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Review

On Aspects of Gradient Elasticity: Green's Functions and Concentrated Forces

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Abstract: In the first part of our review paper, we consider the problem of approximating the Green's function of the Lagrange chain by continuous analogs. It is shown that the use of continuous equations based on the two-point Padé approximants gives good results. In the second part of the paper, the problem of singularities arising in the classical theory of elasticity with affecting concentrated loadings is considered. To overcome this problem, instead of a transition to the gradient theory of elasticity, it is proposed to change the concept of concentrated effort. Namely, the Dirac delta function is replaced by the Whittaker–Shannon–Kotel'nikov interpolating function. The only additional parameter that characterizes the microheterogeneity of the medium is used. An analog of the Flamant problem is considered as an example. The found solution does not contain singularities and tends to the classical one when the microheterogeneity parameter approaches zero. The derived formulas have a simpler form compared to those obtained by the gradient theory of elasticity.

Keywords: lattice; gradient elasticity; Green's function; concentrated force; Flamant problem; Whittaker–Shannon–Kotel'nikov interpolating function

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1. Introduction

The first work in the field of couple stress theory of elasticity is considered to be the book by the Cosserat brothers [1]. In their theory, in addition to ordinary stresses, the presence of couple stresses is admitted [2]. Le Roux [3] was the first to propose taking into account higher displacement gradients in the theory of elasticity.

The couple stress elasticity theories were extensively studied in the papers by Toupin, Mindlin, and others [4–8]. It was expected that the indicated theories could solve the problem of the theoretical determination of the stress concentration coefficients. Comprehensive reviews of the obtained results were presented in [9–16]. However, difficulties arose when determining the couple stress elasticity constants. Initially, it was assumed that these constants would be determined experimentally. Unfortunately, to date such experiments have not been carried out, due to principal difficulties [9,17].

Regarding the theoretical determination of the parameters of this theory, it is worth noting a few important papers.

Gradient moduli are determined for two-phase composites in the case of a low concentration of inclusions in a homogeneous matrix [18,19]. The possibility of estimating the values of the gradient moduli for the crystal lattice of metals using the methods of density functional theory is shown in [20,21].

In [22], the closed-form expressions for five physical constants of the Toupin–Mindlin theory of isotropic gradient elasticity for polycrystalline materials were obtained. The gradient parameters can also be determined through atomistic potentials and ab initio calculations. In [23], three elastic constants and 11 gradient-elastic ones were determined for Cu and Al from atomistic potentials for the cubic case of Mindlin's anisotropic gradient elasticity theory. Nevertheless, these undoubtedly important results are insufficient for the effective use of couple stress elasticity.

A simpler problem is to determine the characteristic size of microinhomogeneity of the given media, which in many cases can be performed theoretically or experimentally [24–28].

In this regard, a number of simplified gradient elasticity theories were proposed using only one characteristic material length [29,30]. In [25], the authors suggested treating these theories as a first-step extension of the classical linear elasticity theory. The essence of such theories is in including scale effects through the introduction of higher-order spatial gradients to the governing equations of the classical elasticity theory. The characteristic material length was usually proposed to be determined experimentally.

Detailed descriptions of such experiments, as a rule, are missing. From a mathematical point of view, the modified equations contain a small parameter with higher derivatives. This provides a regularization of the solution by taking into account the boundary layers that appear in the vicinity of the singularity points of the classical solution.

A generalization of the classical definition of a function and its derivatives, which allows one to obtain regular solutions to singular problems, is proposed in [31,32].

The use of the equations of gradient elasticity allows regularizing the singular solutions of the classical elasticity theory [33–37]. The main interest is directed at the extent to which pathological predictions of the classical theory of elasticity in singular stress concentration problems are altered, mitigated, or possibly even eliminated when couple stresses are taken into account. However, the results obtained are quite complicated. Is it possible to use a simpler method to carry out regularization? In our opinion, when passing to the gradient elasticity theories, it is necessary to change the concept of a concentrated force. This concept must contain the parameter of microheterogeneity of the media. Thus, if the problems of constructing Green's function and the study of the system response to a concentrated load are close in the classical elasticity theory, then they differ in the gradient theories.

It should be noted that links between nonlocal and gradient theories is nontrivial [37–43]. For example, a deep analysis of the connection between the differential and integral formulations of the nonlocal elasticity theory shows that these theories are far from being always equivalent [41,42].

The significance of considering strain gradients when studying the change in the behavior of isotropic micro- and nanostructures and deriving the exact solutions for their static bending response is shown in [44]. As mentioned in [45], Green's tensor of gradient anisotropic elasticity of Helmholtz type can be utilized as a physically based regularization of classical anisotropic Green's tensor. This property makes gradient elasticity relevant to studying nanomechanical phenomena with the effects of anisotropy. The application of the modified nonlocal gradient theory, which does not require higher-order boundary conditions, for modeling single-walled carbon nanotubes can be found in [46].

This paper addresses two issues. The first one is the relations between Green's functions of discrete and continuous media. The second one concerns the concept of concentrated loads in the gradient elasticity theories.

The paper is organized as follows. Section 2 contains preliminary results. Section 3 is devoted to different continuous approximations of dynamic Green's function for the Lagrange lattice. The conception of a concentrated force in the gradient elasticity theories is analyzed in Section 4, based on the generalized Flamant problem. The obtained results are summarized in Section 5.

2. Governing Relations

The original discrete object is the Lagrange lattice [47–49] (for relevant terminology, see Appendix in [50]). The lattice is that of point masses, M, located in equilibrium states at the points of the ξ-axis with coordinates $jh(j = 0, \pm 1, \pm 2, \pm 3, \ldots)$ and coupled with elastic springs of stiffness c (Figure 1).

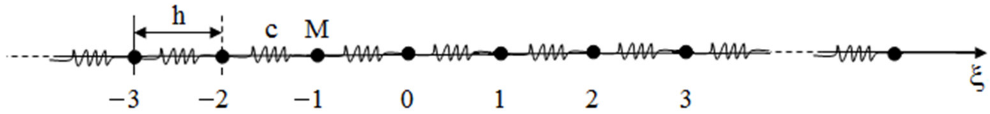

Figure 1. 1D mass-spring lattice. Here M is the value of the point mass; c is the stiffness of spring.

According to Hooke's law, the elastic force acting on the j-th mass is as follows:

$$\sigma_j(t) = c[y_{j+1}(t) - y_j(t)] - c[y_j(t) - y_{j-1}(t)] = c[y_{j-1}(t) - 2y_j(t) + y_{j+1}(t)];\ j = 1, 2, \ldots, n,$$

where $y_j(t)$ is the displacement of the j^{th} material point from its static equilibrium position and t is time.

