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ON ARITHMETIC MACAULAYFICATION OF NOETHERIAN RINGS

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Abstract. The Rees algebra is the homogeneous coordinate ring of a blowing-up. The present paper gives a necessary and sufficient condition for a Noetherian local ring to have a Cohen-Macaulay Rees algebra: A Noetherian local ring has a Cohen-Macaulay Rees algebra if and only if it is unmixed and all the formal fibers of it are Cohen-Macaulay. As a consequence of it, we characterize a homomorphic image of a Cohen-Macaulay local ring. For non-local rings, this paper gives only a sufficient condition. By using it, however, we obtain the affirmative answer to Sharp's conjecture. That is, a Noetherian ring having a dualizing complex is a homomorphic image of a finite-dimensional Gorenstein ring.

1. Introduction

Let A be a commutative ring with identity and $\mathfrak{b}$ an ideal in A. The *Rees algebra* of $\mathfrak{b}$ is the graded ring

$$R(\mathfrak{b}) = \bigoplus_{n \geq 0} (\mathfrak{b}T)^n,$$

where T is an indeterminate. We often regard $R(\mathfrak{b})$ as an A-subalgebra $A[\mathfrak{b}T]$ of the polynomial ring $A[T]$. The Rees algebra is an important object of Algebraic Geometry and Commutative Algebra because the canonical morphism $\operatorname{Proj} R(\mathfrak{b}) \to \operatorname{Spec} A$ is the blowing-up of $\operatorname{Spec} A$ along the closed subscheme $\operatorname{Spec} A/\mathfrak{b}$.

In the present paper, we consider the existence of Cohen-Macaulay Rees algebras. A Rees algebra $R(\mathfrak{b})$ is said to be an *arithmetic Macaulayfication* of A if it is Cohen-Macaulay and $\mathfrak{b}$ is of positive height. The main theorem of this paper is the following.

Theorem 1.1. *Let A be a Noetherian local ring of positive dimension. Then the following statements are equivalent*:

(A) *A has an arithmetic Macaulayfication*;
(B) *A is unmixed and all the formal fibers of A are Cohen-Macaulay.*

Here a Noetherian local ring A is said to be unmixed if $\dim \hat{A}/\mathfrak{p} = \dim \hat{A}$ for every associated prime $\mathfrak{p}$ of the completion $\hat{A}$. The formal fibers of A are the fiber rings of the natural homomorphism $A \to \hat{A}$.

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The studies in the Cohen-Macaulay property of Rees algebras started from Barshay's paper [5]. He gave the defining ideal of $R(\mathfrak{b})$ and its free resolution if $\mathfrak{b}$ is generated by a regular sequence. He also showed that $R(\mathfrak{b})$ is Cohen-Macaulay if A is also and if $\mathfrak{b}$ is generated by a regular sequence. Around 1980, Goto and Shimoda studied several properties of $R(\mathfrak{b})$ in the case where A is a Buchsbaum local ring and $\mathfrak{b}$ a parameter ideal. See [9], [10], [11], and [31]. Summarizing these investigations, Goto and Yamagishi [12] established the theory of unconditioned strong d-sequences. Their theory contains the existence of an arithmetic Macaulayfication in the case where A is unmixed and $\operatorname{Spec} \hat{A}$ is Cohen-Macaulay except for the closed point. See also Brodmann [7] and Schenzel [27]. Recently Kurano [19] proved that a Noetherian local ring A containing a finite field has an arithmetic Macaulayfication if the non-F-rational locus of A is of dimension 1. Independently this was also done by Aberbach [1]. Motivated by Kurano's work, the author [18] also gave some sufficient conditions for A to have an arithmetic Macaulayfication. Theorem 1.1 gives a necessary and sufficient condition for an arithmetic Macaulayfication to exist.

If the Rees algebra $R(\mathfrak{b})$ is a Cohen-Macaulay ring, then the projective scheme $\operatorname{Proj} R(\mathfrak{b})$ is Cohen-Macaulay. However, the converse is not true in general. The author [17] gave an ideal $\mathfrak{b}$ such that $\operatorname{Proj} R(\mathfrak{b})$ is a Cohen-Macaulay scheme for fairly general Noetherian local rings. Theorem 1.1 gives another proof of the result in [17].

In our arithmetic Macaulayfication $R(\mathfrak{b})$, the ideal $\mathfrak{b}$ is generated by monomials of a certain system of parameters, named a p-standard system of parameters. Sections 2 and 3 are devoted to discussing the existence and properties of a p-standard system of parameters. Theorems 2.5 and 3.6 are improvements of Theorems 2.7 and 3.1 of [17], respectively. We give a proof of Theorem 1.1 in Section 4. In our proof the theory of multigraded Rees algebras, which was introduced by Herrmann, Hyry, and Ribbe [15], plays a key role. Our ideal $\mathfrak{b}$ is very complicated. However, their theory makes the proof of Theorem 1.1 simple.

In section 5 we give a consequence of Theorem 1.1.

Corollary 1.2. *A Noetherian local ring is a homomorphic image of a Cohen-Macaulay local ring if and only if it is universally catenary and all the formal fibers of it are Cohen-Macaulay. An excellent local ring is a homomorphic image of a Cohen-Macaulay excellent local ring.*

However, there exists no analogy with the Gorenstein property. In fact, Ogoma [22, Example 1] gave an example of an acceptable local ring which is not a homomorphic image of a Gorenstein ring.

For non-local rings, this paper gives only a sufficient condition for an arithmetic Macaulayfication to exist.

Theorem 1.3. *Let B be a Noetherian ring possessing a dualizing complex. If the codimension function is a constant on the associated primes of B, then B has an arithmetic Macaulayfication.*

We refer the readers to Section 5 for the definition of the codimension function.

By using Theorem 1.3, we give an affirmative answer to Sharp's conjecture [30, Conjecture 4.4].

Corollary 1.4. *A Noetherian ring has a dualizing complex if and only if it is a homomorphic image of a finite-dimensional Gorenstein ring.*

This is a simple criterion for a dualizing complex to exist. Several authors gave partial answers. See [2], [3], [4], [22], and [23]. We give proofs of Theorem 1.3 and Corollary 1.4 in Section 6.

Throughout this paper, A denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$. We assume that the dimension of A is positive. We refer the reader to [13], [14], and [20], for unexplained terminology.

2. A p-standard system of parameters, I

In this section, we give the definition of a p-standard system of parameters and discuss the existence of it. For a finitely generated A-module M, let $\mathfrak{a}^p(M)$ denote the annihilator of the pth local cohomology module $H^p_{\mathfrak{m}}(M)$ of M and let $\mathfrak{a}(M) = \prod_{p<\dim M} \mathfrak{a}^p(M)$.

Definition 2.1. Let M be a finitely generated A-module of dimension $d > 0$, $x_1, \ldots, x_d$ a system of parameters for M and s an integer such that $0 \leq s < d$. We say that $x_1, \ldots, x_d$ is a *p-standard system of parameters* of type s for M if

(1) $x_{s+1}, \ldots, x_d \in \mathfrak{a}(M)$;
(2) $x_i \in \mathfrak{a}(M/(x_{i+1}, \ldots, x_d)M)$ for $1 \leq i \leq s$.

This notion was given by N. T. Cuong [8]. He showed that there exists a p-standard system of parameters of type $d-1$ for M whenever A possesses a dualizing complex. We improve his result. For a finitely generated A-module M, let $\mathrm{NCM}(M)$ denote the non-Cohen-Macaulay locus of M, that is, $\mathrm{NCM}(M) = \{\mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}}$ is not a Cohen-Macaulay $A_{\mathfrak{p}}$-module$\}$. By modifying the proof of [29, Theorem 3.3], we obtain the following lemma.

Lemma 2.2. *Let B and C be Noetherian rings and $B \to C$ a faithfully flat ring homomorphism. We assume that $C_{\mathfrak{p}}/\mathfrak{p}C_{\mathfrak{p}}$ is a Cohen-Macaulay ring for every prime ideal $\mathfrak{p}$ in B. Let M be a finitely generated B-module. If there exists an ideal $\mathfrak{c}$ in C such that* $\mathrm{NCM}(M \otimes_B C) = V(\mathfrak{c})$*, then* $\mathrm{NCM}(M) = V(\mathfrak{c} \cap B)$.

We need the following propositions to choose a p-standard system of parameters.

Proposition 2.3. *Assume that A is universally catenary and that all the formal fibers of A are Cohen-Macaulay. Let M be a finitely generated A-module of dimension $d > 0$. If M is equidimensional, then* $\mathrm{NCM}(M) = V(\mathfrak{a}(M))$*. In particular,* $\dim A/\mathfrak{a}(M) < d$.

Proof. If A has a dualizing complex, then the assertion was given by Schenzel [26, p. 52]. Assume that A has no dualizing complex. The completion $\hat{A}$ of A has a dualizing complex and is a faithfully flat A-algebra. Since A is formally catenary, $M \otimes \hat{A}$ is also equidimensional. Therefore the non-Cohen-Macaulay locus of $M \otimes \hat{A}$ is

$$V(\mathfrak{a}(M \otimes \hat{A})) = V(\mathfrak{a}^0(M \otimes \hat{A}) \cap \cdots \cap \mathfrak{a}^{d-1}(M \otimes \hat{A})).$$

By using Lemma 2.2, we find that the non-Cohen-Macaulay locus of M is

$$V(\mathfrak{a}^0(M \otimes \hat{A}) \cap \cdots \cap \mathfrak{a}^{d-1}(M \otimes \hat{A}) \cap A) = V(\mathfrak{a}^0(M) \cap \cdots \cap \mathfrak{a}^{d-1}(M)).$$

The right-hand side of the equation above is equal to $V(\mathfrak{a}(M))$. Since $\mathrm{NCM}(M)$ contains no minimal prime of M, $\dim A/\mathfrak{a}(M) = \dim \mathrm{NCM}(M) < d$. □

Corollary 2.4. *Assume that A is universally catenary and that all the formal fibers of A are Cohen-Macaulay. Let M be a finitely generated A-module of dimension $d > 0$. If $\dim A/\mathfrak{p} = d$ for every associated prime ideal $\mathfrak{p}$ of M, then $\dim A/\mathfrak{a}(M) < d-1$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of A such that $\dim A/\mathfrak{p} = d-1$ and $M_{\mathfrak{p}} \neq 0$. Then the one-dimensional $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is Cohen-Macaulay because $M_{\mathfrak{p}}$ has no embedded prime. Therefore $\dim A/\mathfrak{a}(M) = \dim \mathrm{NCM}(M) < d-1$. □

The following theorem assures us of the existence of the p-standard system of parameters.

Theorem 2.5. *Assume that A is universally catenary and that all the formal fibers of A are Cohen-Macaulay. Let M be a finitely generated A-module of dimension $d > 0$. If M is equidimensional and s an integer such that $\dim A/\mathfrak{a}(M) \leq s < d$, then there exists a p-standard system of parameters of type s for M.*

Proof. Since $d - \dim A/\mathfrak{a}(M) \geq d-s$, there exist $d-s$ elements $x_{s+1}, \dots, x_d$ in $\mathfrak{a}(M)$ such that $\dim M/(x_{s+1}, \dots, x_d)M = s$. If elements $x_{i+1}, \dots, x_d$ in A such that $\dim M/(x_{i+1}, \dots, x_d)M = i$ are given, then $M/(x_{i+1}, \dots, x_d)M$ is also equidimensional. Therefore $\dim A/\mathfrak{a}(M/(x_{i+1}, \dots, x_d)M) < i$ and hence there exists an element x_i in $\mathfrak{a}(M/(x_{i+1}, \dots, x_d)M)$ such that $\dim M/(x_i, \dots, x_d)M = i-1$. □

3. A p-standard system of parameters, II

In this section, we give some properties of a p-standard system of parameters. First we recall the definition of d-sequences and the one of unconditioned strong d-sequences.

Definition 3.1. Let M be an A-module. A sequence $x_1, \dots, x_d$ of elements in A is said to be a *d-sequence* on M if

$$(x_1, \dots, x_{i-1})M : x_i x_j = (x_1, \dots, x_{i-1})M : x_j$$

for any $1 \leq i \leq j \leq d$. Here we set $(x_1, \dots, x_{i-1}) = (0)$ if $i = 1$.

A sequence $x_1, \dots, x_d$ of elements in A is said to be an unconditioned strong d-sequence (for short, a *u.s.d-sequence*) on M if $x_1^{n_1}, \dots, x_d^{n_d}$ is a d-sequence on M for any positive integers $n_1, \dots, n_d$ and in any order.

The following is one of the important properties of d-sequences. It was first given by Goto and Shimoda [11, Lemma 4.2] for the system of parameters for a Buchsbaum local ring, which is a typical example of d-sequences.

Proposition 3.2 ([12, Theorem 1.3]). *Let M be an A-module and $x_1, \dots, x_d$ a d-sequence on M. If we put $\mathfrak{q} = (x_1, \dots, x_d)$, then*

$$(x_1, \dots, x_{i-1})M : x_i \cap \mathfrak{q}^n M = (x_1, \dots, x_{i-1})\mathfrak{q}^{n-1}M$$

for any $n > 0$ and $1 \leq i \leq d$.

A p-standard system of parameters has several nice properties. The following two properties are given in [17].

Proposition 3.3 ([17, Proposition 2.8]). *Let M be a finitely generated A-module of dimension $d > 0$ and $x_1, \dots, x_d$ a p-standard system of parameters of type s for M. Then $x_{s+1}, \dots, x_d$ is a u.s.d-sequence on $M/(y_1, \dots, y_u)M$ where $y_1, \dots, y_u$ is a subsystem of parameters for $M/(x_{s+1}, \dots, x_d)M$.*

Proposition 3.4 ([17, Theorem 2.9]). *Let M be a finitely generated A-module of dimension $d > 0$, $x_1, \dots, x_d$ a p-standard system of parameters of type s for M, and $y_1, \dots, y_u$ a subsystem of parameters for $M/(x_i, \dots, x_d)M$ where $2 \le i \le d$ and $1 \le u < i$. If $y_u \in \mathfrak{a}(M)$ or $y_u \in \mathfrak{a}(M/(x_i, \dots, x_d)M)$, then*

$$(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v y_u = (y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_u$$

for any $1 \le v \le u$ and $\Lambda \subseteq \{i, \dots, d\}$.

