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ON ASYMPTOTIC NORMALITY OF HILL'S ESTIMATOR FOR THE EXPONENT OF REGULAR VARIATION

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It is shown that Hill's estimator (1975) for the exponent of regular variation is asymptotically normal if the number k_n of extreme order statistics used to construct it tends to infinity appropriately with the sample size n. As our main result, we derive a general condition which can be used to determine the optimal k_n explicitly, provided that some prior knowledge is available on the underlying distribution function with regularly varying upper tail. This condition is simplified under appropriate assumptions and then applied to several examples.

1. Introduction. Consider a distribution function F with regularly varying upper tail, i.e. assume without loss of generality, $F(0) = 0$ and

$$(1) \qquad 1 - F(x) = x^{-\alpha}L(x), \quad x > 0,$$

where $\alpha > 0$ and L is slowly varying at infinity. In recent years the problem of estimating the exponent α (or equivalently α^{-1}) from an independent sample $\xi_1, \cdots, \xi_n$ distributed according to F has received increasing attention; see, for example, de Haan and Resnick (1980), Teugels (1981a) and Hall (1982). Since by (1) only the tail behaviour of F is specified, it is intuitively clear that good estimators should be based on the extreme part of the sample. One important estimator of this kind was proposed by Hill (1975). If $\xi_{1:n}, \cdots, \xi_{n:n}$ denote the order statistics pertaining to $\xi_1, \cdots, \xi_n$, the estimator $H_k^{(n)}$ for α^{-1} is defined by

$$H_k^{(n)} = k^{-1} \textstyle\sum_{i=1}^k \log \xi_{n-i+1:n} - \log \xi_{n-k:n}, \quad 1 \le k < n, \quad n \in \mathbb{N}.$$

Averages of the $k + 1$ largest observations like $H_k^{(n)}$ occur in practice for example in insurance mathematics in connection with the ECOMOR reinsurance policy (cf. Teugels, 1981b).

Originally Hill obtained $H_k^{(n)}$ as a conditional maximum likelihood estimator in the case where L is equal to a constant for large x. Assuming only (1) Weissman (1978) derived $H_k^{(n)}$ for fixed k as the maximum likelihood estimator based on the limiting joint distribution of the k largest order statistics. To give another motivation for $H_k^{(n)}$ being an appropriate estimator under the general model (1) observe that

$$\int_x^\infty \frac{1 - F(y)}{y(1 - F(x))}\, dy = \int_1^\infty \frac{1 - F(xt)}{1 - F(x)} \frac{dt}{t} \to \int_1^\infty t^{-\alpha-1}\, dt = \alpha^{-1} \quad \text{as} \quad x \to \infty$$

by regular variation of $1 - F$ and dominated convergence. If F is replaced by its empirical counterpart, i.e. by the empirical distribution function F_n based on

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$\xi_1, \cdots, \xi_n$, and if x equals $\xi_{n-k:n}$, then an easy calculation shows

$$\int_{\xi_{n-k:n}}^{\infty} \frac{1 - F_n(y)}{y(1 - F_n(\xi_{n-k:n}))} \, dy = H_k^{(n)}.$$

This relation between $H_k^{(n)}$ and F_n suggests $H_k^{(n)}$ is a good estimator for α^{-1} under the general model (1), too.

If k is held fixed as n increases, then $H_k^{(n)}$ converges in law to a gamma distribution (cf. Section 2). Consequently, to get a consistent estimator one has to increase k with n. Mason (1982) proved weak consistency of $H_{k_n}^{(n)}$ for any sequence $k_n \to \infty$ with $k_n = o(n)$ as $n \to \infty$ and strong consistency for $k_n = [n^a]$, $0 < a < 1$. Quite recently, Hall and Welsh (1984) showed that $H_{k_n}^{(n)}$ is optimal w.r.t. rates of convergence provided that k_n is chosen properly (at least for some subclass of the class of all distribution functions defined by (1)).

The problem of asymptotic normality has been studied by Hall (1982) for slowly varying functions which converge to a constant at a polynomial rate. Under appropriate conditions, Hall established the existence of an optimal sequence $\bar{k}_n$ such that $\bar{k}_n^{1/2}(H_{\bar{k}_n}^{(n)} - \alpha^{-1})$ is asymptotically normal where $\bar{k}_n$ is optimal in the sense that $a_n(H_{k_n}^{(n)} - \alpha^{-1})$ for any norming constants a_n never converges in distribution to a nondegenerate limit if k_n tends to infinity faster than $\bar{k}_n$. Other recent approaches to asymptotic normality of $H_{k_n}^{(n)}$ are developed by Csörgő and Mason (1985) and Davis and Resnick (1984). They show that $H_{k_n}^{(n)}$ (centered and normalized suitably) is asymptotically normal for any F satisfying (1) and for all sequences $k_n \to \infty$ such that $k_n = o(n)$. The centering depends on F and n and may even be by random quantities. For statistical purposes, however, it is of importance to center $H_{k_n}^{(n)}$ by the exponent α^{-1} which is aimed to be estimated, i.e. one is interested in results obtained by Hall for his special case of (1). From the asymptotic behaviour of $H_k^{(n)}$ for fixed k it follows immediately that for any F satisfying (1) the sequence $k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1})$, $n \in \mathbb{N}$, converges in law to a centered normal distribution with variance α^{-2} provided that k_n tends to infinity slowly enough. Hall's (1982) result shows that for sequences k_n tending to infinity too fast it is impossible in general to normalize $H_{k_n}^{(n)} - \alpha^{-1}$ in such a way that it converges in distribution. The problem addressed in the present paper is to find conditions for the general model (1) from which all the sequences k_n which make $k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1})$ asymptotically normal can be computed if some prior knowledge about the slowly varying function L is available. This complements the probabilistic results of Csörgő and Mason (1985) and Davis and Resnick (1984) from a statistical point of view. Our basic condition is derived in Section 3. In Section 4 it is simplified into rather manageable forms under appropriate assumptions on L, and concrete examples are considered in Section 5.

