

ON ASYMPTOTIC PROPERTY FOR NONLINEAR DIFFERENCE SYSTEMS

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ABSTRACT. We study asymptotic equivalence between nonlinear difference system

$$\Delta x(n) = f(n, x(n))$$

and its variational system

$$\Delta v(n) = f_x(n, 0)v(n).$$

1. Introduction

We consider the nonlinear difference system

$$(1.1) \quad \Delta x(n) = f(n, x(n)), \quad x(n_0) = x_0,$$

where Δ is the forward difference operator, i.e., $\Delta x(n) = x(n+1) - x(n)$, $f : \mathbb{N}(n_0) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots, n_0 + k, \dots\}$, n_0 is a nonnegative integer, and \mathbb{R}^m is the m -dimensional real Euclidean space.

System (1.1) has *asymptotic equilibrium* if there exists a single $\xi \in \mathbb{R}^m$ such that any solution $x(n) = x(n, n_0, x_0)$ of (1.1) for all $|x_0|$ sufficiently small satisfies

$$(1.2) \quad x(n) = \xi + o(1) \text{ as } n \rightarrow \infty,$$

and, for every $\xi \in \mathbb{R}^m$, there exists a solution of (1.1) such that satisfies (1.2).

Also, for the other nonlinear difference system

$$(1.3) \quad \Delta y(n) = g(n, y(n)),$$

where $g : \mathbb{N}(n_0) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, two systems (1.1) and (1.3) are said to be *asymptotically equivalent* if, for every solution $x(n)$ of (1.1), there exists

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a solution $y(n)$ of (1.3) such that

$$(1.4) \quad x(n) = y(n) + o(1) \text{ as } n \rightarrow \infty,$$

and conversely, for every solution $y(n)$ of (1.3), there exists a solution $x(n)$ of (1.1) such that (1.4) holds.

To study asymptotic relationship between two linear systems, the problem of asymptotic equivalence have been initiated by H. Poincaré (1885) and O. Perron (1921). In [8] Medina and Pinto studied asymptotic equivalence using dichotomy conditions. Also, Medina [7] investigated this problem using some discrete inequalities combined with the Schauder's fixed point theorem.

We established asymptotic equivalence between two nonlinear difference systems via their variational systems in [2]. Also, in [4], we characterized asymptotic equilibrium for nonlinear system by means of n_∞ -similarity, which is defined in [3] as a different discrete analogue of t_∞ -similarity by Conti.

In this paper we study asymptotic equivalence between (1.1) and its variational system

$$(1.5) \quad \Delta v(n) = f_x(n, 0)v(n),$$

where $f_x = \frac{\partial f}{\partial x}$, and give an appropriate example.

2. Main results

We are concerned with the nonlinear difference system

$$(2.1) \quad \Delta x(n) = f(n, x(n)), \quad x(n_0) = x_0.$$

We assume that

- (i) $f_x = \frac{\partial f}{\partial x}$ exists,
- (ii) $I + f_x$ is continuous and invertible on $\mathbb{N}(n_0) \times \mathbb{R}^m$, where I is the $m \times m$ identity real matrix,
- (iii) $f(n, 0) = 0$.

Let $x(n) = x(n, n_0, x_0)$ be the solution of (2.1) with $x(n_0, n_0, x_0) = x_0$.

To study asymptotic behavior for (2.1) it is useful to consider its variational system

$$(2.2) \quad \Delta v(n) = f_x(n, 0)v(n) = A(n)v(n)$$

and rewrite (2.1), by setting $f_x(n, 0) = A(n)$ and using the mean value theorem, as

$$\begin{aligned} \Delta x(n) &= A(n)x(n) + f(n, x(n)) - f_x(n, 0)x(n) \\ (2.3) \quad &= A(n)x(n) + G(n, x(n)), \quad x(n_0) = x_0, \end{aligned}$$

where

$$G(n, x) = \int_0^1 [f_x(n, \theta x) - f_x(n, 0)] d\theta x.$$

Then, by the variational of constants formula [1], the solution of (2.3) is given by

$$(2.4) \quad x(n) = \Phi(n, n_0)x_0 + \sum_{s=n_0}^{n-1} \Phi(n, s+1)G(s, x(s)),$$

where $\Phi(n, n_0)$ is the fundamental matrix solution of the linear difference system $\Delta x(n) = A(n)x(n)$.

