Applied Mathematical Sciences, Vol. 10, 2016, no. 15, 735 - 745
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2016.613

On Approximations by Polynomial and Nonpolynomial Integro-Differential Splines

I. G. Burova

St. Petersburg State University
Mathematics and Mechanics Faculty
St. Petersburg, Russia

A. G. Doronina

St. Petersburg State University
Mathematics and Mechanics Faculty
St. Petersburg, Russia

Abstract

The present article is one of a number of articles in which the authors study the properties of integro-differential splines. The paper deals with the construction of integro-differential polynomial and nonpolynomial splines of the fifth order. The order of approximation with integro-differential polynomial and nonpolynomial splines of the fifth order are given. We use the tensor product of polynomial and non-polynomial splines constructed in this paper for the approximation of functions of two variables. The results of these numerical experiments are given.

Mathematics Subject Classification: 65T40, 41A15, 41A05

Keywords: Polynomial splines, Exponential splines, Integro-Differential Splines, Interpolation

1 Introduction

There are certain problems in the fields of mathematics, mechanics, physics and engineering that can't be solved without using of splines. Nowerdays many splines are known, including cubic, bicubic and biquadratic B-splines, trigonometric, orthogonal splines. Those splines are applied to the construction of curves and surfaces, for the designing of ship hulls, for the transformation of a sound signal's frequency [1, 2, 6–8, 10–19].

Kireev V.I. became the first to use values of one-variable integrals of a function over subintervals for the construction of approximations. As is well known, the one-dimensional case can be extended to multiple dimensions through the use of the tensor product spline constructs [3, 4, 9].

In the present paper we discuss the construction of the polynomial and nonpolynomial splines which use three integrals over subintervals in addition to the values of the function in the nodes. As in previous papers, we construct the approximation separately for each subinterval. As usual, local spline approximation uses values of the approximated function and, sometimes, values of its derivatives.

2 Constructing polynomial integro-differential splines

Suppose that n is natural number, while a, b are real numbers. Let $\{x_j\}$ be points of interpolation, $\{x_j\} \in [a,b]$, $x_j = a+jh$, $j = 0, 1, \ldots, n$, $h = (b-a)/n$.

Suppose that the function $f(x)$ such that $f \in C^5([a,b])$.

For constructing the approximation of the function $f(x)$ in the interval $[x_j, x_{j+1}]$ we need to know the values of the function $f(x_j)$, $f(x_{j+1})$, and $\int_{x_{j-k}}^{x_j} f(x)dx$, $k = 1, 2, 3$.

We denote $\widetilde{f}(x)$ the approximation of the function $f(x)$:

$$\widetilde{f}(x) = f(x_j)\omega_j(x) + f(x_{j+1})\omega_{j+1}(x) + \int\limits_{x_{j-1}}^{x_j} f(x)dx\, \omega_j^{<-1>}(x) +$$

$$+ \int\limits_{x_{j-2}}^{x_j} f(x)dx\, \omega_j^{<-2>}(x) + \int\limits_{x_{j-3}}^{x_j} f(x)dx\, \omega_j^{<-3>}(x),\ x \in [x_j, x_{j+1}]. \quad (1)$$

Here $\omega_j(x)$, $\omega_{j+1}(x)$, $\omega_j^{<-k>}(x)$, $k = 1, 2, 3$ are the basis functions which are obtained from the following system of equations:

$$\widetilde{f}(x) \equiv f(x), \quad f = 1, x, x^2, x^3, x^4. \quad (2)$$

Suppose that $supp\, \omega_j = [x_{j-1}, x_{j+1}]$, $supp\, \omega_j^{<-k>}(x) = [x_j, x_{j+1}]$, $k = 1, 2, 3$. The system of equations can be written in the form:

$\omega_j(x) + \omega_{j+1}(x) + (x_j - x_{j-1})\omega_j^{<-1>}(x) + +(x_j - x_{j-2})\omega_j^{<-2>}(x) + (x_j - x_{j-3})\omega_j^{<-3>}(x) = 1,$