Using Newton's second law of motion, one obtains the initial system of equations describing the system motion [49]:

$$M\frac{d^2y_j(t)}{dt^2} = c[y_{j-1}(t) - 2y_j(t) + y_{j+1}(t)];\ j = 0, \pm1, \pm2, \pm3, \ldots, \quad (1)$$

Let us introduce a new variable $\tau = t\sqrt{c/M}$. Then, DDE (1) can be rewritten in the following form:

$$\frac{d^2y_j(\tau)}{d\tau^2} = D(y_j(\tau)) = y_{j-1}(\tau) - 2y_j(\tau) + y_{j+1}(\tau);\ j = 0, \pm1, \pm2, \pm3, \ldots, \quad (2)$$

where D is the difference operator.

The continuous coordinate ξ is scaled in such a way that $\xi = jh$ at the nodes of the lattice. The values $y_m(\tau)$ can be treated as the discrete approximation to the continuous function $u(mh, \tau)$:

$$y_m(\tau) = u(mh, \tau). \quad (3)$$

In what follows, it is convenient to replace the nonlocal difference operator in Equation (2) by the pseudo-differential one. For this, the relation [51] (sect. 8 in chapter "Introduction to Operational Calculus") is used:

$$\exp\left(-ih\frac{\partial}{\partial\xi}\right)u(\xi, \tau) = u(\xi + h, \tau),$$

where $i = \sqrt{-1}$.

Using the shift operator $\exp\left(h\frac{\partial}{\partial x}\right)$, one can write:

$$D = \exp\left(h\frac{\partial}{\partial x}\right) + \exp\left(-h\frac{\partial}{\partial x}\right) - 2 = -4\sin^2\left(-\frac{ih}{2}\frac{\partial}{\partial x}\right).$$

Then, DDE (2) can be represented as a pseudo-differential equation:

$$\frac{\partial^2 u(\xi, \tau)}{\partial\tau^2} = -4\sin^2\left(-\frac{ih}{2}\frac{\partial}{\partial\xi}\right)u(\xi, \tau), \quad (4)$$

where $\sin^2\left(-\frac{ih}{2}\frac{\partial}{\partial\xi}\right)$ is the pseudo-differential operator given by:

$$4\sin^2\left(-\frac{ih}{2}\frac{\partial}{\partial\xi}\right) = -2\sum_{k=1}\frac{h^{2k}}{(2k)!}\frac{\partial^{2k}}{\partial\xi^{2k}} = -h^2\frac{\partial^2}{\partial\xi^2}\left(1 + \frac{h^2}{12}\frac{\partial^2}{\partial\xi^2} + \frac{h^4}{360}\frac{\partial^4}{\partial\xi^4} + \frac{h^6}{10080}\frac{\partial^6}{\partial\xi^6} + \ldots\right). \quad (5)$$

Using only the first term in expansion (5), one obtains the wave equation, i.e., the partial differential equation (PDE) of hyperbolic type:

$$u_{\tau\tau} - u_{xx} = 0, \quad (6)$$

where $x = \xi/h$.

3. Improved Continuous Approximations and Green's Functions

Green's function in the case of a 1D lattice for the time-harmonic process can be written as:

$$G_d = \frac{i}{2\omega}\exp(2\arcsin(0.5\omega)i|n|), \quad 0 \le \omega \le 2; n = 0, \pm1, \pm2, \pm3, \ldots, \tag{7}$$

where ω is the frequency.

For the classical continuous approximation (6), Green's function is given by the following expression:

$$G_c = \frac{i}{2\omega}\exp(\omega i|x|), \quad 0 \le \omega \le 2. \tag{8}$$

In Figure 2, curves 1 and 2 show the results of calculating $2\omega G_d/i$ and $2\omega G_c/i$, respectively. As can be seen from the graph below, the classical continuous approximation gives satisfactory results only for low frequencies.

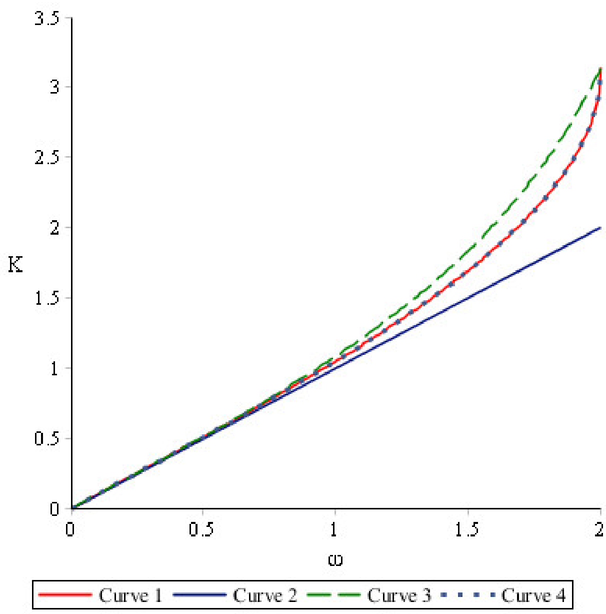

Figure 2. Green's functions for lattice and for various continuous approximations. Here, $K = 2\omega G/i$.

The approximation of Green's functions for lattices by Green's functions of continuous models was considered in [15,16,52,53]. In [15,16], the approximation by the first roots was utilized. The model using the multi-point Padé approximation in the Fourier space of discrete Green's function was described in [52,53].

In the present paper, we analyze the refined continuous models described in [54–56]. These models allow one to properly characterize the vibration frequencies in the entire considered range. Below, we use this model for obtaining Green's functions.

Let us consider the question of more accurate (in comparison with (6)) continuous approximations of DDE (2). Retaining more than one term in expansion (5), we arrive at the so-called intermediate continuous models [57]. The disadvantage of such models is high-order PDEs.

It is possible to increase the accuracy of the approximation significantly while maintaining the PDE order using the Padé approximants or the two-point Padé approximants [54–56,58–60].