The next proposition is not in [17] but we need it to prove Theorem 1.1. The author is inspired by [8, Theorem 2.6].

Proposition 3.5. *Let M be a finitely generated A-module of dimension $d > 0$, $x_1, \dots, x_d$ a p-standard system of parameters of type s for M and $y_1, \dots, y_u$ a subsystem of parameters for $M/(x_i, \dots, x_d)M$ where $1 \le i \le d$ and $1 \le u < i$. Then $x_i, \dots, x_j$ is a d-sequence on $M/(y_1, \dots, y_u, x_{j+1}, \dots, x_d)M$ for any $i \le j \le d$.*

Proof. Let $i \le l \le j$ be an integer. By applying Proposition 3.4 to a subsystem of parameters $y_1, \dots, y_u, x_i, \dots, x_l$ for $M/(x_{l+1}, \dots, x_d)M$ and a subset $\{j+1, \dots, d\}$ of $\{l+1, \dots, d\}$, we obtain

$$\begin{aligned}(y_1, \dots, y_u, x_i, \dots, x_{k-1}, x_{j+1}, \dots, x_d)M &: x_k x_l \\ = (y_1, \dots, y_u, x_i, \dots, x_{k-1}, x_{j+1}, \dots, x_d)M &: x_l\end{aligned}$$

for any $i \le k \le l$. ∎

The following theorem and corollaries are improvements of Theorem 3.1, Corollaries 3.2 and 3.3 of [17], respectively. The old theorems require that all $n_i, \dots, n_j$ are positive but new ones require only that all $n_i, \dots, n_j$ are nonnegative.

Theorem 3.6. *Let M be a finitely generated A-module of dimension $d > 0$ and $x_1, \dots, x_d$ a p-standard system of parameters of type s for M. We put $\mathfrak{q}_i = (x_i, \dots, x_d)$ for all $1 \le i \le d$. Then, for any integers $1 \le i \le j \le d$ and $n_i, \dots, n_j \ge 0$, the following statements hold:*

(A_{ij}) If $y_1, \dots, y_u$ is a subsystem of parameters for $M/\mathfrak{q}_i M$ and if $n_k > 0$ for some integer $i \le k \le j$, then

$$\begin{aligned}(y_1, \dots, y_u, x_k, \dots, x_{l-1})M : x_l \cap [(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \\ = (y_1, \dots, y_u)M + (x_k, \dots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k - 1} \cdots \mathfrak{q}_j^{n_j} M\end{aligned} \tag{3.6.1}$$

for arbitrary integer $k \le l \le d$.

(B_{ij}): If $y_1, \dots, y_u$ is a subsystem of parameters for $M/\mathfrak{q}_i M$ and if $n_k > 0$ for some integer $i \le k \le j$, then

$$\begin{aligned}&[(y_1, \dots, y_{u-1})M + (x_k, \dots, x_l)\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : y_u \\ &\quad = (x_k, \dots, x_l)\{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : y_u\} \\ &\qquad + (y_1, \dots, y_{u-1})M : y_u\end{aligned} \tag{3.6.2}$$

for arbitrary integer $k \le l \le d$. In particular, by letting $l = d$, we have

$$\begin{aligned}&[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k+1} \cdots \mathfrak{q}_j^{n_j} M] : y_u \\ &\quad = \mathfrak{q}_k\{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : y_u\} \\ &\qquad + (y_1, \dots, y_{u-1})M : y_u.\end{aligned}$$

(C_{ij}): *If* $y_1, \dots, y_u$ *is a subsystem of parameters for* $M/\mathfrak{q}_i M$ *and if* $n_i > 0$, *then*

$$[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : y_u \subseteq (y_1, \dots, y_{u-1})M : y_u + \mathfrak{q}_i^{n_i - 1} \cdots \mathfrak{q}_j^{n_j} M. \tag{3.6.3}$$

(D_{ij}): *If* $y_1, \dots, y_u$ *is a subsystem of parameters for* $M/\mathfrak{q}_i M$ *and if* $n_i > 0$, *then*

$$[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : y_u \cap x_i M \subseteq x_i \{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i - 1} \cdots \mathfrak{q}_j^{n_j} M] : y_u\} + (y_1, \dots, y_{u-1})M. \tag{3.6.4}$$

(E_{ij}): *Let* $y_1, \dots, y_u$ *be a subsystem of parameters for* $M/\mathfrak{q}_k M$ *where* $2 \le k \le i$ *and* $1 \le u < k$. *If* $y_u \in \mathfrak{a}(M/\mathfrak{q}_k M)$ *or* $y_u \in \mathfrak{a}(M)$ *and if* $n_i > 0$, *then*

$$[(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : y_v y_u = [(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : y_u \tag{3.6.5}$$

for any $1 \le v \le u$ *and* $\Lambda \subseteq \{k, \dots, i-1\}$.

Proof. We work by induction on $j - i$. First we assume that $i = j$.

(A_{ii}): Since $x_i, \dots, x_d$ is a d-sequence on $M/(y_1, \dots, y_u)M$, (3.6.1) coincides with Proposition 3.2.

(B_{ii}): Let a be an element in the left-hand side of (3.6.2) and put $y_u a = x_l b + c$ with $b \in \mathfrak{q}_i^{n_i} M$ and $c \in (y_1, \dots, y_{u-1})M + (x_i, \dots, x_{l-1})\mathfrak{q}_i^{n_i} M$. By using (A_{ii}), we obtain

$$\begin{aligned} b &\in (y_1, \dots, y_u, x_i, \dots, x_{l-1})M : x_l \cap \mathfrak{q}_i^{n_i} M \\ &\subseteq (y_1, \dots, y_u)M + (x_i, \dots, x_{l-1})\mathfrak{q}_i^{n_i - 1} M. \end{aligned}$$

Let $b = y_u a' + c'$ with $c' \in (y_1, \dots, y_{u-1})M + (x_i, \dots, x_{l-1})\mathfrak{q}_i^{n_i - 1} M$. Then $a' \in [(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} M] : y_u$ and

$$a - x_l a' \in [(y_1, \dots, y_{u-1})M + (x_i, \dots, x_{l-1})\mathfrak{q}_i^{n_i} M] : y_u.$$

By induction on l, we find that a is in the right-hand side of (3.6.2). The opposite inclusion is obvious.

(C_{ii}): By using (B_{ii}) repeatedly, we have

$$\begin{aligned} [(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} M] : y_u &= (y_1, \dots, y_{u-1})M : y_u \\ &\quad + \mathfrak{q}_i^{n_i - 1}\{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i M] : y_u\} \\ &\subseteq (y_1, \dots, y_{u-1})M : y_u + \mathfrak{q}_i^{n_i - 1} M. \end{aligned}$$

(D_{ii}): If $n_i = 1$, then the right-hand side of (3.6.4) equals $(y_1, \dots, y_{u-1}, x_i)M$ and hence contains the left-hand side.

Assume that $n_i > 1$. Let a be an element in M such that $x_i a$ is in the left-hand side of (3.6.4). Then

$$\begin{aligned} y_u x_i a &\in [(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} M] \cap (y_1, \dots, y_{u-1}, x_i)M \\ &= (y_1, \dots, y_{u-1})M + x_i \mathfrak{q}_i^{n_i - 1} M \end{aligned}$$

because of (A_{ii}). Hence

$$\begin{aligned} x_i a &\in [(y_1, \dots, y_{u-1})M + x_i \mathfrak{q}_i^{n_i - 1} M] : y_u \\ &= (y_1, \dots, y_{u-1})M : y_u + x_i\{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i - 1} M] : y_u\}. \end{aligned}$$

Here we used (B_{ii}). By applying Proposition 3.4 to a subsystem of parameters $y_1, \dots, y_u, x_i$ for $M/\mathfrak{q}_{i+1}M$, we have

$$(y_1,\dots,y_{u-1})M : y_u x_i = (y_1,\dots,y_{u-1})M : x_i$$

and hence

$$(y_1,\dots,y_{u-1})M : y_u \cap x_i M = x_i[(y_1,\dots,y_{u-1})M : y_u x_i] \subseteq (y_1,\dots,y_{u-1})M. \tag{3.6.6}$$

Therefore

$$\begin{aligned} x_i a &\in x_i\{[(y_1,\dots,y_{u-1})M + \mathfrak{q}_i^{n_i-1}M] : y_u\} + (y_1,\dots,y_{u-1})M : y_u \cap x_i M \\ &\subseteq x_i\{[(y_1,\dots,y_{u-1})M + \mathfrak{q}_i^{n_i-1}M] : y_u\} + (y_1,\dots,y_{u-1})M. \end{aligned}$$

(E_{ii}): By using (B_{ii}), we have

$$\begin{aligned} &[(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M + \mathfrak{q}_i^{n_i}M] : y_v y_u \\ &\quad = (y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M : y_v y_u \\ &\qquad + \mathfrak{q}_i^{n_i-1}\{[(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M + \mathfrak{q}_i M] : y_v y_u\}. \end{aligned}$$

Applying Proposition 3.4 to a subsystem of parameters $y_1, \dots, y_u$ for $M/\mathfrak{q}_k M$ and two subsets of $\{k,\dots,d\}$: Λ and $\Lambda \cup \{i,\dots,d\}$, we obtain

$$\begin{aligned} &(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M : y_v y_u \\ &\qquad + \mathfrak{q}_i^{n_i-1}\{[(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M + \mathfrak{q}_i M] : y_v y_u\} \\ &\quad = (y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M : y_u \\ &\qquad + \mathfrak{q}_i^{n_i-1}\{[(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M + \mathfrak{q}_i M] : y_u\} \\ &\quad = [(y_1,\dots,y_{u-1},\{x_\lambda \mid \lambda\in\Lambda\})M + \mathfrak{q}_i^{n_i}M] : y_u. \end{aligned}$$

Thus (3.6.5) is shown.

Next we assume that $j > i$ and prove (A_{ij})–(E_{ij}). If $n_i = 0$, then (A_{ij}) and (B_{ij}) are contained in $(A_{i+1,j})$ and $(B_{i+1,j})$, respectively. Therefore we may assume that $n_i > 0$. Similarly we may also assume that $n_j > 0$.

(A_{ij}): Let a be an element in the left-hand side of (3.6.1). If $k = l = i$, then

$$a \in (y_1,\dots,y_u)M : x_i \cap (y_1,\dots,y_u,x_i,\dots,x_d)M = (y_1,\dots,y_u)M.$$

Otherwise, by using $(A_{i+1,j})$, we have

$$\begin{aligned} a &\in (y_1,\dots,y_u,x_i,x_k,\dots,x_{l-1})M : x_l \cap [(y_1,\dots,y_u,x_i)M + \mathfrak{q}_{i+1}^{n_i+n_{i+1}}\cdots\mathfrak{q}_j^{n_j}M] \\ &= \begin{cases} (y_1,\dots,y_u,x_i)M + (x_{i+1},\dots,x_{l-1})\mathfrak{q}_{i+1}^{n_i+n_{i+1}-1}\cdots\mathfrak{q}_j^{n_j}M & \text{if } k \le i+1, \\ (y_1,\dots,y_u,x_i)M + (x_k,\dots,x_{l-1})\mathfrak{q}_{i+1}^{n_i+n_{i+1}}\cdots\mathfrak{q}_k^{n_k-1}\cdots\mathfrak{q}_j^{n_j}M & \text{if } k > i+1 \end{cases} \\ &= (y_1,\dots,y_u,x_i)M + (x_k,\dots,x_{l-1})\mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_k^{n_k-1}\cdots\mathfrak{q}_j^{n_j}M. \end{aligned}$$

Taking the intersection with $(y_1,\dots,y_u)M + \mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_j^{n_j}M$, we obtain

$$\begin{aligned} a &\in (y_1,\dots,y_u)M + (x_k,\dots,x_{l-1})\mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_k^{n_k-1}\cdots\mathfrak{q}_j^{n_j}M \\ &\quad + x_i M \cap [(y_1,\dots,y_u)M + \mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_j^{n_j}M]. \end{aligned}$$

Because of $(C_{i+1,j})$,

$$\begin{aligned}
&x_iM \cap [(y_1,\dots,y_u)M + \mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_j^{n_j}M] \\
&\quad = x_i\mathfrak{q}_i^{n_i-1}\cdots\mathfrak{q}_j^{n_j}M \\
&\qquad + x_iM \cap [(y_1,\dots,y_u)M + \mathfrak{q}_{i+1}^{n_i+n_{i+1}}\cdots\mathfrak{q}_j^{n_j}M] \\
&\quad = x_i\mathfrak{q}_i^{n_i-1}\cdots\mathfrak{q}_j^{n_j}M + x_i\{[(y_1,\dots,y_u)M + \mathfrak{q}_{i+1}^{n_i+n_{i+1}}\cdots\mathfrak{q}_j^{n_j}M] : x_i\} \\
&\quad \subseteq x_i\mathfrak{q}_i^{n_i-1}\cdots\mathfrak{q}_j^{n_j}M + x_i[(y_1,\dots,y_u)M : x_i + \mathfrak{q}_{i+1}^{n_i+n_{i+1}-1}\cdots\mathfrak{q}_j^{n_j}M] \\
&\quad \subseteq (y_1,\dots,y_u)M + x_i\mathfrak{q}_i^{n_i-1}\cdots\mathfrak{q}_j^{n_j}M.
\end{aligned}$$