For further results on asymptotic normality of Hill's estimator we refer to Goldie and Smith (1984) where a slightly differently constructed version is studied.

2. Preliminary results for fixed k. As a starting point we consider the asymptotic behaviour of $H_k^{(n)}$ for fixed k. To simplify the notation and the calculations, it is convenient to get rid of the logarithms in the definition of $H_k^{(n)}$ by means of a time scale transformation. So put $G(x) = F(e^x)$. According to Lemma 1.8 in Seneta (1976) the generalized inverse function $G^{-1}(u) = \inf\{x \in \mathbb{R}: G(x) \geq u\}$ of G is given by

$$G^{-1}(1 - u) = -\alpha^{-1}\log u + \log L'(1/u), \quad 0 < u < 1, \tag{2}$$

where L' is slowly varying at infinity. Let $U_1, U_2, \cdots$ be i.i.d. uniform (0, 1) r.v.'s, and let $U_{1:n}, \cdots, U_{n:n}$ denote the order statistics pertaining to $U_1, \cdots, U_n$. For

$$\bar{H}_k^{(n)} = k^{-1} \sum_{i=1}^{k} G^{-1}(1 - U_{i:n}) - G^{-1}(1 - U_{k+1:n})$$

we have $\bar{H}_k^{(n)} =_{\mathscr{D}} H_k^{(n)}$ for all $1 \leq k < n$ and $n \in \mathbb{N}$. From now on we shall always deal with the versions $\bar{H}_k^{(n)}$, denoting them with $H_k^{(n)}$ for convenience. From (2) one obtains the representation

$$H_k^{(n)} = \alpha^{-1} X_k - R_{n,k} \quad \text{for} \quad 1 \leq k < n \quad \text{and} \quad n \in \mathbb{N} \tag{3}$$

with

$$X_k = k^{-1} \sum_{i=1}^{k} (\log U_{k+1:n} - \log U_{i:n})$$

and

$$R_{n,k} = nk^{-1} \int_0^{U_{k+1:n}} G_n(u)\, d \log L'\left(\frac{1}{u}\right)$$

where G_n is the left continuous empirical distribution function based on $U_1, \cdots, U_n$. All our Riemann-Stieltjes integrals are to be understood in the sense of Apostol (1974), Chapter VII, possibly in the usual improper sense.

As shown in Mason (1982), for each k and n the variable kX_k is distributed as the sum of k i.i.d. exponential r.v.'s with mean 1, i.e. $\alpha^{-1}X_k$ has a gamma distribution with parameters αk and k; especially, the distribution is independent of n which therefore is suppressed in the notation. The Karamata-Representation-Theorem, cf., for example, Seneta (1976), Theorem 1.2, implies

$$\log L'\left(\frac{1}{u}\right) = \eta\left(\frac{1}{u}\right) + \int_1^{1/u} \frac{\varepsilon(t)}{t}\, dt, \quad 0 < u < 1, \tag{4}$$

for some bounded function η: $[1, \infty) \to \mathbb{R}$ with $\eta(x) \to c \in \mathbb{R}$ as $x \to \infty$ and some continuous function ε: $[1, \infty) \to \mathbb{R}$ with $\varepsilon(x) \to 0$ as $x \to \infty$. Hence

$$R_{n,k} = nk^{-1} \int_0^{U_{k+1:n}} G_n(u)\, d\eta\left(\frac{1}{u}\right) - nk^{-1} \int_0^{U_{k+1:n}} \frac{G_n(u)}{u}\, \varepsilon\left(\frac{1}{u}\right) du. \tag{5}$$

Tchebycheff's inequality yields

$$U_{k+1:n} = kn^{-1} + O_p(n^{-1}) \quad \text{as} \quad n \to \infty. \tag{6}$$

Using this fact and $\varepsilon(x) \to 0$ as $x \to \infty$, one obtains by standard arguments that the second summand on the r.h.s. of (5) converges to zero in probability. The first one is equal to

$$nk^{-1}\int_0^{U_{k+1:n}} G_n(u)\, d\left[\eta\left(\frac{1}{u}\right) - \eta\left(\frac{1}{U_{k+1:n}}\right)\right]$$
$$= -nk^{-1}\int_0^{U_{k+1:n}} \left[\eta\left(\frac{1}{u}\right) - \eta\left(\frac{1}{U_{k+1:n}}\right)\right] dG_n(u)$$

by an integration by parts. Now convergence to zero in probability is easily deduced from (6) and $\eta(x) \to c$ as $x \to \infty$. Thus we have shown that $R_{n,k}$ is asymptotically negligible and arrive at the following:

THEOREM 1. *Assume* (1) *holds. Then for fixed* $k \in \mathbb{N}$

$$H_k^{(n)} \to_{\mathscr{D}} Y_{\alpha,k} \quad as \quad n \to \infty,$$

where $Y_{\alpha,k}$ *has a gamma distribution with parameters* αk *and* k *and where* $\to_{\mathscr{D}}$ *denotes convergence in distribution.*

Theorem 1 also follows from the results in Smid and Stam (1975) and in Weissman (1978); cf. also Hall (1978).

As mentioned in the introduction, our main interest is in the asymptotic behaviour of $H_{k_n}^{(n)}$ if k_n tends to infinity with n. A first result following from Theorem 1 by a standard diagonalization argument is:

COROLLARY 1. *If* (1) *holds, then there exists a sequence of integers* $\bar{k}_n \to \infty$ *such that for any sequence of integers* $k_n \to \infty$ *with* $k_n \le \bar{k}_n$ *for each* n *one has* $k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(0, \alpha^{-2})$ *as* $n \to \infty$.