$x_j\omega_j(x) + x_{j+1}\omega_{j+1}(x) + \frac{1}{2}(x_j^2 - x_{j-1}^2)\omega_j^{<-1>}(x) + \frac{1}{2}(x_j^2 - x_{j-2}^2)\omega_j^{<-2>}(x) + (x_j^2 - x_{j-3}^2)\omega_j^{<-3>}(x) = x,$

$x_j^2\omega_j(x) + x_{j+1}^2\omega_{j+1}(x) + \frac{1}{3}(x_j^3 - x_{j-1}^3)\omega_j^{<-1>}(x) + \frac{1}{3}(x_j^3 - x_{j-2}^3)\omega_j^{<-2>}(x) + (x_j^3 - x_{j-3}^3)\omega_j^{<-3>}(x) = x^2,$

$x_j^3\omega_j(x) + x_{j+1}^3\omega_{j+1}(x) + \frac{1}{4}(x_j^4 - x_{j-1}^4)\omega_j^{<-1>}(x) + \frac{1}{4}(x_j^4 - x_{j-2}^4)\omega_j^{<-2>}(x) + (x_j^4 - x_{j-3}^4)\omega_j^{<-3>}(x) = x^3,$

$x_j^4\omega_j(x) + x_{j+1}^4\omega_{j+1}(x) + \frac{1}{5}(x_j^5 - x_{j-1}^5)\omega_j^{<-1>}(x) + \frac{1}{5}(x_j^5 - x_{j-2}^5)\omega_j^{<-2>}(x) + (x_j^5 - x_{j-3}^5)\omega_j^{<-3>}(x) = x^4.$

The value of the determinant of the system is the following: $det = -222h^{13}/5$.

If we put $x = x_j + th, t \in [0,1]$, than we can get the basis functions as follows:

$$\omega_j(x_j + th) = -(t-1)(125t^3 + 557t^2 + +736t + 222)/222, \tag{3}$$

$$\omega_{j+1}(x_j + th) = t(5t^3 + 12 + 33t + 24t^2)/74, \tag{4}$$

$$\omega_j^{<-1>}(x_j + th) = t(t-1)(155t^2 + 603t + 516)/(148\,h), \tag{5}$$

$$\omega_j^{<-2>}(x_j + th) = t(t-1)(55t^2 + 171t + 90)/(148\,h), \tag{6}$$

$$\omega_j^{<-3>}(x_j + th) = t(t-1)(85t^2 + 197t + 92)/(1332\,h). \tag{7}$$

Figures 1–3 show the graphics of the basic functions $\omega_j(x)$, $\omega_{j+1}(x)$, $\omega_j^{<-k>}(x)$, $k = 1, 2, 3$. Figure 3 (right) shows the error of approximation of the Runge function $1/(1 + 25x^2)$ with the polynomial splines, $h = 0.1$, $x \in [-1, 1]$.

Figure 1: Plots of the basic functions: $\omega_j(x)$ (left), $\omega_{j+1}(x)$ (right)

Let us take $\tilde{F}(x)$, $x \in [a,b]$, such that: $\tilde{F}(x) = \tilde{f}(x)$ if $x \in [x_j, x_{j+1}]$, $j = 3, \ldots, n-1$. Now we can obtain the order of approximation of $\widetilde{f}(x) - f(x)$. Suppose $\|f\|_{[a,b]} = \max\limits_{[a,b]} |f(x)|$.

Theorem 2.1 *Let function $f(x)$ be such that $f \in C^5([a,b])$. For approximation $f(x)$, $x \in [x_j, x_{j+1}]$ by (1), (3)-(7), we have:*

$$|\widetilde{f}(x) - f(x)|_{[x_j,x_{j+1}]} \leq Kh^5\|f^V\|_{[x_{j-3},x_{j+1}]}, \ \text{where } K = 0.094, \tag{8}$$

Figure 2: Plots of the basic functions: $\omega_j^{<-1>}(x)$ (left), $\omega_j^{<-2>}(x)$ (right)

Figure 3: Plot of the basic function $\omega_j^{<-3>}(x)$ (left), and the error of approximation of the Runge function with the polynomial splines, $h = 0.1$, $x \in [-1, 1]$ (right)

$$\textit{if } n \geq 4 \textit{ then } |\widetilde{F}(x) - f(x)|_{[x_3,x_n]} \leq Kh^5 \|f^V\|_{[x_0,x_n]}, \textit{ where } K = 0.094. \quad (9)$$