Therefore

$$\begin{aligned}
a \in (y_1,\dots,y_u)M &+ (x_k,\dots,x_{l-1})\mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_k^{n_k-1}\cdots\mathfrak{q}_j^{n_j}M \\
&+ x_i\mathfrak{q}_i^{n_i-1}\cdots\mathfrak{q}_j^{n_j}M.
\end{aligned}$$

If $k = i$, then the proof is completed. If $k > i$, then we work by induction on n_i. Let $a = x_ib + c$ with $b \in \mathfrak{q}_i^{n_i-1}\cdots\mathfrak{q}_j^{n_j}M$ and

$$c \in (y_1,\dots,y_u)M + (x_k,\dots,x_{l-1})\mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_k^{n_k-1}\cdots\mathfrak{q}_j^{n_j}M.$$

If we apply Proposition 3.4 to a subsystem of parameters $y_1, \dots, y_u, x_k, \dots, x_{l-1}, x_i, x_l$ for $M/\mathfrak{q}_{l+1}M$, then we have

$$b \in (y_1,\dots,y_u,x_k,\dots,x_{l-1})M : x_ix_l = (y_1,\dots,y_u,x_k,\dots,x_{l-1})M : x_l.$$

If $n_i = 1$, then $(A_{i+1,j})$ says that

$$\begin{aligned}
b &\in (y_1,\dots,y_u,x_k,\dots,x_{l-1})M : x_l \cap \mathfrak{q}_{i+1}^{n_{i+1}}\cdots\mathfrak{q}_j^{n_j}M \\
&\subseteq (y_1,\dots,y_u)M + (x_k,\dots,x_{l-1})\mathfrak{q}_{i+1}^{n_{i+1}}\cdots\mathfrak{q}_k^{n_k-1}\cdots\mathfrak{q}_j^{n_j}M
\end{aligned}$$

and hence $a = x_ib + c$ is in the right-hand side of (3.6.1). If $n_i > 1$, then we obtain

$$\begin{aligned}
b &\in (y_1,\dots,y_u,x_k,\dots,x_{l-1})M : x_l \cap \mathfrak{q}_i^{n_i-1}\cdots\mathfrak{q}_j^{n_j}M \\
&\subseteq (y_1,\dots,y_u)M + (x_k,\dots,x_{l-1})\mathfrak{q}_i^{n_i-1}\cdots\mathfrak{q}_k^{n_k-1}\cdots\mathfrak{q}_j^{n_j}M
\end{aligned}$$

by the induction hypothesis. Thus $a = x_ib+c$ is also in the right-hand side of (3.6.1).

(B_{ij}): Let a be an element in the left-hand side of (3.6.2) and put $y_ua = x_lb + c$ with $b \in \mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_j^{n_j}M$ and $c \in (y_1,\dots,y_{u-1})M + (x_k,\dots,x_{l-1})\mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_j^{n_j}M$. Then

$$\begin{aligned}
b &\in (y_1,\dots,y_u,x_k,\dots,x_{l-1})M : x_l \cap \mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_j^{n_j}M \\
&\subseteq (y_1,\dots,y_u)M + (x_k,\dots,x_{l-1})\mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_k^{n_k-1}\cdots\mathfrak{q}_j^{n_j}M.
\end{aligned}$$

Here we used (A_{ij}). If we put $b = y_ua' + c'$ with

$$c' \in (y_1,\dots,y_{u-1})M + (x_k,\dots,x_{l-1})\mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_k^{n_k-1}\cdots\mathfrak{q}_j^{n_j}M,$$

then $a' \in [(y_1,\dots,y_{u-1})M + \mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_j^{n_j}M] : y_u$ and

$$a - x_la' \in [(y_1,\dots,y_{u-1})M + (x_k,\dots,x_{l-1})\mathfrak{q}_i^{n_i}\cdots\mathfrak{q}_j^{n_j}M] : y_u.$$

By induction on l, we find that a is in the right-hand side of (3.6.2). The opposite inclusion is obvious.

(C_{ij}): We first show that

$$\begin{aligned}
(y_1,\dots,y_{u-1},x_i)M : y_u &\cap (y_1,\dots,y_{u-1},x_i,\dots,x_l)M \\
&= (y_1,\dots,y_{u-1},x_i)M
\end{aligned} \tag{3.6.7}$$

for all $i \le l \le d$. We work by induction on l. If $l = i$, then there exists nothing to prove. Assume that $l > i$ and let a be an element in the left-hand side of (3.6.7). If we put $a = x_l b + c$ with $c \in (y_1, \ldots, y_{u-1}, x_i, \ldots, x_{l-1})M$, then

$$b \in (y_1, \ldots, y_{u-1}, x_i, \ldots, x_{l-1})M : y_u x_l = (y_1, \ldots, y_{u-1}, x_i, \ldots, x_{l-1})M : x_l.$$

Here we applied Proposition 3.4 to a subsystem of parameters $y_1, \ldots, y_{u-1}, x_i, \ldots, x_{l-1}, y_u, x_l$ for $M/\mathfrak{q}_{l+1}M$. Thus we obtain

$$\begin{aligned} a = x_l b + c &\in (y_1, \ldots, y_{u-1}, x_i)M : y_u \cap (y_1, \ldots, y_{u-1}, x_i, \ldots, x_{l-1})M \\ &= (y_1, \ldots, y_{u-1}, x_i)M \end{aligned}$$

by the induction hypothesis.

Next we show (3.6.3). By using (B_{ij}), we may assume that $n_i = 1$. Let a be an element in the left-hand side of (3.6.3). Then

$$\begin{aligned} a &\in [(y_1, \ldots, y_{u-1}, x_i)M + \mathfrak{q}_{i+1}^{n_{i+1}+1} \cdots \mathfrak{q}_j^{n_j} M] : y_u \\ &\subseteq (y_1, \ldots, y_{u-1}, x_i)M : y_u + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M \end{aligned}$$

because of $(C_{i+1,j})$. On the other hand, since $n_j > 0$, we obtain

$$\begin{aligned} a &\in [(y_1, \ldots, y_{u-1})M + \mathfrak{q}_i^2 M] : y_u \\ &\subseteq (y_1, \ldots, y_{u-1})M : y_u + \mathfrak{q}_i M. \end{aligned}$$

Here we used (C_{ii}). Hence

$$\begin{aligned} a &\in [(y_1, \ldots, y_{u-1}, x_i)M : y_u + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] \cap [(y_1, \ldots, y_{u-1})M : y_u + \mathfrak{q}_i M] \\ &= (y_1, \ldots, y_{u-1})M : y_u + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M + (y_1, \ldots, y_{u-1}, x_i)M : y_u \cap \mathfrak{q}_i M \\ &= (y_1, \ldots, y_{u-1})M : y_u + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M + x_i M. \end{aligned}$$

Here we used (3.6.7). Taking the intersection with

$$[(y_1, \ldots, y_{u-1})M + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] : y_u,$$

we obtain

$$\begin{aligned} a \in\ & (y_1, \ldots, y_{u-1})M : y_u + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M \\ &+ x_i \{[(y_1, \ldots, y_{u-1})M + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] : y_u x_i\}. \end{aligned}$$

By applying $(E_{i+1,j})$ to a subsystem of parameters $y_1, \ldots, y_u, x_i$ for $M/\mathfrak{q}_{i+1}M$, we have

$$[(y_1, \ldots, y_{u-1})M + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] : y_u x_i = [(y_1, \ldots, y_{u-1})M + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] : x_i.$$

Therefore $a \in (y_1, \ldots, y_{u-1})M : y_u + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M$.

(D_{ij}): Let a be an element in M such that $x_i a$ is in the left-hand side of (3.6.4). Then

$$\begin{aligned} y_u x_i a &\in x_i M \cap [(y_1, \ldots, y_{u-1})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \\ &\subseteq (y_1, \ldots, y_{u-1})M + x_i \mathfrak{q}_i^{n_i - 1} \cdots \mathfrak{q}_j^{n_j} M. \end{aligned}$$

Here we used (A_{ij}). We put $y_u x_i a = x_i b + c$ with $b \in \mathfrak{q}_i^{n_i-1} \cdots \mathfrak{q}_j^{n_j} M$ and $c \in (y_1, \ldots, y_{u-1})M$. Then

$$\begin{aligned} b &\in (y_1, \ldots, y_u)M : x_i \cap \mathfrak{q}_j M \\ &\subseteq (y_1, \ldots, y_u)M : x_i \cap \mathfrak{q}_i M \\ &\subseteq (y_1, \ldots, y_u)M \end{aligned}$$

because $n_j > 0$ and $x_i, \dots, x_d$ is a d-sequence on $M/(y_1, \dots, y_u)M$. If we put $b = y_u a' + c'$ with $c' \in (y_1, \dots, y_{u-1})M$, then

$$a' \in [(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i-1} \cdots \mathfrak{q}_j^{n_j} M] : y_u$$

and

$$\begin{aligned} x_i(a - a') &\in (y_1, \dots, y_{u-1})M : y_u \cap x_i M \\ &\subseteq (y_1, \dots, y_{u-1})M. \end{aligned}$$

Here we used (3.6.6) again. Therefore

$$x_i a \in (y_1, \dots, y_{u-1})M + x_i\{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i-1} \cdots \mathfrak{q}_j^{n_j} M] : y_u\}.$$

(E_{ij}): We may assume that $n_i = 1$ in the same way as the proof of (E_{ii}). We divide the proof into two cases.

First we assume that $n_{i+1} + \cdots + n_j = 1$, that is, $n_{i+1} = \cdots = n_{j-1} = 0$ and $n_j = 1$. We show that

$$\begin{aligned} &[(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + (x_i, \dots, x_{l-1}, x_j, \dots, x_d)\mathfrak{q}_i M] : y_v y_u \\ &\quad = [(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + (x_i, \dots, x_{l-1}, x_j, \dots, x_d)\mathfrak{q}_i M] : y_u \end{aligned} \tag{3.6.8}$$

for all $i \le l \le j$ by descending induction on l. If $l = j$, then (3.6.8) coincides with (E_{ii}). Assume that $l < j$ and let a be an element in the left-hand side of (3.6.8). The induction hypothesis says that

$$a \in [(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + (x_i, \dots, x_l, x_j, \dots, x_d)\mathfrak{q}_i M] : y_u.$$

We put $y_u a = x_l b + c$ with $b \in \mathfrak{q}_i M$ and

$$c \in (y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + (x_i, \dots, x_{l-1}, x_j, \dots, x_d)\mathfrak{q}_i M.$$

On the other hand, Proposition 3.4 says that

$$\begin{aligned} a &\in (y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\}, x_i, \dots, x_{l-1}, x_j, \dots, x_d)M : y_v y_u \\ &= (y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\}, x_i, \dots, x_{l-1}, x_j, \dots, x_d)M : y_u. \end{aligned}$$

Hence

$$\begin{aligned} b &\in (y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\}, x_i, \dots, x_{l-1}, x_j, \dots, x_d)M : x_l \cap \mathfrak{q}_i M \\ &\subseteq (y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\}, x_i, \dots, x_{l-1}, x_j, \dots, x_d)M \end{aligned}$$

because $x_i, \dots, x_{j-1}$ is a d-sequence on

$$M/(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\}, x_j, \dots, x_d)M.$$

Therefore

$$y_u a = x_l b + c \in (y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + (x_i, \dots, x_{l-1}, x_j, \dots, x_d)\mathfrak{q}_i M.$$

Thus (3.6.8) is proved. If we put $l = i$, then we obtain

$$\begin{aligned} &[(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i \mathfrak{q}_j M] : y_v y_u \\ &\quad = [(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i \mathfrak{q}_j M] : y_u. \end{aligned}$$

Next we assume that $n_{i+1} + \cdots + n_j > 1$. Let

$$a \in [(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] : y_v y_u.$$

Then $(E_{i+1,j})$ says that

$$\begin{aligned} a &\in [(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\},x_i)M+\mathfrak{q}_{i+1}^{n_{i+1}+1}\cdots\mathfrak{q}_j^{n_j}M]:y_vy_u \\ &= [(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\},x_i)M+\mathfrak{q}_{i+1}^{n_{i+1}+1}\cdots\mathfrak{q}_j^{n_j}M]:y_u. \end{aligned}$$

Therefore

$$\begin{aligned} y_ua &\in [(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M+\mathfrak{q}_i\mathfrak{q}_{i+1}^{n_{i+1}}\cdots\mathfrak{q}_j^{n_j}M]:y_v \\ &\quad\cap [(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\},x_i)M+\mathfrak{q}_{i+1}^{n_{i+1}+1}\cdots\mathfrak{q}_j^{n_j}M] \\ &= (y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M+\mathfrak{q}_{i+1}^{n_{i+1}+1}\cdots\mathfrak{q}_j^{n_j}M \\ &\quad + [(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M+\mathfrak{q}_i\mathfrak{q}_{i+1}^{n_{i+1}}\cdots\mathfrak{q}_j^{n_j}M]:y_v\cap x_iM \\ &= (y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M+\mathfrak{q}_{i+1}^{n_{i+1}+1}\cdots\mathfrak{q}_j^{n_j}M \\ &\quad + x_i\{[(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M+\mathfrak{q}_{i+1}^{n_{i+1}}\cdots\mathfrak{q}_j^{n_j}M]:y_v\}. \end{aligned}$$

Here we used (D_{ij}) to show the second equality. We put $y_ua = x_ib+c$ with

$$b\in[(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M+\mathfrak{q}_{i+1}^{n_{i+1}}\cdots\mathfrak{q}_j^{n_j}M]:y_v \tag{3.6.9}$$

and

$$c\in(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M+\mathfrak{q}_{i+1}^{n_{i+1}+1}\cdots\mathfrak{q}_j^{n_j}M.$$

By applying $(C_{i+1,j})$ to a subsystem of parameters $y_1, \dots, y_{v-1}, y_u, \{x_\lambda \mid \lambda\in\Lambda\}, x_i$ for $M/\mathfrak{q}_{i+1}M$, we obtain

$$\begin{aligned} b &\in [(y_1,\dots,y_{v-1},y_u,\{x_\lambda \mid \lambda\in\Lambda\})M+\mathfrak{q}_{i+1}^{n_{i+1}+1}\cdots\mathfrak{q}_j^{n_j}M]:x_i \\ &\subseteq (y_1,\dots,y_{v-1},y_u,\{x_\lambda \mid \lambda\in\Lambda\})M:x_i+\mathfrak{q}_{i+1}^{n_{i+1}}\cdots\mathfrak{q}_j^{n_j}M. \end{aligned}\tag{3.6.10}$$

On the other hand, since $n_{i+1}+\cdots+n_j>1$, we have

$$\begin{aligned} b &\in [(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M+\mathfrak{q}_{i+1}^2M]:y_v \\ &\subseteq (y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M:y_v+\mathfrak{q}_{i+1}M \end{aligned}\tag{3.6.11}$$

by using $(C_{i+1,i+1})$.