3. Main limit theorem. Theorem 1 explains where asymptotic normality of $k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1})$, for $k_n \to \infty$ slowly enough, comes from, but it yields a mere existence result only. To get results useful in practice, one has to find conditions on F from which it is possible to determine explicitly all sequences k_n for which asymptotic normality holds true. Such a condition is presented in this section. To simplify the formulation we introduce some notation. For each n and k_n and any real x, write $I(n, k_n, x) = (k_n/n)(1 + x/k_n^{1/2})$, and for each distribution function F satisfying (1) and each n, k_n and x put

$$J(F, n, k_n, x) = k_n^{1/2}\int_1^\infty \left\{y^{-\alpha} - \frac{1 - F[F^{-1}(1 - I(n, k_n, x))y]}{I(n, k_n, x)}\right\}\frac{dy}{y}.$$

THEOREM 2. *Assume* (1) *holds and* $k_n \to \infty$ *and* $k_n = o(n)$ *as* $n \to \infty$.

(i) *If* $J(F, n, k_n, x) \to A \in \mathbb{R}$ *uniformly in* x *on compact sets as* $n \to \infty$, *then*

$$k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(-A, \alpha^{-2}) \quad as \quad n \to \infty.$$

(ii) *If* a_n *is a sequence of constants such that* $J(F, n, k_n, x) \sim a_n \to \pm\infty$ *uniformly* *in x on compact sets as $n \to \infty$, then*

$$a_n^{-1} k_n^{1/2} (H_{k_n}^{(n)} - \alpha^{-1}) \to_p -1 \quad \text{as} \quad n \to \infty,$$

where $\to_p$ denotes convergence in probability.

PROOF. For the sake of notational simplicity, we write k instead of k_n. Starting from representation (3), we have $k^{1/2}(H_k^{(n)} - \alpha^{-1}) = \alpha^{-1} k^{1/2}(X_k - 1) - k^{1/2} R_{n,k}$ where $\alpha^{-1}k^{1/2}(X_k - 1)$ converges in distribution to $\mathscr{N}(0, \alpha^{-2})$ by the classical central limit theorem. For the remainder term, we write using (4)

$$\begin{aligned}
&k^{1/2} R_{n,k} \\
&= nk^{-1/2} \int_0^{k/n} (G_n(u) - u)\, d\eta\left(\frac{1}{u}\right) + nk^{-1/2} \int_{k/n}^{U_{k+1:n}} (G_n(u) - u)\, d\eta\left(\frac{1}{u}\right) \\
&\quad - nk^{-1/2} \int_0^{k/n} \frac{G_n(u) - u}{u} \varepsilon\left(\frac{1}{u}\right) du - nk^{-1/2} \int_{k/n}^{U_{k+1:n}} \frac{G_n(u) - u}{u} \varepsilon\left(\frac{1}{u}\right) du \\
&\quad + nk^{-1/2} \int_0^{U_{k+1:n}} u \, d \log L'\left(\frac{1}{u}\right) \\
&= \mathrm{I}_n + \mathrm{II}_n + \mathrm{III}_n + \mathrm{IV}_n + \mathrm{V}_n,
\end{aligned}$$

say. Now $\mathrm{I}_n, \cdots, \mathrm{IV}_n$ converge to zero in probability for each sequence k with $k \to \infty$ and $k = o(n)$. As an example we consider II_n in some detail. An integration by parts yields

$$\begin{aligned}
\mathrm{II}_n &= nk^{-1/2} \int_{k/n}^{U_{k+1:n}} (G_n(u) - u)\, d\left[\eta\left(\frac{1}{u}\right) - \eta\left(\frac{1}{U_{k+1:n}}\right)\right] \\
&= nk^{-1/2} \left[\eta\left(\frac{1}{U_{k+1:n}}\right) - \eta\left(\frac{n}{k}\right)\right]\left(G_n\left(\frac{k}{n}\right) - \frac{k}{n}\right) \\
&\quad - nk^{-1/2} \int_{k/n}^{U_{k+1:n}} \left[\eta\left(\frac{1}{u}\right) - \eta\left(\frac{1}{U_{k+1:n}}\right)\right] dG_n(u) \\
&\quad + nk^{-1/2} \int_{k/n}^{U_{k+1:n}} \left[\eta\left(\frac{1}{u}\right) - \eta\left(\frac{1}{U_{k+1:n}}\right)\right] du.
\end{aligned}$$

Now by Tchebycheff's inequality

(7) $$U_{k+1:n} = k/n + O_p(k^{1/2}/n) \quad \text{as} \quad n \to \infty;$$

hence $\eta(n/k) - \eta(U_{k+1:n}^{-1}) \to_p 0$ since $\eta(x) \to c$ as $x \to \infty$. In view of $nk^{-1/2}(G_n(kn^{-1}) - kn^{-1}) = O_p(1)$, the first summand on the right-hand side of the above equation converges to zero in probability. The absolute value of the third one is bounded by

$$\sup\{|\eta(u^{-1}) - \eta(U_{k+1:n}^{-1})| : u \le \max(kn^{-1}, U_{k+1:n})\} nk^{-1/2} |U_{k+1:n} - kn^{-1}|,$$

which converges to zero in probability because of (7) and $\eta(x) \to c$ as $x \to \infty$. The second summand is handled the same way so that $\mathrm{II}_n \to_p 0$. Similar arguments hold for I_n, III_n and IV_n so that

$$k^{1/2} R_{n,k} = \mathrm{V}_n + o_p(1).$$

Dealing with V_n we employ (2) and a change of variable to get for $0 < a < 1$

$$\begin{aligned} \int_0^a u \, d \log L'\left(\frac{1}{u}\right) &= \frac{a}{\alpha} - \int_0^a u \, d(-G^{-1}(1-u)) \\ &= \frac{a}{\alpha} - \int_1^\infty [1 - F(F^{-1}(1-a)y)] \frac{dy}{y}. \end{aligned} \tag{8}$$

Assume now that the condition in (i) is satisfied. An application of (8) yields

$$nk^{-1/2} \int_0^{I(n,k,x)} u \, d \log L'\left(\frac{1}{u}\right) \to A \quad \text{uniformly in } x \text{ on compact sets.}$$