Specifically, such an approximation is carried out as follows. Use the transformation:

$$-h^2\frac{\partial^2}{\partial\xi^2}\left(1+\frac{h^2}{12}\frac{\partial^2}{\partial\xi^2}\right)\sim -h^2\frac{\partial^2}{\partial\xi^2}/\left(1-\alpha^2\frac{\partial^2}{\partial\xi^2}\right),$$

where the parameter, α^2, is selected in such a way that the resulting continuous equation provides a good approximation of the lattice frequency to the boundary of the first Brillouin zone. As a result, we determine $\alpha^2 = 0.25 - \pi^{-2}$, and the corresponding continuous equation has the form:

$$\left(1-\alpha^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 u(x,\tau)}{\partial\tau^2}-\frac{\partial^2 u(x,\tau)}{\partial x^2}=0. \tag{9}$$

Equation (9) is called the model with modified inertia [61].

Green's function in this case takes the form:

$$G_{c1}=\frac{i}{2\omega}\exp\left(\frac{\omega}{\sqrt{1-\alpha^2\omega^2}}i|x|\right). \tag{10}$$

The technique described above can also be utilized to build higher-order models. The approximation, which includes the fourth-order derivatives with respect to the spatial variable, is given by:

$$\left(1-\alpha_1^2\frac{\partial^2}{\partial x^2}+\alpha_2^2\frac{\partial^4}{\partial x^4}\right)\frac{\partial^2 u(x,\tau)}{\partial\tau^2}-\left(1+\alpha_3^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 u(x,\tau)}{\partial x^2}=0, \tag{11}$$

where $\alpha_1^2=\frac{\pi^2-8}{\pi^2}$, $\alpha_2^2=\frac{16\left(10-\pi^2\right)}{\pi^4}$, $\alpha_3^2=\frac{16\left(\pi^2-9\right)}{\pi^4}$.

For this case, Green's function can be derived analytically:

$$G_{c2}=\frac{i}{2\omega}\exp(\Omega i|x|), \tag{12}$$

where $\Omega=\frac{\pi(B+C)^{\frac{1}{2}}}{A}$,

$A=2\left[\left(10-\pi^2\right)\omega^2+4\pi^2-36\right]$, $B=-\frac{A}{\sqrt{2}}\left[\left(\pi^2-8\right)\omega^2-4\pi^2\right]$,

$C=\left[\left(\pi^4+48\pi^2-576\right)\omega^4-\left(8\pi^4+192\pi^2-2304\right)\omega^2+16\pi^4\right]^{\frac{1}{2}}$.

In Figure 2, curves 3 and 4 demonstrate the functions $2\omega G_{c1}/i$ and $2\omega G_{c2}/i$, respectively. A significant improvement in the approximation compared to the classical continuous case is observed.

The next step in the application of refined continuous models to approximate Green's functions of discrete systems is 2D lattices [62–69]. In this area, a number of exact solutions based, as a rule, on the utilization of the discrete Fourier transforms were obtained [65–69]. Unfortunately, it is not always possible to determine the inverse transforms analytically. However, there exists a significant number of asymptotics of Green's functions that can be used to derive refined continuous models. For the 1D case, one can refer to the successful experience of [50].

4. On the Concept of Concentrated Forces in Media with Microstructure

Consider the solution to the plane problem of the elasticity theory for a half-plane (Figure 3) under the effect of a concentrated force. Replace the Dirac delta function by the Whittaker–Shannon–Kotel'nikov (WSK) interpolating function [70–72]. Note that the Whittaker–Shannon interpolation formula or sinc interpolation definitions are also used.

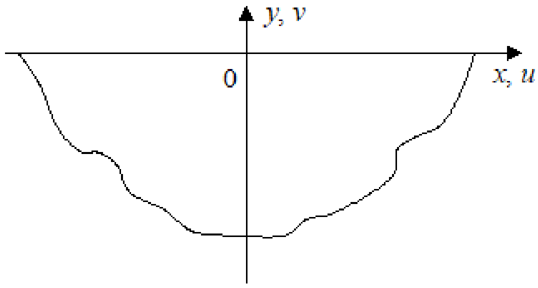

Figure 3. Half-plane under consideration.

Governing equations of the plane elasticity theory are given in the following form:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0; \ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} = 0, \tag{13}$$

$$\varepsilon_x = \frac{\partial u}{\partial x}; \ \varepsilon_y = \frac{\partial v}{\partial y}; \ \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{14}$$

$$\tau = G\varepsilon_{xy}, \tag{15}$$

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y); \ \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \text{for plane stress problem,} \tag{16}$$

$$\begin{matrix} \sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y\right], \\ \sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu\varepsilon_x\right] \end{matrix} \text{ for plane stress problem,} \tag{17}$$

where $G = \frac{E}{2(1+\nu)}$, E is the Young's modulus; ν is the Poisson's ratio.

Boundary conditions are written as:

$$\sigma_y = N(x,l) = P\frac{\sin(\pi x/l)}{\pi x}; \quad \tau = 0 \quad \text{at} y = 0. \tag{18}$$

Here l is the parameter characterizing microheterogeneity of the media. It is important to note that this parameter can be determined analytically for a number of crystalline materials [22].

Since:

$$\lim_{l \to 0} \frac{\sin(\pi x/l)}{\pi x} = \delta(x), \tag{19}$$

at $l \to 0$, one obtains from (13) and (18) the familiar Flamant problem.

Thus, for a microheterogeneous medium, the concentrated force in the form of the Dirac delta function is replaced by the WSK regularized loading [14,15,73].

To solve the boundary value problem (13), (18), we use the complex analysis following [74]. The complex functions are introduced:

$$\Phi(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{N}{t-z}dt = -\frac{iP}{2\pi}\left(\frac{1-\exp(-i\pi z/l)}{z}\right), \tag{20}$$

$$\Psi(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{N}{t-z}dt + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{N}{(t-z)^2}tdt = -\frac{iP}{2\pi z} + P\left(\frac{i}{2\pi z} - \frac{1}{2l}\right)\exp(-i\pi z/l). \tag{21}$$

where $z = x + iy, \ -\infty < x < \infty, \ y \le 0, \ -\pi \le \arg(z) \le 0.$

Antiderivatives for functions (20) and (21) are:

$$\begin{aligned} \varphi(z) &= -\tfrac{iP}{2\pi}\int \tfrac{1-\exp(-i\pi z/l)}{z}dz \\ &= -\tfrac{iP}{2\pi}(\gamma + \ln(i\pi z/l) + E_1(i\pi z/l) + C_1), \ -\pi \le \arg(z) \le 0, \end{aligned} \tag{22}$$

$$\begin{aligned}\psi(z) &= -\tfrac{iP}{2\pi}\int \tfrac{1-\exp(-i\pi z/l)}{z}dz - \tfrac{iP}{2\pi}\exp(-i\pi z/l) \\ &= -\tfrac{iP}{2\pi}[E_1(i\pi z/l)+\gamma+\ln(i\pi z/l)+\exp(-i\pi z/l)+C_2]\,,\ -\pi \le \arg(z) \le 0,\end{aligned} \tag{23}$$

where $E_1(\ldots)$ is the familiar exponential integral [75], and γ is the familiar Euler constant [75].