For system (2.3), we assume that

$$(2.5) \quad |G(n, x)| \leq \omega(n, |x|), \quad n \geq n_0, \quad x \in \mathbb{R}^m,$$

where $\omega : \mathbb{N}(n_0) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous and nondecreasing in $u \in \mathbb{R}_+$ for $n \geq n_0$.

To use the comparison principle we consider the scalar difference equation

$$(2.6) \quad \Delta u(n) = M^2 \omega(n, u(n)), \quad u(n_0) = u_0 > 0,$$

where M is a positive constant, by the function $\omega(n, u)$.

We need the following comparison principle which is a slight modification of [6] :

LEMMA 2.1. [3] *Let $p(n, r)$ be a nondecreasing function in r for any fixed $n \in \mathbb{N}(n_0)$. Suppose that for $n \geq n_0$, nonnegative functions $u(n)$ and $v(n)$ defined on $\mathbb{N}(n_0)$ satisfies the following inequalities :*

$$v(n) - \sum_{l=n_0}^{n-1} p(l, v(l)) < u(n) - \sum_{l=n_0}^{n-1} p(l, u(l)).$$

If $v(n_0) < u(n_0)$, then $v(n) < u(n)$ for all $n \geq n_0$.

Now, we present the result.

THEOREM 2.2. *Assume that (2.5) holds and all solutions of (2.6) are bounded on $\mathbb{N}(n_0)$. If the variational system (2.2) has asymptotic equilibrium, then the original system (2.1) has also asymptotic equilibrium provided $d = M|x_0| < u_0$.*

PROOF. Let $x(n, n_0, x_0)$ be any solution of (2.1). From the variation of constants formula in [1] and conditions (i) and (ii), we obtain

$$\begin{aligned}
 |x(n)| &= |\Phi(n, n_0)[x_0 + \sum_{s=n_0}^{n-1} \Phi^{-1}(s+1, n_0)G(s, x(s))]| \\
 &\leq |\Phi(n, n_0)||x_0| + |\Phi(n, n_0)| \sum_{s=n_0}^{n-1} |\Phi^{-1}(s+1, n_0)||G(s, x(s))| \\
 &\leq M|x_0| + M^2 \sum_{s=n_0}^{n-1} \omega(s, |x(s)|) \\
 &= d + M^2 \sum_{s=n_0}^{n-1} \omega(s, |x(s)|),
 \end{aligned}$$

where $\Phi(n, n_0) \leq M$ for $n, m \geq n_0$ and $d = M|x_0|$. It follows that

$$\begin{aligned}
 |x(n)| - M^2 \sum_{s=n_0}^{n-1} \omega(s, |x(s)|) &= d \\
 &< u_0 = u(n) - M^2 \sum_{s=n_0}^{n-1} \omega(s, u(s)).
 \end{aligned}$$

Applying Lemma 2.1 for $p(n, u) = M^2\omega(n, u)$, we conclude that

$$|x(n)| \leq u(n) \quad \text{for each } n \geq n_0,$$

provided $d < u_0$.

We claim that the solution $x(n)$ of (2.1) which can be written as (2.3) is convergent. If we consider the sequence

$$v(n) = \sum_{s=n_0}^{n-1} \Phi^{-1}(s+1, n_0)G(s, x(s)),$$

then we have

$$\begin{aligned}
 |v(n) - v(m)| &\leq \sum_{s=m}^{n-1} |\Phi^{-1}(s+1, n_0)||G(s, x(s))| \\
 &\leq M \sum_{s=m}^{n-1} \omega(s, |x(s)|)
 \end{aligned}$$

$$\begin{aligned}
&\leq M \sum_{s=m}^{n-1} \omega(s, u(s)) \\
&= M(u(n) - u(m)), \quad n \geq m \geq n_0,
\end{aligned}$$

by the monotonicity of $\omega(n, u)$. It follows that $v(n)$ is Cauchy since $u(n)$ is convergent. Thus there exists a vector $\xi \in \mathbb{R}^m$ such that any solution $x(n, n_0, x_0)$ of (2.3) for all $|x_0|$ sufficiently small satisfies the following asymptotic relationship :

$$x(n) = \xi + o(1) \quad \text{as } n \rightarrow \infty.$$

For the converse asymptotic relationship, we let $\xi \in \mathbb{R}^m$ be any vector and set

$$x_0 = \Phi_{\infty}^{-1} \xi - v_{\infty},$$

where $\Phi_{\infty} = \lim_{n \rightarrow \infty} \Phi(n)$ and $v_{\infty} = \lim_{n \rightarrow \infty} v(n)$. Then any solution $x(n)$ of (2.3) which is equivalent to (2.1) satisfies