Combining with (7) one easily derives $\mathrm{V}_n \to_p A$ which proves part (i) of the theorem. If the condition in (ii) is satisfied, then by (8)

$$nk^{-1/2} \int_0^{I(n,k,x)} u \, d \log L'\left(\frac{1}{u}\right) \sim a_n \quad \text{uniformly in } x \text{ on compact sets,}$$

which together with (7) implies $a_n^{-1} \mathrm{V}_n \to_p 1$. Since $a_n^{-1} k^{1/2}(X_k - 1) = o_p(1)$, this concludes the proof of the theorem. □

If F has a continuous tail, i.e. if it is continuous on (x_0, ∞) for some finite x_0, then it is possible to formulate the conditions of Theorem 1 without explicitly referring to the generalized inverse function. For this, let $z = z(F, n, k_n, x)$ be any solution of the equation

$$1 - F(z) = I(n, k_n, x). \tag{9}$$

For x varying in a fixed compact set, z is well defined uniformly in x if n is large. Furthermore, put

$$\tilde{J}(F, n, k_n, x) = k_n^{1/2} \int_1^\infty \left[y^{-\alpha} - \frac{1 - F(zy)}{1 - F(z)} \right] \frac{dy}{y}.$$

Since $z = F^{-1}(1 - I(n, k_n, x))$ satisfies (9) for n large enough, we obtain from Theorem 2:

COROLLARY 2. *Assume* (1) *and continuity of* F *on* (x_0, ∞) *for some finite* x_0. *Let* $k_n \to \infty$ *such that* $k_n = o(n)$. *Then the assertions of Theorem* 2 *remain valid if* $J(F, n, k_n, x)$ *is replaced by* $\tilde{J}(F, n, k_n, x)$.

The conditions in Theorem 2 and Corollary 2 seem to be clumsy since the convergence involved is assumed to be uniform in x on compact sets. A more appealing theorem would emerge if convergence was required for $x = 0$ only. It can be shown by examples, however, that such a result does not hold; some dependency on additional parameters such as x cannot be avoided. On the other hand, our condition clearly reflects the influence of the slowly varying function L. This is most easily seen if $\tilde{J}(F, n, k_n, x)$ is rewritten, substituting the tail of F given by (1). From

$$\tilde{J}(F, n, k_n, x) = k_n^{1/2} \int_1^\infty y^{-\alpha-1}\left(1 - \frac{L(zy)}{L(z)}\right) dy$$

it becomes clear that the behaviour of $k_n^{1/2}(1 - L(zy)/L(z))$ determines the possible values of k_n in the results on asymptotic normality of $H_{k_n}^{(n)}$.

4. Some special cases. In this section we shall show that the basic conditions of Theorem 2 and Corollary 2 simplify considerably under appropriate assumptions on the slowly varying function L occurring in (1).

To begin with, observe that in the proof of Theorem 2 uniformity in x of the asymptotic behaviour of $J(F, n, k_n, x)$ was needed only when dealing with V_n. If L is a normalized slowly varying function, i.e. if the function η in its Karamata representation, $L(x) = \exp\{\eta(x) + \int_1^x \varepsilon(t)t^{-1}\,dt\}$ with η and ε as in (4), is constant (in a neighbourhood of infinity at least), then the same is true for the slowly varying function L' occurring in (2), i.e. the η in the representation (4) is constant (in a neighbourhood of infinity), too. Then for any sequence k_n with $k_n \to \infty$ and $k_n = o(n)$ we obtain $V_n = J(F, n, k_n, 0) + o_p(1)$ as $n \to \infty$ from (7) and (8). Consequently, it is not necessary to check uniformity in x in the conditions on J and $\tilde{J}$, and we have:

COROLLARY 3. *If the slowly varying function L occurring in* (1) *is normalized, then the $J(F, n, k_n, x)$ in Theorem* 2 *and the $\tilde{J}(F, n, k_n, x)$ in Corollary* 2 *can be replaced by $J(F, n, k_n, 0)$ and $\tilde{J}(F, n, k_n, 0)$, respectively.*

Let us briefly relate this result to some of the results in Csörgő and Mason (1985) and Davis and Resnick (1984). For this, consider $G(x) = F(e^x)$, $x \in \mathbb{R}$. Since L is assumed to be normalized, G belongs to the class $\mathscr{E}_{1/\alpha}^{*}$ introduced by Csörgő and Mason, and it satisfies condition (2.6) of Davis and Resnick (in fact, (2.6) for G is equivalent to L being normalized with possibly discontinuous ε in the Karamata representation). Using the notation

$$b(t) = G^{-1}(1 - 1/t) \quad \text{and} \quad a(t) = t \int_{b(t)}^\infty (1 - G(s))\,ds$$

from Davis and Resnick (1984) an elementary calculation yields

$$\int_1^\infty [1 - F(F^{-1}(1 - k_n/n)y)]\,\frac{dy}{y} = \frac{k_n}{n}\,a\left(\frac{n}{k_n}\right),$$

hence $J(F, n, k_n, 0) = k_n^{1/2}(\alpha^{-1} - a(n/k_n))$ for large n. Theorem 1.7 of Csörgő and

Mason (1985) as well as Theorem 4.1 of Davis and Resnick (1984) shows that

$$k_n^{1/2}[H_{k_n}^{(n)}/a(n/k_n) - 1] \rightarrow_{\mathscr{D}} \mathscr{N}(0, 1) \quad \text{as} \quad n \rightarrow \infty,$$

and this, combined with $a(n/k_n) \rightarrow 1/\alpha$ and $J(F, n, k_n, 0) \rightarrow A$ or $\tilde{J}(F, n, k_n, 0) \rightarrow A$ as $n \rightarrow \infty$, is tantamount to Corollary 3. A sufficient condition for L being normalized is that F satisfies the following classical condition due to von Mises: F has a strictly positive derivative F' on $[x_0, \infty)$ for some $x_0 < \infty$ and $xF'(x)/(1 - F(x)) \rightarrow \alpha$ as $x \rightarrow \infty$; then obviously for $x > x_0$

$$1 - F(x) = (1 - F(x_0))x_0^{\alpha}x^{-\alpha}\exp\left\{\int_{x_0}^{x} \frac{1}{u}\left[\alpha - \frac{uF'(u)}{1 - F(u)}\right] du\right\}.$$

Goldie and Smith (1984) consider their version of Hill's estimator under the assumption that the slowly varying function L satisfies one of the following asymptotic relations (cf. Smith, 1982):

SR1 $L(xy)/L(x) = 1 + O(g(x))$ as $x \rightarrow \infty$ for each $y > 0$,

SR2 $L(xy)/L(x) = 1 + k(y)g(x) + o(g(x))$ as $x \rightarrow \infty$ for each $y > 0$ and some real-valued function k.

