

On Association Cells in Random Heterogeneous Networks

Sarabjot Singh, François Baccelli, and Jeffrey G. Andrews

Abstract—Characterizing user to access point (AP) association strategies in heterogeneous cellular networks (HetNets) is critical for their performance analysis, as it directly influences the load across the network. In this letter, we introduce and analyze a class of association strategies, which we term *stationary association*, and the resulting association cells. For random HetNets, where APs are distributed according to a stationary point process, the area of the resulting association cells are shown to be the marks of the corresponding point process. Addressing the need of quantifying the load experienced by a typical user, a “Feller-paradox” like relationship is established between the area of the association cell containing origin and that of a typical association cell. For the specific case of Poisson point process and max power/SINR association, the mean association area of each tier is derived and shown to increase with channel gain variance and decrease in the path loss exponents of the corresponding tier.

Index Terms—Heterogeneous cellular networks, association cells, load balancing, downlink rate distribution, Feller paradox.

I. INTRODUCTION

DENSIFICATION of wireless cellular infrastructure through deployment of low power APs is a promising approach to meet increasing wireless traffic demands. This complementary infrastructure consists of various classes of APs differing in transmit powers, radio access technologies, backhaul capacities, and deployment scenarios. This increasing heterogeneity and density in wireless networks has provided an impetus to develop new models for their analysis and design. This paper is primarily aimed to analyze such random heterogeneous networks (HetNets), where the AP locations are modeled by a stationary point process.

Using Poisson point processes (PPP) for modeling the irregular AP locations has been shown to be a tractable and accurate approach for characterizing signal-to-interference ratio (SIR) distribution [1]–[3]. Although SIR is one of the key performance metrics for user performance, managing *load* or the number of users sharing the available resources per AP plays an important role in realizing the capacity gains in HetNets [4]. The load at an AP is dictated by the user to AP association strategy adopted in the network. For example, users associating to their nearest AP leads to association cells conforming to a Voronoi tessellation with AP locations as the cell centers and identical load distribution across the APs. However, in HetNets, it is desirable to incorporate the differing base station (BS) capabilities among the classes/tiers of BSs

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The authors are with the Dept. of Electrical and Computer Engineering, University of Texas, Austin (e-mail: sarabjot@utexas.edu, baccelli@math.utexas.edu, jandrews@ece.utexas.edu).

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and the propagation environments in the association strategy. Being able to characterize the resulting complex association cells is one of the goals of this paper.

In this paper, we introduce *stationary* association strategies, which lead to the formation of stationary association cells. Such strategies form a wide class and encompass all association patterns that are invariant by translation, including the earlier studied max SINR association [5]. Leveraging the theory of stationary partitions introduced in [6], we establish a “Feller-paradox” like relation between the association area of the AP containing the origin to that of a typical AP in a HetNet setting, wherein the former is an *area-biased* version of the latter. Such a relation has important practical implications in analyzing the load experienced by a typical user which is served, as we shall see, by an *atypical* AP. The developed theoretical framework also provides rigorous proofs for the arguments used in [7], [8] for load characterization. Further, using the PPP assumption and max-power association, it is shown that the association area of a typical AP of a tier increases with the channel gain variance and decrease in the path loss exponent for the corresponding tier.

II. STATIONARY ASSOCIATION

The locations of the base stations are $\{T_n\}$ and seen as the atoms of a stationary point process (PP) Φ defined on a measurable space $(\Omega, \mathcal{A}, \mathbb{P})$ and having intensity λ . The analysis in this paper is for \mathbb{R}^2 due to the practical implications, but it also extends to \mathbb{R}^d . Further Φ is assumed to be θ_t compatible, where θ_t is a measurable flow on Ω , so that

$$\Phi(\omega, B + x) = \Phi(\theta_x \omega, B), \omega \in \Omega, x \in \mathbb{R}^2, B \in \mathcal{B},$$

where \mathcal{B} denotes the Borel σ -field on \mathbb{R}^2 . The operation $\theta_x \omega$ can also be thought of as $\Phi(\omega)$ shifted by $-x$. Let $\zeta(x) \in \Phi \forall x \in \mathbb{R}^2$ denote the base station to which a user lying at x associates. The mapping $\zeta : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is referred to as an *association strategy*.