Proof. From (3)-(7) we obtain:

$$|\omega_j(x)| \leq 1.691, \ |\omega_{j+1}(x)| \leq 1, \ |\omega_j^{<-1>}(x)| \leq \frac{1.514}{h}, \quad (10)$$

$$|\omega_j^{<-2>}(x)| \leq \frac{0.346}{h}, \ |\omega_j^{<-3>}(x)| \leq \frac{0.044}{h}. \quad (11)$$

We obtain (8) using the Taylor formula in the vicinity of point x_j in the interval $[x_j, x_{j+1}]$, and (10), (11). The inequality (9) follows from (8).

Table 1 shows the errors $R = \max_{x\in[a,b]} |\tilde{F} - f|$, when $[a, b] = [-1, 1]$, $h = 0.1$, $h = 0.01$. Calculations were done in Maple, Digits=25.

Table 2 shows the absolute values of the theoretical errors $R = \max_{x\in[a,b]} |\tilde{F}-f|$, obtained by using Theorem 2.1, when $[a, b] = [-1, 1]$, $h = 0.1, 0.01$.

Table 1.

N	f	$R, h = 0.1$	$R, h = 0.01$
1	$\sin(x)$	$0.180 \cdot 10^{-8}$	$0.190 \cdot 10^{-13}$
2	$\cos(x)$	$0.156 \cdot 10^{-8}$	$0.163 \cdot 10^{-13}$
3	$\sin(2x)$	$0.572 \cdot 10^{-5}$	$0.577 \cdot 10^{-12}$
4	$\cos(2x)$	$0.573 \cdot 10^{-5}$	$0.578 \cdot 10^{-12}$
5	$\sin(3x)$	$0.433 \cdot 10^{-4}$	$0.438 \cdot 10^{-11}$
6	$\cos(3x)$	$0.432 \cdot 10^{-4}$	$0.438 \cdot 10^{-11}$
7	e^x	$0.420 \cdot 10^{-8}$	$0.562 \cdot 10^{-13}$
8	e^{2x}	$0.313 \cdot 10^{-4}$	$0.414 \cdot 10^{-11}$
9	e^{3x}	$0.558 \cdot 10^{-3}$	$0.840 \cdot 10^{-10}$
10	$\frac{1}{1+25x^2}$	$0.225 \cdot 10^{-1}$	$0.559 \cdot 10^{-8}$

Table 2.

N	f	$R, h = 0.1$	$R, h = 0.01$
1	$\sin(x)$	$0.507 \cdot 10^{-6}$	$0.507 \cdot 10^{-11}$
2	$\cos(x)$	$0.789 \cdot 10^{-6}$	$0.789 \cdot 10^{-11}$
3	$\sin(2x)$	$0.125 \cdot 10^{-4}$	$0.125 \cdot 10^{-9}$
4	$\cos(2x)$	$0.273 \cdot 10^{-4}$	$0.300 \cdot 10^{-9}$
5	$\sin(3x)$	$0.226 \cdot 10^{-3}$	$0.226 \cdot 10^{-8}$
6	$\cos(3x)$	$0.228 \cdot 10^{-3}$	$0.228 \cdot 10^{-8}$
7	e^x	$0.255 \cdot 10^{-5}$	$0.255 \cdot 10^{-11}$
8	e^{2x}	$0.222 \cdot 10^{-3}$	$0.222 \cdot 10^{-8}$
9	e^{3x}	$0.458 \cdot 10^{-2}$	$0.458 \cdot 10^{-7}$
10	$\frac{1}{1+25x^2}$	0.294	$294 \cdot 10^{-5}$

3 Non-polynomial integro-differential spline constructing

As in the previous section suppose that we know the values of the function $f(x_j)$, $f(x_{j+1})$ and $\int_{x_{j-k}}^{x_j} f(x)dx$, $k = 1, 2, 3$.