If g is positive and $g(x) \rightarrow 0$ as $x \rightarrow \infty$ (which will always be assumed henceforth), then L is slowly varying with a specified remainder term. We shall discuss how Theorem 2 and Corollary 2 appear under these conditions. Propositions 2.5.1 and 2.5.2 of Goldie and Smith (1984) are crucial for that purpose.

Suppose first that (1) holds and that L fulfills SR1 with g nonincreasing or $g(yx)/g(x) \leq Cy^{\rho}$ for all $y \geq 1$, $x \geq x_0$ and some x_0, $C < \infty$ and $\rho < 0$. Fix $k_n \rightarrow \infty$ with $k_n = o(n)$. If F is continuous and $z = z(F, n, k_n, x)$ is defined by (9), then applying Proposition 2.5.1 of Goldie and Smith (1984) yields

$$(10) \qquad \int_1^{\infty} y^{-\alpha-1}\frac{L(zy)}{L(z)}\,dy = \int_1^{\infty} y^{-\alpha-1}\,dy + O_x(g(z)) \quad \text{as} \quad n \rightarrow \infty,$$

where here and henceforth the subscript x is used to indicate that an order relation holds uniformly in x on compact sets. From (10) one gets $\tilde{J}(F, n, k_n, x) = O_x(k_n^{1/2}g(z))$ as $n \rightarrow \infty$. Therefore, if $k_n^{1/2}g(z) \rightarrow 0$ as $n \rightarrow \infty$ uniformly in x on compact sets, the hypotheses of Corollary 2 hold with $A = 0$. For the case that F is possibly discontinuous, we put $z = F^{-1}(1 - I(F, n, k_n, x))$. The argument leading up to equation (2.2) of Smith (1982) shows that $1 - F(z) = I(n, k_n, x)[1 + O_x(g(z))]$ and, consequently, for all $y \geq 1$

$$\frac{1 - F(zy)}{I(n, k_n, x)} = \frac{1 - F(z)}{I(n, k_n, x)}\,y^{-\alpha}\frac{L(zy)}{L(z)} = [1 + O_x(g(z))]y^{-\alpha}\frac{L(zy)}{L(z)} \quad \text{as} \quad n \rightarrow \infty$$

where the O_x is independent of y. Again (10) is true and implies $J(F, n, k_n, x) = O_x(k_n^{1/2}g(z))$ as $n \rightarrow \infty$, i.e. $k_n^{1/2}g(z) \rightarrow 0$ as $n \rightarrow \infty$ uniformly in x on compact sets suffices for Theorem 2 as well as Corollary 2. Moreover, if g is even regularly varying, then it is easy to see that

$$g(z(F, n, k_n, x)) \sim g(z(F, n, k_n, 0)) \quad \text{as} \quad n \to \infty \quad \text{uniformly in } x \text{ on compact sets} \tag{11}$$

so that it suffices to check the case $x = 0$ only. Summarising we obtain:

COROLLARY 4. *Under the foregoing assumptions* $k_n^{1/2} g(z) \to 0$ *as* $n \to \infty$ *uniformly in x on compact sets implies*

$$k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(0, \alpha^{-2}) \quad as \quad n \to \infty.$$

If g is regularly varying, then $k_n^{1/2} g(z(F, n, k_n, 0)) \to 0$ *as* $n \to \infty$ *is already sufficient.*

Let L occurring in (1) now satisfy SR2. Then under some mild additional restrictions on k the function g is regularly varying with index $\rho \le 0$, and $k(y) = Kh_\rho(y)$, $y > 0$, for some finite K and h_ρ defined by $h_\rho(y) = \int_1^y t^{\rho-1}\,dt$; cf. Goldie and Smith (1984) and references therein. Therefore, from now on we shall assume that SR2 is satisfied in this form with $K \ne 0$. If $\rho = 0$, then we also suppose g to be nonincreasing. Fix $k_n \to \infty$ such that $k_n = o(n)$. If F is continuous and z is defined by (9), then Proposition 2.5.2 of Goldie and Smith (1984) gives

$$\tilde{J}(F, n, k_n, x) \sim -(K/\alpha(\alpha - \rho)) k_n^{1/2} g(z) \tag{12}$$

as $n \to \infty$ uniformly in x on compact sets. Thus Corollary 2 is greatly simplified in the present situation.

To bring Theorem 2 into this framework, consider again $z = F^{-1}(1 - I(n, k_n, x))$ for possibly discontinuous F. Now the argument leading up to equation (3.1) of Smith (1982) is appropriate to obtain $1 - F(z) = I(n, k_n, x)[1 + o_x(g(z))]$ which leads to (12) again (with $\tilde{J}$ replaced by J), and Theorem 2 is simplified in the same manner as is Corollary 2. Moreover, because g is regularly varying, (11) is true, and we arrive at:

COROLLARY 5. *Under the assumptions described before,* $k_n^{1/2} g(z(F, n, k_n, 0))$ $\to A \in \mathbb{R}$ *as* $n \to \infty$ *implies*

$$k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(KA/(\alpha(\alpha - \rho)), \alpha^{-2}) \quad as \quad n \to \infty,$$

and $k_n^{1/2} g(z(F, n, k_n, 0)) \to \pm\infty$ *as* $n \to \infty$ *implies*

$$g(z(F, n, k_n, 0))^{-1}(H_{k_n}^{(n)} - \alpha^{-1}) \to_p K/(\alpha(\alpha - \rho)) \quad as \quad n \to \infty.$$

5. Examples. In this final section, we shall demonstrate the applicability of the previous results by discussing several examples.