Further, we use formula (1) from Ch. 1, sect. 32 of the book [74]:

$$u+iv = \frac{1}{2\mu}\Big(\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\Big), \tag{24}$$

where an overbar denotes the complex conjugate;

$$\kappa = \begin{cases} 3-4\nu & \text{for plane strain problem,} \\ \frac{3-\nu}{1+\nu} & \text{for plane stress problem.}\end{cases}.$$

Then, the solution of the boundary value problem (13) and (18) can be written as follows:

$$\begin{aligned} u+iv &= \tfrac{1}{2G}\Big(\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\Big) \\ &= \tfrac{iP}{4\pi\mu}\Big\{B-(\kappa+1)\gamma - \tfrac{z^2}{|z|^2} + \exp(i\pi\bar{z}/l)\Big(\tfrac{z^2}{|z|^2}-1\Big)\Big\} \\ &-\tfrac{iP}{4\pi\mu}\{\kappa\ln(i\pi z/l) + kE_1(i\pi z/l) + \ln(-i\pi\bar{z}/l) + E_1(-i\pi\bar{z}/l)\},\end{aligned} \tag{25}$$

where $B = \kappa C_1 + \overline{C_2}$.

$$\begin{aligned} u &= \tfrac{P}{4\pi\mu}\Big\{-\mathrm{Im}B + \tfrac{2xy}{x+y} + 2\exp(\pi y/l)\tfrac{y\sin\pi x/l - xy\cos\pi x/l}{x+y}\Big\} \\ &+ \tfrac{P}{4\pi\mu}\Big\{\sum \tfrac{(-1)(\pi\rho/l)(\kappa-1)\cos((2n+1)\theta)}{(2n+1)(2n+1)!} - \sum \tfrac{(-1)(\pi\rho/l)(\kappa-1)\sin(2n\theta)}{(2n)(2n)!}\Big\},\end{aligned} \tag{26}$$

$$\begin{aligned} v &= \tfrac{P}{4\pi\mu}\Big\{\mathrm{Re}B - 1 + \tfrac{2y^2}{x^2+y^2} - 2\exp(\pi y/l)\tfrac{y^2\cos\pi x/l + xy\sin\pi x/l}{x^2+y^2}\Big\} \\ &+ \tfrac{P}{4\pi\mu}\Big\{\sum_{n=0}^{\infty} \tfrac{(-1)^n(\pi\rho/l)^{2n+1}(\kappa+1)\sin((2n+1)\theta)}{(2n+1)(2n+1)!} + \sum_{n=1}^{\infty} \tfrac{(-1)^n(\pi\rho/l)^{2n}(\kappa+1)\cos(2n\theta)}{(2n)(2n)!}\Big\},\end{aligned} \tag{27}$$

where $\rho = \sqrt{x^2+y^2}, \quad -\pi \le \theta = \arg z \le 0$.

The series in (26) and (27) converge absolutely with a high convergence rate.

From the antisymmetric condition:

$$u(x,y) = -u(x,-y),$$

we find $\mathrm{Im}B = 0$.

On the boundary of the half-plane $y=0$, one obtains:

$$u(x,0) = \frac{P}{4\pi\mu}(\kappa-1)\sum_{n=0}^{\infty}\frac{(-1)^n(\pi x/l)^{2n+1}}{(2n+1)(2n+1)!} = \frac{P}{4\pi\mu}(\kappa-1)Si(\pi x/l),$$

$$\begin{aligned} v(x,0) &= \tfrac{P}{4\pi\mu}\Big\{B-1+(\kappa+1)\sum_{n=1}^{\infty}\tfrac{(-1)^n(\pi|x|/l)^{2n}}{(2n)(2n)!}\Big\} \\ &= \tfrac{P}{4\pi\mu}\{B-1+(\kappa+1)(Ci(\pi|x|/l) - \gamma - \ln(\pi|x|/l))\},\end{aligned}$$

where $Si(\ldots)$ and $Ci(\ldots)$ are the familiar sine and cosine integrals [75].

To fix the real constant B, which corresponds to the displacement of the half-plane as a rigid body, we require that $v(1,0)=0$. Then, we have:

$$B = 1-(\kappa+1)(Ci(\pi/l) - \gamma - \ln(\pi/l)).$$

The solution to the Flamant problem is given by:

$$u = \frac{P}{2\pi G}\begin{cases} \frac{xy}{x^2+y^2} - \frac{\kappa-1}{2}\left(\arctan\frac{y}{x} + \frac{\pi}{2}\right), & x \geq 0, y \leq 0, \\ \frac{xy}{x^2+y^2} - \frac{\kappa-1}{2}\left(\arctan\frac{y}{x} - \frac{\pi}{2}\right), & x < 0, y \leq 0, \end{cases} \tag{28}$$

$$v = -\frac{P}{2\pi G}\left[\frac{\kappa+1}{4}\ln\left(x^2+y^2\right) + \frac{x^2}{x^2+y^2}\right], \quad -\infty << x << \infty, \ \ y \leq 0. \tag{29}$$

As $l \to 0$, formulas (26) and (27) turn into the solution of the Flamant problem, (28) and (29), as expected from general considerations [76–78]. Figure 4 demonstrates that the limit transition will not be uniform for the function $u(x,0)$ owing to the familiar Gibbs–Wilbraham phenomenon [79,80].