We denote $\tilde{f}(x)$ the approximation of the function $f(x)$:

$$\widetilde{f}(x) = f(x_j)\,\omega_j(x) + f(x_{j+1})\,\omega_{j+1}(x) + \int_{x_{j-1}}^{x_j} f(t)dt\,\omega_j^{<-1>}(x)+$$

$$\int_{x_{j-2}}^{x_j} f(t)dt\,\omega_j^{<-2>}(x) + \int_{x_{j-3}}^{x_j} f(t)dt\,\omega_j^{<-3>}(x),\ x \in [x_j, x_{j+1}]. \qquad (12)$$

where $\omega_k(x)$, $k = j, j+1$, $\omega_j^{<-s>}(x)$, $s = 1, 2, 3$, we obtain from

$$\widetilde{f}(x) = f(x) \text{ for } f(x) = \varphi_i(x), i = 0, 1, 2, 3, 4. \qquad (13)$$

Let us examine the approximation for the following functions $\varphi_i(x)$:
1) $\varphi_i(x) = 1, x, e^x, e^{-x}, e^{2x}$,
2) $\varphi_i(x) = 1, \sin(x), \cos(x), \sin(2x), \cos(2x)$,
3) $\varphi_i(x) = 1, x, x^2, x^3, e^x$.

Theorem 3.1 *Let function $f(x)$ be such that $f \in C^5([a,b])$. For approximation $f(x)$, $x \in [x_j, x_{j+1}]$ by (12), (13) we have the error estimations:*

$$1)|\widetilde{f}(x) - f(x)|_{[x_j,x_{j+1}]} \leq K_1 h^5 \|f^{(5)} - 2f^{(4)} - f^{(3)} + 2f^{(2)}\|_{[x_{j-3},x_{j+1}]},\ K_1 > 0, \tag{14}$$

where $\varphi_i(x) = 1, x, e^x, e^{-x}, e^{2x}$,

$$2)|\widetilde{f}(x) - f(x)|_{[x_j,x_{j+1}]} \leq K_2 h^5 \|f^{(5)} + 5f^{(3)} + 4f'\|_{[x_{j-3},x_{j+1}]},\ K_2 > 0, \tag{15}$$

where $\varphi_i(x) = 1, \sin(x), \cos(x), \sin(2x), \cos(2x)$,

$$3)|\widetilde{f}(x) - f(x)|_{[x_j,x_{j+1}]} \leq K_3 h^5 e^{4x_{j+1}} \|f^{(5)} - 4f^{(4)}\|_{[x_{j-3},x_{j+1}]},\ K_3 > 0, \tag{16}$$

where $\varphi_i(x) = 1, x, x^2, x^3, e^x$.

Proof. We consider (16) more detail. The proof for the others two are similar. The function $f(x)$ in the third case (as was shown by the author at the conference in Gdansk, May, 2014 [5]) in $[x_j, x_{j+1}]$ can be represented in the form:

$$f(x) = \int_{x_j}^{x} (32(x^3 - t^3) - 24(x^2 + t^2) + 12(x - t) + 96(xt^2 - tx^2) +$$

$$48tx - 3)(4f^{(4)} - f^{(5)})dt + c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{4x}.$$

where c_i, $i = 1, 2, 3, 4, 5$ are arbitrary constants. Using the inequalities

$$|\omega_j(x)| \leq 3.135, |\omega_{j+1}(x)| \leq 1, |\omega_j^{<-1>}(x)| \leq 4.544, \tag{17}$$

$$|\omega_j^{<-2>}(x)| \leq 1.441, |\omega_j^{<-3>}(x)| \leq 0.233 \tag{18}$$

we obtain:

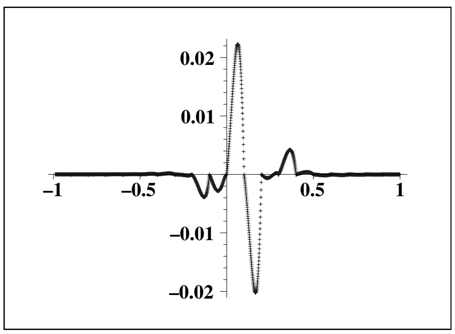
$$|\widetilde{f}(x) - f(x)| \leq K_3 h^5 e^{4x_{j+1}} \|f^{(5)} - 4f^{(4)}\|_{[x_{j-3},x_{j+1}]}.$$