Definition 1. *Stationary Association:* An association strategy $\zeta(x)$ is stationary if the association is translation invariant, i.e.,

$$\zeta(x) = \zeta(0) \circ \theta_x \forall x \in \mathbb{R}^2, \quad (1)$$

where \circ denotes the composition operator.

Further a collection of fields $\{M_n(y)\} \in \mathbb{R}^+ \cup \infty \forall y \in \mathbb{R}^2$ is assumed associated with the atoms $\{T_n\}$ of Φ such that

$$\begin{aligned} M_0(y) \circ \theta_{T_n} &= M_n(y + T_n) \text{ and} \\ M_n(y) &= \infty \text{ if } y = T_n, \end{aligned} \quad (2)$$

and therefore for a given y , the associated field $M_n(y)$ forms a sequence of marks for Φ . Define a mapping $\kappa(y) \triangleq$

$\arg \sup M_n(y)$, where the $\arg \sup$ is assumed to be well defined. Thus, by (1) and (2), $\kappa(y)$ is a stationary association.

Definition 2. *Association Cell:* The association cell $\mathcal{C}(T_n)$ of an AP at T_n is defined as

$$\mathcal{C}(T_n) = \{y \in \mathbb{R}^2 : \kappa(y) = T_n\}$$

and $|\mathcal{C}(T_n)|$ is the corresponding association area.

Lemma 1. *Under stationary association, the area of association cells is a sequence of marks.*

$$\begin{aligned} \text{Proof:} & \text{ It needs to be shown that } |\mathcal{C}(T_n)| = |\mathcal{C}(T_0) \circ \theta_{T_n}|. \\ \mathcal{C}(T_0) \circ \theta_{T_n} &= \{y : M_0(y) \circ \theta_{T_n} > M_m(y) \circ \theta_{T_n} \ \forall m \neq 0\} \\ &\stackrel{(a)}{=} \{y : M_n(y + T_n) > M_{m'}(y + T_n) \ \forall m' \neq n\} \\ &= \mathcal{C}(T_n) - T_n, \end{aligned}$$

where (a) follows from (2). Since area is translation invariant the result follows. ■

Below are listed certain strategies that qualify as a stationary association under certain conditions.

I Max power association: User at y associates with the base station from which it receives the maximum power. Letting $P(n)$ denote the transmit power of AP at T_n , $H_n(y)$ denote the channel power gain and a power law path loss with path loss exponent α_n , the serving AP is

$$\kappa(y) = \arg \sup S_n(y), \quad (3)$$

where $S_n(y) = P(n)H_n(y)\|T_n - y\|^{-\alpha_n}$. The field $S_n(y)$ satisfies (2) if $\{\alpha_n\}$ and $H_n(y)$ are a sequence of marks. Further, if $\arg \sup S_n(y)$ is well defined, then max power association is stationary¹.

II Max SIR association: A user associates with the base station providing the highest SIR. The corresponding field is

$$S_n(y) = \frac{P(n)H_n(y)\|T_n - y\|^{-\alpha_n}}{\sum_{m \neq n} P(m)H_m(y)\|T_m - y\|^{-\alpha_m}}.$$

It can be seen that max SIR association is equivalent to max power association in I. Note that the association cells formed in this case are different than the SINR coverage cells defined in [9].

III Nearest base station association: This results in the classical case of Voronoi cells as association cells, which are stationary.

In this paper, the probability and expectation under the Palm probability are denoted by \mathbb{P}° and $\mathbb{E}^\circ[\cdot]$ respectively.

Proposition 1. *For all measurable functions $f : \Omega \rightarrow \mathbb{R}^+$*

$$\mathbb{E}[f] = \lambda \mathbb{E}^\circ \left[\int_{\mathcal{C}(T_0)} f \circ \theta_u du \right].$$

Proof: Recalling Lemma 1, stationary association leads to association cells that are stationary partitions, i.e., partitions that translate with (or *shadow*) the associated PP. The proof, then, follows using Theorem 4.1 of [6] for stationary partitions. ■

¹If the sum $\sum_{n \geq 1} S_n(y)$ is finite almost surely (a.s.), then there exists no accumulation at $\sup S_n(y)$ a.s. and hence the $\arg \sup S_n(y)$ is well defined a.s.