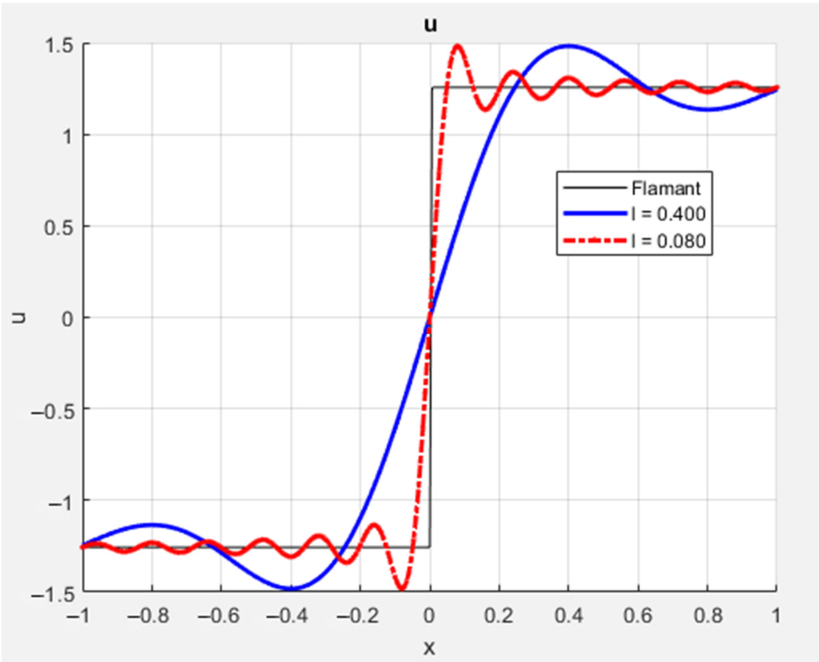

Figure 4. Comparison of the solution to the Flamant problem for $u(x,0)$ with solution (26) for various values of the microinhomogeneity parameter.

The behavior of solutions (26) and (27) at infinity coincides with that of the solutions to the Flamant problem ((28) and (29)).

Let us compare the obtained results with the known solutions within the framework of the couple stress theory or the gradient one [31–37]. The equations of these theories are singularly perturbed in comparison with the equations of the classical elasticity theory [21]. In the case of the couple stress theory, we have two small parameters at higher-order derivatives, l_1, l_2 [21], and for the gradient elasticity theory, one small parameter, l [4]. The difference between solutions based on these theories varies significantly from the solutions of the classical elasticity theory only in a small vicinity of points or lines of singularity. In these zones, solutions of the boundary layer type, which rapidly decay when moving away from the singular point, appear.

Consider the following examples. The singularity of the form $1/r$ at $r \to 0$ is replaced by the expression [4]:

$$(1/r)\ [\exp(r/l) - 1]\ \sim 1/l \quad \text{at}\, l \to 0.$$

The singularity of the form $\ln|x|$ at $x \to 0$ transforms into [21]:

$$\ln|x| + K_0(|x|/l)\ \sim \ln l \quad \text{at}\, l \to 0,$$

where $K_0(\ldots)$ is the familiar modified Bessel function of the second kind [75].

Thus, as the spatial variable tends to some singular point, the infinite values are replaced by finite values approaching infinity as the small parameter tends to zero. At the same time, obtaining analytical solutions within the framework of the couple stress or gradient theories of elasticity does not always succeed. It is associated with the problems of analytical inversion of integral transformations. Let us show that changing the concept of a concentrated force leads to the same result.

Turn to the Flamant problem. The logarithmic singularity at the origin for the function v, present in the Flamant problem, is replaced by a large value tending to infinity at $l \to 0$:

$$v \sim \frac{P(\kappa+1)}{4\pi G} \ln l \text{ at} x \to 0. \tag{30}$$

Some curves for the function $v(x,0)$ are shown in Figure 5. For plane strain elasticity, one obtains from expression (30):

$$v \sim \frac{P(1-v)}{\pi G} \ln l \text{ at} x \to 0. \tag{31}$$

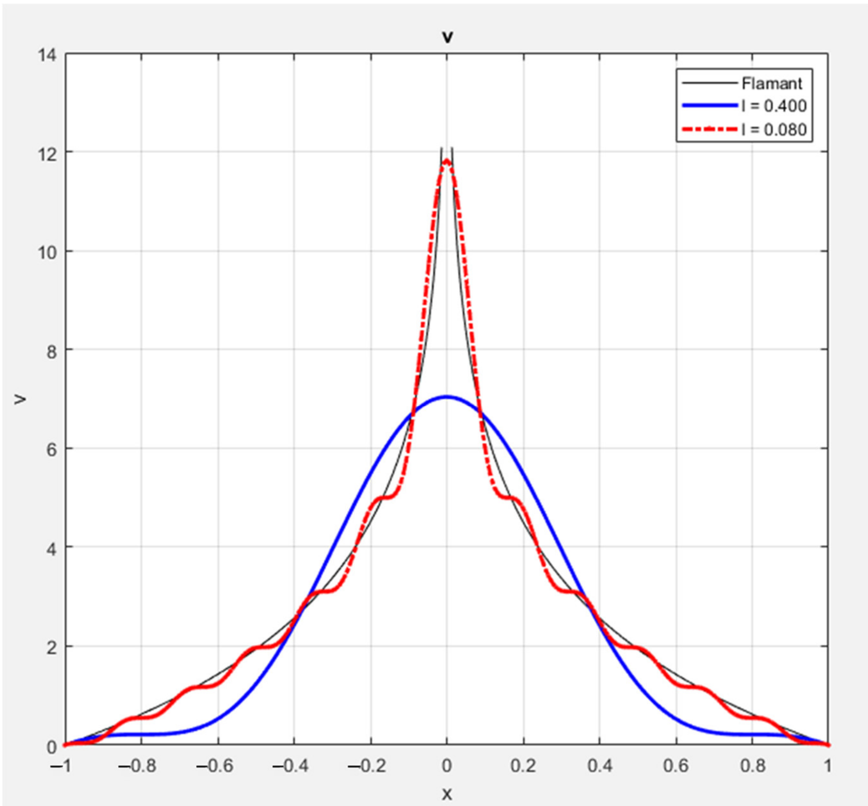

Figure 5. Comparison of the solution to the Flamant problem for $v(x,0)$ with solution (27) for various values of the microinhomogeneity parameter.

The paper [37] examines the effect of concentrated loads within the Toupin–Mindlin elasticity theory. In particular, a solution is found for an analog of the Flamant problem for couple stress elasticity. If we reduce this solution to the simplified gradient elasticity theory [10], which is described by one parameter, then we have:

$$v \sim \frac{P(1-v)}{\pi G}(\ln|x| + K_0|x/l|) \sim \frac{P(1-v)}{\pi G} \ln l \text{ at} x \to 0. \tag{32}$$

A similar result follows from the analysis of papers [34,36]. The paper [34] presents the results of the experiment, quantitative comparison with which is difficult. Qualitatively, experimental values for $u(x, 0)$ at $x \to \infty$ tend to the Flamant solving (28), and this tendency has an oscillating character. Solving (26) has the same behavior.

5. Conclusions

In the first part of this work, the use of the apparatus of two-point Padé approximants allowing one to construct continuous approximations of the Lagrange lattice that well approximate discrete Green's function is shown.