Table 3 shows the actual errors of the approximation of $f(x)$ by the non-polynomial splines in $[-1, 1]$, $h = 0.1$. The approximations were constructed using the functions $\varphi_i(x)$. Here we use the next notations for the errors of approximations:
R^I : $1, x, e^x, e^{-x}, e^{2x}$,
R^{II} : $1, \sin(x), \cos(x), \sin(2x), \cos(2x)$,
R^{III} : $1, x, x^2, x^3, x^4$.

Table 3.

$f(x)$	R^I	R^{II}	R^{III}
x^3	$0.352 \cdot 10^{-5}$	$0.784 \cdot 10^{-5}$	0.
e^{2x}	0.	$0.787 \cdot 10^{-4}$	$0.314 \cdot 10^{-4}$
e^{3x}	$0.172 \cdot 10^{-3}$	$0.898 \cdot 10^{-3}$	$0.558 \cdot 10^{-3}$
$\sin(3x)$	$0.604 \cdot 10^{-4}$	$0.214 \cdot 10^{-4}$	$0.433 \cdot 10^{-4}$
$\sin(5x)$	$0.638 \cdot 10^{-3}$	$0.440 \cdot 10^{-3}$	$0.545 \cdot 10^{-3}$
$\frac{1}{1+25x^2}$	$0.227 \cdot 10^{-1}$	$0.223 \cdot 10^{-1}$	$0.225 \cdot 10^{-1}$

Figure 4 show the error of approximation of the Runge function: with the trigonometrical splines R^{II}, and with the exponential splines R^{III}, where $h = 0.1$, $[a, b] = [-1, 1]$.



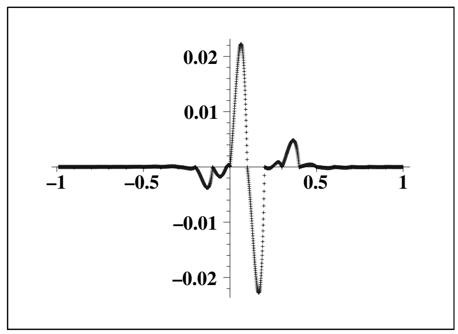


Figure 4: Plots of the error of approximation of the Runge function: with the trigonometrical spline (left), with the exponential spline where $\varphi_i(x) = 1, x, x^2, x^3, e^x$ (right), here $h = 0.1, x \in [-1, 1]$

4 The approximation of functions of two variables

Suppose that n, m are natural numbers, while a, b, c, d are real numbers. Let us build a grid of interpolation nodes: $\{x_j\} \in [a, b]$, $x_j = a+jh$, $j = 0, 1, \ldots, n$, $\{y_k\} \in [c, d]$, $y_k = c + kh_1$, $k = 0, 1, \ldots, m$, $h_x = (b - a)/n$, $h_y = (d - c)/m$.

Consider a rectangular domain Ω where

$$\Omega = \{(x, y)|\ a \leq x \leq b, c \leq y \leq d\}.$$

We introduce a mesh of lines on Ω which divides the domain Ω into the rectangles $\Omega_{j,k}$,

$$\Omega_{j,k} = \{(x, y)|\ x \in [x_j, x_{j+1}], y \in [y_k, y_{k+1}]\}.$$

If $(x,y)\in\Omega_{j,k}$ then we put

$$\tilde{u}(x,y)=\sum_{s=1}^{3}\sum_{i=1}^{3}\int_{y_{k-s}}^{y_k}\int_{x_{j-i}}^{x_j} u(x,y)dxdy\ \omega_k^{<-s>}(y)\ \omega_j^{<-i>}(x)+$$

$$+\sum_{i=1}^{3}\sum_{s=0}^{1}\int_{x_{j-i}}^{x_j} u(x,y_{k+s})dx\ \omega_j^{<-i>}(x)\ \omega_{k+s}(y)+$$

$$+\sum_{p=1}^{3}\sum_{i=0}^{1}\int_{y_{k-p}}^{y_k} u(x_{j+i},y)dy\ \omega_{j+i}(x)\ \omega_k^{<-p>}(y)+$$

$$+\sum_{i=0}^{1}\sum_{p=0}^{1} u(x_{j+i},y_{k+p})\ \omega_{j+i}(x)\ \omega_{k+p}(y). \tag{19}$$