III. ASSOCIATION IN K -TIER NETWORKS

In a K -tier HetNet, the APs are assumed to belong to K distinct *classes*. Assuming independent deployment of APs of different tiers, we define i.i.d. marks mapping the AP index to tier index as $J(T_n) \in \{1 \dots K\}$. The mapping distribution for a typical BS is

$$p_k \triangleq \mathbb{P}^\circ(J(T_0) = k).$$

The location of the APs of k^{th} tier is denoted by the point process Φ_k , where

$$\Phi_k = \sum_{T_i \in \Phi} \delta_{T_i} \mathbf{1}(J(T_i) = k).$$

The following proposition builds up on Prop. 1 to relate the probability of origin being contained in the association cell of tier i to the association area of a typical cell of the corresponding tier.

Proposition 2. *The probability that the origin is contained in the association cell of an atom of Φ_i is*

$$\mathcal{A}_i \triangleq \mathbb{P}(J(\kappa(0)) = i) = \lambda p_i \mathbb{E}^{\circ, i} [|\mathcal{C}(T_0)|]$$

Proof: Using Prop. 1 with $f = \mathbf{1}(J(\kappa(0)) = i)$, we obtain

$$\mathbb{E}[\mathbf{1}(J(\kappa(0)) = i)] = \lambda \mathbb{E}^\circ \left[\int_{\mathcal{C}(T_0)} \mathbf{1}(J(\kappa(0)) \circ \theta_u = i) du \right]$$

$$\mathbb{P}(J(\kappa(0)) = i) \stackrel{(a)}{=} \lambda \mathbb{E}^\circ [\mathbf{1}(J(\kappa(0)) = i) |\mathcal{C}(T_0)|]$$

$$\mathcal{A}_i \stackrel{(b)}{=} \lambda \mathbb{E}^{\circ, i} [|\mathcal{C}(T_0)|] \mathbb{P}^\circ(J(\kappa(0)) = i),$$

where (a) holds, as under palm $J(\kappa(0)) \circ \theta_u = J(\kappa(u)) = J(\kappa(0))$ for $u \in \mathcal{C}(T_0)$ and (b) follows from Bayes theorem. ■

For the case where users in the network form a homogeneous PPP, \mathcal{A}_i also denotes the probability of a typical user associating with the i^{th} tier. The following proposition gives a conditional form of Prop. 1 in a K -tier setting.

Proposition 3. *For all measurable functions $g : \Omega \rightarrow \mathbb{R}^+$*

$$\mathbb{E}[g | J(\kappa(0)) = i] = \frac{\mathbb{E}^{\circ, i} \left[\int_{\mathcal{C}(T_0)} g \circ \theta_u du \right]}{\mathbb{E}^{\circ, i} [|\mathcal{C}(T_0)|]}.$$

Proof: Using $f = g \mathbf{1}(J(\kappa(0)) = i)$ in Prop. 1, the LHS is

$$\mathbb{E}[g \mathbf{1}(J(\kappa(0)) = i)] = \mathbb{E}[g | J(\kappa(0)) = i] \mathcal{A}_i,$$

and the RHS is

$$\lambda \mathbb{E}^\circ \left[\mathbf{1}(J(\kappa(0)) = i) \int_{\mathcal{C}(T_0)} g \circ \theta_u du \right] = \lambda p_i \mathbb{E}^{\circ, i} \left[\int_{\mathcal{C}(T_0)} g \circ \theta_u du \right]$$

Using Prop. 2 in the above, gives the result. ■

Using the above proposition, the distribution (assuming it exists) of association area of the AP of tier i containing origin can be given in terms of that of the area of a typical association cell of the corresponding tier.