EXAMPLE 1 (Hall, 1982).

(a) $$1 - F(x) = Cx^{-\alpha}[1 + O(x^{-\beta})] \quad \text{as} \quad x \to \infty; \quad C, \alpha, \beta > 0.$$

For $L(x) = C[1 + O(x^{-\beta})]$ one has for all $y > 0$ as $x \to \infty$

$$L(xy)/L(x) = [1 + O((xy)^{-\beta})]/[1 + O(x^{-\beta})] = 1 + O(x^{-\beta})$$

so that SR1 is satisfied with the regularly varying function $g(x) = x^{-\beta}$. Consequently, Corollary 4 applies, and one has to examine $k_n^{1/2}g(z) = k_n^{1/2}z^{-\beta}$ for $z = F^{-1}(1 - k_n/n)$. Now $F^{-1}(y) = C^{1/\alpha}(1-y)^{-1/\alpha}[1 + O((1-y)^{\beta/\alpha})]$ as $y \to 1$ implies $z \sim C^{1/\alpha}(k_n/n)^{-1/\alpha}$ and $k_n^{1/2}z^{-\beta} \sim C^{-\beta/\alpha}k_n^{1/2}(k_n/n)^{\beta/\alpha}$ as $n \to \infty$, and this converges to zero if and only if $k_n = o(n^{2\beta/(2\beta+\alpha)})$. Thus we obtain from Corollary 4 for each $k_n \to \infty$ such that $k_n = o(n^{2\beta/(2\beta+\alpha)})$

$$k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(0, \alpha^{-2}) \quad \text{as} \quad n \to \infty.$$

(b) $\quad 1 - F(x) = Cx^{-\alpha}[1 + Dx^{-\beta} + o(x^{-\beta})] \quad \text{as} \quad x \to \infty; \quad C, \alpha, \beta > 0, \quad D \in \mathbb{R}.$

For $L(x) = C[1 + Dx^{-\beta} + o(x^{-\beta})]$ one has for all $y > 0$ as $x \to \infty$

$$\begin{aligned} L(xy)/L(x) &= [1 + Dy^{-\beta}x^{-\beta} + o(x^{-\beta})][1 - Dx^{-\beta} + o(x^{-\beta})] \\ &= 1 - \beta D \int_1^y t^{-\beta-1}\, dt + o(x^{-\beta}). \end{aligned}$$

Thus SR2 is satisfied with $K = -\beta D$, $\rho = -\beta$ and $g(x) = x^{-\beta}$, and Corollary 5 applies. As in part (a) one has $k_n^{1/2}g(z) \sim C^{-\beta/\alpha}k_n^{1/2}(k_n/n)^{\beta/\alpha}$ as $n \to \infty$ for $z = F^{-1}(1 - k_n/n)$, whence according to Corollary 5

(i) if $k_n \to \infty$ and $k_n = o(n^{2\beta/(2\beta+\alpha)})$, then

$$k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(0, \alpha^{-2});$$

(ii) if $k_n \sim \lambda n^{2\beta/(2\beta+\alpha)}$ for $0 < \lambda < \infty$, then

$$k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(-\alpha^{-1}(\alpha + \beta)^{-1}\beta C^{-\beta/\alpha}D\lambda^{(2\beta+\alpha)/2\alpha}, \alpha^{-2});$$

(iii) if $D \neq 0$, $k_n = o(n)$ and $k_n/n^{2\beta/(2\beta+\alpha)} \to \infty$, then

$$(n/k_n)^{\beta/\alpha}(H_{k_n}^{(n)} - \alpha^{-1}) \to_p -\alpha^{-1}(\alpha + \beta)^{-1}\beta C^{-\beta/\alpha}D.$$

Obviously in case (iii) the sequence $a_n(H_{k_n}^{(n)} - \alpha^{-1})$ does not converge in distribution for any norming sequence a_n. The framework of Example 1 covers i.a. Pareto distributions, stable distributions and the extreme value distribution $F(x) = \exp(-x^{-\alpha})$.

EXAMPLE 2.

$$1 - F(x) = Cx^{-\alpha}(\log x)^{\beta}, \quad x \text{ large}, \quad C, \alpha > 0, \quad \beta \in \mathbb{R}\backslash\{0\}.$$

For $L(x) = C(\log x)^{\beta}$ we have for all $y > 0$ as $x \to \infty$

$$L(xy)/L(x) = 1 + \beta \int_1^y t^{-1}\, dt/\log x + o(1/\log x).$$

Thus SR2 is satisfied with $K = \beta$, $\rho = 0$ and $g(x) = 1/\log x$, and Corollary 5 applies. Let z be defined by (9) for $x = 0$, i.e. $k_n/n = 1 - F(z) = Cz^{-\alpha}(\log z)^{\beta}$.

Then $\log z \sim -\alpha^{-1}\log(k_n/n)$ and $k_n^{1/2}g(z) \sim -\alpha k_n^{1/2}/\log(k_n/n)$ as $n \to \infty$. From this and Corollary 5 we infer

(i) if $k_n \to \infty$ and $k_n = o((\log n)^2)$, then

$$k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(0, \alpha^{-2});$$

(ii) if $k_n \sim \lambda(\log n)^2$, $0 < \lambda < \infty$, then

$$k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(\alpha^{-1}\beta\lambda^{1/2}, \alpha^{-2});$$

(iii) if $k_n = o(n)$ and $k_n/(\log n)^2 \to \infty$, then

$$|\log(k_n/n)|(H_{k_n}^{(n)} - \alpha^{-1}) \to_p \alpha^{-1}\beta.$$

EXAMPLE 3.