Furthermore, by applying Proposition 3.4 to a subsystem of parameters $y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda\in\Lambda\}, y_v, x_i$ for $M/\mathfrak{q}_{i+1}M$, we obtain

$$\begin{aligned} &(y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M:y_v \\ &\quad\subseteq (y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M:y_vx_i \\ &\quad= (y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M:x_i. \end{aligned}\tag{3.6.12}$$

Hence, by taking the intersection of (3.6.10) and (3.6.11), we have

$$\begin{aligned} b &\in (y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M:y_v+\mathfrak{q}_{i+1}^{n_{i+1}}\cdots\mathfrak{q}_j^{n_j}M \\ &\quad + (y_1,\dots,y_{v-1},y_u,\{x_\lambda \mid \lambda\in\Lambda\})M:x_i\cap\mathfrak{q}_{i+1}M \\ &\subseteq (y_1,\dots,y_{v-1},\{x_\lambda \mid \lambda\in\Lambda\})M:y_v+y_uM+\mathfrak{q}_{i+1}^{n_{i+1}}\cdots\mathfrak{q}_j^{n_j}M. \end{aligned}$$

Here we apply Proposition 3.2 to a d-sequence $x_i, \dots, x_d$ on

$$M/(y_1,\dots,y_{v-1},y_u,\{x_\lambda \mid \lambda\in\Lambda\})M.$$

Taking the intersection with (3.6.9), we obtain

$$\begin{aligned} b &\in (y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M \\ &\quad + [(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] : y_v \cap y_u M \\ &= (y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M \\ &\quad + y_u\{[(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] : y_v y_u\} \\ &= (y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_v + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M. \end{aligned}$$

Here we used $(E_{i+1,j})$ to show the last equality. By using (3.6.12) again, we find that

$$y_u a = x_i b + c \in (y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M.$$

That is,

$$a \in [(y_1, \dots, y_{v-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] : y_u.$$

The opposite inclusion is obvious. The proof is completed. □

Corollary 3.7. *With the same notation as Theorem* 3.6*, we have*

$$[(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : x_{i-1}^{n_{i-1}} = [(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : \mathfrak{q}_{i-1}$$

for any integers $2 \le i \le j \le d$, $n_{i-1} > 0$, $n_i, \dots, n_j \ge 0$ *and for any subsystem of parameters* $y_1, \dots, y_u$ *for* $M/\mathfrak{q}_{i-1}M$.

Proof. If $n_i = \dots = n_j = 0$, then the equality is trivial. Therefore we may assume that one of $n_i, \dots, n_j$ is positive. We may also assume that $n_{i-1} = 1$ by using Theorem 3.6(E_{ij}). Then we have

$$[(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : x_{i-1} \subseteq (y_1, \dots, y_u)M : x_{i-1} + \mathfrak{q}_i^{n_i - 1} \cdots \mathfrak{q}_j^{n_j} M$$

by applying Theorem 3.6(C_{ij}) to a subsystem of parameters $y_1, \dots, y_u, x_{i-1}$ for $M/\mathfrak{q}_i M$. Since $x_{i-1}, \dots, x_d$ is a d-sequence on $M/(y_1, \dots, y_u)M$,

$$(y_1, \dots, y_u)M : x_{i-1} \subseteq (y_1, \dots, y_u)M : \mathfrak{q}_{i-1}.$$

Therefore

$$\mathfrak{q}_{i-1}\{[(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : x_{i-1}\} \subseteq (y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M.$$

The opposite inclusion is trivial. □

Corollary 3.8. *With the same notation of Theorem* 3.6*, we let* k *be an integer such that* $1 \le k \le d$ *and* $y_1, \dots, y_u$ *a subsystem of parameters for* $M/\mathfrak{q}_k M$. *Assume that*

$$[(y_1, \dots, y_{u-1})M + \mathfrak{q}_k M] : y_u = (y_1, \dots, y_{u-1})M + \mathfrak{q}_k M.$$

Then

$$(y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_u = (y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M \tag{3.8.1}$$

for any $\Lambda \subset \{k, \dots, d\}$. *Furthermore*

$$\begin{aligned} &[(y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : y_u \\ &\qquad = (y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M \end{aligned} \tag{3.8.2}$$

for any integers $k \le i \le j$, $n_i, \dots, n_j \ge 0$, *and* $\Lambda \subseteq \{k, \dots, i-1\}$.

Proof. We first show (3.8.1) by descending induction on the number of elements in Λ. If $\Lambda = \{k, \dots, d\}$, then there exists nothing to prove. Assume that $\Lambda \neq \{k, \dots, d\}$ and let l be an element in $\{k, \dots, d\} \setminus \Lambda$. Let a be an element in the left-hand side of (3.8.1). Then

$$a \in (y_1, \dots, y_{u-1}, x_l, \{x_\lambda \mid \lambda \in \Lambda\})M : y_u = (y_1, \dots, y_{u-1}, x_l, \{x_\lambda \mid \lambda \in \Lambda\})M$$

because of the induction hypothesis. We put $a = x_l b + c$ with

$$c \in (y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M.$$

Since $x_l \in \mathfrak{a}(M)$ or $x_l \in \mathfrak{a}(M/\mathfrak{q}_{l+1}M)$, we obtain

$$b \in (y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_u x_l = (y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : x_l$$

by using Proposition 3.4. Therefore $a = x_l b + c \in (y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M$.

Next we show that (3.8.2). If $n_i = \cdots = n_j = 0$, then the equality is trivial. We assume that n_i, $n_j > 0$ and we work by induction on $j - i$. If $i = j$, then

$$\begin{aligned}
&[(y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i^{n_i} M] : y_u \\
&\quad = (y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M : y_u \\
&\qquad + \mathfrak{q}_i^{n_i - 1}\{[(y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i M] : y_u\} \\
&\quad = (y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i^{n_i} M.
\end{aligned}$$

Here we used Theorem 3.6(B_{ij}) and (3.8.1). Assume that $j > i$. We may assume that $n_i = 1$ by using Theorem 3.6(B_{ij}). Let a be an element of the left-hand side of (3.8.2). The induction hypothesis says that

$$\begin{aligned}
&[(y_1, \dots, y_{u-1}, x_i, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_{i+1}^{n_{i+1}+1} \cdots \mathfrak{q}_j^{n_j} M] : y_u \\
&\quad = (y_1, \dots, y_{u-1}, x_i, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_{i+1}^{n_{i+1}+1} \cdots \mathfrak{q}_j^{n_j} M.
\end{aligned}$$

Therefore

$$\begin{aligned}
a &\in [(y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] : y_u \\
&\quad \cap [(y_1, \dots, y_{u-1}, x_i, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_{i+1}^{n_{i+1}+1} \cdots \mathfrak{q}_j^{n_j} M] \\
&= (y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_{i+1}^{n_{i+1}+1} \cdots \mathfrak{q}_j^{n_j} M \\
&\quad + [(y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] : y_u \cap x_i M \\
&\subseteq (y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_{i+1}^{n_{i+1}+1} \cdots \mathfrak{q}_j^{n_j} M \\
&\quad + x_i\{[(y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] : y_u\} \\
&= (y_1, \dots, y_{u-1}, \{x_\lambda \mid \lambda \in \Lambda\})M + \mathfrak{q}_i \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M.
\end{aligned}$$

Here we used Theorem 3.6(D_{ij}) and the induction hypothesis. □

4. The proof of Theorem 1.1

Before the proof of Theorem 1.1, we give some statements on $\mathbb{Z}^r$-graded rings. Let $R = \bigoplus_{n_1, \dots, n_r \geq 0} R_{(n_1, \dots, n_r)}$ be a Noetherian $\mathbb{Z}^r$-graded ring. For such a ring, let $R_+ = \bigoplus_{(n_1, \dots, n_r) \neq (0, \dots, 0)} R_{(n_1, \dots, n_r)}$.

Proposition 4.1. *Let M be a finitely generated graded R-module and $\mathfrak{b}$ an ideal in $R_{(0, \dots, 0)}$. Then there exists an integer n such that*

$$[H^p_{\mathfrak{b}R + R_+}(M)]_{(n_1, \dots, n_r)} = 0 \quad \textit{unless } n_1, \dots, n_r < n$$

for all $p \geq 0$.

Proof. If $\mathfrak{b} = (0)$, then we can prove the assertion in the same way as [28, no. 66 Théorème 2]. The spectral sequence $E_2^{pq} = H^p_{\mathfrak{b}R} H^q_{R_+}(-) \Rightarrow H^{p+q}_{\mathfrak{b}R+R_+}(-)$ says that the assertion holds in general. □

Let $\varphi\colon \mathbb{Z}^r \to \mathbb{Z}^s$ be a group homomorphism satisfying $\varphi(\mathbb{N}^r) \subseteq \mathbb{N}^s$. We put

$$R^\varphi = \bigoplus_{m_1,\dots,m_s \geq 0} \left(\bigoplus_{\varphi(n_1,\dots,n_r)=(m_1,\dots,m_s)} R_{(n_1,\dots,n_r)} \right),$$

which is a $\mathbb{Z}^s$-graded ring. For a graded R-module M, let

$$M^\varphi = \bigoplus_{m_1,\dots,m_s \in \mathbb{Z}} \left(\bigoplus_{\varphi(n_1,\dots,n_r)=(m_1,\dots,m_s)} M_{(n_1,\dots,n_r)} \right),$$

which is a graded R^φ-module. We know that

$$[H^p_{\mathfrak{b}R+R_+}(M)]^\varphi = H^p_{\mathfrak{b}R^\varphi+(R^\varphi)_+}(M^\varphi)$$

for any ideal $\mathfrak{b}$ in $R_{(0,\dots,0)}$. See Lemma 1.1 of [15].

The following proposition is contained in the proof of [15, Theorem 2.2].

Proposition 4.2. *Let $M = \bigoplus_{n_1,\dots,n_r \geq 0} M_{(n_1,\dots,n_r)}$ be a finitely generated graded R-module and $\mathfrak{b}$ an ideal in $R_{(0,\dots,0)}$. We put*

$$S = \bigoplus_{n_1,\dots,n_{r+1} \geq 0} R_{(n_1,\dots,n_{r-1},n_r+n_{r+1})}$$

and

$$N = \bigoplus_{n_1,\dots,n_{r+1} \geq 0} M_{(n_1,\dots,n_{r-1},n_r+n_{r+1})}.$$

Then S is a Noetherian $\mathbb{Z}^{r+1}$-graded ring and N a finitely generated graded S-module.

If there exists an integer p_0 such that

$$H^p_{\mathfrak{b}R+R_+}(M) = 0 \quad \text{for all } p > p_0, \tag{4.2.1}$$

then

$$H^p_{\mathfrak{b}S+S_+}(N) = 0 \quad \text{for all } p > p_0 + 1.$$

If

$$[H^p_{\mathfrak{b}R+R_+}(M)]_{(n_1,\dots,n_r)} = 0 \quad \text{unless } n_1, \dots, n_r < 0 \tag{4.2.2}$$

for all p, then

$$[H^p_{\mathfrak{b}S+S_+}(N)]_{(n_1,\dots,n_{r+1})} = 0 \quad \text{unless } n_1, \dots, n_{r+1} < 0$$

for all p. If, in addition, there exist integers $p_0 > 0$ and $n_0 < 0$ such that

$$[H^p_{\mathfrak{b}R+R_+}(M)]_{(n_1,\dots,n_r)} = 0 \quad \text{whenever } n_1 + \cdots + n_r \leq n_0 \tag{4.2.3}$$

for all $p < p_0$, then

$$[H^p_{\mathfrak{b}S+S_+}(N)]_{(n_1,\dots,n_{r+1})} = 0 \quad \text{whenever } n_1 + \cdots + n_{r+1} \leq n_0$$

for all $p < p_0 + 1$.

Proof. It is easy to show that S is a $\mathbb{Z}^{r+1}$-graded ring and N a graded S-module. First we show that S is Noetherian. To do this, we may assume that $r=1$ without loss of generality. Since R is Noetherian, R_0 is also and R is generated by finitely generated R_0-modules $R_1, \dots, R_k$ over R_0. Then $S = S_{(0,0)}[S_{(n_1,n_2)} \mid n_1+n_2 \leq k]$. Indeed, if $i+j>k$, then $R_{i+j} = R_1R_{i+j-1} + \cdots + R_kR_{i+j-k}$. Therefore

$$S_{(i,j)} = \begin{cases} \sum_{l=1}^{k} S_{(l,0)}S_{(i-l,j)}, & \text{if } i \geq k; \\ \sum_{l=1}^{i} S_{(l,0)}S_{(i-l,j)} + \sum_{m=1}^{k-i} S_{(i,m)}S_{(0,j-m)}, & \text{if } i < k. \end{cases}$$

We can show that $S_{(i,j)} \subset S_{(0,0)}[S_{(n_1,n_2)} \mid n_1+n_2 \leq k]$ by induction on $i+j$. Similarly we can prove that N is a finitely generated S-module.