Corollary 1. *The distribution of the area of the association cell of tier i containing origin is given by*

$$f_{|\mathcal{C}(\kappa(0))|}^i(c) = \frac{c f_{|\mathcal{C}(T_0)|}^{\circ, i}(c)}{\mathbb{E}^{\circ, i} [|\mathcal{C}(T_0)|]},$$

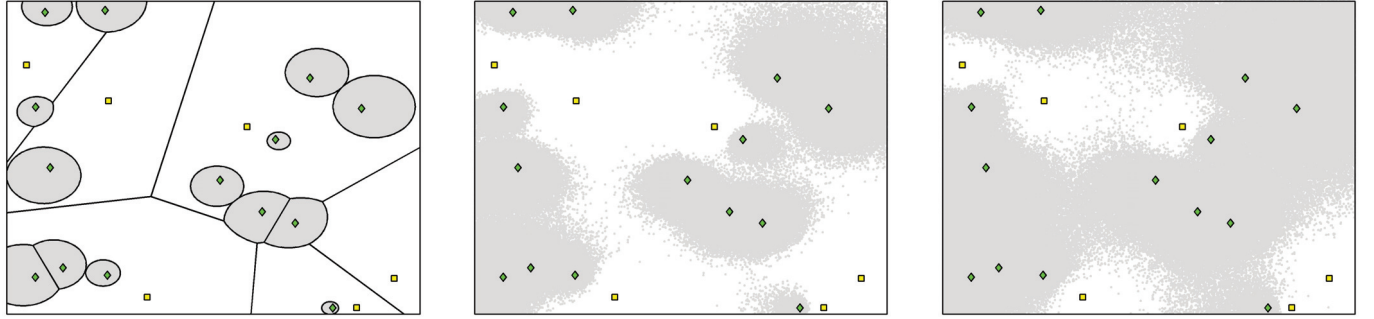
(a) No channel variance, $\sigma_1 = \sigma_2 = 0$ (b) Low channel variance, $\sigma_1 = 1, \sigma_2 = 1$ (c) High channel variance, $\sigma_1 = 1, \sigma_2 = 2$

Fig. 1: The shaded region is served by the APs of tier-2 (diamonds), while the rest of the area is served by tier-1 APs (squares).

where $f_{|\mathcal{C}(T_0)|}^{o,i}$ is the distribution of the area of a typical association cell of tier i .

Proof: Using $g = \mathbf{1}(c \leq |\mathcal{C}(\kappa(0))| \leq c + dc)$ in Prop. 3 we get

$$\begin{aligned} \mathbb{P}(c \leq |\mathcal{C}(\kappa(0))| \leq c + dc | J(\kappa(0)) = i) \\ = \frac{\mathbb{E}^{o,i} \left[\int_{\mathcal{C}(T_0)} \mathbf{1}(c \leq |\mathcal{C}(\kappa(0) \circ \theta_u)| \leq c + dc) du \right]}{\mathbb{E}^{o,i} [|\mathcal{C}(\kappa(0))|]} \end{aligned}$$

$$f_{|\mathcal{C}(\kappa(0))|}^{o,i}(c) \stackrel{(a)}{=} \frac{\mathbb{E}^{o,i} [\mathbf{1}(c \leq |\mathcal{C}(\kappa(0))| \leq c + dc) |\mathcal{C}(\kappa(0))|]}{\mathbb{E}^{o,i} [|\mathcal{C}(\kappa(0))|]},$$

where (a) follows from the fact that under the Palm distribution $|\mathcal{C}(\kappa(0) \circ \theta_u)| = |\mathcal{C}(\kappa(0))|$ for $u \in \mathcal{C}(\kappa(0))$. The final result is obtained using Bayes theorem and the fact that under the Palm distribution $\kappa(0) = T_0$. ■

As a consequence of the above corollary it can be stated that the area of the association cell containing a typical user is larger than that of a typical cell, and the following holds

$$\mathbb{E} [|\mathcal{C}(\kappa(0))|^d | J(\kappa(0)) = i] = \frac{\mathbb{E}^{o,i} [|\mathcal{C}(\kappa(0))|^{d+1}]}{\mathbb{E}^{o,i} [|\mathcal{C}(\kappa(0))|]} \quad \forall d \in \mathbb{R}.$$