The importance of this conclusion is as follows. The construction of Green's functions for ODEs and PDEs is generally easier than for nonlocal equations. This is due to the substantially greater development of continuous mathematics in comparison with discrete. The ability to use the entire arsenal of calculus, functional analysis, and theories of ODE and PDE allows one to obtain analytical expressions of Green's functions [81,82]. However, the problem of such an approximation of nonlocal operators by local ones, for which the main features of the solution of discrete problems are well described by continuous ones, arises. It turns out that, at least in the 1D case, this is possible.

In the second part of this paper, we analyze the concept of a concentrated force in the theory of elasticity. Solutions of the classical elasticity theory near the points of application of concentrated loads may have pathological singularities. To eliminate these features, a transition to the various versions of the gradient elasticity theory is usually used. In this case, regularization occurs due to an increase in the order of the original system. The new system is singularly perturbed and, owing to the solution of the boundary layer type, allows elimination of the features of the solution. However, the expressions obtained using the gradient theories of elasticity are too complicated. In addition, a nontrivial problem of additional boundary conditions must be solved [40].

The Dirac delta function is usually utilized to model the concentrated force or the point connection. This idealization often causes physically incorrect results. This can be avoided by increasing the order of the original system of equations, i.e., passing to couple stress elasticity or gradient elasticity. However, it is easier to change the approximation of the concentrated force itself. For example, in paper [83], a distributed delta function is introduced, and the apt name "a real-world forcing" is suggested for such a function. The authors of paper [62] replace the derivative of the delta function by a regular "pre-delta" function.

For a microheterogeneous medium, in our opinion, the WSK approximation is the most natural.

In our work, it is shown that there exists an alternative possibility of regularizing the solution by changing the concept of a concentrated force. Namely, the Dirac delta function is replaced by the Whittaker–Shannon–Kotel'nikov (WSK) interpolating function. In this case, it is sufficient to utilize only one additional parameter characterizing the microheterogeneity of the medium.

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References

1. Cosserat, E.; Cosserat, F. *Théorie des Corps Déformables*; Libraire Scientifique A. Hermann et Fils: Paris, France, 1909.
2. Vardoulakis, I. *Cosserat Continuum Mechanics with Applications to Granular Media*; Springer Nature: Cham, Switzerland, 2019.
3. Le Roux, M.J. Étude géométrique de la torsion et de la flexion dans la déformation infinitésimale d'un milieu continu. *Ann. Sci. L'école Norm. Supérieure* **1911**, *28*, 523–579. [CrossRef]
4. Mindlin, R.D.; Tiersten, H.F. Effects of couple-stresses in linear elasticity. *Arch. Ration. Mech. Anal.* **1962**, *11*, 415–448. [CrossRef]
5. Toupin, R.A. Elastic materials with couple-stresses. *Arch. Ration. Mech. Anal.* **1962**, *11*, 385–414. [CrossRef]
6. Koiter, W.T. Couple-stresses in the theory of elasticity. *Proc. Koninkl. Nederl. Akad. Wet.* **1964**, *67*, 17–44.
7. Toupin, R.A. Theories of elasticity with couple-stresses. *Arch. Ration. Mech. Anal.* **1964**, *17*, 85–112. [CrossRef]
8. Mindlin, R.D. Microstructure in linear elasticity. *Arch. Ration. Mech. Anal.* **1964**, *16*, 51–78. [CrossRef]
9. Maugin, G.A. *Continuum Mechanics through the Twentieth Century. A Concise Historical Perspective*; Springer: Dordrecht, The Netherlands, 2013.
10. Maugin, G.A. From Piola's manifold to Cosserats' structure. In *Geometry, Continua and Microstructure*; Maugin, G.A., Ed.; Hermann: Paris, France, 1999; pp. 113–120.
11. Goddard, J.D. Microstructural origins of continuum stress fields-a brief history and some unresolved issues. In *Recent Dvelopments in Structured Continua*; DeKee, D., Kaloni, P.M., Eds.; Wiley: New York, NY, USA, 1986; pp. 179–208.
12. Dell'Isola, F.; Andreaus, U.; Placidi, L. At the origins and in the vanguard of peridynamics, non-local and higher-gradient continuum mechanics: An underestimated and still topical contribution of Gabrio Piola. *Math. Mech. Solids* **2015**, *20*, 887–928. [CrossRef]
13. Dell'Isola, F.; Della Corte, A.; Giorgio, I. Higher-gradient continua: The legacy of Piola, Mindlin, Sedov and Toupin and some future research perspectives. *Math. Mech. Solids* **2017**, *22*, 852–872. [CrossRef]
14. Kunin, I.A. Model of an elastic medium of simple structure with three-dimensional dispersion. *J. Appl. Math. Mech.* **1966**, *30*, 642–652. [CrossRef]
15. Kunin, I.A. *Elastic Media with Microstructure. 1. One-Dimensional Models*; Springer: Berlin, Germany, 1982.
16. Kunin, I.A. *Elastic Media with Microstructure. II. Three-Dimensional Models*; Springer: Berlin, Germany, 1983.
17. Vasiliev, V.V.; Lurie, S.A. Correct nonlocal generalized theories of elasticity. *Phys. Mesomech.* **2016**, *19*, 269–281. [CrossRef]
18. Bigoni, D.; Drugan, W.J. Analitical derivation of Cosserat moduli via homogenization of heterogeneous elastic materials. *J. Appl. Mech.* **2007**, *74*, 741–753. [CrossRef]
19. Bacca, M.; Bigoni, D.; Dal Corso, F.; Veber, D. Mindlin second-gradient elastic properties from dilute two-phase Cauchy-elastic composites. Part II: Higher-order constitutive properties and application cases. *Int. J. Solids Struct.* **2013**, *50*, 4020–4029. [CrossRef]
20. Shodja, H.M.; Zaheri, A.; Tehranchi, A. Ab initio calculations of characteristic lengths of crystalline materials in first strain gradient elasticity. *Mech. Mater.* **2013**, *61*, 73–78. [CrossRef]
21. Shodja, H.M.; Moosavian, H.; Ojaghnezhad, F. Toupin–Mindlin first strain gradient theory revisited for cubic crystals of hexoctahedral class: Analytical expression of the material parameters in terms of the atomic force constants and evaluation via ab initio DFT. *Mech. Mater.* **2018**, *123*, 19–29. [CrossRef]
22. Malyi, V.I. Theoretical determination of the five physical constants of the Toupin-Mindlin gradient elasticity for polycrystalline materials. In *Nonlinear Dynamics of Discrete and Continuous Systems*; Abramyan, A., Andrianov, I., Gaiko, V., Eds.; Springer Nature: Cham, Switzerland, 2021; pp. 145–154.
23. Po, G.; Admal, N.C.; Lazar, M. The Green tensor of Mindlin's anisotropic first strain gradient elasticity. *Mater. Theory* **2019**, *3*, 1–16. [CrossRef]
24. Shodja, H.M.; Tehranchi, A. A formulation for the characteristic lengths of fcc materials in first strain gradient elasticity via the Sutton-Chen potential. *Philos. Mag.* **2020**, *90*, 1893–1913, Erratum in Philos. Mag. **2012**, *92*, 1170–1171. [CrossRef]
25. Georgiadis, H.G.; Vardoulakis, I.; Velgaki, E.G. Dispersive Rayleigh-wave propagation in microstructured solids characterized by dipolar gradient elasticity. *J. Elast.* **2004**, *74*, 17–45. [CrossRef]
26. Vasiliev, V.V.; Lurie, S.A. Generalized theory of elasticity. *Mech. Solids* **2015**, *50*, 379–388. [CrossRef]
27. Askes, H.; Aifantis, E.C. Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results. *Int. J. Solids Struct.* **2011**, *48*, 1962–1990. [CrossRef]
28. Rogula, D. Some basic solutions in strain gradient elasticity theory of an arbitrary order. *Arch. Mech.* **1973**, *25*, 43–68.
29. Hadjesfandiari, A.R.; Dargush, G.F. Couple stress theory for solids. *Int. J. Solids Struct.* **2011**, *48*, 2496–2510. [CrossRef]
30. Aifantis, E.C. Update on a class of gradient theories. *Mech. Mater.* **2003**, *35*, 259–280. [CrossRef]
31. Vasiliev, V.V.; Lurie, S.A. Nonlocal solutions to singular problems of mathematical physics and mechanics. *Mech. Solids* **2018**, *53*, 135–144. [CrossRef]
32. Vasiliev, V.V. Singular solutions in the problems of mechanics and mathematical physics. *Mech. Solids* **2018**, *53*, 397–410. [CrossRef]
33. Georgiadis, H.G.; Anagnostou, D.S. Problems of the Flamant–Boussinesq and Kelvin type in dipolar gradient elasticity. *J. Elast.* **2008**, *90*, 71–98. [CrossRef]
34. Vasiliev, V.V.; Lurie, S.A.; Salov, V.A. On the Flamant problem for a half-plane loaded with a concentrated force. *Acta Mech.* **2021**, *232*, 1761–1771. [CrossRef]