Next we consider local cohomology modules. Let

$$I = \bigoplus_{n_1,\dots,n_r \geq 0, n_{r+1} > 0} R_{(n_1,\dots,n_{r-1},n_r+n_{r+1})}$$

and

$$L_1 = \bigoplus_{n_1,\dots,n_r \geq 0, n_{r+1} > 0} M_{(n_1,\dots,n_{r-1},n_r+n_{r+1})}.$$

If we put $\varphi(n_1,\dots,n_r) = (n_1,\dots,n_r,0)$, then $S/I \cong R^{\varphi}$ and $N/L_1 \cong M^{\varphi}$. Therefore

$$[H^p_{\mathfrak{b}S+S_+}(N/L_1)]_{(n_1,\dots,n_{r+1})} = \begin{cases} [H^p_{\mathfrak{b}R+R_+}(M)]_{(n_1,\dots,n_r)}, & \text{if } n_{r+1} = 0; \\ 0, & \text{otherwise} \end{cases}$$

for all p. Similarly we put

$$L_2 = \bigoplus_{n_1,\dots,n_{r-1},n_{r+1} \geq 0, n_r > 0} M_{(n_1,\dots,n_{r-1},n_r+n_{r+1})}.$$

Then

$$[H^p_{\mathfrak{b}S+S_+}(N/L_2)]_{(n_1,\dots,n_{r+1})} = \begin{cases} [H^p_{\mathfrak{b}R+R_+}(M)]_{(n_1,\dots,n_{r-1},n_{r+1})}, & \text{if } n_r = 0; \\ 0, & \text{otherwise} \end{cases}$$

for all p.

There exist two long exact sequences of local cohomology modules

$$\cdots \to H^{p-1}_{\mathfrak{b}S+S_+}(N/L_i) \to H^p_{\mathfrak{b}S+S_+}(L_i) \to H^p_{\mathfrak{b}S+S_+}(N) \to H^p_{\mathfrak{b}S+S_+}(N/L_i) \to \cdots$$

for $i=1$ and 2. On the other hand, $L_1 \cong L_2(0,\dots,0,1,-1)$.

Assume that (4.2.1) holds. If $p > p_0+1$, then

$$\begin{aligned} [H^p_{\mathfrak{b}S+S_+}(N)]_{(n_1,\dots,n_{r+1})} &\cong [H^p_{\mathfrak{b}S+S_+}(L_1)]_{(n_1,\dots,n_{r+1})} \\ &\cong [H^p_{\mathfrak{b}S+S_+}(L_2)]_{(n_1,\dots,n_{r-1},n_r+1,n_{r+1}-1)} \\ &\cong [H^p_{\mathfrak{b}S+S_+}(N)]_{(n_1,\dots,n_{r-1},n_r+1,n_{r+1}-1)} \\ &\cong \cdots = 0. \end{aligned}$$

Here we used Proposition 4.1.

Next we assume that (4.2.2) holds for all p. Unless $n_1, \dots, n_r < 0$, then

$$\begin{aligned}[H^p_{\mathfrak{b}S+S_+}(N)]_{(n_1,\dots,n_{r+1})} &\cong [H^p_{\mathfrak{b}S+S_+}(L_1)]_{(n_1,\dots,n_{r+1})}\\ &\cong [H^p_{\mathfrak{b}S+S_+}(L_2)]_{(n_1,\dots,n_{r-1},n_r+1,n_{r+1}-1)}\\ &\cong [H^p_{\mathfrak{b}S+S_+}(N)]_{(n_1,\dots,n_{r-1},n_r+1,n_{r+1}-1)}\\ &\cong \cdots = 0.\end{aligned}$$

We can also show that $[H^p_{\mathfrak{b}S+S_+}(L)]_{(n_1,\dots,n_{r+1})} = 0$ if $n_{r+1} \geq 0$. In addition, we also assume that (4.2.3) holds for all $p < p_0$. If $p < p_0 + 1$, $n_1 + \cdots + n_{r+1} \leq n_0$, and $n_1, \dots, n_{r+1} < 0$, then

$$\begin{aligned}[H^p_{\mathfrak{b}S+S_+}(N)]_{(n_1,\dots,n_{r+1})} &\cong [H^p_{\mathfrak{b}S+S_+}(L_1)]_{(n_1,\dots,n_{r+1})}\\ &\cong [H^p_{\mathfrak{b}S+S_+}(L_2)]_{(n_1,\dots,n_{r-1},n_r+1,n_{r+1}-1)}\\ &\subseteq [H^p_{\mathfrak{b}S+S_+}(N)]_{(n_1,\dots,n_{r-1},n_r+1,n_{r+1}-1)}\\ &\cong \cdots = 0.\end{aligned}$$

The proof is completed. □

Let $\mathfrak{b}_1, \dots, \mathfrak{b}_r$ be ideals in A. The multigraded Rees algebra of A (for short, the *multi-Rees algebra*) with respect to them is defined to be

$$R(\mathfrak{b}_1, \dots, \mathfrak{b}_r) = A[\mathfrak{b}_1 T_1, \dots, \mathfrak{b}_r T_r],$$

where $T_1, \dots, T_r$ are indeterminates. If $\mathfrak{b}_1, \dots, \mathfrak{b}_r$ are of positive height, then $\dim R(\mathfrak{b}_1, \dots, \mathfrak{b}_r) = \dim A + r$. See Proposition 1.17 of [15]. For an A-module M, let $R_M(\mathfrak{b}_1, \dots, \mathfrak{b}_r)$ denote the $R(\mathfrak{b}_1, \dots, \mathfrak{b}_r)$-module

$$\bigoplus_{n_1,\dots,n_r \geq 0} \mathfrak{b}_1^{n_1} \cdots \mathfrak{b}_r^{n_r} M T_1^{n_1} \cdots T_r^{n_r}.$$

Recently Hyry gives the following theorem.

Theorem 4.3 ([16, Corollary 2.10])**.** *Let $\mathfrak{b}_1, \dots, \mathfrak{b}_r$ be ideals in A of positive height. If the multi-Rees algebra $R(\mathfrak{b}_1, \dots, \mathfrak{b}_r)$ is Cohen-Macaulay, then the ordinary Rees algebra $R(\mathfrak{b}_1 \cdots \mathfrak{b}_r)$ is also Cohen-Macaulay.*

We start to prove Theorem 1.1.

Theorem 4.4. *Let M be a finitely generated A-module and $x_t, \dots, x_d$ elements in A. We fix integers $t \leq s+1 < d$, $\alpha_t, \dots, \alpha_s > 0$, and $\alpha_{s+1} \geq d-s-1$. Let $\mathfrak{q}_i = (x_i, \dots, x_d)$ for all $t \leq i \leq s+1$. We put*

$$S = A[\mathfrak{q}_t T_{t,1}, \dots, \mathfrak{q}_t T_{t,\alpha_t}, \mathfrak{q}_{t+1} T_{t+1,1}, \dots, \mathfrak{q}_s T_{s,\alpha_s}, \mathfrak{q}_{s+1} T_{s+1,1}, \dots, \mathfrak{q}_{s+1} T_{s+1,\alpha_{s+1}}]$$

and N the S-module $R_M(\mathfrak{q}_t, \dots, \mathfrak{q}_{s+1})$. If the sequence $x_t, \dots, x_d$ satisfies the following six conditions:

1. *the sequence $x_i, \dots, x_d$ is a d-sequence on $M/(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M$ for all $t \leq i \leq s+1$, $n_t, \dots, n_{i-1} > 0$, and $\Lambda \subseteq \{t, \dots, i-1\}$;*
2. *the sequence $x_i, \dots, x_{d-1}$ is a d-sequence on $M/(\{x_\lambda \mid \lambda \in \Lambda\}, x_d)M$ for all $t \leq i \leq s+1$, $n_t, \dots, n_{i-1} > 0$, and $\Lambda \subseteq \{t, \dots, i-1\}$;*
3. *the sequence $x_{s+1}, \dots, x_d$ is a u.s.d-sequence on $M/(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M$ for all $n_t, \dots, n_s > 0$ and $\Lambda \subseteq \{t, \dots, s\}$;*

(4) *the equality*

$$(\{x_\lambda^{n_\lambda} \mid \lambda \in \Lambda\}, x_k, \ldots, x_{l-1})M : x_l \cap [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_{s+1}^{n_{s+1}} M] = (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M + (x_k, \ldots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k - 1} \cdots \mathfrak{q}_{s+1}^{n_{s+1}} M$$

holds for any integers $t \le i \le k \le s+1$, $k \le l \le d$, $n_t, \ldots, n_{i-1}, n_k > 0$, $n_i, \ldots, n_{k-1}, n_{k+1}, \ldots, n_{s+1} \ge 0$, *and* $\Lambda \subseteq \{t, \ldots, i-1\}$;

(5) *the equality*

$$[(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_{s+1}^{n_{s+1}} M] : x_{i-1}^{n_{i-1}} = [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_{s+1}^{n_{s+1}} M] : \mathfrak{q}_{i-1}$$

holds for any $t < i \le s+1$, $n_t, \ldots, n_{i-1} > 0$, $n_i, \ldots, n_{s+1} \ge 0$, *and* $\Lambda \subset \{t, \ldots, i-2\}$;

(6) $0 :_M x_d \subseteq 0 :_M x_t$,

then

$$H^0_{\mathfrak{q}_t S + S_+}(N) = 0 \underset{M}{:} x_d, \tag{4.4.1}$$

$$H^p_{\mathfrak{q}_t S + S_+}(N) = 0 \quad \text{for } p \ne 0,\ d - t + 1 + \alpha_t + \cdots + \alpha_{s+1}, \tag{4.4.2}$$

and

$$[H^{d-t+1+\alpha_t + \cdots + \alpha_{s+1}}_{\mathfrak{q}_t S + S_+}(N)]_{(n_{t,1}, \ldots, n_{s+1,\alpha_{s+1}})} = 0, \tag{4.4.3}$$

unless $n_{t,1}, \ldots, n_{s+1,\alpha_{s+1}} < 0$.

Proof. We show that (4.4.1)–(4.4.3) by descending induction on t. First we note that $d - s \ge 2$ because of the assumption. Furthermore $0 :_M x_t \subset \cdots \subset 0 :_M x_d$ because $x_t, \ldots, x_d$ is a d-sequence on M. Therefore (1) and (6) say that $0 :_M x_t = \cdots = 0 :_M x_d$. Without loss of generality, we may assume that $0 :_M x_d = 0$. Indeed, assumptions (1)–(6) hold on $\overline{M} = M/0 :_M x_d$. For example,

$$\begin{aligned}
&[(\{x_\lambda^{n_\lambda} \mid \lambda \in \Lambda\}, x_k, \ldots, x_{l-1})M + 0 \underset{M}{:} x_t] : x_l \\
&\quad = (\{x_\lambda^{n_\lambda} \mid \lambda \in \Lambda\}, x_k, \ldots, x_{l-1})M : x_l^2 \\
&\quad = (\{x_\lambda^{n_\lambda} \mid \lambda \in \Lambda\}, x_k, \ldots, x_{l-1})M : x_l
\end{aligned}$$

because $0 :_M x_t \subset 0 :_M x_l$. Hence

$$\begin{aligned}
&(\{x_\lambda^{n_\lambda} \mid \lambda \in \Lambda\}, x_k, \ldots, x_{l-1})M] : x_l \cap [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_{s+1}^{n_{s+1}} M + 0 \underset{M}{:} x_t] \\
&\quad = (\{x_\lambda^{n_\lambda} \mid \lambda \in \Lambda\}, x_k, \ldots, x_{l-1})M : x_l \cap [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_{s+1}^{n_{s+1}} M] \\
&\qquad + 0 \underset{M}{:} x_t \\
&\quad = (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M + (x_k, \ldots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k - 1} \cdots \mathfrak{q}_{s+1}^{n_{s+1}} M + 0 \underset{M}{:} x_t.
\end{aligned}$$

Thus (4) holds on $\overline{M}$. Similarly we can show that (1)–(3) and (5) hold on $\overline{M}$. Of course $0 :_{\overline{M}} x_t = 0 :_{\overline{M}} x_d = 0$. On the other hand, if $\overline{N}$ denotes the S-module $R_{\overline{M}}(\mathfrak{q}_t, \ldots, \mathfrak{q}_{s+1})$, then there exists an exact sequence of S-modules

$$0 \to 0 \underset{M}{:} x_t \to N \to \overline{N} \to 0.$$

Since $0 :_M x_t$ is annihilated by $\mathfrak{q}_t S + S_+$,

$$0 \to 0 \underset{M}{:} x_t \to H^0_{\mathfrak{q}_t S + S_+}(N) \to H^0_{\mathfrak{q}_t S + S_+}(\overline{N}) \to 0$$

is exact and

$$H^p_{\mathfrak{q}_t S+S_+}(\overline{N}) \cong H^p_{\mathfrak{q}_t S+S_+}(N) \quad \text{for all } p > 0.$$

Thus if the assertion holds for $\overline{M}$, then the one holds for M.

From now on we assume that $0 :_M x_t = \cdots = 0 :_M x_d = 0$. Because of Proposition 4.2, we may assume that $\alpha_t = \cdots = \alpha_s = 1$ and $\alpha_{s+1} = d - s - 1$. For the simplicity, we write $T_t = T_{t,1}, \dots, T_{s+1} = T_{s+1,1}, T_{s+2} = T_{s+1,2}, \dots, T_{d-1} = T_{s+1,d-s-1}$.