IV. MEAN ASSOCIATION AREA IN PPP HETNETS

In this section, the mean association area is derived for the case, where the base station process Φ is assumed to be a PPP. The general max power/SINR association given in (3) is considered. It is further assumed that the APs of k^{th} tier have the same constant power and path loss exponents, and have independent but identical channel gain distribution, i.e., $P(n) = P_k$, $\alpha_n = a_k$, and $H_n(y) \stackrel{(d)}{=} H_k \forall T_n \in \Phi_k$. Due to the i.i.d. assumption on J marks, by the thinning theorem [10], each tier process Φ_k is a PPP with density $\lambda_k \triangleq p_k \lambda$ for $k = 1 \dots K$. For illustration, Fig. 1 shows the association cells in a two tier setup with $P_1 = 53$ dBm, $P_2 = 33$ dBm, $a_1 = a_2 = 4$, and the channel gain is lognormal $H_k \sim \ln \mathcal{N}(0, \sigma_k)$. As seen from the plots, increasing the variance in the channel gain for the second tier increases the corresponding association areas (the shaded areas). This observation is made rigorous by the following analysis.

A. Analysis

Lemma 2. Under the max power association, the mean association area of a typical base station of the i^{th} tier is

$$2\pi \int_0^\infty r \mathbb{E}_{H_i} \left[\exp \left(-\pi \sum_{k=1}^K \tilde{\lambda}_k r^{2a_i/a_k} P_i^{-2/a_k} H_i^{-2/a_k} \right) \right] dr,$$

where $\tilde{\lambda}_k = \lambda_k P_k^{2/a_k} \mathbb{E} [H_k^{2/a_k}]$ and $\mathbb{E} [H_k^{2/a_k}] < \infty$.

Proof: The mean association area of a typical cell of the i^{th} tier is $\mathbb{E}^{o,i} [|\mathcal{C}(T_0)|] = \int_{\mathbb{R}^2} \mathbb{P}^{o,i} (u \in \mathcal{C}(T_0)) du$, which $\forall n \neq 0$

$$\begin{aligned} &= \mathbb{P}^{o,i} \left(\|u\|^{a_i/\alpha_n} (H_0(u) P_i)^{-1/\alpha_n} < \|T_n - u\| (H_n(u) P_n)^{-1/\alpha_n} \right) \\ &= \mathbb{P}^{o,i} \left(\bigcap_{k=1}^K \tilde{\Phi}_k \left(B^o(0, \|u\|^{a_i/a_k} (H_0(u) P_i)^{-1/a_k} \right) = 0 \right), \end{aligned}$$

where $B^o(0, r)$ is the open ball of radius r around origin, $\tilde{\Phi}_k$ denotes the PPP formed by transforming the atoms of $\Phi_k: T_n \rightarrow (T_n - u)(H_n(u) P_k)^{-1/a_k}$. By the i.i.d. displacement theorem [10], $\tilde{\Phi}_k$ is a homogeneous PPP with $\tilde{\lambda}_k = \lambda_k P_k^{2/a_k} \mathbb{E} [H_k^{2/a_k}]$, given $\mathbb{E} [H_k^{2/a_k}] < \infty$, [5], [11]. Thus

$$\begin{aligned} \mathbb{P}^{o,i} (u \in \mathcal{C}(T_0)) &= \int_0^\infty \mathbb{P}^{o,i,h} (u \in \mathcal{C}(T_0)) f_{H_i}(h) dh \\ &= \mathbb{E}_{H_i} \left[\prod_{k=1}^K \mathbb{P}^{o,i} \left(\tilde{\Phi}_k \left(B^o \left(0, \|u\|^{a_i/a_k} (H_i P_i)^{-1/a_k} \right) \right) = 0 \right) \right] \\ &= \mathbb{E}_{H_i} \left[\exp \left(-\pi \sum_{k=1}^K \tilde{\lambda}_k \|u\|^{2a_i/a_k} (H_i P_i)^{-2/a_k} \right) \right] \end{aligned} \quad (4)$$

For the case with the path loss exponents of each tier being the same: $a_k \equiv a$, the mean association area simplifies to

$$\mathbb{E}^{o,i} [|\mathcal{C}(T_0)|] = \frac{P_i^{2/a} \mathbb{E} [H_i^{2/a}]}{\sum_{k=1}^K \lambda_k P_k^{2/a} \mathbb{E} [H_k^{2/a}]} \quad (5)$$

and thus depends on only the $\frac{2}{a}$ th moment of the channel gain. Using Prop. 2 for association probability leads to the earlier derived result in [5], which used *propagation invariance*.