(a) $$1 - F(x) = Cx^{-\alpha}[1 + O((\log x)^{-\beta})] \quad \text{as} \quad x \to \infty; \quad C, \alpha, \beta > 0.$$

For $L(x) = C[1 + O((\log x)^{-\beta})]$ we have for all $y > 0$ as $x \to \infty$

$$L(xy)/L(x) = 1 + O((\log x)^{-\beta}).$$

Thus SR1 is satisfied with the slowly varying function $g(x) = (\log x)^{-\beta}$, and Corollary 4 applies. Since $F^{-1}(y) \sim C^{1/\alpha}(1 - y)^{-1/\alpha}$ as $y \to 1$, we obtain

$$\log z \sim -\alpha^{-1}\log(k_n/n) \quad \text{as} \quad n \to \infty \quad \text{for} \quad z = F^{-1}(1 - k_n/n),$$

whence $k_n^{1/2}g(z) -\alpha^{\beta}k_n^{1/2}/(\log(k_n/n))^{\beta}$ which converges to zero if and only if $k_n = o((\log n)^{2\beta})$. Thus for $k_n \to \infty$ with $k_n = o((\log n)^{2\beta})$ Corollary 4 implies

$$k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(0, \alpha^{-2}) \quad \text{as} \quad n \to \infty.$$

Let us construct an example showing that the exponent 2β in our condition on k_n cannot be increased. For this, consider

$$1 - F(x) = x^{-\alpha}[1 + U(x)(\log x)^{-\beta}], \quad x \text{ large},$$

where $U(x) = \sin(\log x)$. By elementary calculus, it is easy to see that F is strictly increasing for large x so that $1 - F$ is the tail of a continuous distribution function belonging to the class under consideration. For $n \in \mathbb{N}$ put $\ell_n = [(2\pi\alpha)^{-1}(-\log n + 2\beta \log\log n)]$ and $k_n = [ne^{2\pi\alpha\ell_n}]$ where $[x]$ denotes the integer part of x. Then one has

(13) $$\liminf_{n\to\infty} k_n(\log n)^{-2\beta} = e^{-2\pi\alpha} \quad \text{and} \quad \limsup_{n\to\infty} k_n(\log n)^{-2\beta} = 1.$$

Define z as in (9). Then an elementary calculation shows

(14) $$\begin{aligned} &\tilde{J}(F, n, k_n, x) \\ &= k_n^{1/2}(\log z)^{-\beta}[(\alpha(1 + \alpha^2))^{-1}\sin(\log z) - (1 + \alpha^2)^{-1}\cos(\log z) + o_x(1)] \end{aligned}$$

whereas $\log z = \alpha^{-1}|\log(k_n/n)| + o_x(1) = -2\pi\ell_n + o_x(1)$ from the definitions of z, ℓ_n and k_n. Substituting into (14) we arrive at

$$\tilde{J}(F, n, k_n, x) = -\alpha^{\beta}(1 + \alpha^2)^{-1}k_n^{1/2}(\log n)^{-\beta}[1 + o_x(1)].$$

Because of (13) we have $k_{n'}(\log n')^{-2\beta} \to e^{-2\pi\alpha}$ as $n' \to \infty$ for some subsequence n' of the positive integers. Consequently, $\tilde{J}(F, n', k_{n'}, x) \to -\alpha^\beta(1+\alpha^2)^{-1}e^{-\pi\alpha}$ as $n' \to \infty$ uniformly in x on compact sets, and Corollary 2, applied along the sequence n', yields

$$k_{n'}^{1/2}(H_{k_{n'}}^{(n')} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(\alpha^\beta(1+\alpha^2)^{-1}e^{-\pi\alpha}, \alpha^{-2}) \quad \text{as} \quad n' \to \infty.$$

In the same way, (13) implies the existence of another subsequence n'' with

$$k_{n''}^{1/2}(H_{k_{n''}}^{(n'')} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(\alpha^\beta(1+\alpha^2)^{-1}, \alpha^{-2}) \quad \text{as} \quad n'' \to \infty$$

so that the whole sequence $k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1})$, $n \in \mathbb{N}$, does not converge in distribution. On the other hand, we have $k_n = O((\log n)^{2\beta})$ by (13). This shows that $k_n = o((\log n)^{2\beta})$ is the optimal condition on k_n in the present case.

(b) In contrast to Example 1, within the class

(15) $$1 - F(x) = Cx^{-\alpha}[1 + D(\log x)^{-\beta} + o((\log x)^{-\beta})] \quad \text{as} \quad x \to \infty$$

where $C, \alpha, \beta > 0$ and $D \neq 0$ the sequences $k_n \sim \lambda(\log n)^{2\beta}$ need not be optimal. To demonstrate this, we consider

(16) $$1 - F(x) = Cx^{-\alpha}[1 + D(\log x)^{-\beta}], \quad x \text{ large}, \quad C, \alpha, \beta > 0, \quad D \neq 0.$$

For $L(x) = C[1 + D(\log x)^{-\beta}]$ an elementary calculation shows that for all $y > 0$ as $x \to \infty$

$$L(xy)/L(x) = 1 - \beta D \log y(\log x)^{-\beta-1} + o((\log x)^{-\beta-1})$$

so that SR2 is satisfied with $K = -\beta D$, $\rho = 0$ and $g(x) = (\log x)^{-\beta-1}$, and Corollary 5 applies again. From $\log z \sim -\alpha^{-1}\log(k_n/n)$ for z defined as before, we obtain $k_n^{1/2}g(z) \sim -\alpha^{\beta+1}k_n^{1/2}/(\log(k_n/n))^{\beta+1}$, whence by Corollary 5

(i) if $k_n \to \infty$ and $k_n = o((\log n)^{2\beta+2})$, then

$$k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(0, \alpha^{-2});$$

(ii) if $k_n \sim \lambda(\log n)^{2\beta+2}$, $0 < \lambda < \infty$, then

$$k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(-D\alpha^{\beta-1}\beta\lambda^{1/2}, \alpha^{-2});$$

(iii) if $D \neq 0$, $k_n = o(n)$ and $k_n/(\log n)^{2\beta+2} \to \infty$, then

$$|\log(k_n/n)|^{\beta+1}(H_{k_n}^{(n)} - \alpha^{-1}) \to_p - D\alpha^{\beta-1}\beta.$$

So we see that within the class (16) the sequences $k_n \sim \lambda(\log n)^{2\beta+2}$ are optimal. The difference between the classes (15) and (16) seems to be explained by the fact that the L occurring in (16) satisfies SR2 with $g(x) = (\log x)^{-\beta-1}$ whereas the one occurring in (15) in general does not.