Assume that $t = s+1$ and put $R = A[\mathfrak{q}_{s+1}T_{s+1}]$. Then we know that

$$[H^p_{\mathfrak{q}_{s+1}R+R_+}(R_M(\mathfrak{q}_{s+1}))]_n = 0 \quad \text{unless } 2 - p \le n \le -1$$

for all $p < d - s + 1$,

$$[H^{d-s+1}_{\mathfrak{q}_{s+1}R+R_+}(R_M(\mathfrak{q}_{t+1}))]_n = 0 \quad \text{unless } n < 0,$$

and

$$H^p_{\mathfrak{q}_{s+1}R+R_+}(R_M(\mathfrak{q}_{t+1})) = 0 \quad \text{for all } p > d - s + 1.$$

See [12, Theorem 4.1]. By using Proposition 4.2, repeatedly, we find that

$$H^p_{\mathfrak{q}_{s+1}S+S_+}(N) = 0 \quad \text{for } p \ne 2d - 2s - 1$$

and

$$[H^{2d-2s-1}_{\mathfrak{q}_{s+1}S+S_+}(N)]_{(n_{s+1},\dots,n_{d-1})} = 0 \quad \text{unless } n_{s+1}, \dots, n_{d-1} < 0.$$

Thus we obtain (4.4.1)–(4.4.3).

Next we assume that $t < s+1$. Then $x_t^m M : x_{t+1} = x_t^m M : x_d$ for any $m > 0$. Indeed, if $a \in x_t^m M : x_d$ and we put $x_d a = x_t^m b$, then $b \in x_d M : x_t^m \subseteq x_d M : x_{t+1}$ because of (2). Let $x_{t+1} b = x_d c$. Then $x_{t+1} x_d a = x_t^m x_{t+1} b = x_t^m x_d c$. Therefore $x_{t+1} a - x_t^m c \in 0 :_M x_d = 0$ and hence $a \in x_t^m M : x_{t+1}$. Thus the sequence $x_{t+1}, \dots, x_d$ satisfies (1)–(6) on M and on $M/x_t^m M$ for any $m > 0$.

Let $R = A[\mathfrak{q}_{t+1}T_{t+1}, \dots, \mathfrak{q}_{s+1}T_{s+1}, \dots, \mathfrak{q}_{s+1}T_{d-1}]$ and

$$Y = \bigoplus_{n_{t+1},\dots,n_{d-1} \ge 0} [\mathfrak{q}_{t+1}^{n_{t+1}} \cdots \mathfrak{q}_{s+1}^{n_{s+1}+\cdots+n_{d-1}} M : \mathfrak{q}_t] T_{t+1}^{n_{t+1}} \cdots T_{d-1}^{n_{d-1}}.$$

Then assumption (5) gives an exact sequence of R-modules

$$0 \to Y \xrightarrow{x_t^m} R_M(\mathfrak{q}_{t+1} \cdots \mathfrak{q}_{s+1}) \to R_{M/x_t^m M}(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1}) \to 0$$

and hence Y is finitely generated over R. The induction hypothesis says that

$$H^p_{\mathfrak{q}_{t+1}R+R_+}(R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1})) = 0 \quad \text{for } p \ne 2d - 2t - 1,$$
$$[H^{2d-2t-1}_{\mathfrak{q}_{t+1}R+R_+}(R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1}))]_{(n_{t+1},\dots,n_{d-1})} = 0$$

unless $n_{t+1}, \dots, n_{d-1} < 0$,

$$H^p_{\mathfrak{q}_{t+1}R+R_+}(R_{M/x_t^m M}(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1})) = 0 \quad \text{for } p \ne 0,\ 2d - 2t - 1,$$

and

$$[H^{2d-2t-1}_{\mathfrak{q}_{t+1}R+R_+}(R_{M/x_t^m M}(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1}))]_{(n_{t+1},\dots,n_{d-1})} = 0$$

unless $n_{t+1}, \dots, n_{d-1} < 0$. The spectral sequence

$$E_2^{pq} = H^p_{x_t} H^q_{\mathfrak{q}_{t+1}R+R_+}(-) \Rightarrow H^{p+q}_{\mathfrak{q}_t R+R_+}(-)$$

gives a short exact sequence

$$0 \to H^1_{x_t} H^{p-1}_{\mathfrak{q}_{t+1}R+R_+}(-) \to H^p_{\mathfrak{q}_t R+R_+}(-) \to H^0_{x_t} H^p_{\mathfrak{q}_{t+1}R+R_+}(-) \to 0.$$

By using it, we obtain

$$H^p_{\mathfrak{q}_t R+R_+}(R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1})) = 0 \quad \text{for } p \neq 2d-2t-1,\, 2d-2t,$$
$$[H^{2d-2t}_{\mathfrak{q}_t R+R_+}(R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1}))]_{(n_{t+1},\dots,n_{d-1})} = 0$$

unless $n_{t+1}, \dots, n_{d-1} < 0$,

$$H^p_{\mathfrak{q}_t R+R_+}(R_{M/x_t^m M}(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1})) = 0 \quad \text{for } p \neq 0,\, 2d-2t-1,$$

and

$$[H^{2d-2t-1}_{\mathfrak{q}_t R+R_+}(R_{M/x_t^m M}(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1}))]_{(n_{t+1},\dots,n_{d-1})} = 0$$

unless $n_{t+1}, \dots, n_{d-1} < 0$. Therefore

$$H^p_{\mathfrak{q}_t R+R_+}(Y) = 0 \quad \text{for } p \neq 1,\, 2d-2t-1,\, 2d-2t,$$
$$[H^{2d-2t}_{\mathfrak{q}_t R+R_+}(Y)]_{(n_{t+1},\dots,n_{d-1})} = 0 \quad \text{unless } n_{t+1}, \dots, n_{d-1} < 0,$$

and

$$0 \to H^{2d-2t-1}_{\mathfrak{q}_t R+R_+}(Y) \to H^{2d-2t-1}_{\mathfrak{q}_t R+R_+}(R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1}))$$

is exact. We show that $H^{2d-2t-1}_{\mathfrak{q}_t R+R_+}(Y) = 0$. Let $E = H^{2d-2t-1}_{\mathfrak{q}_t R+R_+}(R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1}))$. Because of (5),

$$\mathfrak{q}_t Y \subseteq R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1}) \subseteq Y.$$

Therefore

$$H^p_{\mathfrak{q}_t R+R_+}(Y/R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1})) \cong H^p_{R_+}(Y/R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1})).$$

Let

$$\begin{array}{lclcccccc}
f_{2t+2} & = & x_{t+1}T_{t+1}, & & & & & & \\
f_{2t+3} & = & x_{t+2}T_{t+1}, & & & & & & \\
f_{2t+4} & = & x_{t+3}T_{t+1} & + & x_{t+2}T_{t+2}, & & & & \\
 & \vdots & & & & & & & \\
f_{d+t+1} & = & x_dT_{t+1} & + & x_{d-1}T_{t+2} & + & \cdots, & & \\
f_{d+t+2} & = & & & x_dT_{t+1} & + & \cdots, & & \\
 & \vdots & & & & & & & \\
f_{2d-2} & = & & & & & x_dT_{d-2} & + & x_{d-1}T_{d-1}, \\
f_{2d-1} & = & & & & & & & x_dT_{d-1}.
\end{array}$$

Then $\sqrt{R_+} = \sqrt{(f_{2t+2}, \dots, f_{2d-1})R}$. The proof is quite similar to [11, Lemma 3.2]. We omit it. Therefore

$$H^p_{\mathfrak{q}_t R+R_+}(Y/R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1})) = 0 \quad \text{for } p > 2d-2t-2$$

and hence

$$H^{2d-2t-2}_{\mathfrak{q}_t R+R_+}(Y/R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1})) \to E \to H^{2d-2t-1}_{\mathfrak{q}_t R+R_+}(Y) \to 0$$

is exact. Thus

$$H^{2d-2t-1}_{\mathfrak{q}_t R+R_+}(Y/R_M(\mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1})) \to E \xrightarrow{x_t^m} E \tag{4.4.4}$$

is exact. Since the first term of (4.4.4) is annihilated by x_t, we obtain $0 :_E x_t^m = 0 :_E x_t$. Therefore $x_t E = 0$ and hence $H^{2d-2t-1}_{\mathfrak{q}_t R + R_+}(Y) = 0$ because $E = \bigcup_{m>0} 0 :_E x_t^m$. Since $R = S/\mathfrak{q}_t T_t S$, Y is also an S-module and

$$\text{(4.4.5)} \qquad \begin{aligned} &H^p_{\mathfrak{q}_t S + S_+}(Y) = 0 \quad \text{for } p \neq 1,\, 2d-2t, \\ &[H^{2d-2t}_{\mathfrak{q}_t S + S_+}(Y)]_{(n_t, \dots, n_{d-1})} = 0 \quad \text{unless } n_t = 0,\, n_{t-1},\, \dots,\, n_{d-1} < 0. \end{aligned}$$

Let $S' = A[\mathfrak{q}_{t+1} T_t, \mathfrak{q}_{t+1} T_{t+1}, \dots, \mathfrak{q}_s T_s, \mathfrak{q}_{s+1} T_{s+1}, \dots, \mathfrak{q}_{s+1} T_{d-1}]$. Then the induction hypothesis says that

$$H^p_{\mathfrak{q}_{t+1} S' + S'_+}(R_{M/x_t M}(\mathfrak{q}_{t+1}, \mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1})) = 0 \quad \text{for } p \neq 0,\, 2d-2t$$

and

$$[H^{2d-2t}_{\mathfrak{q}_{t+1} S' + S'_+}(R_{M/x_t M}(\mathfrak{q}_{t+1}, \mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1}))]_{(n_t, \dots, n_{d-1})} = 0$$

unless $n_t, \dots, n_{d-1} < 0$. Since S' is an A-subalgebra of S, we can regard the S-module $R_{M/x_t M}(\mathfrak{q}_t, \dots, \mathfrak{q}_{t+1})$ as an S'-module and there exists an S'-isomorphism

$$R_{M/x_t M}(\mathfrak{q}_t, \mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1}) \cong R_{M/x_t M}(\mathfrak{q}_{t+1}, \mathfrak{q}_{t+1}, \dots, \mathfrak{q}_{s+1}).$$

Since $(x_t, x_t T_t) R_{M/x_t M}(\mathfrak{q}_t, \dots, \mathfrak{q}_{s+1}) = 0$,

$$\text{(4.4.6)} \qquad \begin{aligned} &H^p_{\mathfrak{q}_t S + S_+}(R_{M/x_t M}(\mathfrak{q}_t, \dots, \mathfrak{q}_{s+1})) \\ &\quad = H^p_{(\mathfrak{q}_{t+1} S' + S'_+) S}(R_{M/x_t M}(\mathfrak{q}_t, \dots, \mathfrak{q}_{s+1})) \\ &\quad = H^p_{\mathfrak{q}_{t+1} S' + S'_+}(R_{M/x_t M}(\mathfrak{q}_t, \dots, \mathfrak{q}_{s+1})) = 0 \end{aligned}$$

for $p \neq 0,\, 2d-2t$ and

$$[H^{2d-2t}_{\mathfrak{q}_t S + S_+}(R_{M/x_t M}(\mathfrak{q}_t, \dots, \mathfrak{q}_{s+1}))]_{(n_t, \dots, n_{d-1})} = 0 \quad \text{unless } n_t,\, \dots,\, n_{d-1} < 0.$$

Let X be the kernel of the natural epimorphism $N \to R_{M/x_t M}(\mathfrak{q}_t, \dots, \mathfrak{q}_{s+1})$. Then there exists an exact sequence of S-modules

$$0 \to X \to N \to R_{M/x_t M}(\mathfrak{q}_t, \dots, \mathfrak{q}_{s+1}) \to 0.$$

Since

$$x_t M \cap \mathfrak{q}_t^{n_t} \cdots \mathfrak{q}_{s+1}^{n_{s+1} + \cdots + n_{d-1}} M = x_t \mathfrak{q}_t^{n_t - 1} \cdots \mathfrak{q}_{s+1}^{n_{s+1} + \cdots + n_{d-1}} M$$

if $n_t > 0$,

$$\bigoplus_{n_t > 0} X_{(n_t, \dots, n_{d-1})} = x_t T_t N$$

and there exists an exact sequence

$$0 \to N(-1, 0, \dots, 0) \xrightarrow{x_t T_t} X \xrightarrow{x_t^{-1}} Y \to 0.$$

Because of (4.4.5) and (4.4.6),

$$0 \to H^p_{\mathfrak{q}_t S + S_+}(N)(-1, 0, \dots, 0) \xrightarrow{x_t T_t} H^p_{\mathfrak{q}_t S + S_+}(N)$$

is exact if $3 \leq p < 2d - 2t + 1$ or $p > 2d - 2t + 1$. Since $H^p_{\mathfrak{q}_t S + S_+}(N)$ is annihilated by some power of $x_t T_t$ elementwise,

$$H^p_{\mathfrak{q}_t S + S_+}(N) = 0 \quad \text{if } 3 \leq p < 2d - 2t + 1 \text{ or } p > 2d - 2t + 1.$$

Furthermore

$$H^{2d-2t}_{\mathfrak{q}_t S + S_+}(Y) \to H^{2d-2t+1}_{\mathfrak{q}_t S + S_+}(N)(-1, 0, \dots, 0) \to H^{2d-2t+1}_{\mathfrak{q}_t S + S_+}(X) \to 0$$

and

$$H^{2d-2t}_{\mathfrak{q}_t S+S_+}(R_{M/x_tM}(\mathfrak{q}_t,\dots,\mathfrak{q}_{s+1})) \to H^{2d-2t+1}_{\mathfrak{q}_t S+S_+}(X) \to H^{2d-2t+1}_{\mathfrak{q}_t S+S_+}(N) \to 0$$

are exact. Unless $n_t, \dots, n_{d-1} < 0$, then we obtain

$$\begin{aligned}[H^{2d-2t+1}_{\mathfrak{q}_t S+S_+}(N)]_{(n_t,\dots,n_{d-1})} &\cong [H^{2d-2t+1}_{\mathfrak{q}_t S+S_+}(X)]_{(n_t+1,n_{t+1},\dots,n_{d-1})}\\ &\cong [H^{2d-2t+1}_{\mathfrak{q}_t S+S_+}(N)]_{(n_t+1,n_{t+1},\dots,n_{d-1})}\\ &\cong \cdots = 0.\end{aligned}$$

Thus (4.4.2) is proved.