$$\begin{aligned}
x(n, n_0, x_0) &= \Phi(n, n_0) \left[x_0 + \sum_{s=n_0}^{\infty} \Phi^{-1}(s+1, n_0) G(s, x(s)) \right] \\
&- \Phi(n, n_0) \sum_{s=n}^{\infty} \Phi^{-1}(s+1, n_0) G(s, x(s)) \\
&= \xi + o(1) \quad \text{as } n \rightarrow \infty,
\end{aligned}$$

since

$$\Phi(n, n_0) \sum_{s=n}^{\infty} \Phi^{-1}(s+1, n_0) G(s, x(s)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This completes the proof. \square

As an illustration of Theorem 2.2 we give the following example.

EXAMPLE 2.3. We consider the nonlinear difference equation

$$(2.7) \quad \Delta x(n) = f(n, x(n)) = \frac{a^n x(n)}{\sqrt{1 + 2x^2(n)}}$$

and its associated variational difference equation

$$(2.8) \quad \Delta v(n) = f_x(n, 0)v(n) = a^n v(n),$$

where a is a constant with $0 < |a| < 1$. Thus

$$f(n, x) = \frac{a^n x}{\sqrt{1 + 2x^2}} \quad \text{and} \quad f_x(n, x) = \frac{a^n}{(1 + 2x^2)^{\frac{3}{2}}}.$$

First, we can rewrite (2.7) as (2.3) :

$$(2.9) \quad \Delta x(n) = a^n x(n) + a^n \left[\frac{1}{\sqrt{1+2x^2(n)}} - 1 \right] x(n).$$

Since

$$G(n, x) = \int_0^1 [f_x(n, \theta x) - f_x(n, 0)] d\theta x,$$

we have

$$\begin{aligned} |G(n, x)| &\leq |a^n [\int_0^1 \frac{d\theta}{\sqrt{1+2(\theta x)^2}} - 1] d\theta x| \\ &= |a^n [\frac{1}{\sqrt{1+2x^2}} - 1] x| \\ &\leq a^n |x|. \end{aligned}$$

So $\omega(n, |x|) = a^n u$ and it is continuous nondecreasing in $u > 0$.

Now, (2.6) becomes

$$(2.10) \quad \Delta u(n) = M^2 a^n u(n), \quad u(0) = u_0 > 0.$$

Since any solution $u(n)$ of (2.10) is given by

$$\begin{aligned} u(n) &= u_0 + M^2 \sum_{s=0}^{n-1} a^s \\ &= u_0 + M^2 \left(\frac{1-a^n}{1-a} \right), \quad n \geq n_0 = 0, \end{aligned}$$

$u(n)$ is bounded on $\mathbb{N}(n_0)$. Thus, by Theorem 2.2, (2.7) has asymptotic equilibrium by setting

$$d = \lim_{n \rightarrow \infty} \prod_{s=0}^{n-1} (1+a^s) |x_0|.$$

THEOREM 2.4. *Two systems (2.1) and (2.2) are asymptotically equivalent under the conditions in Theorem 2.2.*

PROOF. Let $v(n)$ be any solution of (2.2). Then we have $\lim_{n \rightarrow \infty} v(n) = v_\infty$ by asymptotic equilibrium of (2.2). Setting

$$x_0 = \Phi_\infty^{-1} v_\infty - p_\infty,$$

where $\lim_{n \rightarrow \infty} \Phi(n) = \Phi_\infty$ and $p_\infty = \sum_{s=n_0}^{\infty} \Phi^{-1}(s+1, n_0)G(s, x(s))$, then there exists a solution $x(n, n_0, x_0)$ of (2.1) such that

$$\begin{aligned} \lim_{n \rightarrow \infty} [x(n) - v(n)] &= \lim_{n \rightarrow \infty} [\Phi(n)(\Phi_\infty^{-1}v_\infty - p_\infty \\ &\quad + \sum_{s=n_0}^{n-1} \Phi^{-1}(s+1, n_0)G(s, x(s))) - \Phi(n, n_0)v_0] \\ &= \Phi_\infty(\Phi_\infty^{-1}v_\infty) - v_\infty \\ &= 0. \end{aligned}$$

Conversely, we easily see that the asymptotic relationship also holds by setting $v_0 = x_0 + p_\infty$. This completes the proof. \square

EXAMPLE 2.5. Two difference equations (2.7) and (2.8) in Example 2.3 are asymptotically equivalent by Theorem 2.4.

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