Distribution functions satisfying (1) with slowly varying functions converging to a constant at a logarithmic rate occur in connection with certain transformations of the standard normal distribution (cf. Smith, 1982).

In our final example, we consider a case for which the results of Section 4 do not apply, so that one has to rely upon Theorem 2 itself.

EXAMPLE 4.

(a) $$1 - F(x) = Cx^{-\alpha}[1 + O(e^{-\beta x})] \quad \text{as} \quad x \to \infty; \quad C, \alpha, \beta > 0.$$

Then $F^{-1}(y) = C^{1/\alpha}(1-y)^{-1/\alpha}[1 + O(\exp(-\beta C^{1/\alpha}(1-y)^{-1/\alpha}))]$ as $y \to 1$. Writing $R(x)$ and $S(1-y)$ for the remainder terms in $1 - F(x)$ and $F^{-1}(y)$, respectively, it is easy to see that for any sequence $k_n \to \infty$ such that $k_n = o(n)$ as $n \to \infty$

$$J(F, n, k_n, x) = k_n^{1/2} S(I(n, k_n, x))[1 + o_x(1)] - U(F, n, k_n, x), \tag{17}$$

where

$$U(F, n, k_n, x) = k_n^{1/2} \int_1^\infty y^{-\alpha-1} R(F^{-1}(1 - I(n, k_n, x))y)\, dy.$$

In the present case, one gets

$$O_x(\exp(\tfrac{1}{2}(n/k_n)^{1/\alpha}[(k_n/n)^{1/\alpha} \log k_n - 2\beta C^{1/\alpha} + o_x(1)]))$$

as a bound for $k_n^{1/2} S(I(n, k_n, x))$ and $U(F, n, k_n, x)$. In view of $n/k_n \to \infty$, this bound is $o_x(1)$ if $(k_n/n)^{1/\alpha} \log k_n = o(1)$, being equivalent to $k_n = o(n(\log n)^{-\alpha})$, or if $(k_n/n)^{1/\alpha} \log k_n \to \lambda \in (0, 2\beta C^{1/\alpha})$ being equivalent to $k_n \sim \lambda^\alpha n(\log n)^{-\alpha}$. Consequently, in these cases we have $k_n^{1/2}(H_{k_n}^{(n)} - \alpha^{-1}) \to_{\mathscr{D}} \mathscr{N}(0, \alpha^{-2})$.

(b) The case

$$1 - F(x) = Cx^{-\alpha}[1 + De^{-\beta x} + o(e^{-\beta x})] \quad \text{as} \quad x \to \infty; \quad C, \alpha, \beta > 0, \quad D \neq 0$$

can also be treated via (17) by a detailed examination of the asymptotic behaviour of $S(I(n, k_n, x))$ and $U(F, n, k_n, x)$. It turns out that for $k_n/n(\log n)^{-\alpha} \to \infty$ or $k_n \sim \lambda n(\log n)^{-\alpha}$ with $\lambda > (2\beta)^\alpha C$ the sequence $a_n(H_{k_n}^{(n)} - \alpha^{-1})$ does not converge in distribution for any norming sequence a_n. The remaining case $k_n \sim (2\beta)^\alpha Cn(\log n)^{-\alpha}$ is more complicated. It splits up into several subcases which can be further investigated.

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REFERENCES

APOSTOL, T. M. (1974). *Mathematical Analysis*. 2nd ed. Addison-Wesley, Reading, Massachusetts.

CSÖRGŐ, S. and MASON, D. M. (1985). Central limit theorems for sums of extreme values. To appear in *Math. Proc. Cambridge Philos. Soc.*

DAVIS, R. and RESNICK, S. I. (1984). Tail estimates motivated by extreme value theory. *Ann. Statist.* **12** 1467–1468.

DE HAAN, L. and RESNICK, S. I. (1980). A simple asymptotic estimate for the index of a stable distribution. *J. Roy. Statist. Soc. Ser. B* **42** 83–87.

GOLDIE, C. M. and SMITH, R. L. (1984). Slow variation with remainder: A survey of the theory and its applications. Preprint.

HALL, P. (1978). Representations and limit theorems for extreme value distributions. *J. Appl. Probab.* **15** 639–644.

HALL, P. (1982). On some simple estimates of an exponent of regular variation. *J. Roy. Statist. Soc. Ser. B* **44** 37–42.

HALL, P. and WELSH, A. H. (1984). Best attainable rates of convergence for estimates of parameters of regular variation. *Ann. Statist.* **12** 1079–1084.

HILL, B. M. (1975). A simple general approach to inference about the tail of a distribuition. *Ann. Statist.* **3** 1163–1174.

MASON, D. M. (1982). Laws of large numbers for sums of extreme values. *Ann. Probab.* **10** 754–764.

SENETA, E. (1976). Regularly varying functions. *Lecture Notes in Math.* **508** Springer-Verlag, Berlin.

SMID, B. and STAM, A. J. (1975). Convergence in distribution of quotients of order statistics. *Stochastic Process. Appl.* **3** 287–292.

SMITH, R. L. (1982). Uniform rates of convergence in extreme-value theory. *Adv. Appl. Probab.* **14** 600–622.

TEUGELS, J. L. (1981a). Limit theorems on order statistics. *Ann. Probab.* **9** 868–880.

TEUGELS, J. L. (1981b). Remarks on large claims. *Bull. Inst. Internat. Statist.* **49** 1490–1500.

WEISSMAN, I. (1978). Estimation of parameters and large quantiles based on the k largest observations. *J. Amer. Statist. Assoc.* **73** 812–815.

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