Finally we show that x_sT_s, $x_{s+1}T_{s+1}$, x_{s+2} is a regular sequence on N. Since x_s is regular on M, x_sT_s is regular on N.

Let $aT_t^{n_t}\cdots T_{d-1}^{n_{d-1}} \in x_sT_sN : x_{s+1}T_{s+1}$. If $n_s = 0$, then $x_{s+1}a = 0$ and hence $a = 0$. If $n_s > 0$, then

$$a \in x_sM : x_{s+1} \cap \mathfrak{q}_t^{n_t}\cdots\mathfrak{q}_{s+1}^{n_{s+1}+\cdots+n_{d-1}}M = x_s\mathfrak{q}_t^{n_t}\cdots\mathfrak{q}_s^{n_s-1}\mathfrak{q}_{s+1}^{n_{s+1}+\cdots+n_{d-1}}M.$$

Here we used (4). Hence $aT_t^{n_t}\cdots T_{d-1}^{n_{d-1}} \in x_sT_sN$.

Let $aT_t^{n_t}\cdots T_{d-1}^{n_{d-1}} \in (x_sT_s, x_{s+1}T_{s+1})N : x_{s+2}$. If $n_s = n_{s+1} = 0$, then $x_{s+2}a = 0$ and hence $a = 0$. If $n_s > 0$ and $n_{s+1} = 0$, then $a \in x_sM : x_{s+2}$. Because of (3), we have $x_sM : x_{s+1} = x_sM : x_{s+2}$. Hence

$$a \in x_sM : x_{s+1} \cap \mathfrak{q}_t^{n_t}\cdots\mathfrak{q}_{s+1}^{n_{s+1}+\cdots+n_{d-1}}M = x_s\mathfrak{q}_t^{n_t}\cdots\mathfrak{q}_s^{n_s-1}\mathfrak{q}_{s+1}^{n_{s+1}+\cdots+n_{d-1}}M,$$

that is, $aT_t^{n_t}\cdots T_{d-1}^{n_{d-1}} \in x_sT_sN$. If $n_s = 0$ and $n_{s+1} > 0$, then

$$a \in x_{s+1}M : x_{s+2} \cap \mathfrak{q}_t^{n_t}\cdots\mathfrak{q}_{s+1}^{n_{s+1}+\cdots+n_{d-1}}M = x_{s+1}\mathfrak{q}_t^{n_t}\cdots\mathfrak{q}_{s+1}^{n_{s+1}+\cdots+n_{d-1}-1}M$$

and hence $aT_t^{n_t}\cdots T_{d-1}^{n_{d-1}} \in x_{s+1}T_{s+1}N$. If n_s, $n_{s+1} > 0$, then

$$\begin{aligned}a &\in (x_s,x_{s+1})M : x_{s+2} \cap \mathfrak{q}_t^{n_t}\cdots\mathfrak{q}_{s+1}^{n_{s+1}+\cdots+n_{d-1}}M\\ &= (x_s,x_{s+1})\mathfrak{q}_t^{n_t}\cdots\mathfrak{q}_s^{n_s-1}\mathfrak{q}_{s+1}^{n_{s+1}+\cdots+n_{d-1}}M\\ &= x_s\mathfrak{q}_t^{n_t}\cdots\mathfrak{q}_s^{n_s-1}\mathfrak{q}_{s+1}^{n_{s+1}+\cdots+n_{d-1}}M + x_{s+1}\mathfrak{q}_t^{n_t}\cdots\mathfrak{q}_{s+1}^{n_{s+1}+\cdots+n_{d-1}-1}M.\end{aligned}$$

Therefore $aT_t^{n_t}\cdots T_{d-1}^{n_{d-1}} \in (x_sT_s, x_{s+1}T_{s+1})N$.

Thus we obtain

$$H^p_{\mathfrak{q}_t S+S_+}(N) = 0 \quad \text{for } p < 3.$$

The proof is completed. □

Corollary 4.5. *Let A be a Noetherian local ring of dimension $d \geq 2$ and $x_1, \dots, x_d$ a p-standard system of parameters of type s for A. We put $\mathfrak{q}_i = (x_i, \dots, x_d)$ for all $1 \leq i \leq s+1$. If $s < d-1$ and $(0):x_d = 0$, then the Rees algebra $R(\mathfrak{q}_1\cdots\mathfrak{q}_s\mathfrak{q}_{s+1}^{d-s-1})$ is a Cohen-Macaulay ring. If, in addition, $A/\mathfrak{q}_t$ is Cohen-Macaulay for some $1 < t \leq s+1$, then $R(\mathfrak{q}_t\cdots\mathfrak{q}_s\mathfrak{q}_{s+1}^{d-s-1})$ is a Cohen-Macaulay ring.*

Proof. In this case Propositions 3.3, 3.5, Theorem 3.6, and Corollary 3.7 say that $x_1, \dots, x_d$ satisfies assumptions (1)–(5) of Theorem 4.4. Moreover $(0):x_1 \supseteq (0):x_d = 0$. Thus we find that $A[\mathfrak{q}_1T_1, \dots, \mathfrak{q}_sT_s, \mathfrak{q}_{s+1}T_{s+1}, \dots, \mathfrak{q}_{s+1}T_{d-1}]$ is Cohen-Macaulay by using Theorem 4.4. Hyry's theorem says that $R(\mathfrak{q}_1\cdots\mathfrak{q}_s\mathfrak{q}_{s+1}^{d-s-1})$ is Cohen-Macaulay.

Assume that $A/\mathfrak{q}_t$ is Cohen-Macaulay. That is, $x_1, \dots, x_{t-1}$ is a regular sequence on $A/\mathfrak{q}_t$. We show that

$$(x_1, \dots, x_i) : x_d = (x_1, \dots, x_i) \quad \text{for } 1 \le i \le t-1$$

by induction on i. If $i = 0$, then there exists nothing to prove. Assume that $i > 0$ and let $a \in (x_1, \dots, x_i) : x_d$. If we put $x_d a = b + x_i c$ with $b \in (x_1, \dots, x_{i-1})$, then

$$\begin{aligned} c &\in (x_1, \dots, x_{i-1}, x_d) : x_i \\ &= (x_1, \dots, x_{i-1}, x_d). \end{aligned}$$

Here we used Corollary 3.8. Let $c = b' + x_d a'$ with $b' \in (x_1, \dots, x_{i-1})$. Then

$$a - x_i a' \in (x_1, \dots, x_{i-1}) : x_d = (x_1, \dots, x_{i-1})$$

because of the induction hypothesis. Therefore $a \in (x_1, \dots, x_i)$. Thus $x_t, \dots, x_d$ satisfies the assumptions of Theorem 4.4 on $\bar{A} = A/(x_1, \dots, x_{t-1})$. Therefore

$$\bar{A}[\mathfrak{q}_t \bar{A} T_t, \dots, \mathfrak{q}_s \bar{A} T_s, \mathfrak{q}_{s+1} \bar{A} T_{s+1}, \dots, \mathfrak{q}_{s+1} \bar{A} T_{d-1}]$$

is a Cohen-Macaulay ring and hence $R(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1} \bar{A})$ is also. Corollary 3.8 also says that $x_1, \dots, x_{t-1}$ is a regular sequence on A and on $A/(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1})^n$ for all $n > 0$. Taking Koszul cohomology of a short exact sequence

$$0 \to R(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1}) \to A[T] \to \bigoplus_{n>0} (A/(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1})^n) T^n \to 0$$

with respect to $x_1, \dots, x_{t-1}$, we obtain that

$$H^p(x_1, \dots, x_{t-1}; R(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1})) = 0 \quad \text{for } p < t-1$$

and

$$H^{t-1}(x_1, \dots, x_{t-1}; R(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1})) \cong R(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1} \bar{A}).$$

That is, $x_1, \dots, x_{t-1}$ is a regular sequence on $R(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1})$ and

$$R(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1} \bar{A}) \cong R(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1}) / (x_1, \dots, x_{t-1}) R(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1}).$$

Therefore $R(\mathfrak{q}_t \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1})$ is a Cohen-Macaulay ring. □

Proof of Theorem 1.1. Let A be a Noetherian local ring of dimension $d > 0$. First we prove that (B) implies (A). Assume that A satisfies (B). If $d = 1$, then A is Cohen-Macaulay because A has no embedded prime. Let a be a system of parameters for A. Then $R(aA)$ is a polynomial ring over A and hence Cohen-Macaulay.

Assume that $d \ge 2$. Since A is unmixed, $\dim A/\mathfrak{p} = d$ for any associated prime $\mathfrak{p}$ of A. Thus $s = \dim A/\mathfrak{a}(A) < d-1$ because of Corollary 2.4. Theorem 2.5 assures us that there exists a p-standard system of parameters $x_1, \dots, x_d$ of type s for A. Since A is unmixed, $x_1, \dots, x_d$ are non-zero divisors on A. Therefore Corollary 4.5 gives an arithmetic Macaulayfication of A.

Next we show that (A) implies (B). Let $\mathfrak{b}$ be an ideal in A of positive height such that $R = A[\mathfrak{b}T]$ is a Cohen-Macaulay ring. Then A is a homomorphic image of a Cohen-Macaulay local ring $R_{\mathfrak{m}R+R_+}$ and hence all the formal fibers of A are Cohen-Macaulay. Next we show that A is unmixed. By passing through the completion, we may assume that A is complete. Since $\mathfrak{b}$ is of positive height, $\dim R = d+1$. See [32, Corollary 1.6]. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the associated primes of A. Then

$$\mathfrak{p}_i^* = \mathfrak{p}_i A[T] \cap R \quad \text{where } i = 1, \dots, s$$

are the associated primes of R. Since R is a Cohen-Macaulay ring of dimension $d+1$, $\dim R/\mathfrak{p}_i^* = d+1$ and hence $\dim A/\mathfrak{p}_i = d$; see [32, Corollary 1.6] again, for all i. $\square$

To close this section, we give an example.

Example 4.6. Let k be a field, B an affine semigroup ring

$$k[a, b, c, d, e^2, e^3, ade, bde, cde, d^2e]$$

and $\mathfrak{n}$ the homogeneous maximal ideal of B. Then $A = B_{\mathfrak{n}}$ is a Noetherian local ring of dimension 5. The sequence $x_1 = a^4$, $x_2 = b^4$, $x_3 = c^4$, $x_4 = d^4$, $x_5 = e^4$ is a p-standard system of parameter of type 3 for A. See [17, Appendix B].

Let $\mathfrak{q}_i = (x_i, \dots, x_d)$ for $i = 1, \dots, 4$. Then the proof of Corollary 4.5 says that the multi-Rees algebra $A[\mathfrak{q}_1T_1, \dots, \mathfrak{q}_4T_4]$ is a Cohen-Macaulay ring of dimension 9. However, we can verify that it is a Cohen-Macaulay ring by using a computer [6]. Indeed the sequence x_1, $x_1T_1 + x_2$, $x_2T_1 + x_3$, $x_2T_2 + x_3T_1 + x_4$, $x_3T_2 + x_4T_1 + x_5$, $x_3T_3 + x_4T_2 + x_5T_1$, $x_4T_3 + x_5T_2$, $x_4T_4 + x_5T_3$, x_5T_4 is a regular sequence on $A[\mathfrak{q}_1T_1, \dots, \mathfrak{q}_4T_4]$ of length 9.

5. The proof of Corollary 1.2

Before proving Corollary 1.2, we state the definition of the codimension function.

Definition 5.1. Let B be a Noetherian ring. An integer-valued function t_B defined on $\operatorname{Spec} B$ is said to be a *codimension function* of B if

$$\operatorname{ht} \mathfrak{p}_1/\mathfrak{p}_2 = t_B(\mathfrak{p}_1) - t_B(\mathfrak{p}_2) \quad \text{whenever } \mathfrak{p}_1 \supseteq \mathfrak{p}_2.$$

A codimension function of B is not unique even if it exists. In fact, if $t(\mathfrak{p})$ is a codimension function, then $t(\mathfrak{p})+c$ is also a codimension function for any constant c. However, the codimension function is unique up to constant if $\operatorname{Spec} B$ is connected.

Proposition 5.2. (1) *A catenary local ring has a codimension function.*

(2) *A catenary integral domain has a codimension function.*
(3) *A Cohen-Macaulay ring has a codimension function even if it is neither a local ring nor an integral domain.*
(4) *If a Noetherian ring has a codimension function, then its homomorphic image does also.*
(5) *If a Noetherian ring has a codimension function, then its localization does also.*
(6) *A Noetherian ring possessing a dualizing complex has a codimension function.*

Proof. Let B be a Noetherian ring.

(1) Let $t(\mathfrak{p}) = -\dim B/\mathfrak{p}$. If B is a cantenary local ring, then $t(\mathfrak{p})$ is a codimension function of B.

(2) Let $t(\mathfrak{p}) = \dim B_{\mathfrak{p}}$. If B is a catenary integral domain, then $t(\mathfrak{p})$ is a codimension function of B.

(3) Let $t(\mathfrak{p}) = \dim B_{\mathfrak{p}}$. Then $t(\mathfrak{p})$ is the codimension function of B. See the proof of [20, Theorem 17.4(ii)].

(4) and (5) Obvious.

(6) See [14, Chapter 5, §7]. $\square$

A Noetherian ring is catenary if it has a codimension function. But the converse is not necessarily true. Moreover the universally catenarity is independent of the existence of a codimension function.

Example 5.3. (1) Ogoma [24, §5 I] gave a Noetherian, universally catenary ring with no codimension function.

(2) Nagata [21, Example 2] gave a two-dimensional local integral domain which is not quasi-unmixed. It has a codimension function but is not universally catenary.

If a Noetherian ring B is universally catenary and has a codimension function, then the polynomial ring over B does also.