Figure 5 shows the plot of the approximation of function $u_1(x,y)=1/((1+9x^2)(1+9y^2))$ by (19) with the polynomial splines and the error of the approximation, when $h_x=h_y=0.2,\Omega=[-1,1]\times[-1,1]$. Figure 6 shows the plot of the approximation of function $(x-y)^2(x+y)^2$ by (19) with the polynomial splines and the error of the approximation, when $h_x=h_y=0.2,\Omega=[-1,1]\times[-1,1]$, Digits=15.

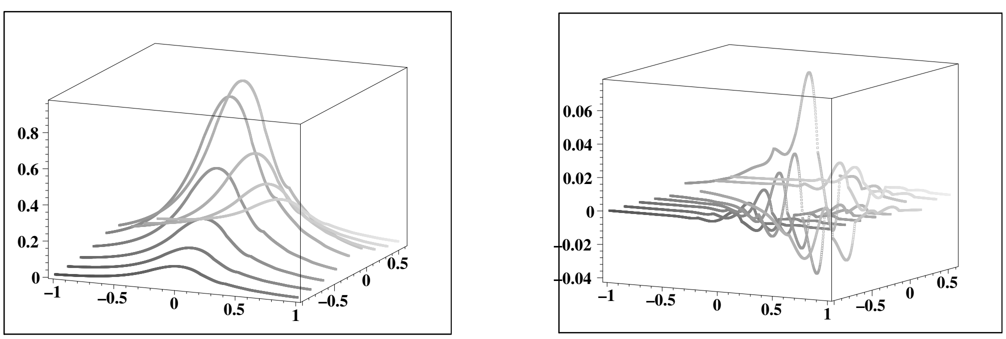


Figure 5: Plots of the approximation of the function $u_1(x,y)$ with the polynomial splines (19) (left), the error of the approximation (right)

5 Conclusion

Here we constructed the approximation using the values of integrals of the function over the subintervals immediately to the left of this subinterval. Further, we will investigate in detail the construction of the approximation of a function in subinterval using the values of the function in the ends of the

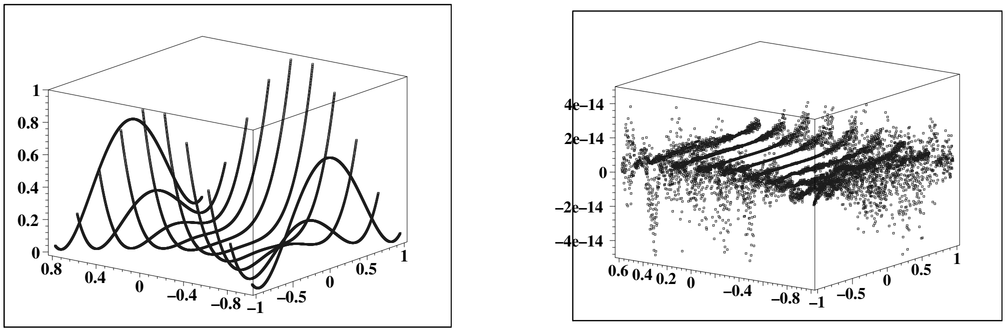


Figure 6: Plots of the approximation of the function $(x-y)^2(x+y)^2$ with the polynomial splines (left), the error of the approximation (right)

subinterval and the values of the integrals of function over the subintervals immediately to the left of this subinterval. If the values of integral of the function are unknown, we will use quadrature formulae with the fifth order of approximation in polynomial and non-polynomial cases.

References

[1] M. Abbas, A. A. Majid, M. N. H. Awang, J. Md. Ali, Shape-preserving rational bi-cubic spline for monotone surface data, *WSEAS Transactions on Mathematics,* **11** (2012), no. 7, 644 - 657.

