{f} = \min f(A, b, c) \quad \textit{subject to} \quad A \in \mathbf{A},\ b \in \mathbf{b},\ c \in \mathbf{c}$$

is strongly NP-hard even in the class of problems with interval objective function coefficients and real constraint coefficients.

Proposition

We can put

$$\mathbf{b} := \overline{b}.$$

Proof.

$Ax \leq b$ implies $Ax \leq \overline{b}$ for any $A \in \mathbf{A}$ and $b \in \mathbf{b}$. □

The best case optimal value

Proposition (Computation of $\underline{f}$)

We have

$$\underline{f} = \min_{s \in \{\pm 1\}^n} f_s,$$

where

$$\begin{aligned} f_s = {} & \min (c_c - \operatorname{diag}(s)\, c_\Delta)^T x \\ & \textit{subject to } \ (A_c - A_\Delta \operatorname{diag}(s))x \leq b, \ \operatorname{diag}(s)\, x \geq 0. \end{aligned}$$

Remarks

- It requires solving 2^n linear programs.
- If variables are a priori non-negative, then just one LP.
- It suffices to inspect orthants with feasible solutions only.

Upper bound on $\underline{f}$

Definition (Feasible set)

$$\mathcal{F} := \{x : \exists A \in \mathbf{A} : Ax \leq b\}$$

Algorithm (Upper bound on $\underline{f}$)

1. Start with the orthant corresponding to $f(A_c, b, c_c)$.
2. Then check the neighboring connected orthants.

Proposition

The algorithm computes $\underline{f}$ provided $\mathcal{F}$ is connected.

Example

The feasible set to

$$[-1, 1]x + y \leq -1, \;\; y \leq 0, \;\; -y \leq 0$$

consists of two disjoint sets $(-\infty, -1] \times \{0\}$ and $[1, \infty) \times \{0\}$.

Connectivity of $\mathcal{F}$

Proposition

If $b \geq 0$, then $\mathcal{F}$ is connected.

Proof.

$0 \in \mathcal{F}$, so $\mathcal{F}$ is connected via the origin. $\square$

Proposition

If the linear system of inequalities

$$\overline{A}u - \underline{A}v \leq b, \;\; u, v \geq 0 \qquad (\star)$$

is feasible, then $\mathcal{F}$ is connected.

Proof.

If u, v solves $(\star)$, then $x^* := u - v$ solves $Ax \leq b$ for every $A \in \mathbf{A}$. $\square$

Another upper bound on $\underline{f}$

Algorithm (Another upper bound on $\underline{f}$)

1. Put $A := A_c$, $c := c_c$.
2. Let x^* be a solution to $f^* := f(A, b, c)$
3. Put $s := \operatorname{sgn}(x^*)$.
4. Let x^s be a solution to

$$f^s \equiv \min (c_c - \operatorname{diag}(s)\, c_\Delta)^T x \quad \text{subject to} \quad (A_c - A_\Delta \operatorname{diag}(s))x \leq b.$$

5. Update $f^* := \min(f^*, f^s)$.
6. Put $s := \operatorname{sgn}(x^s)$.
7. Go to step 3. (repeat while f^* improves)

Lower bound on $\underline{f}$

Algorithm (Lower bound on $\underline{f}$)

1. Let B be an optimal basis corresponding to $f(A_c, b, c_c)$.
2. Let $\mathbf{y}$ be an enclosure to the interval linear system

$$A_B^T y = c, \quad c \in \mathbf{c},\ A_B \in \mathbf{A}_B.$$

3. Provided $\overline{y} \leq 0$, we have a lower bound

$$b_B^T y^* \leq \underline{f},$$

where $y_i^* = \underline{y}_i$ if $b_{B_i} \geq 0$, and $y_i^* = \overline{y}_i$ otherwise.

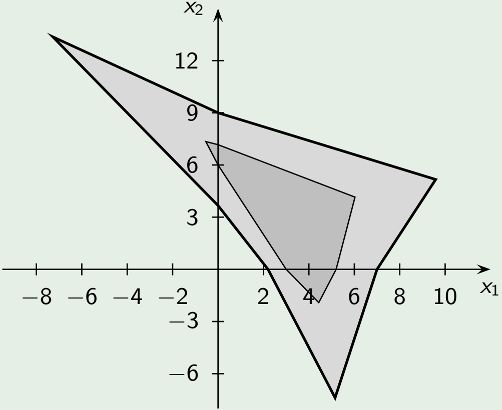
Proof.

$\overline{y} \leq 0$ implies that B is an optimal basis of the dual problem, so it gives a lower bound on the primal objective. □

Example

Example

$$\min\ 1x_1 + 2x_2 \ \text{ subject to } \begin{pmatrix} -[4,5] & -[2,3] \\ [4,5] & -[1,2] \\ [2,3] & [5,6] \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} -[11,12] \\ [26,28] \\ [43,45] \end{pmatrix}$$



Example

Results:

- The exact best case optimal value

$$\underline{f} = -9.6154.$$

- Optimal solution for the selection $A := A_c$, $c := c_c$:

$$x^* = (4.8056,\ -4.2500)^T, \quad f^* = -3.6944.$$

Optimal solution in the orthant $s = (1, -1)$:

$$x^s = (5.1538,\ -7.3846)^T, \quad f^s = -9.6154.$$

- Enclosure to the dual system $A_B^T y = c$, $A_B \in \mathbf{A}_B$, $c \in \mathbf{c}$:

$$\mathbf{y} = ([-0.8340, -0.3326],\ [-0.6536, -0.0686])^T$$

which yields a lower bound -14.6402.

Conclusion and future work

Conclusion

- Not necessarily exponential algorithm for $\underline{f}$.
- Lower and upper bounds for $\underline{f}$.
- By duality in LP, we have analogous results for the worst case of

$$\min c^T x \ \text{ subject to } \ Ax = b,\ x \geq 0,$$

where $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$.

Future work

- Improve the lower bound on $\underline{f}$.
- Extension to more complex forms
(mixed equations and inequalities, ...)
- Handling dependencies.

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