Theorem 5.4. *Let B be a Noetherian, universally catenary ring and C an essentially of finite type B-algebra. If B has a codimension function, then C does also.*

Proof. We may assume that C is a polynomial ring over B. Let t_B be a codimension function. We put

$$t_C(\mathfrak{q}) = t_B(\mathfrak{p}) + \operatorname{ht} \mathfrak{q}/\mathfrak{p}C \quad \text{where } \mathfrak{p} = \mathfrak{q} \cap B$$

for each prime ideal $\mathfrak{q}$ in C. Then t_C is a codimension function of C. □

The following is the key lemma for the proof of Corollary 1.2.

Lemma 5.5. *Let B be a Noetherian, universally categnary ring which has a codimension function. Then it is a homomorphic image of a finite type B-algebra C such that the codimension function of C is a constant on the associated primes of C. If, in addition, B is a local ring, then there exists a maximal ideal $\mathfrak{n}$ of C such that B is a homomorphic image of $C_{\mathfrak{n}}$.*

Proof. Let t_B be a codimension function of B and

$$(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$$

the irredundant primary decomposition of (0) in B. We may assume that

$$\sup\{t_B(\sqrt{\mathfrak{q}_i}) \mid i = 1, \ldots, s\} = 0.$$

We put $n = -\inf\{t_B(\sqrt{\mathfrak{q}_i}) \mid i = 1, \ldots, s\}$ and $n_i = -t_B(\sqrt{\mathfrak{q}_i})$ for all i. Then

$$C = B[T_1, \ldots, T_n] \Big/ \bigcap_{i=1}^{s} (\mathfrak{q}_i, T_1, \ldots, T_{n_i}) B[T_1, \ldots, T_n]$$

has the required property. If B is a local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{n} = \mathfrak{m}C + (T_1, \ldots, T_n)C$ has the required property. □

Proof of Corollary 1.2. The only if part is obvious. We prove the if part. Let A be a Noetherian, universally catenary local ring with maximal ideal $\mathfrak{m}$ and assume that all the formal fibers of A are Cohen-Macaulay. If $\dim A = 0$, then A itself is Cohen-Macaulay.

We assume that $\dim A > 0$. By modifying the proof of [29, Theorem 5.7], we find that all the formal fibers of an essentially of finite type A-algebra are Cohen-Macaulay. By using this fact and Lemma 5.5, we may assume that $\dim A/\mathfrak{p} = \dim A$ for each associated prime $\mathfrak{p}$ of A. It implies that A is unmixed because A is formally catanary and all the formal fibers of A are Cohen-Macaulay. Theorem 1.1 says that there exists an arithmetic Macaulayfication R of A. Thus A is a homomorphic image of a Cohen-Macaulay local ring $R_{\mathfrak{m}R+R_+}$.

If A is excellent, then any essentially of finite type A-algebra is also. Therefore we obtain the second assertion. □

We should mention that Corollary 1.2 is not true for non-local rings. Indeed, all the formal fibers of all the localization of Ogoma's example above are Cohen-Macaulay. But it is not a homomorphic image of a Cohen-Macaulay ring because it has no codimension function.

6. Non-local rings

First we prove Theorem 1.3. Let B be a Noetherian ring with dualizing complex D. Then there exists a codimension function t of B such that

$$H^p(\mathrm{Hom}_B(B/\mathfrak{p}, D)_\mathfrak{p}) = 0 \quad \text{if } p \neq t(\mathfrak{p})$$

for each prime ideal $\mathfrak{p}$ in B. The following lemma is an analogue of Proposition 2.3 and Corollary 2.4. We can prove them by using the local duality theorem. Here $\operatorname{ann} M$ denotes the annihilator of a B-module M.

Lemma 6.1. *Let M be a finitely generated B-module and $\mathfrak{p}$ a prime ideal in B. Assume that $t(\mathfrak{q}) = 0$ for all minimal prime $\mathfrak{q}$ of M. Then $M_\mathfrak{p}$ is Cohen-Macaulay if and only if $\mathfrak{p} \not\supseteq \prod_{j>0} \operatorname{ann} H^j(\mathrm{Hom}(M, D))$.*

In particular, if $\mathfrak{p} \supseteq \prod_{j>0} \operatorname{ann} H^j(\mathrm{Hom}(M, D))$, then $t(\mathfrak{p}) > 0$. If $t(\mathfrak{q}) = 0$ for all associated prime $\mathfrak{q}$ of M, then $\mathfrak{p} \supseteq \prod_{j>0} \operatorname{ann} H^j(\mathrm{Hom}(M, D))$ implies that $t(\mathfrak{p}) \geq 2$.

We start the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $d = \dim B$ and assume that $t(\mathfrak{q}) = 0$ for all associated primes $\mathfrak{q}$ of B. Then $s_0 = \inf\{t(\mathfrak{p}) \mid B_\mathfrak{p} \text{ is not Cohen-Macaulay}\} \geq 2$. If s is an integer such that $d - s_0 \leq s < d - 1$, then there exist elements $x_1, \dots, x_d$ in B satisfying the following conditions:

(1) if $\mathfrak{p}$ is a minimal prime of $B/(x_i, \dots, x_d)B$, then $t(\mathfrak{p}) = d - i + 1$;
(2) $x_{s+1}, \dots, x_d \in \prod_{j>0} \operatorname{ann} H^j(D)$;
(3) $x_i \in \prod_{j>d-i} \operatorname{ann} H^j(\mathrm{Hom}(B/(x_{i+1}, \dots, x_d), D))$ for $i \leq s$.

We note that (1) implies $(0) : x_d = 0$. Let $\mathfrak{q}_i = (x_i, \dots, x_d)$ for $1 \leq i \leq s+1$ and $R = R(\mathfrak{q}_1 \cdots \mathfrak{q}_s \mathfrak{q}_{s+1}^{d-s-1})$.

We show that $R_\mathfrak{p}$ is Cohen-Macaulay for all prime ideal $\mathfrak{p}$ in B. If $\mathfrak{q}_1 \cdots \mathfrak{q}_{s+1}^{d-s-1} \not\subseteq \mathfrak{p}$, then $\prod_{j>0} \operatorname{ann} H^j(D) \not\subseteq \mathfrak{p}$. Therefore $R_\mathfrak{p}$ is a polynomial ring over a Cohen-Macaulay ring $B_\mathfrak{p}$.

Assume that $\mathfrak{q}_1 \cdots \mathfrak{q}_{s+1}^{d-s-1} \subseteq \mathfrak{p}$. Then $x_t, \dots, x_d \in \mathfrak{p}$ and $x_{t-1} \notin \mathfrak{p}$ for some $1 \leq t \leq s+1$, where we put $x_0 = 1$. Taking localization of (1)–(3), we find that

(1) $\dim B_\mathfrak{p}/(x_t, \dots, x_d) = \dim B_\mathfrak{p} - (d - t + 1)$;
(2) $x_{s+1}, \dots, x_d \in \mathfrak{a}(B_\mathfrak{p})$;
(3) $x_i \in \mathfrak{a}(B_\mathfrak{p}/(x_{i+1}, \dots, x_d))$ for $t \leq i \leq s+1$;
(4) $\mathfrak{a}(B_\mathfrak{p}/(x_t, \dots, x_d)) = B_\mathfrak{p}$ if $t > 1$.

Hence $x_t, \dots, x_d$ is a subsystem of a p-standard system of parameters for $B_\mathfrak{p}$ and $B_\mathfrak{p}/(x_t, \dots, x_d)$ is Cohen-Macaulay if $t > 1$. We find that $R_\mathfrak{p} = R(\mathfrak{q}_t \cdots \mathfrak{q}_{s+1}^{d-s-1} B_\mathfrak{p})$ is Cohen-Macaulay by using Corollary 4.5. □

Now Corollary 1.4 becomes trivial.

Proof of Corollary 1.4. Let B be a Noetherian ring with dualizing complex. We may assume that the codimension function of B is a constant on the associated primes of B because of [23, Theorem 3.5]. Then B has an arithmetic Macaulayfication R. Since R also has a dualizing complex and is Cohen-Macaulay, R is a homomorphic image of a finite-dimensional Gorenstein ring. See [25] and [30, Theorem 4.3]. Therefore B is also. □

References

1. Ian M. Aberbach, *Arithmetic Macaulayfications using ideals of dimension one*, Illinois J. Math. **40** (1996), 518–526. MR **97h:**13002
2. Yoichi Aoyama and Shiro Goto, *A brief summary of the elements of the theory of dualizing complexes and Sharp's conjecture*, The Curve Seminar at Queen's, Vol. 4, Queen's Papers in Pure and Appl. Math., vol. 76, 1986. MR **88g:**13013
3. ———, *Some special cases of a conjecture of Sharp*, J. Math. Kyoto Univ. **26** (1986), 613–634. MR **88h:**13013
4. ———, *A conjecture of Sharp—the case of local rings with* $\dim \text{non-CM} \leq 1$ *or* $\dim \leq 5$, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, Kinokuniya, 1988, pp. 27–34. MR **90b:**13018
5. Jacob Barshay, *Graded algebras of powers of ideals generated by A-sequences*, J. Algebra **25** (1973), 90–99. MR **48:**11074
6. Dave Bayer and Michael Stillman, *Macaulay: A system for computation in algebraic geometry and commutative algebra*, 1982–1994, Source and object code available for Unix and Macintosh computers. Contact the authors, or download from `math.harvard.edu` via anonymous ftp.
7. Markus Brodmann, *Local cohomology of certain Rees- and form-rings I*, J. Algebra **81** (1983), 29–57. MR **85b:**13030
8. Nguyen Tu Cuong, *P-standard systems of parameters and p-standard ideals in local rings*, Acta Math. Vietnam. **20** (1995), 145–161. MR **96h:**13064
9. Shiro Goto, *Blowing-up of Buchsbaum rings*, Proceedings, Durham symposium on Commutative Algebra, London Math. Soc. Lect. Notes, vol. 72, Cambridge Univ. Press, 1982, pp. 140–162. MR **84h:**13032
10. ———, *On the associated graded rings of Buchsbaum rings*, J. Algebra **85** (1983), 490–534. MR **85d:**13032
11. Shiro Goto and Yasuhiro Shimoda, *On Rees algebras over Buchsbaum rings*, J. Math. Kyoto Univ. **20** (1980), 691–708. MR **82c:**13028
12. Shiro Goto and Kikumichi Yamagishi, *The theory of unconditioned strong d-sequences and modules of finite local cohomology*, preprint.
13. Alexander Grothendieck, *Local cohomology*, Lecture Notes in Math., vol. 41, Springer-Verlag, Berlin, Heiderberg, New-York, 1967. MR **37:**219
14. Robin Hartshorne, *Residue and duality*, Lecture Notes in Math., vol. 20, Springer-Verlag, Berlin, Heidelberg, New York, 1966. MR **36:**5145
15. Manfred Herrmann, Eero Hyry, and Jürgen Ribbe, *On the Cohen-Macaulay and Gorenstein properties of multigraded Rees algebra*, Manuscripta Math. **79** (1993), 343–377. MR **94h:**13003
16. Eero Hyry, *The diagonal subring and the Cohen-Macaulay property of a multigraded ring*, Trans. Amer. Math. Soc. **351** (1999), 2213–2232. MR **99i:**13005
17. Takesi Kawasaki, *On Macaulayfication of Noetherian schemes*, Trans. Amer. Math. Soc. **352** (2000), 2517–2552. MR **2000j:**14077
18. ———, *On arithmetic Macaulayfication of certain local rings*, Comm. Algebra **26** (1998), 4385–4396. MR **99m:**13007
19. Kazuhiko Kurano, *On Macaulayfication obtained by a blow-up whose center is an equi-multiple ideal*, J. Algebra **190** (1997), 405–434, with an appendix by Yamagishi, Kikumich. MR **98f:**13004
20. Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Math., vol. 8, Cambridge University Press, 1986, First paperback edition, 1989. MR **90i:**13001; MR **88h:**13001
21. Masayoshi Nagata, *Local rings*, Interscience Tracts, vol. 13, Interscience, New York-London-Sydney, 1962. MR **27:**5790
22. Tetsushi Ogoma, *Existence of dualizing complexes*, J. Math. Kyoto Univ. **24** (1984), 27–48. MR **85j:**13028
23. ———, *Associated primes of fibre product rings and a conjecture of Sharp in lower dimensional cases*, Mem. Fac. Sci. Kochi Univ. (Math.) **6** (1985), 1–9. MR **86e:**13009

24. ———, *Fibre products of Noetherian rings and their applications*, Math. Proc. Cambridge Philos. Soc. **97** (1985), 231–241. MR **86e:**13008
25. Idun Reiten, *The converse to a theorem of Sharp in Gorenstein modules*, Proc. Amer. Math. Soc. **32** (1972), 417–420. MR **45:**5128
26. Peter Schenzel, *Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe*, Lecture Notes in Math., vol. 907, Springer, Berlin-Heidelberg-New York, 1982. MR **83i:**13013
27. ———, *Standard systems of parameters and their blowing-up rings*, J. Reine Angew. Math. **344** (1983), 201–220. MR **84m:**13025
28. Jean-Pierre Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278.
29. Rodney Y. Sharp, *Acceptable rings and homomorphic images of Gorenstein rings*, J. Algebra **44** (1977), 246–261. MR **56:**348
30. ———, *Necessary conditions for the existence of dualizing complexes in commutative algebra*, Sém. Algèbre P. Dubreil 1977/78, Lecture Notes in Mathematics, vol. 740, Springer-Verlag, 1979, pp. 213–229. MR **81d:**13013
31. Yasuhiro Shimoda, *A note on Rees algebras of two dimensional local domains*, J. Math. Kyoto Univ. **19** (1979), 327–333. MR **80k:**13011
32. Giuseppe Valla, *Certain graded algebras are always Cohen-Macaulay*, J. Algebra **42** (1976), 537–548. MR **54:**10240

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