[2] S. A. Abdul Karim, M. A. Mohd Rosli, M. I. Mohd Mustafa, Cubic spline interpolation for petroleum engineering data, *Applied Mathematical Sciences,* **8** (2014), no. 102, 5083 - 5098.
http://dx.doi.org/10.12988/ams.2014.44284

[3] C. de Boor, Efficient computer manipulation of tensor products, *ACM Trans. Math. Software,* **5** (1979), no. 2, 173 - 182.
http://dx.doi.org/10.1145/355826.355831

[4] C. de Boor, *A Practical Guide to Splines,* Springer, New York, USA, 1978.

[5] Irina Burova, On Integro-Differential Splines Construction, *Advances in Applied and Pure Mathematics, Proceedings of the 7th International Conference on Finite Differences, Finite Elements, Finite Volumes, Boundary Elements (F-and-B'14)*, Gdansk, Poland, May 15-17, (2014), 57 - 61.

[6] Fengmin Chen, Patricia J. Y. Wong, On periodic discrete spline interpolation: Quintic and biquintic cases, *Journal of Computational and Applied Mathematics*, **255** (2014), 282 - 296.
http://dx.doi.org/10.1016/j.cam.2013.05.005

[7] C. K. Chui, *Multivariate Splines,* Society for Industrial and Applied Mathematics, Pensylvania, USA, 1988.
http://dx.doi.org/10.1137/1.9781611970173

[8] David F. Rogers, Steven G. Satterfield, B-spline surfaces for ship hull design, *SIGGRAPH'80, Proceedings of the 7th annual conference on Computer graphics and interactive techniques*, **14** (1980), no. 3, 211 - 217.
http://dx.doi.org/10.1145/800250.807494

[9] Eric Grosse, Tensor spline approximation, *Linear Algebra and its Applications*, **34** (1980), 29 - 41. http://dx.doi.org/10.1016/0024-3795(80)90157-3

[10] T. Kuragano, Quintic B-spline curve generation using given points and gradients and modification based on specified radius of curvature, *WSEAS Transactions on Mathematics,* **9** (2010), no. 2, 79 - 89.

[11] A. Lamnii, F. Oumellal, A method for local interpolation with tension trigonometric spline curves and surfaces, *Applied Mathematical Sciences,* **9** (2015), no. 61, 3019 - 3035. http://dx.doi.org/10.12988/ams.2015.52154

[12] H. T. Nguyen, H. D. Nguyen, J. Y. Uan, D. A. Wang, A nonrational B-spline profiled horn with high displacement amplification for ultrasonic welding, *Ultrasonics,* **54** (2014), no. 8, 2063 - 2071.
http://dx.doi.org/10.1016/j.ultras.2014.07.003

[13] S. Safak, On the trivariate polynomial interpolation, *WSEAS Transactions on Mathematics,* **11** (2012), no. 8, 722 - 730.

[14] S. Saini, H. K. Mishra, A new quartic B-spline method for third-order self-adjoint singularly perturbed boundary value problems, *Applied Mathematical Sciences,* **9** (2015), no. 8, 399 - 408.
http://dx.doi.org/10.12988/ams.2015.48654

[15] M. Sarfraz, Generating outlines of generic shapes by mining feature points, *WSEAS Transactions on Systems,* **13** (2014), 584 - 595.

[16] M. Sarfraz, N. Al-Dabbous, Curve representation for outlines of planar images using multilevel coordinate search, *WSEAS Transactions on Computers,* **12** (2013), no. 2, 62 - 73.

[17] V. Skala, Fast interpolation and approximation of scattered multidimensional and dynamic data using radial basis functions, *WSEAS Transactions on Mathematics,* **12** (2013), no. 5, 501 - 511.

[18] M. Zamani, A new, robust and applied model for approximation of huge data, *WSEAS Transactions on Mathematics,* **12** (2013), no. 6, 727 - 735.

[19] X. Zhuang, N. E. Mastorakis, A Model of Virtual Carrier Immigration in Digital Images for Region Segmentation, *WSEAS Transactions on Computers,* **14** (2015), 708 - 718.

Received: January 18, 2016; Published: March 9, 2016