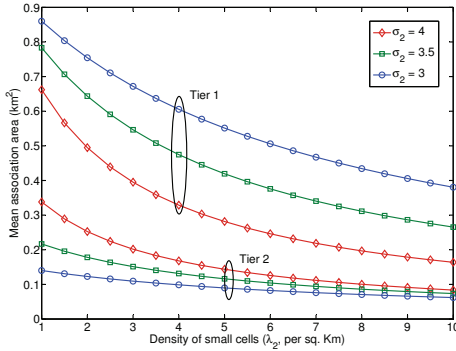


Fig. 2: Variation of mean association areas of two tiers with density for different channel gain variances of second tier.

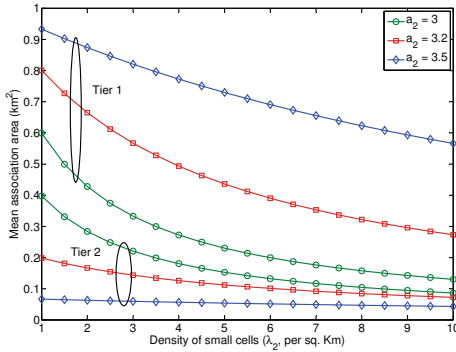


Fig. 3: Variation of mean association areas of two tiers with density for different path loss exponents of second tier.

Remark 1. The framework developed above can be used to compute additive functionals over association cells. Using Campbell's theorem [10], the mean of an additive characteristic g associated with a typical association cell of tier i and defined on an independent PPP Φ_u of intensity λ_u is

$$\begin{aligned} \bar{S} &= \mathbb{E}^{o,i} \left[\mathbb{E}_{\Phi_u} \left[\sum_{Y_j \in \Phi_u} g(Y_j) \mathbf{1}(Y_j \in \mathcal{C}(T_0)) \right] \right] \\ &= \int_{\mathbb{R}^2} g(y) \mathbb{P}^{o,i}(y \in \mathcal{C}(T_0)) \lambda_u dy. \end{aligned}$$

For example, if Φ_u represents the user point process and $g(x) = \|x\|^{-a}$ (a path loss function), then \bar{S} represents the mean total power received at a typical AP of tier i from all the users served by it and is given by (using (4))

$$2\pi\lambda_u \int_{r>0} r^{-a+1} \mathbb{E}_{H_i} \left[\exp \left(-\pi \sum_{k=1}^K \tilde{\lambda}_k r^{2a_i/a_k} (H_i P_i)^{-2/a_k} \right) \right] dr$$

B. Numerical Results

We consider a two tier (macro and pico, say) setup along with max power association with respective transmit powers:

$P_1 = 53$ dBm and $P_2 = 33$ dBm. The variation in the mean association area with the variation in density of small cells (second tier) is shown in Fig. 2, where $\sigma_1 = 2$, $a_1 = a_2 = 3.5$, $\lambda_1 = 1$ BS/sq. Km, and the channel gains are assumed lognormal with $H_k \sim \ln\mathcal{N}(0, \sigma_k)$. It can be seen that with increasing variance in the channel propagation, the corresponding mean association area increases. This follows from (5) and the fact that $\mathbb{E} \left[H_k^{2/a} \right] = \exp(0.5(2/a)^2 \sigma_k^2)$. The effect of path loss exponent on the mean association area is shown in Fig. 3. For the plot, $\sigma_1 = 2$, $\sigma_2 = 4$, and $a_1 = 3$. As can be seen, with decreasing path loss exponent of small cells, the corresponding association area increases. Intuitively, the lower the path loss exponent, the lower the decay rate of the corresponding AP's transmission power and hence there will be a larger number of users associating with the same.

V. CONCLUSION

We introduce the notion of stationary association for random HetNets with the resulting association areas shown to be the marks of the corresponding point process. Analogous to a Voronoi tessellation, an inversion formula relating the association area of the cell containing origin to a typical association area for each tier is proved. It is shown that with max power association, the mean association area of small cells decreases with path loss exponent and increases with channel gain variance.

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