35. Hadjesfandiari, A.R.; Dargush, G.F. Fundamental solutions for isotropic size-dependent couple stress elasticity. *Int. J. Solids Struct.* **2013**, *50*, 1253–1265. [CrossRef]
36. Lazar, M.; Maugin, G.A. A note on line forces in gradient elasticity. *Mech. Res. Commun.* **2006**, *33*, 674–680. [CrossRef]
37. Gourgiotis, P.A.; Zisis, T.; Georgiadis, H.G. On concentrated surface loads and Green's functions in the Toupin-Mindlin theory of strain-gradient elasticity. *Int. J. Solids Struct.* **2018**, *130*, 153–171. [CrossRef]
38. Eringen, A.C. *Nonlocal Continuum Field Theories*; Springer: New York, NY, USA, 2002.
39. Borino, G.; Polizzotto, C. A method to transform a nonlocal model into a gradient one within elasticity and plasticity. *Eur. J. Mech.-A Solids* **2014**, *46*, 30–42. [CrossRef]
40. Chebakov, R.; Kaplunov, J.; Rogerson, G.A. Refined boundary conditions on the free surface of an elastic half-space taking into account non-local effects. *Proc. R. Soc. A* **2016**, *472*, 20150800. [CrossRef]
41. Kaplunov, J.; Prikazchikov, D.A.; Prikazchikova, L. On integral and differential formulations in nonlocal elasticity. *arXiv* **2021**, arXiv:2107.09925v1.
42. Romano, G.; Barretta, R.; Diaco, M.; de Sciarra, F.M. Constitutive boundary conditions and paradoxes in nonlocal elastic nanobeams. *Int. J. Mech. Sci.* **2017**, *121*, 151–156. [CrossRef]
43. Vaccaro, M.S.; Pinnola, F.P.; de Sciarra, F.M.; Barretta, R. Limit behaviour of Eringen's two-phase elastic beams. *Eur. J. Mech. A Solids* **2021**, *89*, 104315. [CrossRef]
44. Sidhardh, S.; Ray, M.C. Exact solutions for elastic response in micro- and nano-beams considering strain gradient elasticity. *Math. Mech. Solids* **2019**, *24*, 895–918. [CrossRef]
45. Lazar, M.; Po, G. The non-singular Green tensor of gradient anisotropic elasticity of Helmholtz type. *Eur. J. Mech. A Solids* **2015**, *50*, 152–162. [CrossRef]
46. Barretta, R.; Čanadija, M.; Marotti de Sciarra, F. Modified nonlocal strain gradient elasticity for nano-rods and application to carbon nanotubes. *Appl. Sci.* **2019**, *9*, 514. [CrossRef]
47. Lagrange, J.L. *Méchanique Analitique*; Chez la Veuve Desaint: Paris, France, 1788; Engl. Transl.: Analytical Mechanics; Springer: Dor-drecht, The Netherlands, 1997.
48. Pinney, E. *Ordinary Difference-Differential Equations*; University California Press: Berkeley, CA, USA, 1958.
49. Myshkis, A.D. Mixed functional differential equations. *J. Math. Sci.* **2005**, *129*, 4111–4226. [CrossRef]
50. Andrianov, I.; Koblik, S.; Starushenko, G. Transition from discrete to continuous media: The impact of symmetry changes on asymptotic behavior of waves. *Symmetry* **2021**, *13*, 1008. [CrossRef]
51. Maslov, V.P. *Operational Methods*; Mir: Moscow, Russia, 1976.
52. Charlotte, M.; Truskinovsky, L. Towards multi-scale continuum elasticity theory. *Contin. Mech. Thermodyn.* **2008**, *20*, 133–161. [CrossRef]
53. Charlotte, M.; Truskinovsky, L. Lattice dynamics from a continuum viewpoint. *J. Mech. Phys. Solids* **2012**, *60*, 1508–1544. [CrossRef]
54. Andrianov, I.V.; Awrejcewicz, J.; Weichert, D. Improved continuous models for discrete media. *Math. Probl. Eng.* **2010**, *2010*, 986242. [CrossRef]
55. Andrianov, I.V.; Danishevskyy, V.V.; Kaplunov, J.D.; Markert, B. Wide frequency higher-order dynamic model for transient waves in a lattice. In *Problems of Nonlinear Mechanics and Physics of Materials*; Andrianov, I.V., Manevich, A.I., Mikhlin, Y.V., Gendelman, O.V., Eds.; Springer Nature: Cham, Switzerland, 2018; pp. 3–12.
56. Andrianov, I.V.; Awrejcewicz, J.; Danishevskyy, V.V. *Linear and Nonlinear Waves in Microstructured Solids: Homogenization and Asymptotic Approaches*; CRC Press, Taylor & Francis: Boca Raton, FL, USA, 2021.
57. Filimonov, A.M. Continuous approximations of difference operators. *J. Differ. Equ. Appl.* **1996**, *2*, 411–422. [CrossRef]
58. Collins, M.A. A quasicontinuum approximation for solitons in an atomic chain. *Chem. Phys. Lett.* **1981**, *77*, 342–347. [CrossRef]
59. Rosenau, P. Dynamics of dense lattices. *Phys. Rev. B* **1987**, *36*, 5868–5876. [CrossRef]
60. Wattis, J.A.D. Approximations to solitary waves on lattices. II. Quasi-continuum methods for fast and slow waves. *J. Phys. A* **1993**, *26*, 1193–1209. [CrossRef]
61. Kaplunov, J.D.; Kossovich, L.Y.; Nolde, E.V. *Dynamics of Thin Walled Elastic Bodies*; Academic Press: San Diego, CA, USA, 1998.
62. Movchan, A.B.; Slepyan, L.I. Band gap Green's functions and localized oscillations. *Proc. R. Soc. A* **2007**, *463*, 2709–2727. [CrossRef]
63. Movchan, A.B.; Slepyan, L.I. Resonant waves in elastic structured media: Dynamic homogenisation versus Green's functions. *Int. J. Solids Struct.* **2014**, *51*, 2254–2260. [CrossRef]
64. Mishuris, G.S.; Movchan, A.B.; Slepyan, L.I. Waves in elastic bodies with discrete and continuous dynamic microstructure. *Phil. Trans. R. Soc. A* **2019**, *378*, 20190313. [CrossRef] [PubMed]
65. Poulton, C.G.; McPhedran, R.C.; Nicorovici, N.A.; Botten, L.C. Localized Green's functions for a two-dimensional periodic material. In *IUTAM Symp. on Asymptotics, Singularities and Homogenisation in Problems of Mechanics*; Movchan, A.B., Ed.; Kluwer: Dordrecht, The Netherlands, 2003; pp. 181–190.
66. Martinsson, P.-G.; Rodin, G. Asymptotic expansions of lattice Green's functions. *Proc. R. Soc. A* **2002**, *458*, 2609–2622. [CrossRef]
67. Martin, P.A. Discrete scattering theory: Green's function for a square lattice. *Wave Motion* **2006**, *43*, 619–629. [CrossRef]
68. Martinsson, P.G.; Movchan, A.B. Vibrations of lattice structures and phononic band gaps. *Q. J. Mech. Appl. Math.* **2003**, *56*, 45–64. [CrossRef]
69. Movchan, A.B.; Movchan, N.V.; Jones, I.S.; Colquitt, D.J. *Mathematical Modelling of Waves in Multi-Scale Structured Media*; Taylor and Francis: London, UK, 2018.

70. Whittaker, E.T. On the functions which are represented by the expansion of the interpolation theory. *Proc. R. Soc. Edinburgh A* **1915**, *35*, 181–194. [CrossRef]
71. Kotel'nikov, V.A. On the carrying capacity of the "ether" and wire in telecommunications. In *Material for the first All-Union Conference on Questions of Communications*; Izd. Red. Upr. Svyazi RKKA: Moscow, Russia, 1933; Engl. transl. in *Modern Sampling Theory: Mathematics and Applications*; Benedetto, J.J., Ferreira, P.J.S.J., Eds.; Birkhäuser: Boston, USA, 2000; pp. 27–45.
72. Shannon, C.E. Communication in the presence of noise. *Proc. IRE* **1949**, *137*, 10–21. [CrossRef]
73. Rogula, D. Introduction to nonlocal theory of material media. In *Nonlocal Theory of Elastic Media*; Rogula, D., Ed.; Springer: Wien, Austria, 1982; pp. 123–222.
74. Muskhelishvili, N.I. *Some Basic Problems of the Mathematical Theory of Elasticity. Fundamental Equations Plane Theory of Elasticity, Torsion and Bending*; Springer: Dordrecht, The Netherlands, 1977.
75. Abramowitz, M. *, Stegun, I.A., Eds. Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*; Dover: New York, NY, USA, 1965.
76. Sternberg, E.; Eubanks, R.A. On the concept of concentrated loads and an extension of the uniqueness theorem in the linear theory of elasticity. *J. Rat. Mech. Anal.* **1955**, *4*, 135–168. [CrossRef]
77. Muki, R.; Sternberg, E. The influence of couple-stresses on singular stress concentrations in elastic solids. *ZAMP* **1965**, *16*, 611–648. [CrossRef]
78. Bogy, D.B.; Sternberg, E. The effect of couple-stresses on singularities due to discontinuous loadings. *Int. J. Solids Struct.* **1967**, *3*, 757–770. [CrossRef]
79. Hewitt, E.; Hewitt, R.E. The Gibbs-Wilbraham phenomenon: An episode in Fourier analysis. *Arch. Hist. Exact Sci.* **1979**, *21*, 129–160. [CrossRef]
80. Jerri, A.J. *The Gibbs Phenomenon in Fourier Analysis, Splines and Wavelet Approximations*; Springer: Dordrecht, The Netherlands, 1998.
81. Kupradze, V.D. (Ed.) *Three-Dimensional Problems of Elasticity and Thermoelasticity*; North-Holland: Amsterdam, The Netherlands, 1979.
82. Sutton, A.P. *Green's Function in Linear Elasticity*; Oxford University Press: Oxford, UK, 2020.
83. Wootton, P.; Kaplunov, J. Bridging waves on a membrane: An approach to preserving wave patterns. In *Modern Trends in Structural and Solid Mechanics 2*; Challamel, N., Kaplunov, J., Takewaki, I., Eds.; Wiley: New York, NY, USA, 2021; pp. 203–229.